then H_{0_ε} and Φ_0 are not independent.
By using again the non-degeneracy, this allows to conclude that also H and Φ are not independent.

— ∘ —

Comments. The genericness condition is very hard to check and it can be violated (at order ε) by many Fourier harmonics $\underline{k}$.
However, the perturbative procedure allowing to compute the first integrals is such that the Fourier harmonics are ~~generic~~ regenerated at every perturbative order in a generic way, unless there are some simmetries.
Thus, if a Fourier harmonic is missing at order ε in the expansion of H_1, it will appear in the r.h.s. of the equations for determining Φ_2 at order ε^2 or Φ_3 at order ε^3 and so on.

— ∘ —

- Main ideas behind the KAM theorem and a first statement of it

The (sketch of the) proof of the Poincaré theorem highlights that the main obstruction to the existence

of the first integrals is due to the fact that we look for their existence on open sets (that ~~will~~ are undoubtly filled by the resonant manifolds).
Thus, in order to find solutions of the Hamilton equations it is convenient to focus on non-resonant manifold such that

$$\underline{k} \cdot \underline{\omega}(p) \neq 0 \, .$$

In particular ~~the~~ most comfortable choice is to work with fixed ~~resonance~~ frequencies that are farthest away from the resonances. Therefore, we look for quasiperiodic solutions related to a frequency vector $\underline{\omega}$ that is diophantine i.e., there are $\gamma > 0$ and $\tau \ (\geq n-1)$ such that

$$|\underline{k} \cdot \underline{\omega}| \geq \frac{\gamma}{|\underline{k}|^{\tau}} \quad \forall \ \underline{k} \in \mathbb{Z}^n \setminus \{0\} \, .$$

Putting all together the previous remarks it is natural to imagine that a major role can be played by the Kolmogorov normal form (introduced at pag. 4) for quasiperiodic invariant tori of Diophantine type.
This idea is fully supported by the KAM theorem,

a general statement of which is reported in the following.

<u>Theorem (KAM)</u>: Let us consider the general problem of the dynamics, described by a Hamiltonian

$$H(\underline{p},\underline{q};\varepsilon) = \underline{\omega}\cdot\underline{p} + h(\underline{p}) + \varepsilon f(\underline{p},\underline{q};\varepsilon),$$

where the "integrable part" (corresponding to $\varepsilon = 0$) is already in Kolmogorov normal form, because $h(\underline{p}) = \mathcal{O}(\|\underline{p}\|^2)$. Let us assume the following hypotheses:

(1) H is analytic in all its arguments in a set $B_\varrho(\underline{0})\times\mathbb{T}^n\times B_r(0)$, being $B_\varrho(\underline{0})\subset\mathbb{R}^n$ and $B_r(0)\subset\mathbb{C}$ two open balls centered about the origin;

(2) $\underline{\omega}$ is diophantine, i.e. there are two constants $\gamma>0$ and $\tau\ge n-1$ such that $|\underline{k}\cdot\underline{\omega}|\ge\gamma/|\underline{k}|^\tau$ $\forall\,\underline{k}\in\mathbb{Z}^n,\ \underline{k}\neq\underline{0}$;

(3) h is non degenerate, i.e., $\det\left(\frac{\partial^2 h(\underline{p})}{\partial p_i\,\partial p_j}\right)_{i,j=1,\ldots,n}\neq 0$ $\forall\,\underline{p}\in B_\varrho(\underline{0})$;

(4) ε is small enough, i.e. $|\varepsilon|<\varepsilon^*$ (whose definition is complicate).

Therefore, there exists an invariant torus ~~that~~ is close to $\{(\underline{p},\underline{q}) : \underline{p}=\underline{0},\ \underline{q}\in\mathbb{T}^n\}$. The <u>flow on that torus</u> is quasiperiodic with frequencies $\underline{\omega}$.

In other words, the thesis means that there exists a canonical transformation $\underline{\mathcal{C}}_{\varepsilon}(\underline{P},\underline{Q})=(\underline{p},\underline{q})$ (for any fixed value of $|\varepsilon|<\varepsilon^*$) such that the Hamiltonian is conjugated to a new one that is in Kolmogorov normal form, i.e.,

$$K(\underline{P},\underline{Q})=H(\underline{\mathcal{C}}_{\varepsilon}(\underline{P},\underline{Q}))=\underline{\omega}\cdot\underline{P}+\mathcal{R}(\underline{P},\underline{Q};\varepsilon)$$

with $\mathcal{R}=\mathcal{O}(\|\underline{P}\|^2)$. Such canonical transformation is close to the identity in the sense that

$$\lim_{\varepsilon\to 0}\|\underline{\mathcal{C}}_{\varepsilon}(\underline{P},\underline{Q})-(\underline{P},\underline{Q})\|=0\,.$$

<u>Other comments</u> Of course, we can imagine as many normal form as we want, but this does not mean that all the corresponding solutions exist for rather generic Hamiltonian.
In order to claim that a generic Hamiltonian can be conjugated to a particular normal form, a deep understanding of the dynamics is needed.
The approach of this exposition, starting from the sketch of the proof of the Poincaré theorem on the non-existence of the first integrals, should help in finding motivations explaining why KAM

theorem should work. Other motivations are of historical character: the Siegel center problem (solved for diophantine frequencies about 15 years before the birth of KAM), the fact that at the epoch only quasiperiodic motions were observed in Celestial Mechanics planetary problems, etc. Also Poincaré raised the question about the convergence of the Lindstedt series (allowing to find solutions by a perturbative approach) when the frequencies are non-resonant but his conclusion was that such a convergence looked "fort invraisemblable" to him. A way to convince that it is worth to investigate the existence of quasiperiodic motion with diophantine frequencies is related to the following remark: <u>almost all the frequency vectors in $\mathbb{R}^n$ are diophantine</u>. This will be proved in the following.
Therefore, <u>quasiperiodic motions are generically diophantine</u>, when they exist.

• Basics of diophantine theory: nearly all vectors in $\mathbb{R}^n$ are diophantine

It is convenient to introduce the following definition for the Diophantine vectors in $\mathbb{R}^n$:

$$\Omega_{\gamma,\tau} = \left\{ \underline{\omega} \in \mathbb{R}^n : |\underline{k}\cdot\underline{\omega}| \geq \frac{\gamma}{|k|^\tau} \ \forall\, \underline{k} \in \mathbb{Z}^n \setminus \{0\} \right\}.$$

We are going to prove the following

Theorem : The Lebesgue measure of $\mathbb{R}^n \setminus \bigcup_{\gamma>0} \Omega_{\gamma,\tau}$ is zero $\forall\, \tau > n-1$.

— • —

Such a theorem is an immediate corollary of the following ~~lemma~~

Lemma: Let D an open and bounded subset of $\mathbb{R}^n$, then for any $\tau > n-1$ the ~~total~~ corresponding set of Diophantine vectors (for any positive and small enough γ) has non-zero Lebesgue measure, because

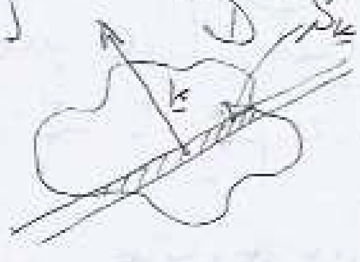
$$\mathrm{meas}\left(D \setminus \Omega_{\gamma,\tau} \right) \lesssim O(\gamma) \quad \text{when } \gamma \to 0^+.$$

— o —

Proof: let us start by considering a more generic non-resonance relation, i.e., $|\underline{k}\cdot\underline{\omega}| \geq \Psi(|k|)\ \forall\, \underline{k} \in \mathbb{Z}^n \setminus \{0\}$

with some suitable positive function Ψ.

Consider $S_{\underline{k}} := \{\underline{\omega}\in D : |\underline{k}\cdot\underline{\omega}| < \Psi(|\underline{k}|)\}$ for any fixed $\underline{k}\in\mathbb{Z}^n\setminus\{0\}$. We can easily estimate the volume of $S_{\underline{k}}$ as follows:



$$\mathrm{meas}\left(S_{\underline{k}}\right) \le 2\left(\mathrm{diam}\, D\right)^{n-1} \frac{\Psi(|\underline{k}|)}{\|\underline{k}\|},$$

where we used the fact that the maximal width of $S_{\underline{k}}$ in the $\frac{\underline{k}}{\|\underline{k}\|}$ direction is $2\Psi(|\underline{k}|)$.

Therefore, we can provide an upper bound for the volume occupied by the union of the strips:

$$\mathrm{meas}\left(\bigcup_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} S_{\underline{k}}\right) \le \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} \mathrm{meas}\, S_{\underline{k}}$$

$$\le \sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} 2\left(\mathrm{diam}\, D\right)^{n-1}\sqrt{n}\,\frac{\Psi(|\underline{k}|)}{|\underline{k}|},$$

where we used the well known inequality $\frac{|\cdot|}{\sqrt{n}} \le \|\cdot\| \le |\cdot|$ between the l_1 and l_2 norm. It is convenient to split the previous series by separating all the harmonics with the same l_1 norm, i.e.

$$\mathrm{meas}\left(\bigcup_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}} S_{\underline{k}}\right) \le \sum_{s=1}^{+\infty}\sum_{|\underline{k}|=s} 2\left(\mathrm{diam}\, D\right)^{n-1}\sqrt{n}\,\frac{\Psi(s)}{s}$$

$$\le 2^{n+1}\sqrt{n}\left(\mathrm{diam}\, D\right)^{n-1}\sum_{s=1}^{+\infty}\frac{s^{n-1}\,\Psi(s)}{s}$$

The previous inequality makes evident that the measure of the volume of all the resonant strips is of order γ when $\psi(|\underline{k}|) = \frac{\gamma}{|\underline{k}|^{\tau}}$ $\forall\, \tau > n-1$.

Therefore, we have that

$$\mathrm{meas}\left(D \setminus \Omega_{\gamma,\tau}\right) \leq \gamma\, 2^{n+1}\sqrt{n}\left(\mathrm{diam}\, D\right)^{n-1} \sum_{s=1}^{+\infty} \frac{1}{s^{\tau-(n-2)}},$$

where the coefficient of γ is a finite number for any fixed $\tau > n-1$.

Thus, $\forall\, \tau > n-1$ and γ small enough (in such a way that the upper bound of $\mathrm{meas}(D\setminus\Omega_{\gamma,\tau})$ is smaller than $\mathrm{meas}(D)$) we know for sure that the measure of the (nonempty!) set $\Omega_{\gamma,\tau}$ is positive and its order of magnitude is $\sim \gamma$.

In order to prove the theorem, it is enough to remark that

$$\lim_{m\to+\infty} \mathrm{meas}\left(B_R(0) \setminus \Omega_{\frac{1}{m},\tau}\right) = 0^+ \quad \forall\, \tau > n-1 \text{ and } \forall\, R > 0.$$

It is important to recall a theorem (that we will not prove) due to Neishtadt: the ~~fm~~ measure of the phase space that is not occupied by KAM tori (when KAM theorem applies) is $O(\varepsilon^{1/2})$ when $\varepsilon \to 0^+$. In other words, the relative measure of the KAM tori is going to ~~tend~~ 1 (the full measure) for small perturbations.

• Lie series - A ~~first~~ purely formal introduction

Lie series are useful in order to write the solutions (as a flow) of differential equations. Here, we limit ourselves to use lie series in order to express canonical transformations in a very comfortable way. The treatment, during this part of the lecture, is purely formal in the sense that we do not consider the problem of the convergence of the series at this stage. The analysis of the problem of the convergence is deferred to the next lecture (or to another part of the present lecture).
First, let us focus on some interesting remarks.
Let $\mathcal{X}$ a generic Hamiltonian, then it is very easy to check that $\dot{f} = \{f, \mathcal{X}\}$ for any dynamical function f (i.e. any function defined on the phase space), including the canonical variables themselves.

Therefore, by expanding the Taylor formula we obtain

$$f(p(t), q(t)) = f(p(0), q(0)) + t\{f, \chi\} + \frac{t^2}{2}\{\{f, \chi\}, \chi\} + \ldots$$

Obviously, it is convenient to introduce the so called Lie derivative operator, i.e.

$$L_\chi \cdot = \{\cdot, \chi\} .$$

Thus, we can write the Taylor series giving us the value of the dynamical function f at time t as follows:

$$f(p(t), q(t)) = \sum_{j=0}^{+\infty} \frac{t^j}{j!} L_\chi^j f .$$

Of course, when $j=0$, $L_\chi^0 f = f$ that is evaluated by inserting $p(0)$ and $q(0)$ as arguments; the same is done also for $L_\chi^j f$ $\forall\, j>0$.

In order to write easily the r.h.s. of the previous equation, ~~even~~ let us introduce the Lie series operator, i.e.,

$$\exp(\varepsilon L_\chi) \cdot = \sum_{j=0}^{+\infty} \frac{\varepsilon^j}{j!} L_\chi^j \cdot ;$$

in words, the Lie series operator maps a dynamical function from its initial value at time $t=0$ to its value at time ε, along the flow induced by the Hamiltonian χ.

As it has been anticipated before, the same definitions hold (and are interesting too) for the canonical variables; therefore, the equations

$$p_j(\varepsilon) = \exp(\varepsilon L_\chi)\, p_j \qquad q_j(\varepsilon) = \exp(\varepsilon L_\chi)\, q_j \,.$$

give the values of the canonical coordinates $(\underline{p}(\varepsilon), \underline{q}(\varepsilon))$ corresponding to any initial conditions $(\underline{p}(0), \underline{q}(0))$ along the flow induced by χ for a time ε.

Since it is well known (in Hamiltonian mechanics) that ~~the~~ a Hamiltonian flow defines a canonical transformation, then we already ~~have~~ obtained a remarkable result: the transformation

$$(\underline{p}, \underline{q}) \longmapsto \left(\exp(\varepsilon L_\chi)\underline{p}, \exp(\varepsilon L_\chi)\underline{q}\right)$$

is canonical for any Hamiltonian χ (that we assume to be an analytic function) and for ε small enough to ensure the convergence of the Lie series (this was actually the reason to denote the time ~~in~~ ruling the flow with the symbol ε).

Let us also remark that such canonical transformations are (by definition) close to the identity in the sense that

$$\left(\exp(\varepsilon L_\chi)\underline{p}, \exp(\varepsilon L_\chi)\underline{q}\right) - (\underline{p}, \underline{q}) = O(\varepsilon) \,.$$

The last of the previous remarks clearly shows why the Lie series fit so well in the framework of perturbative proof schemes.
As it has already been explained, the ~~moth~~ normal form method requires ~~that~~ to determine the expression of the Hamiltonian in the new coordinates. This can be made easily, in the formalism of Lie series, by using the so called exchange theorem due to Gröbner, which can be stated as follows.

<u>Theorem</u>: let f and χ be two analytic dynamical functions. Therefore, the following equality holds true:

$$f(p,q)\Big|_{\substack{p=\exp(\varepsilon L_{\chi})p' \\ q=\exp(\varepsilon L_{\chi})q'}} = \exp(\varepsilon L_{\chi}) f\Big|_{\substack{p=p' \\ q=q'}}$$

The theorem claims that the new Hamiltonian can be calculated by applying directly the Lie series to the Hamiltonian and then, if needed, by changing the name to the dummy variables.
Let us remark that, by using the exchange theorem, we don't need any substitution of variables, the algorithm

just requires to implement sums and products. This gives big advantages when the algorithm is implemented in a programming code so to compute explicitly normal forms and the corresponding solutions (the substitutions of variables are not easy to implement, and they are demanding for what concerns the CPU-time requests).

During these lectures we will not prove the exchange theorem. We limit ourselves to remark that it is a consequence of the following properties of the Lie series.

(I) The Lie series are linear operators (i.e. they commute with sums and multiplication by scalars);

(II) The Lie series commute with products.

(III) The Lie series commute with Poisson brackets.

The properties (I)-(II), joined with suitable hypotheses on the convergence of series, allow to prove the exchange theorem, because the Lie series commute with the series defining the analytic functions.

Property (III) allows to give a direct proof that the transformations induced by the Lie series are canonical.