Chapter 2

Local behavior

By local behavior we mean the study of the motion in a neighborhood of a point. As we have seen in the linear case, the motion can leave the neighborhood in a fixed time but it is also possible that it stays in the neighborhood for an unlimited time. In the latter case we will have the first example of how to tackle one of our stated goals: the study of the motion for long times. We start with a trivial case.

2.1 Flow box theorem

Let us consider the differential equation

$$\dot{x} = V(x) \tag{2.1.1}$$

where $V \in \mathcal{C}^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$. By the results of the previous chapter there exist $\delta_-, \delta_+ : \mathbb{R}^d \to \mathbb{R}_+$ and $\phi : \{(z,t) \in \mathbb{R}^d \times \mathbb{R} \ : \ t \in (-\delta_-(z), \delta_+(z))\} =: D \to \mathbb{R}^d$ such that $\phi(z,t)$ is the solution of (2.1.1) with initial condition z. We would like to study the solution in a neighborhood of $x_0 \in \mathbb{R}^d$ such that $V(x_0) \neq 0$.

Theorem 2.1.1 (Flow box Theorem) *In the hypotheses above there exists a neighborhood U of x_0 and a change of variables $\Theta \in \mathcal{C}^1(U, \mathbb{R}^d)$ such that $\Theta(\phi(x,t)) = \Theta(x) + t(0,\dots,0,1)$, for each $x \in U$, $(x,t) \in D$.*

Proof. Let $S = \{x \in \mathbb{R}^d \ : \ \langle x - x_0, V(x_0)\rangle = 0\}$ and $\{e_i\}_{i=1}^{d-1} \subset S$ the an orthonormal base.[1] For $r > 0$ small enough let $D_r = \{z \in \mathbb{R}^d \mid |z_i| \le r\}$. Then define $\Xi : D_r \to U$ by $\Xi(\xi) = \phi(x_0 + \sum_{i=1}^{d-1} \xi_i e_i, \xi_d)$. Note that Ξ is invertible since if $\Xi(\xi) = \Xi(\xi')$, $\xi'_d \le \xi_d$, it would be

$$\phi(x_0 + \sum_{i=1}^{d-1} \xi_i e_i, \xi_d - \xi'_d) = x_0 + \sum_{i=1}^{d-1} \xi'_i e_i.$$

[1] That is $\langle e_i, e_j\rangle = \delta_{ij}$.

That is there would be $x \in S$ and $\tau = \xi_d - \xi'_d \in (0, 2r)$ such that $\phi(x, \tau) \in S$. But $\langle V(x_0), \phi(x, 0)\rangle = \langle V(x_0), \phi(x, \tau)\rangle = 0$ by definition and, for $t \in [0, 2r]$,

$$\frac{\langle V(x_0), \phi(x, t)\rangle}{dt} = \langle V(x_0), V(\phi(x, t))\rangle > 0$$

provided that r is chosen small enough. Hence $\xi_d = \xi'_d$ and, consequently, $\xi = \xi'$. We can then define $\Theta = \Xi^{-1}$ and, for each $x = \Xi(\xi)$,

$$\begin{aligned}
\Theta(\phi(x, t)) &= \Theta(\phi(\phi(x_0 + \sum_{i=1}^{d-1} \xi_i e_i, \xi_d), t) = \Theta(\phi(x_0 + \sum_{i=1}^{d-1} \xi_i e_i, \xi_d + t)) \\
&= \Theta(\Xi(\xi + (0, \dots, 0, t))) = \xi + (0, \dots, 0, t) \\
&= \Theta(x) + (0, \dots, 0, t).
\end{aligned}$$

□

2.2 Behavior close to a fixed point

In this section we will consider a more interesting situation: the study of the solutions of (2.1.1) in a neighborhood of a point x_0 such that $V(x_0) = 0$ and $\det(D_{x_0} V) \neq 0$.

Problem 2.1 *Note that the condition* $\det(D_{x_0} V) \neq 0$ *can always be achieved by a small* $\mathcal{C}^1$ *change of the vector field. On the contrary, a zero of the vector field cannot be eliminated by small* $\mathcal{C}^1$ *changes of the vector field: prove that if* $V(x_0) = 0$ *and* W *is a vector field* $\mathcal{C}^1$ *close enough to* V*, then there exists a* x_* *close to* x_0 *such that* $W(x_*) = 0$*, and* $D_{x_*} W$ *is close to* $D_{x_0} V$*. In this sense we will say that the above conditions are* generic *(more on this concept later).*

It is then necessary to understand the behavior of the equation in the vicinity of the point x_0. First of all, by a translation, we can assume without loss of generality $x_0 = 0$. Then we can develop V by the Taylor formula to obtain

$$\dot{x} = Ax + R(x) \tag{2.2.2}$$

where $\|R(x)\| \leq C\|x\|^2$ and $\|D_x R\| \leq C\|x\|$, for all $\|x\| \leq 1$.

Problem 2.2 *Show that, by a linear change of variable, one can transform* A *in its Jordan canonical form. Show then that, by an arbitrary small* $\mathcal{C}^1$ *change of the vector field one can eliminate all the Jordan blocks and insure that all the eigenvalues have real part different from zero: this is called the* hyperbolic case*.*

For now, in view of Problem 2.2, we limit ourselves to the hyperbolic case.

We will start by considering the case in which all the eigenvalue of A have real part strictly smaller than zero.

Problem 2.3 *Prove that if A is diagonal with eigenvalues with real part strictly smaller than zero, then there exists $\sigma > 0$ such that, for all $x \in \mathbb{C}^n$,*[2]

$$\langle x, (A + A^*)x\rangle \leq -\sigma\langle x, x\rangle \tag{2.2.3}$$

Prove that if A has only simple (that is with algebraic multiplicity one) eigenvalues, then there exists a positive matrix B (that is $B^* = B$ and $\langle x, Bx\rangle > 0$ for all $x \neq 0$) such that

$$\langle x, B(A + A^*)x\rangle \leq -\sigma\langle x, Bx\rangle$$

In other words (2.2.3) still holds provided one redefines the scalar product conveniently. Prove the same for a general matrix A with all the eigenvalues with real part strictly smaller than zero.

Till the end of this section we assume that all the eigenvalue of A are strictly negative, hence we assume (2.2.3). In this case it is well known that the linear part of (2.2.2) has solutions that tend to zero exponentially fast, the question is: does the same holds true for the solutions of the equation (2.2.2)?

To see it, consider $z := \langle x, x\rangle$. By Problem 2.3,

$$\begin{aligned}\frac{d}{dt}z &= \langle x, Ax + R(x)\rangle + \langle Ax + R(x), x\rangle \\ &= \langle x, (A + A^*)x\rangle + \mathcal{O}(\|x\|^3) \leq -\sigma z + \mathcal{O}(z^{\frac{3}{2}}).\end{aligned}$$

If we assume $\|x\| \leq \frac{\sigma}{2}$, then we have

$$\frac{d}{dt}z \leq -\frac{\sigma}{2}z$$

which implies that also the solutions of (2.2.2) tend exponentially fast to zero.

Remark 2.2.1 *What we have just seen is that, locally, $F(x) := \langle x, x\rangle$ is a Lyapunov function for (2.2.2). Given a differential equation like (2.1.1), where 0 is a fixed point, a Lyapunov function is any $\mathcal{C}^1$ function L such that $L(0) = 0$, $L \geq 0$ and $\langle \nabla_x L, V(x)\rangle < 0$ for all $x \neq 0$. This implies that, for each solution $x(t)$ of (2.1.1) holds*

$$\frac{dL(x(t))}{dt} = \langle \nabla_{x(t)} L, V(x(t))\rangle < 0.$$

This readily implies that $\lim_{t\to\infty} x(t) = 0$. (Prove it !).

[2] As usual $\langle x, y\rangle := \sum_{i=1}^n \bar{x}_i y_i$ where $\bar{a}$ is the complex conjugate of a. Moreover by A^* we mean the adjoint of A.

Yet, the above result is far from being satisfactory: it is possible to tend to zero in many different ways and it would be nice to understand better how this happens.

Let us start with a very simple example: $x \in \mathbb{R}$, $A = -1$, $R(x) = bx^2$. Then the equation reads

$$\dot{x} = -x + bx^2. \tag{2.2.4}$$

If we consider the change of variables

$$z = \Psi(x) = \frac{x}{1 - bx}$$

we have

$$\dot{z} = \frac{-x + bx^2}{1 - bx} + \frac{bx(-x + bx^2)}{(1 - bx)^2} = -\frac{x}{1 - bx} = -z.$$

We have just seen that in a neighborhood of size smaller than b^{-1} of zero we have a diffeomorphism that conjugate the solution of (2.2.4) with its linear part.

One can then suspect that this is always the case. This is not so: consider

$$\begin{aligned} \dot{x} &= -2x + cy^2 \\ \dot{y} &= -y \end{aligned} \tag{2.2.5}$$

Let us consider a change of variables

$$\begin{aligned} z &= x + \alpha x^2 + \beta xy + \gamma y^2 + q(x, y) \\ \eta &= y + p(x, y) \end{aligned}$$

where q is of third order and p of second. Substituting in (2.2.5) one can see that it is always possible to choose $p \equiv 0$, while the first of the (2.2.5) yields

$$\dot{z} = -2x + cy^2 - 2x(2\alpha x + \beta y) - y(\beta x + 2\gamma y) + \mathcal{O}(3)$$

where by $\mathcal{O}(3)$ we designate third order terms. If we try to impose the right hand side of the above equation equal to $-2z$ (up to second order) we obtain

$$-2\alpha x^2 - 2\beta xy - 2\gamma y^2 = -4\alpha x^2 - 3\beta xy - (2\gamma + c)y^2$$

that does not admit any solutions if $c \neq 0$.

So there is no hope of finding an analytic conjugation with the linear part.

What can be salvaged?

2.2.1 Grobman–Hartman

One can look for a less regular change of variables. This may not make sense for the o.d.e. itself but it is meaningful for the associated flows.

Thus let us fix some small $r > 0$ and consider a smooth non increasing function $g : \mathbb{R}_+ \to [0,1]$ such that $g(x) = 1$ for $x \leq r$ and $g(x) = 0$ for $x \geq 2r$, with $-g' \leq C$. We can then define the functions $\varphi : \mathbb{R}^d \to [0,1]$ $F_0, F : \mathbb{R}^d \to \mathbb{R}^d$ as $\varphi(x) := g(\|x\|)$ and

$$\begin{aligned} &F_0(x) := e^A x \\ &F(x) := e^A x + \varphi(x)\left[\phi_1(x) - e^A x\right] =: F_0(x) + \Delta(x), \end{aligned}$$

where ϕ_1 is the time one flow associated to (2.2.2). Remember that we are still considering the case in which all the eigenvalues of A have strictly negative real part. Clearly, for $\|x\| \leq r$ the two functions are simply the time one map of the linear flow and the time one map of (2.2.2), moreover they are globally Lip. Since we will be interested only in x in the ball of radius r the modification outside such a ball is totally irrelevant and it has been done only to facilitate the exposition of the following argument.

Problem 2.4 *Show that, for r small enough, F is a diffeomorphism. Prove that $\|\Delta\|_\infty < \infty$.*

The idea is to consider the maps $F_0, F : \mathbb{R}^d \to \mathbb{R}^d$ and to show that they can be conjugated, that is there exists an homeomorphism $\tilde{\Phi} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\tilde{\Phi} \circ F = F_0 \circ \tilde{\Phi}$.

Let us look for a solution in the form $\tilde{\Phi}(x) = x + \Phi(x)$, then we have

$$F_0(x + \Phi(x)) = F(x) + \Phi(F(x))$$

or, setting $\xi = F(x)$,

$$\Phi(\xi) = F_0(F^{-1}(\xi) + \Phi \circ F^{-1}(\xi)) - \xi.$$

We define then the operator $K : \mathcal{C}^0(\mathbb{R}^d) \to \mathcal{C}^0(\mathbb{R}^d)$ defined by

$$K(\Phi)(\xi) := F_0(F^{-1}(\xi) + \Phi \circ F^{-1}(\xi)) - \xi$$

then our problem boils down to establishing the existence of a fixed point for K. First of all notice that, for each $\|\xi\| \geq 2r + \|\phi\|_\infty$,

$$\|K(\Phi)(\xi)\| = \|F_0(\Phi \circ F^{-1}(\xi)\| \leq e^{-\sigma/2}\|\Phi\|_\infty.$$

Thus $\|K(\Phi)\|_\infty \leq 4rC + e^{-\sigma/2}\|\Phi\|_\infty$. Thus the set $\{h \in \mathcal{C}^0 \ : \ \|h\|_\infty \leq 4rC(1 - e^{-\sigma/2})^{-1}\}$ is invariant for the operator K.

Now, given two functions $h, g \in \mathcal{C}^0(\mathbb{R}^d)$, holds

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^d} \|K(h)(\xi) - K(g)(\xi)\| &= \sup_{x\in\mathbb{R}^d} \|F_0(x + h(x)) - F_0(x + g(x))\| \\ &\leq \frac{e^{-\sigma}}{2}\|h - g\|_\infty\end{aligned}$$

Thus the contracting mapping theorem yields the wanted result.

Problem 2.5 *What can be done if all the eigenvalues of A have strictly positive real part?*

We have then, topologically, the behavior of a source, a node or a stable or unstable focus are the same as the one of the linear part of the equation. But the generic case is the one in which both eigenvalues with positive and negative real part are present, does the same conclusions hold for such a more general situation? The answer is yes. To see it consider that in such a case $\mathbb{R}^d$ is naturally split into two spaces $V \oplus W$, invariant for A and such that A restricted to V has only eigenvalues with negative real part while restricted to W has eigenvalues with positive real part. Then the spaces are invariant for F_0 as well, on one F_0 contracts, on the other expands. Call d_s the dimension of V and d_u the dimension of W. Clearly $d_s + d_u = d$.

Then each $e \in \mathbb{R}^d$ has a unique splitting as $e = v + w$, $v \in V$, $w \in W$. It is then convenient to define the projections $p_1 : \mathbb{R}^d \to V$ and $p_2 : \mathbb{R}^d \to W$ $p_1(e) = v$, $p_2(e) = w$. Moreover we can split $\mathcal{C}^0(\mathbb{R}^d)$ as $\mathbb{V} \oplus \mathbb{W}$ where $\mathbb{V} := \{f \in \mathcal{C}^0(\mathbb{R}^d) \ : \ p_2 \circ f = 0\}$ and $\mathbb{W} := \{f \in \mathcal{C}^0(\mathbb{R}^d) \ : \ p_1 \circ f = 0\}$. We can then write canonically f as $(f_1, f_2) := (p_1 \circ f, p_2 \circ f)$.

Accordingly our conjugation equation $F_0 \circ \tilde{\Phi} = \tilde{\Phi} \circ F$, becomes

$$\begin{aligned}B\tilde{\Phi}_1 &= \tilde{\Phi}_1 \circ F \\ D\tilde{\Phi}_2 &= \tilde{\Phi}_2 \circ F\end{aligned}$$

where $F_0((x_1, x_2)) =: (Bx_1, Dx_2)$. We transform the first equation as we did for the contracting case, while on the second we act as you probably did if you solved Problem 2.5:

$$\begin{aligned}\tilde{\Phi}_1 &= B\tilde{\Phi}_1 \circ F^{-1} \\ \tilde{\Phi}_2 &= D^{-1}\tilde{\Phi}_2 \circ F.\end{aligned}$$

Note that, if we apply the above reasoning directly to such equations we obtain that they have only one bounded solution: $\tilde{\Phi} = 0$, yet we are not looking for bounded solutions but rather for solutions of the form $\tilde{\Phi}(x) = x + \Phi(x)$, where Φ is bounded. Substituting such a form for $\tilde{\Phi}$ one can see that bounded function are mapped into bounded functions (thanks to Problem 2.4), hence the contracting map argument applies and the existence of a unique conjugation is established.

Remark 2.2.2 *By the way, what we just proved is known as the Grobman-Hartman Theorem.*

2.3 Dominated Splitting and center manifold

Let $U \subset \mathbb{R}^d$ be an open set containing zero and let us consider a vector field $V \in \mathcal{C}^k(U, \mathbb{R}^d)$, $k \geq 1$, such that $V(0) = 0$ and $A := D_0V$ has a spectrum that *splits* into two disjoint parts. More precisely, assume there exists real numbers $\alpha < \beta$, such that $\sigma(A) = \Sigma_1 \cup \Sigma_2$ where $\mu \in \Sigma_1$ implies $\Re(\mu) \geq \beta$ and $\mu \in \Sigma_2$ implies $\Re(\mu) \leq \alpha$. Let $\mathbb{V}_1, \mathbb{V}_2$ be the eigenspaces associated to Σ_1, Σ_2, respectively.

We say that a manifold W is locally invariant at zero under the flow ϕ_t generated by the vector field V if there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, there exists $\delta_t \in (0, \delta]$ such that $\phi_t(W \cap B(0, \delta_t)) \subset W$.

Note that, letting $\widetilde{R}(x) := V(x) - Ax$, we can then write the differential equation as

$$\dot{x} = Ax + \widetilde{R}(x). \tag{2.3.6}$$

In the special case $\tilde{R} \equiv 0$, the differential equation is linear and the subspaces $\mathbb{V}_i$ are invariant manifolds for the above differential equation. It is then natural to wonder if there exists invariant manifolds also for the non linear case. Note that the nonlinearity is small only in a neighborhood of zero, it is then natural to look for local invariant manifolds at zero.

We are thus interested in the solutions of (2.3.6) only in a neighborhood of zero. It is then convenient to modify the equation outside the ball $B(0, \delta)$ so that the dynamics is linear outside such a ball. This will allow us to look for a globally invariant manifold for the modified dynamics with the property of bein locally invariant for the original one.

Namely, let $\varphi \in \mathcal{C}^\infty(\mathbb{R}_+, [0, 1])$, be a decreasing function such that $\varphi(t) = 1$ for $t \leq \delta$ and $\varphi(t) = 0$ for $t \geq 2\delta$. We then define $R(x) = \widetilde{R}(x)\varphi(\|x\|)$. Clearly, if we construct an invariant manifold for the differential equation

$$\dot{x} = Ax + R(x),$$

then it is a locally invariant manifold for (2.3.6) as well. By the variation of constant formula we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}R(x(s))ds.$$

To put the problem into a more general context it is convenient to define, for a given τ large enough, the map $F \in \mathcal{C}^k$ such that $F(x(0)) = x(\tau)$.

Problem 2.6 *Prove that*

1. *F is invertible;*
2. *we can choose $\delta > 0$ such that $F(B(0, 3e^{\|A\|\tau}\delta)) \supset B(0, 2\delta)$;*
3. *$F(0) = 0$, $D_0F = e^{A\tau}$ and $D_xF = e^{A\tau}$ for $\|x\| \geq 3e^{\|A\|\tau}\delta$;*
4. *for each $\varepsilon > 0$ we can chose δ such that $\|D_xF - e^{A\tau}\|_\infty \leq \varepsilon$;*
5. *for each $\beta > \beta' > \alpha' > \alpha \geq 0$ there is τ such that $\|e^{-A\tau}|_{\mathbb{V}_1}\| \leq e^{-\beta'\tau}$ and $\|e^{A\tau}|_{\mathbb{V}_2}\| \leq e^{\alpha'\tau}$.*

Problem 2.7 *Show that a manifold W is locally invariant at zero for (2.3.6) if and only if it is so for F.*

The above shows the relevance of the following theorem

Theorem 2.3.1 *Let $F \in \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$, $k \geq 1$, be an invertible map from $\mathbb{R}^d$ to itself such that it enjoys the properties of Problem 2.6 and, for a sufficiently small ε, $\|D_xF - D_0F\|_\infty \leq \varepsilon$. Then, there exists a $\mathcal{C}^{k-1}$ locally invariant manifold W. In addition, W is dim($\mathbb{V}_1$) dimensional and tangent to $\mathbb{V}_1$ at zero.*

Proof. By the hypotheses $\sigma(D_0F)$ splits in two parts $\tilde{\Sigma}_1, \tilde{\Sigma}_2$. Let $\mathbb{V}_1, \mathbb{V}_2$ be the associated eigenspaces. By a change of variable we can assume that $\mathbb{V}_1 = \{(\xi, 0)\}_{\xi\in\mathbb{R}^{d_1}}$ and $\mathbb{V}_2 = \{(0, \eta)\}_{\xi\in\mathbb{R}^{d_2}}$. Also, let $\Pi_1(\xi, \eta) = (\xi, 0)$, $\Pi_2 = \mathbb{1} - \Pi_1$, $\Pi_1 D_0F\Pi_1 = \Lambda$ and $\Pi_2 D_0F\Pi_2 = \Gamma$. In addition,[3] the hypotheses imply that $\|\Lambda^{-1}\| \leq e^{-\beta}$ and $\|\Gamma\| \leq e^{\alpha}$ with $\alpha < \beta$.

The basic idea is to consider manifolds that can be described by a function $G : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ via $W = \{(\xi, G(\xi)\}_{\xi\in\mathbb{R}^{d_1}}$. Obviously we need to limit the set to which G might belong. To this end we define,

$$\Omega = \{G \in \mathcal{C}^k(\mathbb{R}^{d_1}, \mathbb{R}^{d_1}) \;:\; G(0) = 0,\ \|DG\|_\infty \leq 1\}.$$

Let

$$F(\xi, \eta) = (\Lambda\xi + A(\xi, \eta), \Gamma\eta + B(\xi, \eta)).$$

If $\|\eta\| \leq \|\xi\|$ and ε is small enough, we have that there exists $\beta' > \alpha$ such that

$$\|\Lambda\xi + A(\xi, \eta)\| \geq e^{\beta'}\|\xi\|.$$

Thus, for each $G \in \Omega$ the map $T_G(\xi) = \Lambda\xi + A(\xi, G(\xi))$ is invertible. Moreover, for $\|\xi\| \geq C\delta$ we have $T_G(\xi) = \Lambda\xi$. We can then describe the evolution of the manifolds of interest:

$$F(\xi, G(\xi)) = (T_G(\xi), S_G \circ T_G^{-1}(T_G(\xi)))$$

[3] For convenience I am renaming the constants α, β and, possibly, substituting F^n to F in order to offsets the constants coming from the equivalence of the norms in the new coordinates.

where $S_G(\xi) = \Gamma G(\xi) + B(\xi, G(\xi))$. Again note that, for $\|\xi\| \geq C\delta$ we have $S_G(\xi) = \Gamma G(\xi)$. It follows that the image manifold is described by the operator $K : \Omega \to \mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$

$$K(G)(\xi) = S_G \circ T_G^{-1}(\xi).$$

For $G \in \Omega$, $K(G)(0) = 0$. Also

$$D[K(G)] = \left[(\Gamma DG + \partial_\xi A + \partial_\eta ADG)(\Lambda + \partial_\xi B + \partial_\eta BDG)^{-1}\right] \circ T_G^{-1}.$$

Note that, if $DG(0) = 0$, then also $D(K(G))(0) = 0$.

From the above computations it follows that, for ε small enough, there exists $\sigma \in [0, 1]$ such that

$$\|D[K(G)]\|_\infty \leq \sigma \|DG\|_\infty + C\varepsilon < \|DG\|_\infty. \tag{2.3.7}$$

Accordingly, $K(\Omega) \subset \Omega$. A direct computation shows that, for $G_1, G_2 \in \Omega$,

$$\begin{aligned}
&\|T_{G_1} - T_{G_2}\|_\infty \leq C_\# \varepsilon \|G_1 - G_2\|_\infty \\
&\|S_{G_1} - S_{G_2}\|_\infty \leq (e^\alpha + C_\# \varepsilon)\|G_1 - G_2\|_\infty.
\end{aligned}$$

On the other hand, for all $\xi \in \mathbb{R}^{d_1}$,

$$\begin{aligned}
\|T_{G_1}^{-1}(\xi) - T_{G_2}^{-1}(\xi)\| &= \|T_{G_2}^{-1} \circ T_{G_2} \circ T_{G_1}^{-1}(\xi) - T_{G_2}^{-1}(\xi)\| \\
&\leq (e^{-\beta} + C_\# \varepsilon)\|T_{G_2} \circ T_{G_1}^{-1}(\xi) - T_{G_1} \circ T_{G_1}^{-1}(\xi)\| \\
&\leq C_\#(e^{-\beta} + C_\# \varepsilon)\varepsilon \|G_1 \circ T_{G_1}^{-1}(\xi) - G_2 \circ T_{G_1}^{-1}(\xi)\|.
\end{aligned}$$

To conclude we introduce the norm[4]

$$|||G||| = \sup_{\xi \in \mathbb{R}^{d_1}} \|G(\xi)\| \cdot \|\xi\|^{-1}.$$

Remark that if $G \in \Omega$, then $|||G||| \leq 1$. Next, note that

$$\begin{aligned}
\|K(G_1)(\xi) - K(G_2)(\xi)\| &\leq \|S_{G_1} \circ T_{G_1}^{-1}(\xi) - S_{G_1} \circ T_{G_2}^{-1}(\xi)\| \\
&\quad + \|S_{G_1} \circ T_{G_2}^{-1}(\xi) - S_{G_2} \circ T_{G_2}^{-1}(\xi)\| \\
&\leq (e^\alpha + C_\# \varepsilon)\|T_{G_1}^{-1}(\xi) - T_{G_2}^{-1}(\xi)\| + (e^\alpha + C_\# \varepsilon)\|G_1 \circ T_{G_2}^{-1}(\xi) - G_2 \circ T_{G_2}^{-1}(\xi)\|.
\end{aligned}$$

Accordingly,

$$|||K(G_1)(\xi) - K(G_2)||| \leq \left[C_\#(e^{-\beta} + \varepsilon)\varepsilon e^{-\beta'} + (e^\alpha + C_\# \varepsilon)e^{-\beta'}\right] |||G_1 - G_2|||$$

[4] This norm is necessary only because we do not assume $\alpha < 0$. If we would do so, then the usual sup norm would work perfectly.

Hence, provided ε is small enough, there exists $\sigma \in (0,1)$, such that for each $G_1, G_2 \in \Omega$

$$\|K(G_1) - K(G_2)\| \leq \sigma \|G_1 - G_2\| .$$

The above implies that K has a unique fixed point $G = \lim_{n\to\infty} K^n(0)$. In addition, G is of the form $G(\xi) = \|\xi\|\hat{G}(\xi)$ with $\hat{G} \in \mathcal{C}^0$.

We leave to the reader the task of checking that the contraction takes place in $\mathcal{C}^{k-1}$ as well. In particular, if $k \geq 2$, it is trivial to check that $DG(0) = 0$. □

From the above we directly obtain the following very useful result.

Theorem 2.3.2 (Center Manifold Theorem) *Let $F \in \mathcal{C}^k$ be an invertible map from $\mathbb{R}^d$ to itself such that it enjoys the properties (1-4) of Problem 2.6. Moreover assume that the spectrum of the matrix A now splits into three disjoint parts $\Sigma_- \cup \Sigma_0 \cup \Sigma_+$ such that $\mu \in \Sigma_-$ implies $\Re(\mu) \leq \alpha < 0$, $\mu \in \Sigma_0$ implies $\alpha < \Re(\mu) < \beta$ and $\mu \in \Sigma_+$ implies $\Re(\mu) \geq \beta > 0$. Let $\mathbb{V}_0$ be the eigenspace associated to Σ_0 and d_0 be its dimension. Then, there exists a $\mathcal{C}^{k-1}$ d_0 dimensional locally invariant manifold W. In addition, W is tangent to $\mathbb{V}_0$ at zero.*

Proof. Let $\mathbb{V}_+, \mathbb{V}_0, \mathbb{V}_-$ be the eigenspaces associated to the splitting of the spectrum and d_+, d_0, d_- be their dimensions. Simply apply Theorem 2.3.1 to F with the splittings $\Sigma_1 = \Sigma_+ \cup \Sigma_0$, $\Sigma_2 = \Sigma_-$ and to F^{-1} with the splitting $\Sigma_1 = \Sigma_+$, $\Sigma_2 = \Sigma_- \cup \Sigma_0$. In such a way we obtain two invariant manifolds: W^+ (the weak unstable manifold) and W^- (the weak stable manifold) respectively of dimension $d_+ + d_0$ and $d_- + d_0$. The reader can easily check that the hypotheses of the implicit function theorem apply and prove that $W = W^+ \cap W^-$ is a d_0 dimensional $\mathcal{C}^{k-1}$ locally invariant manifold tangent to $\mathbb{V}_0$ in zero.[5] □

2.4 Hadamard-Perron

Theorem 2.3.2 is quite general but it has a couple of disadvantages: a slightly annoying loss of regularity (from $\mathcal{C}^k$ to $\mathcal{C}^{k-1}$) and, most importantly, it does not provides any information on the dynamics when restricted to the invariant manifold which, in fact, can be pretty much anything. To eliminate such shortcoming it is necessary to consider situations in which there are no eighevalues with zero real part. This gives rise to a sharper results: the Hadamard-Perron theorem. We will discuss it in the simplest possible setting, also we will repeat several arguments to make this section independent on the previous one.

[5] To show that the matrix at zero is invertible, remember (2.3.7) which says that the manifolds are graphs of functions with derivative strictly less than one.

Definition 2.4.1 *Given a smooth map $T : X \to X$, X being a Riemannian manifold, and a fixed point $p \in X$ (i.e. $Tp = p$) we call (local) stable manifold (of size δ) a manifold $W^s(p)$ such that*[6]

$$W^s(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^n x, p) = 0\}.$$

Analogously, we will call (local) unstable manifold (of size δ) a manifold $W^u(p)$ such that

$$W^u(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^{-n} x, p) = 0\}.$$

It is quite clear that $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$ (Problem 2.8). Less clear is that these sets deserve the name "manifold." Yet, if one thinks of a linear map it is obvious that the stable and unstable manifolds at zero are just segments in the stable and unstable direction, the next Theorem shows that this is a quite general situation.

Theorem 2.4.2 (Hadamard-Perron) *Consider an invertible map $T : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $T \in \mathcal{C}^1(U, \mathbb{R}^2)$, such that $T0 = 0$ and*

$$D_0 T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \tag{2.4.8}$$

where $0 < \mu < 1 < \lambda$.[7] *That is, the map T is hyperbolic at the fixed point 0. Then there exists unique $\mathcal{C}^1$ stable and unstable manifolds at 0. Moreover, $\mathcal{T}_0 W^{s(u)} = E^{s(u)}$ where $E^{s(u)}$ are the expanding and contracting subspaces of $D_0 T$.*[8]

Remark 2.4.3 *There is an issue not completely addresses in our formulation of Hadamard-Perron theorem: the uniqueness of the manifolds.*[9] *It is not hard to prove that the $W^{s(u)}$ are indeed the only sets satisfying Definition 2.4.1 (see Problem 2.11).*

The proof of Theorem 2.4.2 will be done in two steps: first we will show the existence of the invariant manifolds and then we will prove the regularity.

[6]Sometime we will write $W^s_\delta(p)$ when the size really matters. By $B_\delta(x)$ we will always mean the open ball of radius δ centered at x.

[7]Notice that if $D_0 T$ has eigenvalues $0 < \mu < 1 < \lambda$ then one can always perform a change of variables such that (2.4.8) holds.

[8]By $\mathcal{T}_0 W^{s(u)}$ I mean the tangent space to the manifold (curve) W^u (or W^s) at the point zero.

[9]Namely the doubt may remain that a less regular set satisfying Definition 2.4.1 exists.

2.4.1 Existence of the invariant manifold: a fixed point argument

We will deal explicitly only with the unstable manifold since the stable one can be treated exactly in the same way by considering T^{-1} instead of T.

Proof of existence of the unstable manifold. Since the map is continuously differentiable for each $\varepsilon > 0$ we can choose $\delta > 0$ so that, in a 2δ-neighborhood of zero, we can write

$$T(x) = D_0Tx + R(x) \tag{2.4.9}$$

where $\|R(x)\| \leq \varepsilon\|x\|$, $\|D_xR\| \leq \varepsilon$.

The first step is to decide how to represent manifolds. In the present case, since we deal only with curves, it seems very reasonable to consider the set of curves $\Gamma_{\delta,c}$ passing through zero and "close" to being horizontal, that is the differentiable functions $\gamma : [-\delta, \delta] \to \mathbb{R}^2$ of the form

$$\gamma(t) = \begin{pmatrix} t \\ u(t) \end{pmatrix}$$

and such that $\gamma(0) = 0$; $\|(1, 0) - \gamma'\|_\infty \leq c$. It is immediately clear that any smooth curve passing through zero and with tangent vector, at each point, in the cone $\mathcal{C} := \{(a, b) \in \mathbb{R}^2 \mid |\frac{b}{a}| \leq c\}$, can be associated to a unique element of $\Gamma_{\delta,c}$, just consider the part of the curve contained in the strip $\{(x, y) \in \mathbb{R}^2 \mid |x| \leq \delta\}$. Moreover, if $\gamma \in \Gamma_{\delta,c}$ then $\gamma \subset B_{2\delta}(0)$, provided $c \leq 1/2$.

Notice that it suffices to specify the function u in order to identify uniquely an element in $\Gamma_{\delta,c}$. It is then natural to study the evolution of a curve through the change in the associated function.

To this end let us investigate how the image of a curve in $\Gamma_{\delta,c}$ under T looks like.

$$T\gamma(t) = \begin{pmatrix} \lambda t + R_1(t, u(t)) \\ \mu u(t) + R_2(t, u(t)) \end{pmatrix} := \begin{pmatrix} \alpha_u(t) \\ \beta_u(t) \end{pmatrix}.$$

At this point the problem is clearly that the image it is not expressed in the way we have chosen to represent curves, yet this is easily fixed. First of all, $\alpha_u(0) = \beta_u(0) = 0$. Second, by choosing $\varepsilon < \lambda$, we have $\alpha'_u(t) > 0$, that is, α_u is invertible. In addition, $\alpha_u([-\delta, \delta]) \supset [-\lambda\delta + \varepsilon\delta, \lambda\delta - \varepsilon\delta] \supset [-\delta, \delta]$, provided $\varepsilon \leq \lambda^{-1}$. Hence, α_u^{-1} is a well defined function from $[-\delta, \delta]$ to itself. Finally,

$$|\frac{d}{dt}\beta_u \circ \alpha_u^{-1}(t)| = \left|\frac{\beta'_u(\alpha_u^{-1}(t))}{\alpha'_u(\alpha_u^{-1}(t))}\right| \leq \frac{\mu c + \varepsilon}{\lambda - \varepsilon} \leq c$$

where, again, we have chosen $\varepsilon \leq \frac{c(\lambda-\mu)}{1+c}$.

We can then consider the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta,c}$ defined by

$$\tilde{T}\gamma(t) := \begin{pmatrix} t \\ \beta_u \circ \alpha_u^{-1}(t) \end{pmatrix} \tag{2.4.10}$$

which associates to a curve in $\Gamma_{\delta,c}$ its image under T written in the chosen representation. It is now natural to consider the set of functions $B_{\delta,c} = \{u \in \mathcal{C}^1([-\delta,\,\delta]) \mid u(0) = 0,\ |u'|_\infty \le c\}$ in the vector space $\mathrm{L}ip([-\delta,\,\delta])$.[10] As we already noticed $B_{\delta,c}$ is in one-one correspondence with $\Gamma_{\delta,c}$, we can thus consider the operator $\hat{T} : \mathrm{L}ip([-\delta,\,\delta]) \to \mathrm{L}ip([-\delta,\,\delta])$ defined by

$$\hat{T}u = \beta_u \circ \alpha_u^{-1} \tag{2.4.11}$$

From the above analysis follows that $\hat{T}(B_{\delta,c}) \subset B_{\delta,c}$ and that $\hat{T}u$ determines uniquely the image curve.

The problem is then reduced to studying the map $\hat{T}$. The easiest, although probably not the most productive, point of view is to show that $\hat{T}$ is a contraction in the sup norm. Note that this creates a little problem since $\mathcal{C}^1$ it is not closed in the sup norm (and not even $\mathrm{L}ip([-\delta,\,\delta])$ is closed). Yet, the set $B^*_{\delta,c} = \{u \in \mathrm{L}ip([-\delta,\,\delta]) \mid u(0) = 0, \sup_{t,s\in[-\delta,\,\delta]} \frac{|u(s)-u(t)|}{|t-s|} < c\}$ is closed (see Problem 2.9). Thus $\overline{B_{\delta,c}} \subset B^*_{\delta,c}$. This means that, if we can prove that the sup norm is contracting, then the fixed point will belong to $B^*_{\delta,c}$ and we will obtain only a Lipschitz curve. We will need a separate argument to prove that the curve is indeed smooth.

Let us start to verify the contraction property. Notice that

$$\alpha_u^{-1}(t) = \lambda^{-1}t + \lambda^{-1}R_1(\alpha_u^{-1}(t), u(\alpha_u^{-1}(t))),$$

thus, given $u_1, u_2 \in B_{\delta,c}$, by Lagrange Theorem

$$\begin{aligned} |\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| &\le \lambda^{-1}|\langle \nabla_\zeta R_1,\, (\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t), u_1(\alpha_{u_1}^{-1}(t)) - u_2(\alpha_{u_2}^{-1}(t)))\rangle| \\ &\le \frac{\varepsilon}{\lambda}\left\{|\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| + |u_1(\alpha_{u_2}^{-1}(t)) - u_2(\alpha_{u_2}^{-1}(t))|\right\}. \end{aligned}$$

This implies immediately

$$|\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| \le \frac{\lambda^{-1}\varepsilon}{1-\lambda^{-1}\varepsilon}\|u_1 - u_2\|_\infty. \tag{2.4.12}$$

On the other hand

$$\begin{aligned} |\beta_{u_1}(t) - \beta_{u_2}(t)| &\le \mu|u_1(t) - u_2(t)| + |\langle \nabla_\zeta R_2,\, (0, u_1(t) - u_2(t))\rangle| \\ &\le (\mu + \varepsilon)\|u_1 - u_2\|_\infty. \end{aligned} \tag{2.4.13}$$

[10] This are the Lipschitz functions on $[-\delta,\,\delta]$, that is the functions such that $\sup_{t,s\in[-\delta,\,\delta]} \frac{|u(s)-u(t)|}{|t-s|} < \infty$.

Moreover,

$$|\beta_u'(t)| \le \mu + \varepsilon. \tag{2.4.14}$$

Collecting the estimates (2.4.12, 2.4.13, 2.4.14) readily yields

$$\begin{aligned}\|\hat{T}u_1 - \hat{T}u_2\|_\infty &\le \|\beta_{u_1} \circ \alpha_{u_1}^{-1} - \beta_{u_1} \circ \alpha_{u_2}^{-1}\|_\infty + \|\beta_{u_1} \circ \alpha_{u_2}^{-1} - \beta_{u_2} \circ \alpha_{u_2}^{-1}\|_\infty \\ &\le \left\{ [\mu + \varepsilon] \frac{\lambda^{-1}\varepsilon}{1 - \lambda^{-1}\varepsilon} + (\mu + \varepsilon) \right\} \|u_1 - u_2\|_\infty \\ &\le \sigma \|u_1 - u_2\|_\infty,\end{aligned}$$

for some $\sigma \in (0,1)$, provided ε is chosen small enough.

Clearly, the above inequality immediately implies that there exists a unique element $\gamma_* \in \Gamma_{\gamma,c}$ such that $\tilde{T}\gamma_* = \gamma_*$, this is the *local* unstable manifold of 0. □

2.4.2 Regularity of invariant manifolds–a cone field argument

As already mentioned, a separate argument is needed to prove that γ_* is indeed a $\mathcal{C}^1$ curve.

To prove this, one possibility could be to redo the previous fixed point argument trying to prove contraction in $\mathcal{C}^1_{Lip}$ (the $\mathcal{C}^1$ functions with Lipschitz derivative); yet this would require to increase the regularity requirements on T. A more geometrical, more instructive and more inspiring approach is the following.

Proof of the regularity of the unstable manifold. Let $\delta > 0$ such that the arguments of section 2.4.1 apply. We want to define local cone fields in the region $\{\xi = (\xi_x, \xi_y) \in \mathbb{R}^2 \ : \ |\xi_x| < \delta\}$. For each $|u| \le c\delta$ and $0 < \theta \le c\delta$ we define the affine cone field $\mathcal{C}_\theta(\xi, u) := \{\xi + (a,b) \in \mathbb{R}^2 \ : \ |b - au| \le \theta\,|a|\}$.[11] As we need to perform a local argument we must localise the cones. To this end we will intersect them with cylinders of the form $D_h(\xi) = \{\xi + (a,b) \in \mathbb{R}^2 \ : \ |a| \le h\}$. We define thus a local affine cone field (that in the following we will simply call *cone field*) by

$$\mathcal{C}_{\theta,h}(\xi, u) = \mathcal{C}_\theta(\xi, u) \cap D_h(\xi) = \{\xi + (a,b) \in \mathbb{R}^2 \ : \ |a| \le h;\ |b - au| \le \theta\,|a|\}.$$

By the construction in Section 2.4.1, $D_h(\xi) \cap \gamma_* \subset \mathcal{C}_{c\delta,h}(\xi, 0)$ for each $\xi \in \gamma_*$. We will study the evolution of such a cone field on γ_*.

For all $\eta \in \mathcal{C}_{\theta,h}(\xi, u)$, if $(a,b) = \eta - \xi$ and $(\alpha, \beta) = T\eta - T\xi$, it holds

$$(\alpha, \beta) = D_0T(a,b) + \mathcal{O}(\varepsilon|a|) = (\lambda a, \mu b) + \mathcal{O}(\varepsilon|a|).$$

[11] A set $\mathcal{C}$ is a cone iff, for all $y \in \mathcal{C}$ and $\alpha \in \mathbb{R}$, $\alpha y \in \mathcal{C}$. A set $\mathcal{C}$ is an affine cone if there exists z such that $\{y - z \ : \ y \in \mathcal{C}\}$ is a cone.

and, at the same time, since T is $\mathcal{C}^1$, $\|(\alpha,\beta)-D_\xi T(a,b)\|\leq \varepsilon\theta|a|$ provided $h\leq h_\theta$ for some h_θ small enough. Thus, setting $(\alpha',\beta')=D_\xi T(a,ua)$ and $u'=\frac{\beta'}{\alpha'}$, one can compute

$$\begin{aligned}\|(\alpha,\beta)-(\alpha',\beta')-(0,\mu(b-ua))\| &\leq \|(D_\xi T-D_0T)(0,b-ua)\|+\theta\varepsilon|a|\\ &\leq C\theta\varepsilon|a|.\end{aligned}$$

Hence,

$$\left|\frac{\beta}{\alpha}-u'\right|\leq\left|\frac{\beta}{\alpha}-\frac{\beta'}{\alpha}\right|+\left|\frac{\beta'}{\alpha}\right|\left|1-\frac{\alpha}{\alpha'}\right|\leq\frac{\mu\theta}{\lambda-C\varepsilon}+\frac{(\mu+C\varepsilon)C\theta\varepsilon}{(\lambda-C\varepsilon)^2}.$$

Accordingly, if $h\leq h_\theta$, then there exists $\sigma\in(0,1)$ such that

$$D_h(T\xi)\cap T\mathcal{C}_{\theta,h}(\xi,u)\subset\mathcal{C}_{\sigma\theta,h}(T\xi,u'). \tag{2.4.15}$$

A similar, but rougher, computation yields

$$D_h(T\xi)\cap T\mathcal{C}_{\theta,h}(\xi,u)\subset\mathcal{C}_{\theta,h}(T\xi,0). \tag{2.4.16}$$

Finally, let $\xi\in\gamma_*$, then, for each $n\in\mathbb{N}$, $T^{-n}\xi\in\gamma_*$ and $\gamma_*\cap D_{h_n}(T^{-n}\xi)\subset\mathcal{C}_{c\delta,h_n}(T^{-n}\xi,0)$. Thus, for all $h_n\leq h_{\sigma^nc\delta}$, (2.4.15) implies[12]

$$\begin{aligned}\gamma_*\cap D_{h_n}(\xi)&\subset T^n\mathcal{C}_{c\delta,h_n}(T^{-n}\xi,0)\cap D_{h_n}(\xi)\\ &=T^{n-1}\left(T\mathcal{C}_{c\delta,h_n}(T^{-n}\xi,0)\cap D_{h_n}(T^{n-1}\xi)\right)\cap D_{h_n}(\xi)\\ &\subset T^{n-1}\mathcal{C}_{\sigma c\delta,h_n}(T^{-n+1}\xi,v_{n,1})\cap D_{h_n}(\xi)\\ &\subset\mathcal{C}_{\sigma^nc\delta,h_n}(\xi,v_n)\end{aligned} \tag{2.4.17}$$

where $(a,av_{n,k}(\xi))=D_{T^{-n}\xi}T^k(1,0)$, for some $a\in\mathbb{R}_+$, and $v_n(\xi)=v_{n,n}(\xi)$. The last relevant fact is that the limit

$$v_*=\lim_{n\to\infty}v_n \tag{2.4.18}$$

exists. The proof of this fact is left as an entertainment for the reader (see Problem 2.10). Using (2.4.17), (2.4.18) and remembering that γ_* admits the parametrization $\gamma_*(t)=(t,u_*(t))$ we can compute the derivative. Indeed, let τ so that $(\tau,u_*(\tau))=\xi\in\gamma_*$, then for each $\varepsilon>0$ let m so that $\sigma^mc\delta\leq\frac{\varepsilon}{2}$ and $|v_m-v_*|\leq\frac{\varepsilon}{2}$, then for each $h\leq h_m$ holds

$$\begin{aligned}\left|\frac{u_*(\xi+h)-u_*(\xi)-v_*h}{h}\right|&\leq\left|\frac{u_*(\xi+h)-u_*(\xi)-v_mh}{h}\right|+\frac{\varepsilon}{2}\\ &\leq c\sigma^m\delta+\frac{\varepsilon}{2}\leq\varepsilon.\end{aligned}$$

[12] Remember that the map T expands in the first coordinate, hence $TD_h(\xi)\supset D_h(T\xi)$ provided $\xi\in\mathcal{C}_{c\delta,\delta}(0,0)$ and h is small enough.

That is, γ_* is differentiable and

$$\gamma_*'(\tau) = (1, v_*). \tag{2.4.19}$$

□

There is another point of view that can be adopted in the study of stable and unstable manifolds: to "grow" the manifolds. This is done by starting with a very short curve in $\Gamma_{\delta,c}$, e.g. $\gamma_0(t) = (t, 0)$ for $t \in [\lambda^{-n}\delta, \lambda^n\delta]$, and showing that the sequence $\gamma_n := T^n\gamma_0$ converges to a curve in the strip $[-\delta, \delta]$, independent of γ_0. From a mathematical point of view, in the present case, it corresponds to spell out explicitly the proof of the fixed point theorem. Nevertheless, it is a more suggestive point of view and it is more convenient when the hyperbolicity is non uniform. For example consider the map[13].

$$T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 2x - \sin x + y \\ x - \sin x + y \end{pmatrix} \tag{2.4.20}$$

then 0 is a fixed point of the map but

$$D_0T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not hyperbolic, yet, due to the higher order terms, there exist stable and unstable manifolds (see Problems 2.13, 2.14, 2.15).

Problems

2.8. Show that, if p is a fixed point, then $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$.

2.9. Prove that the set $B^*_{\delta,c}$ in section 2.4.1 is closed with respect to the sup norm $\|u\|_\infty = \sup_{t\in[-\delta,\delta]} |u(t)|$.

2.10. Prove that the limit in (2.4.19) is well defined and depend continuously on ξ.

2.11. Prove that, in the setting of Theorem 2.4.2, the unstable manifold is unique.

2.12. Show that Theorem 2.4.2 holds assuming only $T \in \mathcal{C}^1(U, U)$.

2.13. Consider the Lewowicz map (2.4.20), show that, given the set of curves $\Gamma_{\delta,c} := \{\gamma : [-\delta, \delta] \to \mathbb{R}^2 \mid \gamma(t) = (t, u(t));\ \gamma(0) = 0;\ |u'(t)| \in [c^{-1}t, ct]\}$, it is possible to construct the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta(1+c^{-1}\delta),\, c}$ in analogy with (2.4.10).

[13] Some times this is called *Lewowicz map*

2.14. In the case of the previous problem show that for each $\gamma_i \in \Gamma_{\delta,c}$ holds $d(\tilde{T}\gamma_1, \tilde{T}\gamma_2) \leq (1 - c\delta)d(\gamma_1, \gamma_2)$.

2.15. Show that for the Lewowicz map zero has a unique unstable manifold.

Hints to solving the Problems

2.1. Use the implicit function theorem on the one parameter vector fields $V(\lambda) = V + \lambda(W - V)$.

2.4. By the variation of constant method follows that

$$\phi_t(x) = e^{At}x + \int_0^t e^{A(t-s)} R(\phi_s(x))ds.$$

2.10 By (2.4.16) and arguing as in (2.4.17) it follows

$$T^n \mathcal{C}_{c\delta,h_n}(T^{-n}\xi, 0) \cap D_{h_n}(\xi) \subset T^{n-1}\mathcal{C}_{c\delta,h_n}(T^{-n+1}\xi, 0) \cap D_{h_n}(\xi) \subset \mathcal{C}_{\sigma^{n-1}c\delta,h_n}(\xi, v_{n-1}(\xi)).$$

Since, for a small enough, $T^n(T^{-n}\xi + (a, 0)) = \xi + aD_{T^{-n}\xi}T^n(1, 0) + o(a)$, it follows that $(a, v_n(\xi)a) \in \mathcal{C}_{\sigma^{n-1}c\delta,h_n}(\xi, v_{n-1}(\xi))$. Hence $|v_n(\xi) - v_{n-1}(\xi)| \leq \sigma^{n-1}c\delta$. From this the Problem easily follows.

2.11. This amounts to showing that the set of points that are attracted to zero are exactly the manifolds constructed in Theorem 2.4.2. Use the local hyperbolicity to show that.

2.14. Grow the manifolds, that is, for each $n > 1$ define $\delta_n := \frac{\rho}{n}$. Show that one can choose ρ such that $\delta_{n-1} \geq \delta_n(1 + c^{-1}\delta_n)$. according to Problem 2.13 it follows that $\tilde{T} : \Gamma_{\delta_n,c} \to \Gamma_{\delta_{n-1},c}$. Moreover,

$$d(\tilde{T}^{n-1}\gamma_1, \tilde{T}^{n-1}\gamma_2) \leq \prod_{i=1}^{n}(1 - c\delta_i)d(\gamma_1, \gamma_2).$$

Finally, show that, setting $\gamma_n(t) = (0, t) \in \Gamma_{\delta_n,c}$, the sequence $\tilde{T}^{n-1}\gamma_n$ is a Cauchy sequence that converges in $\mathcal{C}^0$ to a curve in $\Gamma_{1,c}$ invariant under $\tilde{T}$.

Notes

The content of this section is quite standard and rather sketchy, it is intended only to introduce the reader to some basic ideas and techniques. The treatment of the Hadamard-Perron Theorem follows mostly [HK95].