# BEHAVIOUR CLOSE TO A GENERIC FIXED POINT

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Let us consider the differential equation

$$\dot{x} = V(x) \tag{0.1}$$

where $x \in \mathbb{R}^n$ and $V \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Suppose that $V(x_0) = 0$ and $\det(D_{x_0}V) \neq 0$.

**Exercize 1.** *Note that the condition* $\det(D_{x_0}V) \neq 0$ *can always be achieved by a small* $\mathcal{C}^1$ *change of the vector field. On the contrary, the existence of a zero of the vector field cannot be avoided by small* $\mathcal{C}^1$ *changes of the vector field: prove that if* $W$ *is a vector field* $\mathcal{C}^1$ *close to* $V$*, then there exists a* $x_*$ *close to* $x_0$ *such that* $W(x_*) = 0$*, and* $D_{x_*}W$ *is close to* $D_{x_0}V$.[1] *In this sense we will say that the above conditions are* generic *(more on this concept later).*

It is then necessary to understand the behavior of the equation in the vicinity of the point $x_0$. First of all, by a translation, we can assume without loss of generality $x_0 = 0$. Then we can develop $V$ by the Taylor formula to obtain

$$\dot{x} = Ax + R(x) \tag{0.2}$$

where $\|R(x)\| \leq C\|x\|^2$ and $\|D_xR\| \leq C\|x\|$, for all $\|x\| \leq 1$.

**Exercize 2.** *Show that, by a linear change of variable, one can transform* $A$ *in its Jordan canonical form. Show then that, by an arbitrary small* $\mathcal{C}^1$ *change of the vector field one can eliminate all the Jordan blocks and insure that all the eigenvalues have real part different from zero: this is called the* hyperbolic case.

Since the hyperbolic case is generic, we will limit to it our considerations. We will start by considering the case in which all the eigenvalue of $A$ have real part strictly smaller than zero.

**Exercize 3.** *Prove that if* $A$ *is diagonal with eigenvalues with real part strictly smaller than zero, then there exists* $\sigma > 0$ *such that, for all* $x \in \mathbb{C}^n$,[2]

$$\langle x, (A + A^*)x\rangle \leq -\sigma\langle x, x\rangle$$

*Prove that if* $A$ *has only simple (that is with algebraic multiplicity one) eigenvalues, then there exists a positive matrix* $B$ *(that is* $B^* = B$ *and* $\langle x, Bx\rangle > 0$ *for all* $x \neq 0$*) such that*

$$\langle x, B(A + B^{-1}A^*B)x\rangle \leq -\sigma\langle x, Bx\rangle$$

*Prove the same for a general matrix* $A$ *with all the eigenvalues with real part strictly smaller than zero.*

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[1] Hint: Use the implicit function theorem on the one parameter vector fields $V(\lambda) = V + \lambda(W - V)$.

[2] As usual $\langle x, y\rangle := \sum_{i=1}^n \bar{x}_i y_i$ where $\bar{a}$ is the complex conjugate of $a$. Moreover by $A^*$ we mean the adjoint of $A$.

It is well known that the linear part of (0.2) has solutions that tend to zero exponentially fast, the question is: does the same holds true for the solutions of the equation (0.2)?

To see it, consider $z := \langle x, x\rangle$,

$$\frac{d}{dt}z = \langle x, Ax+R(x)\rangle + \langle Ax+R(x), x\rangle = \langle x, (A+A^*)x\rangle + \mathcal{O}(\|x\|^3) \leq -\sigma z + \mathcal{O}(z^{\frac{3}{2}}).$$

If we assume $\|x\| \leq \frac{\sigma}{2}$, then we have

$$\frac{d}{dt}z \leq -\frac{\sigma}{2}z$$

which implies that also the solutions of (0.2) tend exponentially fast to zero.[3]

Yet, the above result is far from being satisfactory: it is possible to tend to zero in many different ways and it would be nice to understand better how this happens.

Let us start with a very simple example: $x \in \mathbb{R}$, $A = -1$, $R(x) = bx^2$. Then the equation reads

$$\dot{x} = -x + bx^2. \tag{0.3}$$

If we consider the change of variables

$$z = \Psi(x) = \frac{x}{1 - bx}$$

we have

$$\dot{z} = \frac{-x + bx^2}{1 - bx} + \frac{bx(-x + bx^2)}{(1 - bx)^2} = -\frac{x}{1 - bx} = -z.$$

We have just seen that in a neighborhood of size smaller than $b^{-1}$ of zero we have a diffeomorphism that conjugate the solution of (0.3) with its linear part.

One can then suspect that this is always the case. This is not so: consider

$$\begin{aligned} \dot{x} &= -2x + cy^2 \\ \dot{y} &= -y \end{aligned} \tag{0.4}$$

Let us consider a change of variables

$$\begin{aligned} z &= x + \alpha x^2 + \beta xy + \gamma y^2 + q(x, y) \\ \eta &= y + p(x, y) \end{aligned}$$

where $q$ is of third order and $p$ of second. Substituting in (0.4) one can see that it is always possible to choose $p \equiv 0$, while the first of the (0.4) yields

$$\dot{z} = -2x + cy^2 - 2x(2\alpha x + \beta y) - y(\beta x + 2\gamma y) + \mathcal{O}(3)$$

where by $\mathcal{O}(3)$ we designate third order terms. If we try to impose the right hand side of the above equation equal to $-2z$ (up to second order) we obtain

$$-2\alpha x^2 - 2\beta xy - 2\gamma y^2 = -4\alpha x^2 - 3\beta xy - (2\gamma + c)y^2$$

that does not admit any solutions if $c \neq 0$.

[3] What we have just seen is that, locally, $F(x) := \langle x, x\rangle$ is a Lyapunov function for (0.2). Given a differnetial eqaution like (0.1), where 0 is a fixed point, a Lyapunov function is any $\mathcal{C}^1$ function $L$ such that $L(0) = 0$, $L \geq 0$ and $\langle \nabla_x L, V(x)\rangle < 0$ for all $x \neq 0$. This implies that, for each solution $x(t)$ of (0.1) holds

$$\frac{dL(x(t))}{dt} = \langle \nabla_{x(t)} L, V(x(t))\rangle < 0.$$

This readily implies that $\lim_{t\to\infty} x(t) = 0$. (Prove it !).

So there is no hope to find a smooth conjugation.

*What can be salvaged?*

One can look for a less regular change of variables. This may not make sense for the o.d.e. itself but it is meaningful for the associated flows.

Thus let us fix some small $r > 0$ and consider a smooth non increasing function $g : \mathbb{R}_+ \to [0,1]$ such that $g(x) = 1$ for $x \leq r$ and $g(x) = \frac{3r}{2x}$ for $x \geq 2r$, with $-g' \leq C$. We can then define the functions $\varphi : \mathbb{R}^n \to [0,1]$ $F_0, F : \mathbb{R}^d \to \mathbb{R}^d$ as $\varphi(x) := g(\|x\|)$ and

$$\begin{aligned} F_0(x) &:= e^A x \\ F(x) &:= e^A x + \phi_1(\varphi(x)x) - \varphi(x)e^A x =: F_0(x) + \Delta(x), \end{aligned}$$

where $\phi_1$ is the time one flow associated to (0.2). Clearly, for $\|x\| \leq r$ the two functions are simply the time one map of the linear flow and the time one map of (0.2), moreover they are globally Lip. Since we will be interested only in $x$ in the ball of radius $r$ the modification outside such a ball is totally irrelevant and it has been done only to facilitate the exposition of the following argument.

**Exercize 4.** *Show that, for $r$ small enough, $F$ is a diffeomorphism. Prove that $\|\Delta\|_\infty < \infty$.*[4]

The idea is to consider the maps $F_0, F : \mathbb{R}^n \to \mathbb{R}^n$ and to show that they can be conjugated, that is there exists an homeomorphism $\tilde{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{\Phi} \circ F = F_0 \circ \tilde{\Phi}$.

Let us look for a solution in the form $\tilde{\Phi}(x) = x + \Phi(x)$, then we have

$$F_0(x + \Phi(x)) = F(x) + \Phi(F(x))$$

or, setting $\xi = F(x)$,

$$\Phi(\xi) = F_0(F^{-1}(\xi) + \Phi \circ F^{-1}(\xi)) - \xi.$$

We define then the operator $K : \mathcal{C}^0(\mathbb{R}^n) \to \mathcal{C}^0(\mathbb{R}^n)$ defined by

$$K(\Phi)(\xi) := F_0(F^{-1}(\xi) + \Phi \circ F^{-1}(\xi)) - \xi$$

then our problem boils down to establishing the existence of a fixed point for $K$. Now, given two functions $h, g \in \mathcal{C}^0(\mathbb{R}^n)$, holds

$$\sup_{\xi \in \mathbb{R}^n} \|K(h)(\xi) - K(g)(\xi)\| = \sup_{x \in \mathbb{R}^n} \|F_0(x + h(x)) - F_0(x + g(x))\| \leq e^{-\sigma}\|h - g\|_\infty$$

Thus the contracting mapping theorem yields the wanted result.

**Exercize 5.** *What can be done if all the eigenvalues of $A$ have strictly positive real part?*

We have then, topologically, the behaviour of a source, a node or a stable or unstable focus are the same as the one of the linear part of the equation. But the generic case is the one in which both eigenvalues with positive and negative real part are present, does the same conclusions hold for such a more general situation? The answer is yes. To see it consider that in such a case $\mathbb{R}^n$ is naturally split into two

[4]Hint: By the variation of constant method follows that

$$\phi_t(x) = e^{At}x + \int_0^t e^{A(t-s)} R(\phi_s(x)) ds.$$

spaces $V \oplus W$, invariant for $A$ and such that $A$ restricted to $V$ has only eigenvalues with negative real part while restricted to $W$ has eigenvalues with positive real part. Then the spaces are invariant for $F_0$ as well, on one $F_0$ contracts, on the other expands. Call $d_s$ the dimension of $V$ and $d_u$ the dimension of $W$. Clearly $d_s + d_u = d$.

Then each $e \in \mathbb{R}^n$ has a unique splitting as $e = v + w$, $v \in V$, $w \in W$. It is then convenient to define the projections $p_1 : \mathbb{R}^d \to V$ and $p_2 : \mathbb{R}^d \to W$ $p_1(e) = v$, $p_2(e) = w$. Moreover we can split $\mathcal{C}^0(\mathbb{R}^n)$ as $\mathbb{V} \oplus \mathbb{W}$ where $\mathbb{V} := \{f \in \mathcal{C}^0(\mathbb{R}^d) \ : \ p_2 \circ f = 0\}$ and $\mathbb{W} := \{f \in \mathcal{C}^0(\mathbb{R}^d) \ : \ p_1 \circ f = 0\}$. We can then write canonically $f$ as $(f_1, f_2) := (p_1 \circ f, p_2 \circ f)$.

Accordingly our conjugation equation $F_0 \circ \tilde{\Phi} = \tilde{\Phi} \circ F$, becomes

$$
\begin{aligned}
B\tilde{\Phi}_1 &= \tilde{\Phi}_1 \circ F \\
D\tilde{\Phi}_2 &= \tilde{\Phi}_2 \circ F
\end{aligned}
$$

where $F_0((x_1, x_2)) =: (Bx_1, Dx_2)$. We transform the first equation as we did for the contracting case, while on the second we act as you probably did if you solved Exercise 5:

$$
\begin{aligned}
\tilde{\Phi}_1 &= B\tilde{\Phi}_1 \circ F^{-1} \\
\tilde{\Phi}_2 &= D^{-1}\tilde{\Phi}_2 \circ F.
\end{aligned}
$$

Note that, if we apply the above reasoning directly to such equations we obtain that they have only one bounded solution: $\tilde{\Phi} = 0$, yet we are not looking for bounded solutions but rather for solutions of the form $\tilde{\Phi}(x) = x + \Phi(x)$, where $\Phi$ is bounded. Substituting such a form for $\tilde{\Phi}$ one can see that bounded function are mapped into bounded functions (thanks to exercise 4), hence the contracting map argument applies and the existence of a unique conjugation is established.

**Remark 0.1.** *By the way, what we just proved is known as the Grobman-Hartman Theorem.*

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