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"The KAM theorem, following the original approach designed by Kolmogorov"

Mini-course in the framework of the Pisa - Hokkaido - Roma2 summer school on Mathematics and its Applications (2018, from Mon. 27 Aug. to Fri. 7 Sep.) -

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- Very short introduction to the Hamiltonian formalism

The Newtonian mechanics

$$\begin{cases} m_j \ddot{\underline{x}}_j = \underline{F}_j(\underline{x}_1, \ldots, \underline{x}_N) & \underline{x}_j \in \mathbb{R}^3 \ \forall j = 1, \ldots, N \\ f_i(\underline{x}_1, \ldots, \underline{x}_N) = 0 & \forall i = 1, \ldots, r \\ \sum_{j=1}^{N} \underline{\Phi}_j \cdot \delta \underline{x}_j = 0 & \forall \delta \underline{x}_1, \ldots, \delta \underline{x}_N \text{ compatible with the constraints} \end{cases}$$

ideality of the constraints

is equivalent to the Lagrangian mechanics

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad \forall j = 1, \ldots, n = 3N - r \text{ degrees of freedom}$$

with $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$

Moreover, Lagrangian mechanics is equivalent to Hamiltonian mechanics

$$\begin{cases} \dot{p}_j = -\dfrac{\partial H}{\partial q_j} & \forall j=1,\dots,n \\ \dot{q}_j = \dfrac{\partial H}{\partial p_j} \end{cases}$$

where $H(\underline{p},\underline{q}) = \sum_{j=1}^{n} p_j \dot{q}_j - L(\underline{q},\dot{\underline{q}})\Big|_{\dot{\underline{q}}=\dot{\underline{q}}(\underline{p},\underline{q})}$ and $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$. For conservative mechanical systems, we have $H(\underline{p},\underline{q}) = T(\underline{p},\underline{q}) + U(\underline{q})$

The Hamilton equations can be rewritten in the elegant form $\begin{cases} \dot{p}_j = \{p_j, H\} \\ \dot{q}_j = \{q_j, H\} \end{cases}$ $\forall j=1,\dots,n$, where the Poisson brackets $\{f,g\}$ is defined so that

$$\{f,g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j},$$

being $f = f(\underline{p},\underline{q})$, $g = g(\underline{p},\underline{q})$ 2 generic dynamical variables.

• Canonical transformations (definition)

let $(\underline{p},\underline{q}) = \underline{C}(\underline{P},\underline{Q})$ a diffeomorphism such that

$$\{p_i, p_j\}_{(\underline{P},\underline{Q})} = 0, \ \{q_i, q_j\}_{(\underline{P},\underline{Q})} = 0, \ \{p_i, q_j\}_{(\underline{P},\underline{Q})} = -\delta_{ij} \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases},$$

where the Poisson brackets are calculated with respect to the new variables $(\underline{P},\underline{Q})$, e.g. $\{p_i(\underline{P},\underline{Q}), q_j(\underline{P},\underline{Q})\}_{(\underline{P},\underline{Q})} = \sum_{\ell=1}^{n} \frac{\partial p_i}{\partial Q_\ell}\frac{\partial q_j}{\partial P_\ell} - \frac{\partial p_i}{\partial P_\ell}\frac{\partial q_j}{\partial Q_\ell}$

Then, ~~we say that~~ $(\underline{p},\underline{q}) = \underline{\mathcal{C}}(\underline{P},\underline{Q})$ is said to be a canonical transformation.

- Fundamental property of the canonical transformations

Let $K(\underline{P},\underline{Q}) = H\left(\underline{\mathcal{C}}(\underline{P},\underline{Q})\right)$ be the Hamiltonian expressed in the new variables $(\underline{P},\underline{Q})$, or, for short, the new Hamiltonian.

Let $t \mapsto (\underline{P}(t),\underline{Q}(t))$ ~~the~~ a solution of the new Hamilton equations that are $\begin{cases} \dot{P}_j = -\dfrac{\partial K}{\partial Q_j} \\ \dot{Q}_j = \dfrac{\partial K}{\partial P_j} \end{cases}$ $\forall j = 1, \ldots, n$. Therefore, the law of motion

$$t \mapsto (\underline{p}(t),\underline{q}(t)) = \underline{\mathcal{C}}(\underline{P}(t),\underline{Q}(t))$$

is a solution of the original Hamilton equations $\begin{cases} \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \\ \dot{q}_j = \dfrac{\partial H}{\partial p_j} \end{cases}$

- Normal forms (a very short, informal, introduction)

The fundamental property of the canonical transformations (as described above) opens the possibility to solve Hamilton equations, by using an approach based on normal forms.

The basic idea can be formulated as follows: let us imagine to have found a canonical transformation

$(\underline{p},\underline{q}) = \underline{\mathcal{C}}(\underline{P},\underline{Q})$, such that it is easy to find a solution of the Hamilton equations for $K(\underline{P},\underline{Q}) = H(\mathcal{C}(\underline{P},\underline{Q}))$, then this ~~new~~ solution, transferred in the ~~new~~ old coordinates $(\underline{p},\underline{q})$, solves also the equations for $H(\underline{p},\underline{q})$ (i.e., the original problem); in such a case, K is said to be in normal form.

Example

$$K(\underline{P},\underline{Q}) = \underline{\omega}\cdot\underline{P} + \mathcal{R}(\underline{P},\underline{Q}) \quad , \text{ where } \underline{Q}\in\mathbb{T}^n \text{ and } \mathcal{R}(\underline{P},\underline{Q}) = O(\|\underline{P}\|^2)$$

K is in Kolmogorov* normal form.

It is easy to check that $t \mapsto \left(\underline{P}(t) = \underline{0}\ ,\ \underline{Q}(t) = \underline{Q}_0 + \underline{\omega}t\right)$ is a solution of the Hamilton equation related to K.

In fact,

$$\begin{cases} \dot{P}_j = \dfrac{\partial \mathcal{R}}{\partial Q_j} = O(\|P\|^2) \\[2ex] \dot{Q}_j = \omega_j + \dfrac{\partial \mathcal{R}}{\partial P_j} = \omega_j + O(\|P\|) \end{cases}$$

and

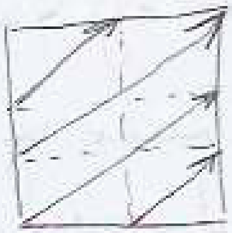
become two identities when $\underline{P} = \underline{0}$ and $\underline{\dot{Q}} = \underline{\omega}$!

Since $\underline{Q}\in\mathbb{T}^n$, we say that the torus $\{(\underline{P},\underline{Q}) : \underline{P} = \underline{0}\}$ is invariant.

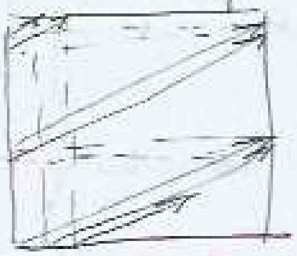
The flow on the invariant torus is strictly periodic if there are $\alpha \in \mathbb{R}_+$ and $\underline{v} \in \mathbb{Z}^n$ such that

$$\underline{\omega} = \alpha \underline{v} ;$$

otherwise, that flow is said to be quasi-periodic.



periodic flow on $\mathbb{T}^2$



quasi-periodic flow on $\mathbb{T}^2$ (It fills densely $\mathbb{T}^2$)

If a Hamiltonian $H(\underline{p},\underline{q})$ can be brought in Kolmogorov normal form ~~if there~~ by a canonical transformation $(\underline{p},\underline{q}) = \mathcal{C}(\underline{P},\underline{Q})$, then the law of motion

$$t \mapsto \mathcal{C}(\underline{0}, \underline{Q} = \underline{\omega} t + \underline{Q}_0)$$

is a solution of the Hamilton equations related to H.

• Integrability, Liouville theorem

The strongest result about integrability of Hamiltonian systems (i.e. solvability of the Hamilton equations) is the celebrated Liouville theorem:

Theorem: let $H(\underline{p},\underline{q})$ be a Hamiltonian describing a dynamical system. Let $I_1(\underline{p},\underline{q}), \ldots, I_n(\underline{p},\underline{q})$ be n independent first integrals that are in involution. This means that

$$\det\left(\frac{\partial I_i}{\partial p_j}\right)_{i,j=1,\dots,n} \neq 0 ,$$

it is not restrictive that the derivatives are only with respect to p!

H is a constant of motion, thus "only" n-1 are needed

$$\{I_j, H\} = 0 \quad \text{and} \quad \{I_i, I_j\} = 0 \quad \forall\, i,j = 1,\dots,n .$$

Therefore, the system is integrable by quadrature.

Comments The proof ensure the existence of a canonical transformation (expressed in mixed coordinates)

$$(p, q) = \underline{C}(\underline{I}, \underline{Q})$$

not necessarily angles!

such that the new momenta are the first integrals $(I_1, \dots, I_n)$. The new Hamiltonian

$$K(\underline{I}, \underline{Q}) = H(\mathcal{C}(\underline{I}, \underline{Q})) = \mathcal{H}(\underline{I})$$

actually depends just on such new momenta.
Thus, the new Hamilton equations can be written as follows:

$$\begin{cases} \dot{I}_j = -\dfrac{\partial \mathcal{H}}{\partial Q_j} = 0 \implies I_j(t) = I_j(0) & \text{constant of motion (as we already knew)} \\ \dot{Q}_j = \dfrac{\partial \mathcal{H}}{\partial I_j} = \omega_j & \text{constant because of} \end{cases}$$

$$\implies Q_j(t) = \omega_j t + Q_j(0) .$$

As usual, the solution of the Hamilton equations expressed in the old coordinates is given by

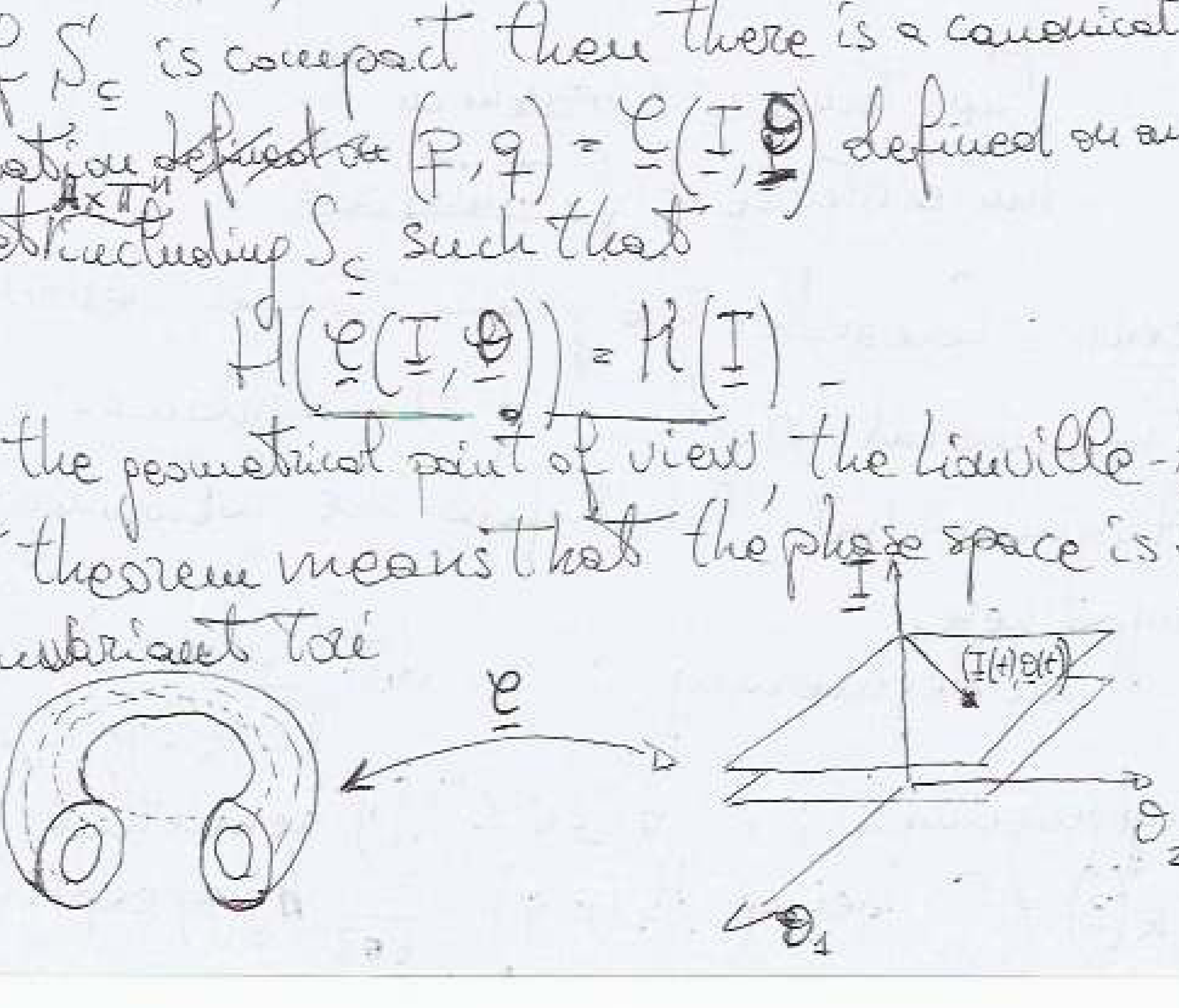
$$t \mapsto \underline{C}\big(I_j(t) = I_j(0),\ Q_j(t) = \omega_j t + Q_j(0)\big) .$$

In the particular case that there is at least a connected component of the set defined by the integrals of motion that is compact, then we can locally introduce action angle coordinates. This is ensured by the following extension of the Liouville theorem due to Arnold (and Jost).

<u>Theorem</u>: let us assume the same hypotheses of the Liouville theorem. Moreover, let $S_{\underline{c}}$ be a connected component that is a subset of $\left\{ (\underline{p}, \underline{q}) : I_j(\underline{p}, \underline{q}) = c_j ,\ j=1,\dots,n \right\}$ where $c_1, \dots, c_n$ are real constant values.
If $S_{\underline{c}}$ is compact then there is a canonical transformation ~~defined on~~ $(\underline{p}, \underline{q}) = \underline{\mathcal{C}}(\underline{I}, \underline{\theta})$ defined on an open set $A \times \mathbb{T}^n$ including $S_{\underline{c}}$ such that

$$H\left(\underline{\mathcal{C}}(\underline{I}, \underline{\theta})\right) = K(\underline{I}) .$$

From the geometrical point of view, the Liouville-Arnold-Jost theorem means that the phase space is filled by invariant Tori



• General problem of dynamics

According Poincaré the general problem of dynamics is a perturbation of an integrable system. In fact, the general problem of the dynamics is represented by ~~We say that~~ a dynamical system governed by a Hamiltonian of type

$$H(\underline{p},\underline{q};\varepsilon) = H_0(\underline{p}) + \sum_{s=1}^{+\infty} \varepsilon^s H_s(\underline{p},\underline{q}),$$

where $\underline{p} \in \mathcal{A} \subseteq \mathbb{R}^n$, being $\mathcal{A}$ an open set, $\underline{q} \in \mathbb{T}^n$, ~~and~~ ε a small parameter defined in $\mathcal{B}(0)$, H an analytic function in all its arguments.

• The Poincaré theorem on the non-existence of first integrals

<u>Theorem</u>: Consider the Hamiltonian describing the general problem of the dynamics. Assume that it satisfies the following hypotheses:
non-degeneracy, i.e. $\det\left(\frac{\partial^2 H_0}{\partial p_i \partial p_j}\right)_{i,j=1,..,n} \neq 0$,

and genericness, i.e. $\forall\, \underline{k} \in \mathbb{Z}^n \setminus \{\underline{0}\}$ we have that $h_{\underline{k}}(\underline{p}) \neq 0$ being $H_1(\underline{p},\underline{q}) = \sum_{\underline{k}\in\mathbb{Z}^n} h_{\underline{k}}(\underline{p}) \exp(i\underline{k}\cdot\underline{q})$

then there is not any analytic function
that is a first integral that does not depend on H.

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It is convenient to give a sketch of the proof.
Let us consider a series expansion of a first integral

$$\Phi(p,q;\varepsilon) = \Phi_0(p) + \sum_{s=1}^{+\infty} \varepsilon^s \Phi_s(p,q),$$

that is analogous to that of the Hamiltonian.
Let us try to determine Φ, by solving perturbatively the equation $\{H,\Phi\}=0$. This means that we have

$$\{H_0,\Phi_0\} = 0 \qquad (\text{order } \varepsilon^0)$$

$$\{H_0,\Phi_1\} = -\{H_1,\Phi_0\} \qquad (\text{order } \varepsilon)$$

$$\{H_0,\Phi_2\} = -\{H_2,\Phi_0\} - \{H_1,\Phi_1\} \qquad (\text{order } \varepsilon^2)$$

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In the r.h.s. we have put the terms that would be fixed at the previous perturbative steps.
The equation at order ε^0 joined with the non-degeneracy condition, allows to verify that Φ_0 cannot depend on the angles; moreover by the Liouville theorem it is natural to look for first integrals that are perturbations of the momenta.

Now, we consider the $\mathcal{O}(\varepsilon)$ equation, i.e.

$$\left\{H_0, \Phi_1\right\} = -\left\{H_1, \Phi_0\right\}.$$

let us focus on the Fourier expansion of that equation

$$-i\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}}\left(\sum_{j=1}^{n}\frac{\partial H_0}{\partial p_j}k_j\right)\varphi_{\underline{k}}(\underline{p})\exp(i\underline{k}\cdot\underline{q}) = -i\sum_{\underline{k}\in\mathbb{Z}^n\setminus\{0\}}\left(\sum_{j=1}^{n}k_j\frac{\partial\Phi_0}{\partial p_j}\right)h_{\underline{k}}(\underline{p})e^{i\underline{k}\cdot\underline{q}}$$

$\Rightarrow \forall\, \underline{k}\in\mathbb{Z}^n\setminus\{0\}$ we have $\underline{k}\cdot\underline{\omega}(\underline{p})\,\varphi_{\underline{k}}(\underline{p}) = \left(\sum_{j=1}^{n}k_j\frac{\partial\Phi_0}{\partial p_j}\right)h_{\underline{k}}(\underline{p})$

being $\Phi_1(\underline{p},\underline{q}) = \sum_{\underline{k}\in\mathbb{Z}^n}\varphi_{\underline{k}}(\underline{p})\exp(i\underline{k}\cdot\underline{q})$ and

$\omega_j(\underline{p}) = \frac{\partial H_0}{\partial p_j}$.

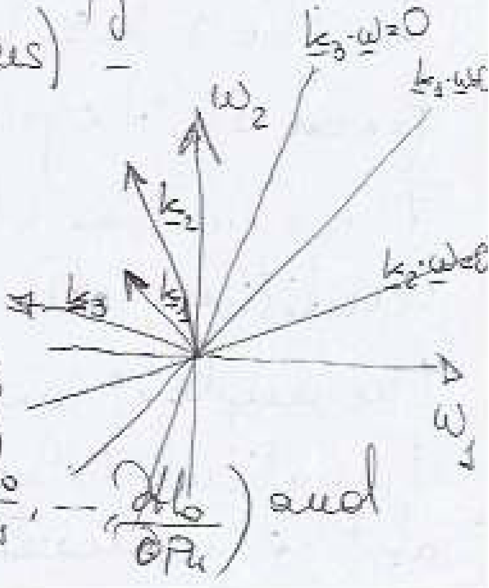
We now consider the so called resonant manifold $\underline{k}\cdot\underline{\omega}(\underline{p}) = 0$, that admits solutions in force of the non-degeneracy condition.

$\forall\, \underline{k}\in\mathbb{Z}^n\setminus\{0\}$, $\forall\, \underline{p}: \underline{k}\cdot\underline{\omega}(\underline{p}) = 0 \Rightarrow \sum_{j=1}^{n}k_j\frac{\partial\Phi_0}{\partial p_j} = 0$

(here we used the genericness conditions).

Thus on the resonant modules we have

that $\cancel{\det}$ rank $\dfrac{\partial(H_0,\Phi_0)}{\partial(p_1,\dots,p_n)} = 1$.



Since the resonant modules fill densely the set of the angular velocities $\underline{\omega} = \left(\frac{\partial H_0}{\partial p_1}, \dots, \frac{\partial H_0}{\partial p_n}\right)$ and (by the non-degeneracy) the action space I

then H_0 and Φ_0 are not independent.
By using again the non-degeneracy, this allows to conclude that also H and Φ are not independent.

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Comments. The genericness condition is very hard to check and it can be violated (at order ε) by many Fourier harmonics $\underline{k}$.
However, the perturbative procedure allowing to compute the first integrals is such that the Fourier harmonics are ~~generic~~ regenerated at every perturbative order in a generic way, unless there are some simmetries.
Thus, if a Fourier harmonic is missing at order ε in the expansion of H_1, it will appear in the r.h.s. of the equations for determining Φ_2 at order ε^2 or Φ_3 at order ε^3 and so on.

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- Main ideas behind the KAM theorem and a first statement of it

The (sketch of the) proof of the Poincaré theorem highlights that the main obstruction to the existence