Proposizione: Sia $S(\underline{P}, \underline{q}, t)$ una funz. t.c. $\det\left(\frac{\partial^2 S}{\partial P_i \partial q_j}\right)_{i,j=1,\dots,n} \neq 0$, allora

la trasf. $Q_i = \frac{\partial S}{\partial P_i}(\underline{P}, \underline{q}, t)$ e $p_i = \frac{\partial S}{\partial q_i}(\underline{P}, \underline{q}, t)$

def. un cambio di coord. $(\underline{p}, \underline{q}) = \mathcal{C}(\underline{P}, \underline{Q}, t)$

che è canonico e la nuova Ham.

$$K(\underline{P}, \underline{Q}, t) = H(\underline{p}, \underline{q}, t)\Big|_{(\underline{p},\underline{q})=\mathcal{C}(\underline{P},\underline{Q},t)} + \frac{\partial S}{\partial t}\Big|_{\underline{q}=\underline{q}(\underline{P},\underline{Q},t)}$$

Proposizione: Sia $S(\underline{\alpha}, \underline{q}, t)$ t.c.

$$\det\left(\frac{\partial^2 S}{\partial \alpha_i \, \partial q_j}\right)_{ij} \neq 0$$

soluzione della eq. di Hamilton-Jacobi, cioè

$$H\left(\frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, q_1, \dots, q_n, t\right) + \frac{\partial S}{\partial t}\Bigg|_{(\underline{p}, \underline{q}) = \varphi(\underline{\alpha}, \underline{\beta}, t)} = 0 ,$$

allora la sol. delle eq. di Hamilton per $H = H(\underline{p}, \underline{q}, t)$ sono espresse in forma implicita

$$p_j = \frac{\partial S}{\partial q_j}(\underline{\alpha}, \underline{q}, t) \quad , \quad \beta_j = \frac{\partial S}{\partial \alpha_j}(\underline{\alpha}, \underline{q}, t) \quad \forall j = 1, \dots, n$$

dove $\underline{\alpha}, \underline{\beta}$ sono costanti.

Dim. Si osservi che le Ham. nelle nuove coord. $(\underline{\alpha}, \underline{\beta})$ è $K(\underline{\alpha}, \underline{\beta}, t) = 0 \Rightarrow \dot{\alpha}_j = \frac{\partial K}{\partial \beta_j} = \dot{\beta}_j = \frac{\partial K}{\partial \alpha_j} = 0$

$\Rightarrow \alpha(t) = \alpha_0$, $\beta(t) = \beta_0 \Rightarrow p_j = \frac{\partial S}{\partial q_j}(\underline{\alpha}, \underline{q}, t)$,

$\beta_j = \frac{\partial S}{\partial \alpha_j}(\underline{\alpha}, \underline{q}, t) \Rightarrow \underline{q} = \underline{q}(\underline{\alpha}, \underline{\beta}, t)$, $\underline{p} = \underline{p}(\underline{\alpha}, \underline{\beta}, t)$ sol. delle eq. di Ham. originali

C.V.D.

Esempio: il moto libero in 3D.

$H(p_x, p_y, p_z, x, y, z) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$. Dobbiamo determinare S t.c.

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{\partial S}{\partial t} = 0$$

Oss. che se scriviamo $\frac{\partial}{\partial x}$ dell'eq. prec. abbiamo

$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x}\right)^2 = 0$$

se imponiamo $S = X(x) + Y(y) + Z(z) + T(t)$

analogamente abbiamo $\frac{\partial}{\partial y}\left(\frac{\partial S}{\partial y}\right)^2 = \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right)^2 = \frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}\right)^2 = 0$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} = \frac{\partial^2 Y}{\partial y^2} = \frac{\partial^2 Z}{\partial z^2} = \frac{\partial^2 T}{\partial t^2} = 0 \Rightarrow X, Y, Z, T \text{ sono lineari}$$

in x, y, z e t. Ne segue che $\underbrace{X(x) = \alpha_x \cdot x}$, $\underbrace{Y(y) = \alpha_y \cdot y}$, $\underbrace{Z(z) = \alpha_z \cdot z}$

$\Rightarrow$ l'eq. di H.-J. $\Rightarrow$ $$\frac{1}{2m}\left[\underbrace{\left(\frac{dX}{dx}\right)^2} + \underbrace{\left(\frac{dY}{dy}\right)^2} + \underbrace{\left(\frac{dZ}{dz}\right)^2}\right] + \frac{dT}{dt} = 0$$

$$\Rightarrow \frac{1}{2m}\left(\alpha_x^2 + \alpha_y^2 + \alpha_z^2\right) + \frac{dT}{dt} = 0 \Rightarrow T(t) = -\frac{1}{2m}\left(\alpha_x^2 + \alpha_y^2 + \alpha_z^2\right)t$$

$$\Rightarrow \beta_x = \frac{\partial S}{\partial \alpha_x} = x - \frac{\alpha_x \cdot t}{m} \Rightarrow \boxed{x = \beta_x + \frac{\alpha_x \cdot t}{m}} = \beta_x + \frac{p_x}{m} t$$

$$\Rightarrow \beta_y = \frac{\partial S}{\partial \alpha_y} = y - \frac{\alpha_y \cdot t}{m} \Rightarrow \boxed{y = \beta_y + \frac{\alpha_y \cdot t}{m}} = \beta_y + \frac{p_y}{m} \cdot t$$

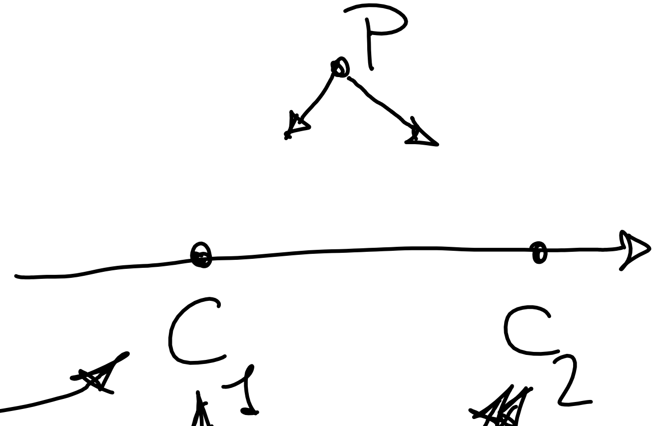
$$\Rightarrow \beta_z = \frac{\partial S}{\partial \alpha_z} = z - \frac{\alpha_z \cdot t}{m} \Rightarrow \boxed{z = \beta_z + \frac{\alpha_z \cdot t}{m}} = \beta_z + \frac{p_z}{m} t$$

$$\Rightarrow \boxed{p_x = \frac{\partial S}{\partial x} = \alpha_x \, , \quad p_y = \frac{\partial S}{\partial y} = \alpha_y \, , \quad p_z = \frac{\partial S}{\partial z} = \alpha_z}$$

Cioè abbiamo trovato che $x(t) = \underbrace{\beta_x}_{x(0) - v_x t_0} + v_x \cdot t = x(0) + v_x (t - t_0)$

e $y(t) = \beta_y + v_y \cdot t = y(0) + v_y (t - t_0)$, $z(t) = z(0) + v_z (t - t_0)$.

Si risolve l'eq. di Ham.-Jacobi per il problema dei due centri fissi



hanno massa (oppure carica).

Per il resto, soluzioni interessanti dell'eq. di Ham.-Jacobi derivano dal th. di Liouville.

Def.: Si dice che $\Phi_1(p,q), \dots, \Phi_n(p,q)$ formano un sistema <u>completo</u> in involuzione, quando

$$\det\left(\frac{\partial \Phi_i}{\partial p_j}\right)_{i,j=1,\dots,n} \neq 0 \quad \text{e} \quad \{\Phi_i, \Phi_j\} = 0 \quad \forall\, i,j = 1,\dots,n$$

questa ultima relazione è equivalente a richiedere che $\operatorname{rank}\left(\frac{\partial \Phi_i}{\partial z_j}\right)_{\substack{i=1,\dots,n \\ j=1,\dots,2n}} = n$ con $\underline{z} = (p, q)$

Teorema: Sia H t.c. ammette n integrali primi $\Phi_1(p,q), \dots, \Phi_n(p,q)$ che formano un sistema completo in involuzione, allora le eq. di Ham. $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$ $\forall j = 1, \dots, n$ sono integrabili "per quadrature" -

Lemma: Se $\{\Phi_j(p,q)\}_{j=1,\dots,n}$ è un sistema completo in involuzione, allora $\exists$ una trasf. canonica che possiamo esprimere tramite $S=S(\Phi,q)$, cioè $\alpha_j = \frac{\partial S}{\partial \Phi_j}$, $p_j = \frac{\partial S}{\partial q_j}$ $\forall j=1,\dots,n$.

Tale generatrice è $S(\Phi,q) = \sum_{j=1}^{n} \int p_j(\Phi,q)\, dq_j$

— o —

Dim. Osserviamo che siccome $\det\left(\frac{\partial \Phi_i}{\partial p_j}\right)_{i,j=1,\dots,n} \neq 0$

$\Rightarrow p = p(\Phi,q)$. Dobbiamo dimostrare che

$\sum_{j=1}^{n} p_j(\Phi,q)\, dq_j$ è esatta

È conveniente considerare

$$d\Phi_j = d\Phi_j\big(P(\Phi,q),q\big) = \sum_{l,i=1}^{n}\left(\frac{\partial \Phi_j}{\partial P_i}\frac{\partial P_i}{\partial \Phi_l}\,d\Phi_l + \frac{\partial \Phi_j}{\partial p_i}\frac{\partial p_i}{\partial q_l}\,dq_l\right) + \sum_{l=1}^{n}\frac{\partial \Phi_j}{\partial q_l}\,dq_l$$

$$\Rightarrow \sum_{i=1}^{n}\frac{\partial \Phi_j}{\partial P_i}\frac{\partial P_i}{\partial \Phi_l} = \delta_{j,l}$$

(ovvia perché è associata al cambio da $\Phi = \Phi(P,q)$ a $P = P(\Phi,q)$ e viceversa)

$$\Rightarrow \boxed{\frac{\partial \Phi_j}{\partial q_l} = -\sum_{i=1}^{n}\frac{\partial \Phi_j}{\partial p_i}\frac{\partial p_i}{\partial q_l}}$$

Consideriamo $\forall\, i,j = 1,\dots,n$

$$0 = \{\Phi_i,\Phi_j\} = \sum_{m=1}^{n}\frac{\partial \Phi_i}{\partial q_m}\frac{\partial \Phi_j}{\partial p_m} - \frac{\partial \Phi_i}{\partial p_m}\frac{\partial \Phi_j}{\partial q_m} = \sum_{m,l=1}^{n}\left(\frac{\partial \Phi_j}{\partial p_m}\frac{\partial \Phi_i}{\partial p_l}\frac{\partial p_l}{\partial q_m} - \frac{\partial \Phi_i}{\partial p_m}\frac{\partial \Phi_j}{\partial p_l}\frac{\partial p_l}{\partial q_m}\right)$$

$$= \sum_{m=1}^{n} \frac{\partial \Phi_j}{\partial p_m} \left(\sum_{l=1}^{n} \frac{\partial \Phi_i}{\partial p_l} \frac{\partial p_l}{\partial q_m} - \frac{\partial \Phi_i}{\partial p_l} \cdot \frac{\partial p_m}{\partial q_l} \right)$$

abbiamo dovuto scambiare l e m

$$= \sum_{m=1}^{n} \frac{\partial \Phi_j}{\partial p_m} \sum_{l=1}^{n} \frac{\partial \Phi_i}{\partial p_l} \left(\frac{\partial p_l}{\partial q_m} - \frac{\partial p_m}{\partial q_l} \right) =$$

osservo che posso scriverlo in forma $U \cdot A$ dove

$$U = \left(\frac{\partial \Phi_j}{\partial p_m} \right)_{\substack{j=1,\dots,n \\ m=1,\dots,n}} \quad e \quad A = \left[\left(\sum_{l=1}^{n} \frac{\partial \Phi_i}{\partial p_l} \left(\frac{\partial p_l}{\partial q_m} - \frac{\partial p_m}{\partial q_l} \right) \right)_{i,m=1,\dots,n} \right]^T,$$

ma $\det U \neq 0 \Rightarrow A = 0$ (A matr. $n \times n$) _

$A = U \cdot B^T$ dove $B = \left(\frac{\partial p_l}{\partial q_m} - \frac{\partial p_m}{\partial q_l} \right)_{l,m=1,\dots,n}$, ma siccome $\det U \neq 0$,

$\Rightarrow B = 0 \Rightarrow \dfrac{\partial p_l}{\partial q_m} = \dfrac{\partial p_m}{\partial q_l}$ dove $p = p(\Phi, q)$

cioè $d\left(\sum_{j=1}^{n} p_j(\Phi, q)\, dq_j\right) = 0$ cioè $\sum_{j=1}^{n} p_j\, dq_j$ è chiusa $\Rightarrow$ è esatta $\exists\, S(\Phi, q)$ t.c. $dS = \sum_{j=1}^{n} p_j\, dq_j$

cioè $\exists\, S(\Phi, q) = \sum_{j=1}^{n} \int p_j(\Phi, q)\, dq_j$ t.c. la trasf.

$\alpha_i = \dfrac{\partial S}{\partial \Phi_i}$, $p_j = \dfrac{\partial S}{\partial q_j} = p_j(\Phi, q)$

$\Downarrow$ da cui si ricava $q = q(\Phi, \alpha)$ poiché $\left(\dfrac{\partial^2 S}{\partial \Phi_i \partial q_j}\right)_{i,j} = \left(\dfrac{\partial p_i}{\partial \Phi_j}\right)_{i,j=1,\dots,n}$ è invertibile

C.V.D. (del lemma).

Dim. (del teorema di Liouville); per il lemma, sappiamo che possiamo esprimere $(\underline{p}, \underline{q}) = \mathcal{C}(\underline{\Phi}, \underline{\alpha})$ dove $\underline{\Phi} = \underline{\Phi}(\underline{p}, \underline{q})$ è il sist. completo in involuzione e $\mathcal{C}$ è canonica.

Calcoliamo $\dfrac{\partial H(\underline{\Phi}, \underline{\alpha})}{\partial \alpha_i} = -\{H, \Phi_i\}_{\underline{\Phi}, \underline{\alpha}} = -\{H, \Phi_i\}_{\underline{p}, \underline{q}} = 0$

$\Rightarrow H = H(\Phi_1, \dots, \Phi_n)$ e $\dot{\Phi}_j = -\dfrac{\partial H}{\partial \alpha_j} = 0 \Rightarrow \Phi_j(t) = \Phi_j(0)$

$\forall j = 1, \dots, n$; $\dot{\alpha}_j = \dfrac{\partial H}{\partial \Phi_j}(\Phi_1, \dots, \Phi_n) = \dfrac{\partial H}{\partial \Phi_j}(\Phi_1(0), \dots, \Phi_n(0)) = c_j$ (costante)

$\Rightarrow \alpha_j(t) = \alpha_j(0) + c_j(t - t_0)$

$\Rightarrow$ le eq. di Ham. nelle vecchie coordinate sono risolte

da $\left(\underline{p}(t), \underline{q}(t)\right) = \mathcal{C}\left(I_1(0), \dots, I_n(0), \alpha_1(0)+c_1(t-t_0), \dots, \alpha_n(0)+c_n(t-t_0)\right)$

C.V.D.