# Appendix A

# Fixed Points Theorems (an idiosincratic selection)

In this appendinx I provide some standard and less standard Fixed poins theorems. These constitute a very partial introduction to the subject. The choice of the topics if motivated by the needs of the previous chapters.

## A.1 Banach Fixed Point Theorem

**Theorem A.1.1 (Fixed point contraction)** *Given a Banach space $\mathcal{B}$, a bounded closed set $A \subset \mathcal{B}$ and a map $K : A \to \mathcal{B}$ if*

*i)* $K(A) \subset A$,

*ii) there exists* $\sigma \in (0,1)$ *such that* $\|K(v) - K(w)\| \leq \sigma\|v - w\|$ *for each* $v, w \in A$,

*then there exists a unique* $v_* \in A$ *such that* $Kv_* = v_*$.

Proof. Since $A$ is bounded $\sup_{x,y\in A} \|x-y\| = L < \infty$, i.e. it has a finite diameter. Let $a_0 \in A$ and consider the sequence of points defined recursively by $a_{n+1} = K(a_n)$ and the sequence of sets $A_0 = A$ and $A_{n+1} = K(A_n) \subset A$. Let $d_n := \sup_{x,y\in A_n} \|x - y\|$ be the diameter of $A_n$. Then if $x, y \in A_n$, we have

$$\|K(y) - K(x)\| \leq \sigma\|x - y\| \leq \sigma d_n.$$

That is $d_{n+1} \leq \sigma d_n \leq \sigma^n L$. This means that, for each $n, m \in \mathbb{N}$, $a_n, a_0 \in A$ and $a_m, a_{n+m} \in A_m$, hence $\|a_{n+m} - a_m\| \leq \sigma^m L$. That is $\{a_n\} \subset A$ is a Cauchy sequence and, being $\mathcal{B}$ a Banach space, it must have an accumulation point $v_* \in \mathcal{B}$. Moreover since $A$ is closed it must be $v_* \in A$. Clearly

$$\begin{aligned}\|Kv_* - v_*\| &= \lim_{n\to\infty} \|Kv_* - a_n\| = \lim_{n\to\infty} \|Kv_* - Ka_{n-1}\| \\ &\leq \lim_{n\to\infty} \sigma\|v_* - a_{n-1}\| = 0.\end{aligned}$$

Hence, $v_*$ is a fixed point. Next, suppose there exist $u \in A$, such that $Ku = u$. Then

$$\|u - v_*\| = \|K(u - v_*)\| \leq \sigma \|u - v * \|$$

implies $u = v_*$. □

**Corollary A.1.2** *Given a Banach space $\mathcal{B}$ and a map $K : \mathcal{B} \to \mathcal{B}$ with the property that there exists $\sigma \in (0, 1)$ such that $\|K(v) - K(w)\| \leq \sigma \|v - w\|$ for each $v, w \in \mathcal{B}$, then there exists a unique $v_* \in \mathcal{B}$ such that $Kv_* = v_*$.*

PROOF. To prove the theorem, for each $L \in \mathbb{R}_+$ consider the sets $B_L := \{v \in \mathcal{B} : \|v\| \leq L\}$. Then $\|K(v)\| \leq \|K(v) - K(0)\| + \|K(0)\| \leq \sigma\|v\| + \|K(0)\| \leq \sigma L + \|K(0)\|$. Thus, for each $L \geq (1 - \sigma)^{-1}\|K(0)\|$ we have that $K(B_L) \subset B_L$. The existence follows by applying Theorem A.1.1. The uniqueness follows by the same argument used at end of the proof of Theorem A.1.1. □

## A.2 Hilbert metric and Birkhoff theorem

In this section we will see that the Banach fixed point theorem can produce unexpected results if used with respect to an appropriate metric: projective metric.

Projective metrics are widely used in geometry, not to mention the importance of their generalizations (e.g. Kobayashi metrics) for the study of complex manifolds [IK00]. It is quite surprising that they play a major rôle also in our situation, [Liv95].

Here we limit ourselves to a few word on the Hilbert metric, a quite important tool in hyperbolic geometry.

### A.2.1 Projective metrics

Let $C \in \mathbb{R}^n$ be a strictly convex compact set. For each two point $x, y \in C$ consider the line $\ell = \{\lambda x + (1 - \lambda y) \mid \lambda \in \mathbb{R}\}$ passing through $x$ and $y$. Let $\{u, v\} = \partial C \cap \ell$ and define[1]

$$\Theta(x, y) = \left| \ln \frac{\|x - u\| \|y - v\|}{\|x - v\| \|y - v\|} \right|$$

(the logarithm of the cross ratio). By remembering that the cross ratio is a projective invariant and looking at Figure A.1 it is easy to check that $\Theta$ is indeed a metric. Moreover the distance of an inner point from the boundary is always infinite. One can also check that if the convex set is a disc then the disc with the Hilbert metric is nothing else than the Poincaré disc.

[1]Remark that $u$, $v$ can also be $\infty$.

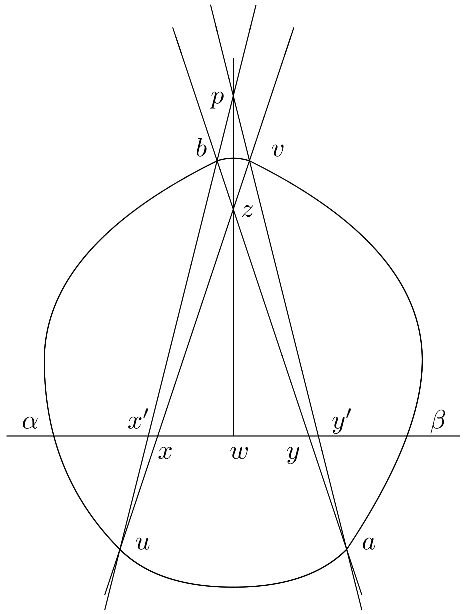


Figure A.1: Hilbert metric

The object that we will use in our subsequent discussion are not convex sets but rather convex cones, yet their projectivization is a convex set and one can define the Hilbert metric on it (whereby obtaining a semi-metric for the original cone). It turns out that there exists a more algebraic way of defining such a metric, which is easier to use in our context. Moreover, there exists a simple connection between vector spaces with a convex cone and vector lattices (in a vector lattice one can always consider the positive cone). This justifies the next digression in lattice theory.[2]

Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[3] We require the partial order to be "continuous," i.e. given $\{f_n\} \in \mathbb{V}$ $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for each $n$, then $f \succeq g$. We call such vector lattices "integrally closed." [4]

[2]For more details see [Bir57], and [Nus88] for an overview of the field.

[3]We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $f, g \in \mathbb{V}$ $f \succeq g \iff f - g \succeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+\backslash\{0\}$ $f \succeq 0 \Longrightarrow \lambda f \succeq 0$; for each $f \in \mathbb{V}$ $f \succeq 0$ and $f \preceq 0$ imply $f = 0$ (antisymmetry of the order relation).

[4]To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $f, g \in \mathbb{V}$, if

We define the closed convex cone [5] $\mathcal{C} = \{f \in \mathbb{V} \mid f \neq 0,\, f \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ iff there exists $\lambda \in \mathbb{R}^+ \backslash \{0\}$ such that $f = \lambda g$. If we call $\widetilde{\mathcal{C}}$ the quotient of $\mathcal{C}$ with respect to $\sim$, then $\widetilde{\mathcal{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa. The reader must therefore be advised that "$\preceq$" will mean different things in different contexts.

It is then possible to define a projective metric $\Theta$ (Hilbert metric),[6] in $\mathcal{C}$, by the construction:

$$\begin{aligned}
\alpha(f,\, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\} \\
\beta(f,\, g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\} \\
\Theta(f,\, g) &= \log\left[\frac{\beta(f,\, g)}{\alpha(f,\, g)}\right]
\end{aligned}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The relevance of the above metric in our coontex is due to the following Theorem by Garrett Birkhoff.

**Theorem A.2.1** *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed vector lattices; $\mathcal{L} : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$ with $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$. Let $\Theta_i$ be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,\, g \in T(\mathcal{C}_1)} \Theta_2(f,\, g)$ we have*

$$\Theta_2(\mathcal{L}f,\, \mathcal{L}g) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(f,\, g) \qquad \forall f,\, g \in \mathcal{C}_1$$

*(*$\tanh(\infty) \equiv 1$*).*

Proof. The proof is provided for the reader convenience.

Let $f,\, g \in \mathcal{C}_1$, on the one hand if $\alpha = 0$ or $\beta = \infty$, then the inequality is obviously satisfied. On the other hand, if $\alpha \neq 0$ and $\beta \neq \infty$, then

$$\Theta_1(f,\, g) = \ln \frac{\beta}{\alpha}$$

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$\alpha_n f \succeq g$, then $\alpha f \succeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[5] Here, by "cone," we mean any set such that, if $f$ belongs to the set, then $\lambda f$ belongs to it as well, for each $\lambda > 0$.

[6] In fact, we define a semi–metric, since $f \sim g \Rightarrow \Theta(f,\, g) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$.

where $\alpha f \preceq g$ and $\beta f \succeq g$, since $\mathbb{V}_1$ is integrally closed. Notice that $\alpha \geq 0$, and $\beta \geq 0$ since $f \succeq 0$, $g \succeq 0$. If $\Delta = \infty$, then the result follows from $\alpha \mathcal{L} f \preceq \mathcal{L} g$ and $\beta \mathcal{L} f \succeq \mathcal{L} g$. If $\Delta < \infty$, then, by hypothesis,

$$\Theta_2\left(\mathcal{L}(g - \alpha f),\, \mathcal{L}(\beta f - g)\right) \leq \Delta$$

which means that there exist $\lambda$, $\mu \geq 0$ such that

$$\begin{aligned} \lambda \mathcal{L}(g - \alpha f) &\preceq \mathcal{L}(\beta f - g) \\ \mu \mathcal{L}(g - \alpha f) &\succeq \mathcal{L}(\beta f - g) \end{aligned}$$

with $\ln \frac{\mu}{\lambda} \leq \Delta$. The previous inequalities imply

$$\begin{aligned} \frac{\beta + \lambda\alpha}{1 + \lambda} \mathcal{L} f &\succeq \mathcal{L} g \\ \frac{\mu\alpha + \beta}{1 + \mu} \mathcal{L} f &\preceq \mathcal{L} g. \end{aligned}$$

Accordingly,

$$\begin{aligned} \Theta_2(\mathcal{L} f,\, \mathcal{L} g) &\leq \ln \frac{(\beta + \lambda\alpha)(1 + \mu)}{(1 + \lambda)(\mu\alpha + \beta)} = \ln \frac{e^{\Theta_1(f,\, g)} + \lambda}{e^{\Theta_1(f,\, g)} + \mu} - \ln \frac{1 + \lambda}{1 + \mu} \\ &= \int_0^{\Theta_1(f,\, g)} \frac{(\mu - \lambda) e^{\xi}}{(e^{\xi} + \lambda)(e^{\xi} + \mu)} d\xi \leq \Theta_1(f,\, g) \frac{1 - \frac{\lambda}{\mu}}{\left(1 + \sqrt{\frac{\lambda}{\mu}}\right)^2} \\ &\leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\, g). \end{aligned}$$

□

**Remark A.2.2** *If $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, then it follows that $\Theta_2(\mathcal{L} f,\, \mathcal{L} g) \leq \Theta_1(f,\, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of $\mathcal{L}$ (see [Bir79] for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the strength of the Hilbert metric compared to other, more usual, metrics. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

**Lemma A.2.3** *Let $\|\cdot\|$ be a norm on the vector lattice $\mathbb{V}$, and suppose that, for each $f$, $g \in \mathbb{V}$,*

$$-f \preceq g \preceq f \Longrightarrow \|f\| \geq \|g\|.$$

*Then, given $f$, $g \in \mathcal{C} \subset \mathbb{V}$ for which $\|f\| = \|g\|$,*

$$\|f - g\| \leq \left(e^{\Theta(f,\,g)} - 1\right) \|f\|.$$

Proof. We know that $\Theta(f,\,g) = \ln \frac{\beta}{\alpha}$, where $\alpha f \preceq g$, $\beta f \succeq g$. This implies that $-g \preceq 0 \preceq \alpha f \preceq g$, i.e. $\|g\| \geq \alpha\|f\|$, or $\alpha \leq 1$. In the same manner it follows that $\beta \geq 1$. Hence,

$$\begin{aligned} g - f \preceq&(\beta - 1)f \preceq (\beta - \alpha)f \\ g - f \succeq&(\alpha - 1)f \succeq -(\beta - \alpha)f \end{aligned}$$

which implies

$$\|g - f\| \leq (\beta - \alpha)\|f\| \leq \frac{\beta - \alpha}{\alpha}\|f\| = \left(e^{\Theta(f,\,g)} - 1\right) \|f\|.$$

□

Many normed vector lattices satisfy the hypothesis of Lemma 1.3 (e.g. Banach lattices[7]); nevertheless, we will see that some important examples treated in this paper do not.

## A.2.2 An application: Perron-Frobenius

Consider a matrix $L : \mathbb{R}^n \to \mathbb{R}^n$ of all strictly positive elements: $L_{ij} \geq \gamma > 0$. The Perron-Frobenius theorem states that there exists a unique eigenvector $v^+$ such that $v_i^+ > 0$, in addition the corresponding eigenvalue $\lambda$ is simple, maximal and positive. There quite a few proofs of this theorem a possible one is based on Birkhoff theorem. Consider the cone $\mathcal{C}^+ = \{v \in \mathbb{R}^2 \mid v_i \geq 0\}$, then obviously $L\mathcal{C}^+ \subset \mathcal{C}^+$. Moreover an explicit computation (see

**Problem A.1** *shows that*

$$\Theta(v, w) = \ln \sup_{ij} \frac{v_i w_j}{v_j w_i}. \tag{A.2.1}$$

[7]A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\|\,|f|\,\| = \|f\|$ for each $f \in \mathbb{V}$, where $|f|$ is the least upper bound of $f$ and $-f$. For this definition to make sense it is necessary to require that $\mathbb{V}$ is "directed," i.e. any two elements have an upper bound.

Then, setting $M = \max_{ij} L_{ij}$, it follows that

$$\Theta(Lv, Lw) \leq 2 \ln \frac{M}{\gamma} := \Delta < \infty.$$

We have then a contraction in the Hilbert metric and the result follows from usual fixed points theorems. Note that, since $\Theta(v, \lambda v) = 0$, for all $\lambda \in \mathbb{R}^+$, the fixed point $v_+ \in \mathbb{R}^n$ is only projective, that is $Lv_+ = \lambda v_+$ for some $\lambda \in \mathbb{R}$; in other words, we have an eigenvalue.

Remark that $L^*$ satisfies the same conditions as $L$, thus there exists $w^+ \in \mathcal{C}^+$, $\mu \in \mathbb{R}^+$, such that $L^* w^+ = \mu w^+$. Next, define $\rho_1(v) = |\langle w^+, v \rangle|$ and $\rho_2(v) = \|v\|$. It is easy to check that they are two homogeneous forms of degree one adapted to the cone.

In addition, if $\rho_1(v) = \rho_2(v)$, then $\rho_1(L^n v) = \rho_1(L^n w)$. Hence, by Lemma A.2.3

$$\begin{aligned} \|L^n v - L^n w\| &\leq \left(e^{\Theta(L^n v, L^n w)} - 1\right) \min\{\|L^n v\|, \|L^n w\|\} \\ &\leq K \Lambda^n \min\{\|L^n v\|, \|L^n w\|\}, \end{aligned} \tag{A.2.2}$$

for some constant $K$ depending only on $v, w$. The estimate A.2.2 means that all the vectors in the cone grow at the same rate. In fact, for all $v \in \text{int}\mathcal{C}$,

$$\|\lambda^{-n} L^n v - \lambda^{-n} L^n w\| \leq K \Lambda^n.$$

Hence, $\lim_{n\to\infty} \lambda^{-n} L^n v = v_+$.

Finally, consider $\mathbb{V}_1 = \{v \in \mathbb{V} \mid \langle w^+, v \rangle = 0\}$. Clearly $L\mathbb{V}_1 \subset \mathbb{V}_1$ and $\mathbb{V}_1 \oplus \text{span}\{v_+\} = \mathbb{V}$. Let $w \in \mathbb{V}_1$, clearly there exists $\alpha \in \mathbb{R}^+$ such that $\alpha v_+ + w \in \mathcal{C}$,[8] thus

$$\|L^n w\| \leq \|L^n(\alpha v_+ + w) - \alpha L^n v_+\| \leq L \Lambda^n \lambda^n.$$

This immediately implies that $L$ restricted to the subspace $\mathbb{V}_1$ has spectral radius less that $\lambda\Lambda$. In other words, $\lambda$ is the maximal eigenvalue, it is simple and any other eigenvalue must be smaller than $\lambda\Lambda$. We have thus obtained an estimate of the spectral gap between the first and the second eigenvalue.

# Notes

For more details on Hilbert metrics see [Bir79], and [Nus88] for an overview of the field.

[8]this is a special case of the general fat that any vector can be written as the linear combination of two vectors belonging to the cone.