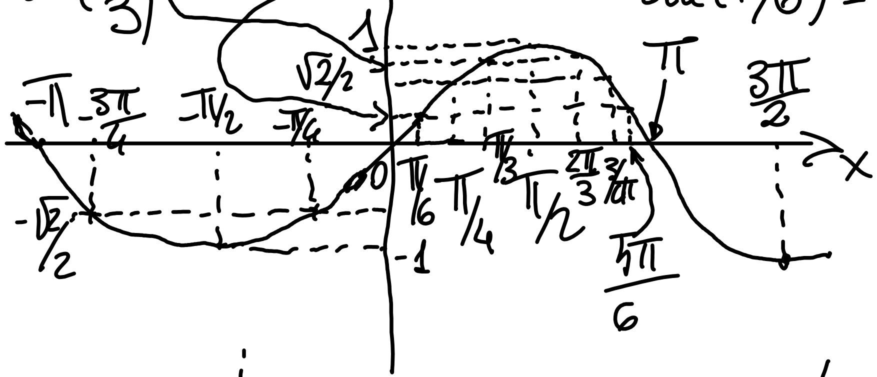
$\sin(\pi/3) = \sqrt{3}/2 = \sin\left(\frac{2\pi}{3}\right)$

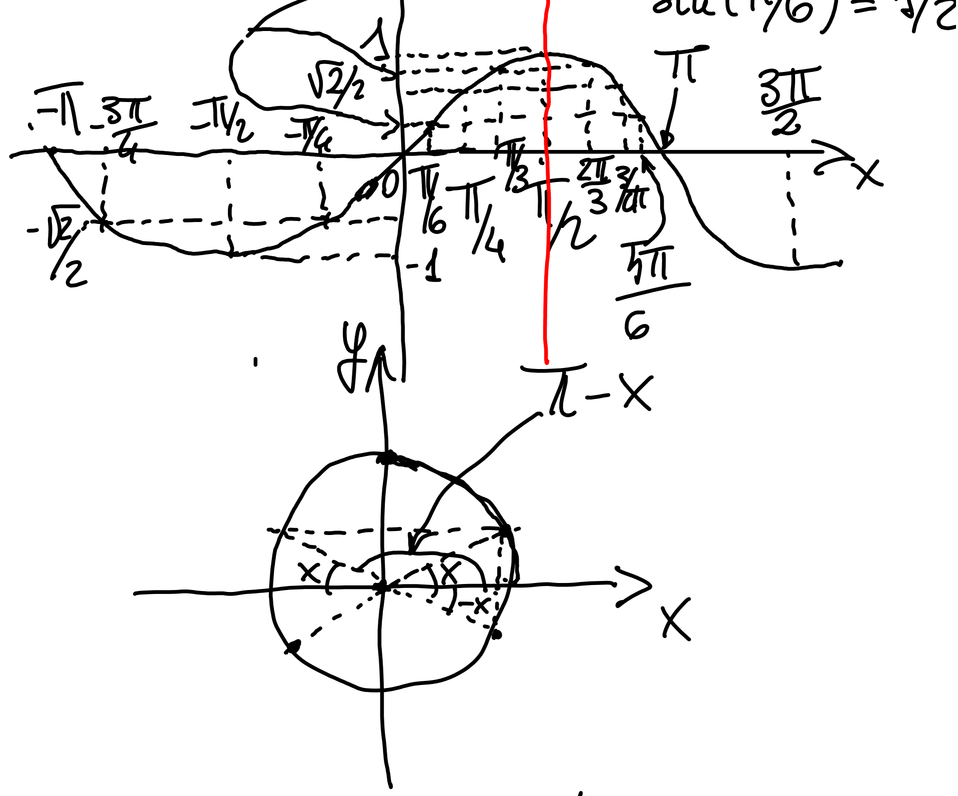
$\sin(\pi/4) = \sqrt{2}/2 = \sin(3\pi/4)$

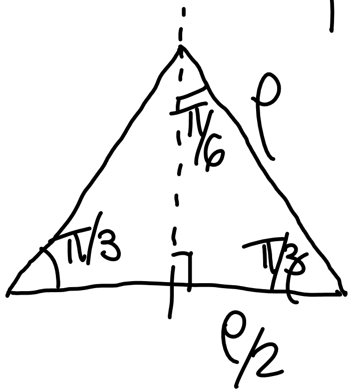
$\sin(\pi/6) = 1/2 = \sin(5\pi/6)$



$\sin(\pi/4) = \sqrt{2}/2$

$\sin(\pi/6) = 1/2$



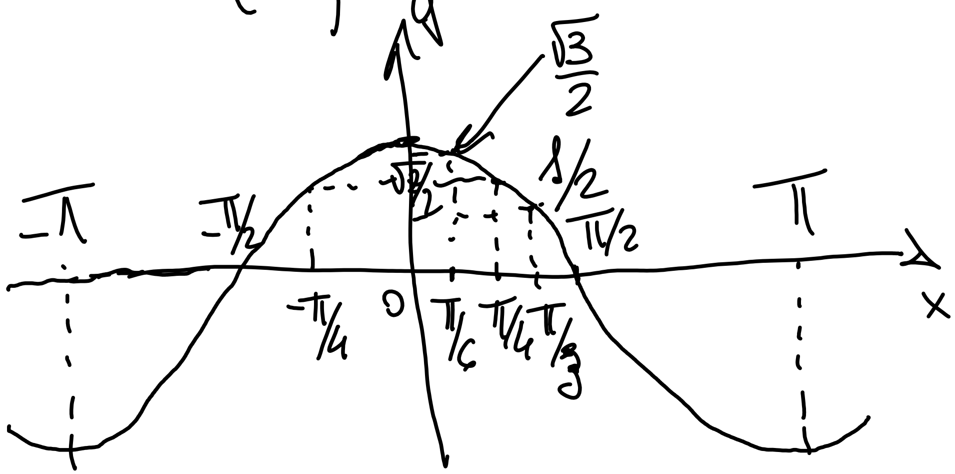


$$\sin(\pi/6) = \frac{\ell/2}{\cancel{\ell}}$$

$$\sin(\pi/3) = \frac{\sqrt{\ell^2 - \ell^2/4}}{\ell} = \frac{\sqrt{3}}{2}$$

$$\sin(-x) = -\sin(x)$$

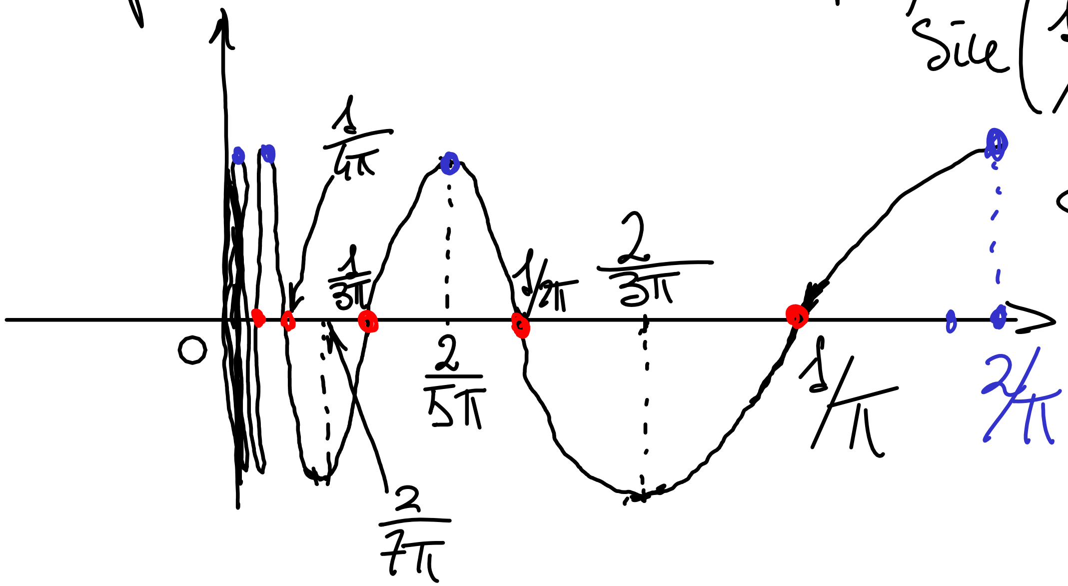
Osserviamo che $\sin\left(\frac{\pi}{2} + x\right) = \overset{1}{\cancel{\sin\left(\frac{\pi}{2}\right)}}\cos(x) + \cancel{\cos\left(\frac{\pi}{2}\right)\sin(x)} = \cos(x)$



$$\cos(-x) = \cos(x)$$

Proposizione: Se $f(x)$, $f: D \to \mathbb{R}$ e $D \subseteq \mathbb{R}$, è t.c. $\exists \lim_{x \to c} f(x) = \ell$ allora il limite è unico.

Esempio: consideriamo $f(x) = \sin(1/x)$ e $\lim_{x \to 0} \sin(1/x) = ?$



$$\sin\left(\frac{1}{1/\pi}\right) = 0 \qquad \sin\left(\frac{1}{\frac{2}{5\pi}}\right) = 1$$

$$\sin\left(\frac{1}{\left(\frac{2}{3\pi}\right)}\right) = -1 \qquad \sin\left(\frac{1}{\left(\frac{1}{3\pi}\right)}\right) = 0$$

$$\sin\left(\frac{1}{(1/2\pi)}\right) = 0 \qquad \sin\left(\frac{1}{\frac{2}{7\pi}}\right) = -1$$

$$\sin\left(\frac{1}{(1/4\pi)}\right) = 0$$

Verifichiamo che $\lim_{x \to 0} \sin(1/x) \nexists$

Definiamo $\xi_n = \frac{1}{n\pi}$ $\forall n > 0$ (così prendo i punti rossi).

e definiamo $\eta_n = 1/(\pi/2 + 2n\pi)$ $\forall n > 0$ (" " " " blu).

Se $\exists\ \ell = \lim_{x \to 0} \sin 1/x$ allora $\left(\text{siccome } \lim_{n\to+\infty} \xi_n = 0\right)$

$$\ell = \lim_{n\to+\infty} \sin\left(\frac{1}{\xi_n}\right) = \lim_{n\to+\infty} \sin\left(\frac{1}{\left(\frac{1}{n\pi}\right)}\right) = \lim_{n\to+\infty} \sin(n\pi) = 0$$

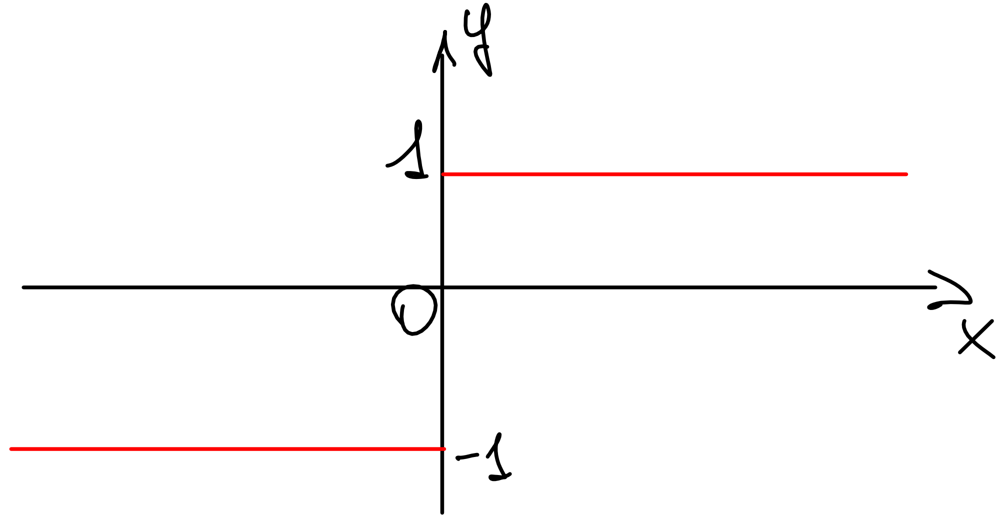
Purtroppo, però anche $\lim_{n\to+\infty} \eta_n = 0$, quindi se esiste ℓ deve essere $\ell = \lim_{n\to+\infty} \sin\left(\frac{1}{\eta_n}\right) = \lim_{n\to+\infty} \sin\left(\frac{1}{\frac{1}{\left(\frac{\pi}{2}+2n\pi\right)}}\right) = 1$,

quindi $\nexists$ limite perché non è unico.

Def.: Sia $c \in \mathbb{R}$ diciamo che ℓ è il limite destro [sinistro] di $f(x)$ quando x tende a c^+ $[c^-]$, se $\forall$ successione $\{x_n\}_{n \geq n_0}$ t.c. $\lim_{n\to+\infty} x_n = c$ e $\forall n \geq n_0$ $x_n > c$ $[x_n < c]$ si ha che $\lim_{n\to+\infty} f(x_n) = \ell$.

Def.: diciamo che una funzione $f: D \to \mathbb{R}$ ha una discontinuità con salto in c, quando $\exists \lim_{x \to c^-} f(x) = \ell_1$ e $\lim_{x \to c^+} f(x) = \ell_2$, ma $\ell_1 \neq \ell_2$

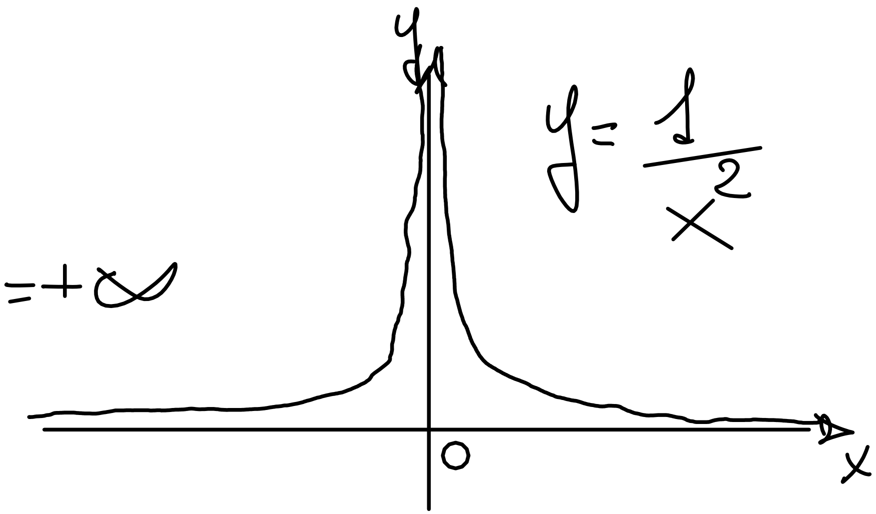
Esempio: $\operatorname{sign}(x) = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$

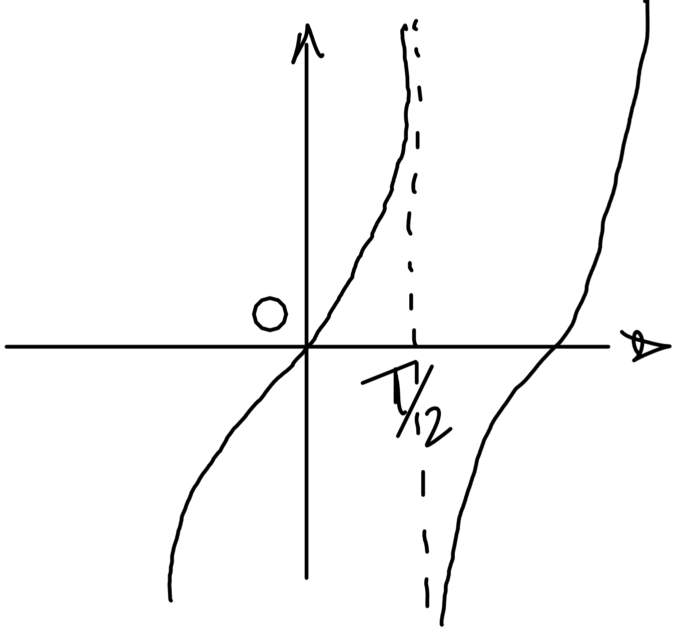


$\lim_{x \to 0^+} \operatorname{sign}(x) = 1 \neq \lim_{x \to 0^-} \operatorname{sign}(x) = -1$.

Diciamo che $f(x)$ ammette un asintoto verticale in c, quando $\lim_{x \to c} f(x) = \pm\infty$.

Esempi: $\lim_{x \to 0} \frac{1}{x^2} = +\infty$





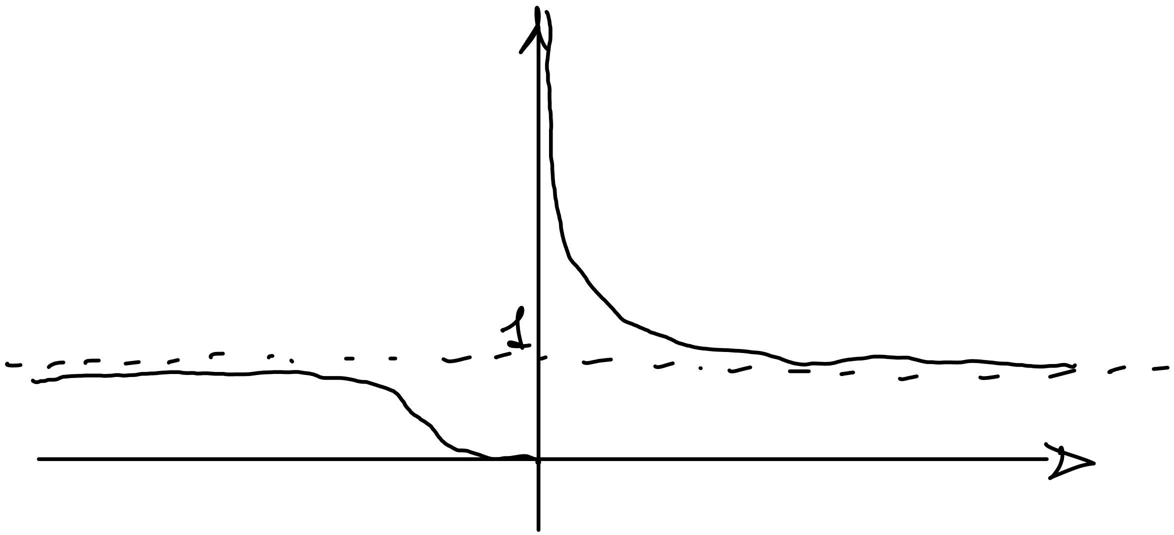
$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$$

$$\lim_{x \to \pi/2^+} \tan(x) = -\infty$$

$f(x) = \exp(1/x)$

$$\lim_{x \to 0^-} e^{1/x} = 0^+$$

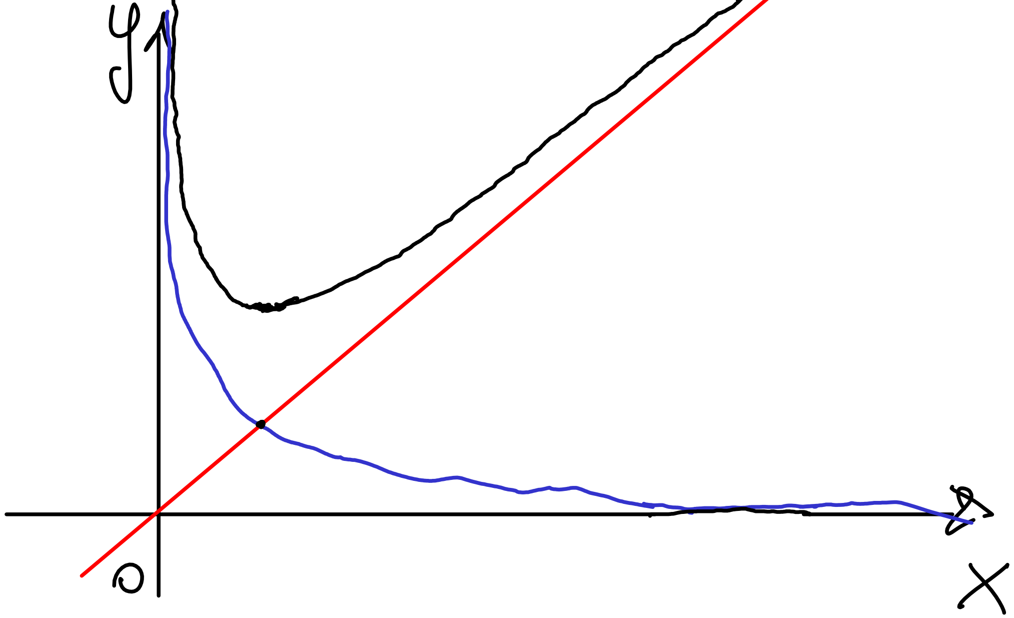
$$\lim_{x \to 0^+} e^{1/x} = +\infty$$



Def.: diciamo che $f(x)$ ammette un asintoto obliquo per $x \to +\infty$ [$-\infty$] quando $\exists\, m, q \in \mathbb{R}$ t.c.

$$\begin{cases} \lim_{x \to +\infty} \frac{f(x)}{x} = m \\ \lim_{x \to +\infty} f(x) - mx = q \end{cases} \qquad \left[\begin{cases} \lim_{x \to -\infty} \frac{f(x)}{x} = m \\ \lim_{x \to -\infty} f(x) - mx = q \end{cases} \right]$$

Esempio: $f(x) = x + 1/x$

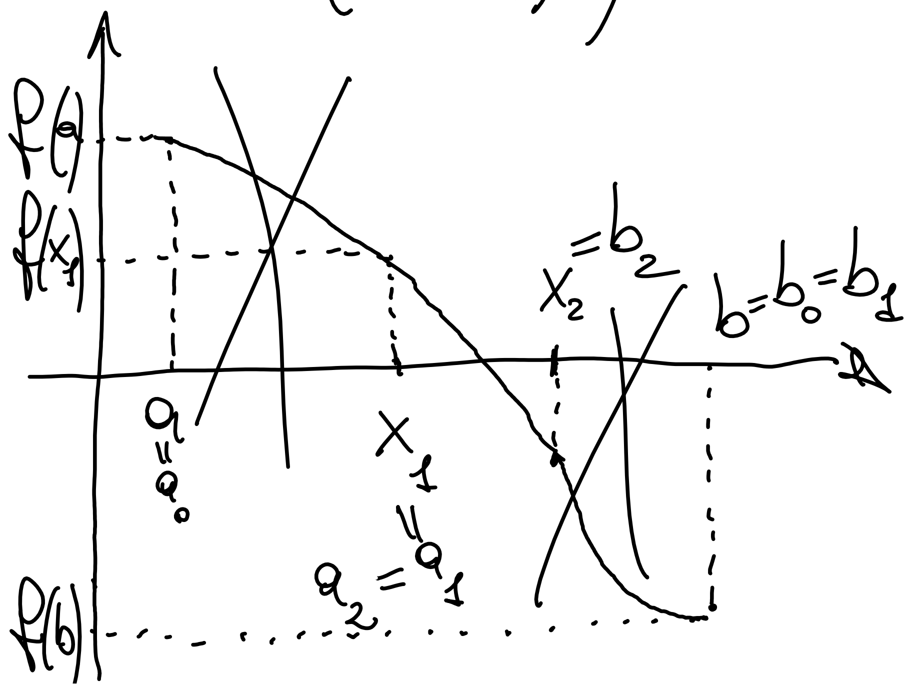


Teorema [degli zeri]:

Sia $f : [a,b] \to \mathbb{R}$ t.c.

- f è continua su tutto $[a,b]$ $\left(f \in \mathcal{C}([a,b])\right)$
- $f(a) \cdot f(b) < 0$

Allora $\exists\, c \in (a,b)$ t.c. $f(c) = 0$



Dim.: illustriamo il metodo di bisezione per risolvere $f(x) = 0$

definiamo $x_1 = \dfrac{a_0+b_0}{2}$, valuto $f(x_1)$

Se $f(x_1)\cdot f(a_0) < 0$, definiamo $a_1 = a_0$ e $b_1 = x_1$

altrimenti se $f(x_1)\cdot f(a_0) > 0$, " $a_1 = x_1$ e $b_1 = b_0$

In generale, dati a_n e b_n t.c. $f(a_n)\cdot f(b_n) < 0$,

definisco $x_{n+1} = \dfrac{a_n+b_n}{2}$ e se $f(x_n) = 0$ (stoppo e pongo $c = x_n$)

oppure pongo $a_{n+1} = a_n$ e $b_{n+1} = x_{n+1}$ se $f(a_n)\cdot f(x_{n+1}) < 0$

infine " $\underline{a_{n+1} = x_{n+1}}$ e $\underline{b_{n+1} = b_n}$ se $f(a_n)\cdot f(x_{n+1}) > 0$.

Osservo che a_n è monotona crescente, mentre b_n è monotona decrescente.

$$\Rightarrow \exists \lim_{n \to +\infty} a_n = \bar{a} \in \mathbb{R} \quad e \ \exists \lim_{n \to +\infty} b_n = \bar{b} \in \mathbb{R}$$