SIEGEL CENTER PROBLEM, KOLMOGOROV STYLE

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1. The problem

For each $r > 0$, let $D_r := \{z \in \mathbb{C} \ : \ |z| \leq r\}$ and, given $R > 0$, let $T, T_* : D_R \to \mathbb{C}$ be defined by

$$
\begin{aligned}
T_*(z) &= e^{i\alpha} z \\
T(z) &= T_*(z) + f(z),
\end{aligned}
$$

where f is holomorphic in a neighborhood of D_R, and $f(0) = f'(0) = 0$.

In addition we require that α is a Diophantine irrational, that is that there exists $p \in \mathbb{N}$ such that[1].

$$|e^{i\alpha n} - e^{i\alpha}| \geq C n^{-p} \quad \forall n \in \mathbb{N}.$$

The question is if there exists $r > 0$ and an invertible function $H : D_r \to \mathbb{C}$, holomorphic in D_r, such that $H^{-1} \circ T \circ H = T_*$. That is if T is analytically conjugate to its linear part. In fact, we will show that such an H exists with the form $H(z) = z + h(z)$ with $h(0) = h'(0) = 0$.

Hence we are looking for h such that

$$e^{i\alpha} h(z) - h(e^{i\alpha} z) + f(z + h(z)) = 0. \tag{1.1}$$

To solve a functional equation it is usually necessary to specify the space of function in which we are looking for a solution.

2. Some Banach spaces

Consider the function holomorphic in a neighborhood of some disk D_r. Clearly each such function can be written as $g(z) = \sum_{k=0}^{\infty} g_k z^k$. We can then define the two norms

$$
\begin{aligned}
\|g\|_r &:= \sup_{|z| \leq r} |g(z)| \\
|g|_r &:= \sum_{k=0}^{\infty} |g_k| r^k.
\end{aligned}
\tag{2.1}
$$

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[1] A natural question is: do such number exists? To see it consider the sets $I_{m,n} := [\frac{2\pi m}{n} - C n^{-p}, \frac{2\pi m}{n} - C n^{-p}]$. If $m \leq n$, then $I_{m,n} \subset [0, 2\pi]$. Clearly if $\alpha \notin I_{m,n}$ for all $n \geq m \in \mathbb{N}$, then α satisfies the Diophantine condition. But $\sum_{n \geq m} |\, I_{m,n}| \leq C \sum_{n=1}^{\infty} n^{-p+1}$ which converges provided $p > 2$ and can be made arbitrarily small by choosing C small. Accordingly, almost all numbers are Diophantine for some $p > 2$. In fact, there exists also numbers that are diophantine with $p = 2$, this are the one that can be represented by *continuous fracitons* with bounded entries. In particular, the quadratic irrationals have periodic continous fractions hence they are Diophanine with $p = 2$. If you are unaware of these facts, look at the notes *Numeri razionali e reali*, sezione 3, e *Una digressione sul calendario*, sezioni 2.1, 2.3 at the web page `http://www.mat.uniroma2.it/~liverani/Inform06/didattica.html`

In this way we have two normed spaces, completing with respect to the norms we obtain two Banach spaces $\bar{\mathcal{B}}_r$ and $\mathcal{B}_r$, respectively.

Note that the two norms have the simple relation:

$$\|g\|_r \leq |g|_r.$$

The other direction is a bit less simple:

$$|g_n| \leq \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{g(\xi)}{\xi^{n+1}} d\xi \right| \leq r^{-n} \|g\|_r$$

thus

$$|g|_{e^{-\tau}r} \leq \sum_{k=0}^{\infty} e^{-k\tau} \|g\|_r \leq \tau^{-1} e^{\tau} \|g\|_r. \tag{2.2}$$

3. Kolmogorov's idea

The basic idea is to solve the equation

$$e^{i\alpha} h(z) - h(e^{i\alpha} z) + f(z) = 0, \tag{3.1}$$

instead of equation (1.1). Let us consider some $r \in (0, R)$ to be chosen later, then

$$h_n = \frac{f_n}{e^{i\alpha n} - e^{i\alpha}}.$$

Remark 3.1. *In the following we will use C to designate a generic constant that depends only on α. Note that different occurrences of C may stand for different numerical values.*

Then

$$|h|_{e^{-\tau}r} = \sum_{n=2}^{\infty} |h_n| r^n e^{-n\tau} \leq C \sum_{n=2}^{\infty} |f_n| r^n e^{-m\tau} n^p \leq C |f|_r \tau^{-p}. \tag{3.2}$$

We have thus a solution albeit on a smaller disk. Form now on, since we are interested in small losses in the radius, we will assume

$$\tau \in (0, 1). \tag{3.3}$$

The next step is to see to what T is conjugated by H, to do so we need to compute H^{-1}, provided it is invertible. Let us look for an inverse of the type $H^{-1}(\zeta) = \zeta + g(\zeta)$. Clearly g must satisfy the functional equation[2]

$$g(\zeta) = -h(\zeta + g(\zeta)).$$

In other words we look for the fixed point of the operator $K(g)(\zeta) := -h(\zeta + g(\zeta))$. To study such an operator it is more convenient to use the space $\bar{\mathcal{B}}_{e^{-3\tau}r}$.

For each $\delta > 0$ let $B_\delta := \{g \in \bar{\mathcal{B}}_{e^{-3\tau}r} \; : \; \|g\|_{e^{-3\tau}r} \leq \delta\}$, then for each $|\zeta| \leq e^{-2\tau} r$ and $g \in B_\delta$ holds

$$|\zeta + g(\zeta)| \leq e^{-3\tau} r + \delta \leq e^{-2\tau} r$$

provided

$$\delta < e^{-\tau} r \tau. \tag{3.4}$$

With such a condition we have

$$\|Kg\|_{e^{-3\tau}r} \leq \|h\|_{e^{-2\tau}r} \leq |h|_{e^{-\tau}r} \leq C |f|_r \tau^{-p} \leq \delta$$

[2] Simply compute $H^{-1} \circ H = \text{identity}$.

provided

$$\delta \geq C|f|_r \tau^{-p}. \tag{3.5}$$

Since we are interested in the smallest possible δ we therefore choose

$$\delta = C|f|_r \tau^{-p}. \tag{3.6}$$

Thus (3.4) is satisfied if

$$\tau \geq (CLr^{-1}|f|_r)^{\frac{1}{p+1}} \tag{3.7}$$

for some large L to be chosen later.[3] In addition, for each $g, \bar{g} \in B_\delta$ we have

$$\|Kg - K\bar{g}\|_{e^{-3\tau}r} \leq \|h'\|_{e^{-2\tau}r}\|g - \bar{g}\|_{e^{-3\tau}r}.$$

To estimate the derivative we can use again Cauchy formula, for each $|z| \leq e^{-2\tau}r$, holds

$$|h'(z)| \leq \left|\frac{1}{2\pi i}\int_{|z|=re^{-\tau}} \frac{h(\zeta)}{z-\zeta}d\zeta\right| \leq |h|_{e^{-\tau}r}e^{tau}\tau^{-1}. \tag{3.8}$$

Accordingly,

$$\|Kg - K\bar{g}\|_{e^{-3\tau}r} \leq C|f|_r\tau^{-p-1}\|g - \bar{g}\|_{e^{-3\tau}r} \leq L^{-1}r\|g - \bar{g}\|_{e^{-3\tau}r}.$$

That is we have a contraction provided L is chosen large enough.

We can finally set

$$H^{-1} \circ T \circ H(z) =: e^{i\alpha}z + f_1(z) =: T_1(z).$$

Indeed,

$$\|H\|_{e^{-5\tau r}} \leq e^{-5\tau} + C|f|_r\tau^{-p} \leq e^{-4\tau}r,$$

provided L has been chosen large enough, it follows $\|T \circ H\|_{e^{-5\tau r}} \leq \|T\|_{e^{-4\tau r}} \leq e^{-4\tau} + |f|_r \leq e^{-3\tau}r$ provided r is sufficiently small. Hence $H^{-1} \circ T \circ H(z)$, and thus f_1, is well defined for $|z| \leq e^{-5\tau r}$. To estimate the size of f_1 it is convenient to use the equation $T \circ H = H \circ T_1$, that is

$$\begin{aligned} f_1(z) &= e^{i\alpha}h(z) - h(e^{i\alpha}z + f_1(z)) + f(z + h(z)) \\ &= h(e^{i\alpha}z) - h(e^{i\alpha}z + f_1(z)) + f(z + h(z)) - f(z) =: K_1(f_1)(z). \end{aligned}$$

where we have used the fact that h satisfy (3.1). We are thus interested in the action of the operator K_1 on $\bar{\mathcal{B}}_{e^{-5\tau}r}$. If $\varphi \in \bar{\mathcal{B}}_{e^{-5\tau}r}$, $\|\varphi\|_{e^{-5\tau}r} \leq \frac{1}{2e}|f|_r\tau \leq e^{-4\tau}\tau$,[4]

$$\begin{aligned} \|K_1(\varphi)\|_{e^{-5\tau}r} &\leq C\tau^{-2}|f|_r|h|_{e^{-\tau}r} + C\tau^{-2}|h|_{e^{-\tau}r}\|\varphi\|_{e^{-5\tau}r} \\ &\leq C\tau^{-p-2}|f|_r^2 + C\tau^{-p-2}|f|_r\|\varphi\|_{e^{-5\tau}r}, \end{aligned}$$

where we have estimated the derivatives similarly to (3.8) and used (3.2). We can now finally choose

$$\tau = (CLr^{-1}|f|_r)^{\frac{1}{p+3}}. \tag{3.9}$$

Clearly $\tau \geq (CLr^{-1}|f|_r)^{\frac{1}{p+1}}$, as requested by (3.7), provided $CLr^{-1}|f|_r < 1$ which can always be achieved by choosing r small enough. Thus,

$$\|K_1(\varphi)\|_{e^{-5\tau}r} \leq rL^{-1}\tau|f|_r + rL^{-1}\tau\|\varphi\|_{e^{-5\tau}r},$$

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[3]Note that, by definition, if $r \leq R/2$, we have $|f|_r \leq Cr^2$, thus $\tau < 1$ can always be achieved by choosing r sufficiently small.

[4]Indeed, $|f|_r\tau \leq 2e^{-4\tau+1}\tau$ always holds for r small enough.

we can thus choose L such that $rL^{-1} \leq \frac{1}{2e+1}$ and obtain

$$\|K_1(\varphi)\|_{e^{-5\tau}r} \leq \frac{1}{2e+1}|f|_r\tau + \frac{1}{2e+1}\|\varphi\|_{e^{-5\tau}r}\tau \leq \frac{1}{2e}|f|_r\tau.$$

This means that the fixed points of K_1 must belong to the ball we are considering, that is $\|f_1\|_{e^{-5\tau}} \leq \frac{1}{2e}|f|_r\tau$. Hence,

$$|f_1|_{e^{-6\tau}r} \leq e\tau^{-1}\|f_1\|_{e^{-5\tau}} \leq \frac{1}{2}|f|_r. \tag{3.10}$$

4. The iteration scheme

We are finally ready to set up an iteration scheme of which the previous section will constitute a generic step. Let $f_0 := f$, and let $r < R$ be chosen so that all the requirements of the above section are satisfied. Then set $r_0 := r$ and $r_{n+1} := e^{-6\tau_n}r_n$ where $\tau_n = (CLr_n^{-1}|f_n|_{r_n})^{\frac{1}{p+3}}$. Next define $H_n(z) := z + h_n(z)$ and f_{n+1} as solutions of the equations

$$e^{i\alpha}h_n(z) - h_n(e^{i\alpha}z) + f_n(z) = 0,$$

and

$$f_{n+1}(z) = h_n(e^{i\alpha}z) - h_n(e^{i\alpha}z + f_{n+1}(z)) + f_n(z + h_n(z)) - f_n(z).$$

Finally set $T_n(z) := e^{i\alpha}z + f_n(z)$. By the results of the previous section holds

$$H_n^{-1} \circ T_n \circ H_n = T_{n+1}.$$

Hence, setting $\Psi_n := H_0 \circ H_1 \circ \cdots \circ H_{n-1}$, holds

$$\Psi_n^{-1} \circ T_0 \circ \Psi_n = T_n.$$

Note that (3.10) implies $|f_n|_{r_n} \leq \frac{1}{2}|f_{n-1}|_{r_{n-1}} \leq 2^{-n}|f|_r$. Thus

$$\lim_{n\to\infty} T_n(z) = e^{i\alpha}$$

for each $|z| \leq r_n$, for each $n \in \mathbb{N}$. Let us verify that such a set is not empty. By definition holds $\tau_n \leq C(2^{-n}r_n)^{\frac{1}{p+3}}$. Consider n such that $r_k \geq \frac{r}{2}$ for all $k < n$, then

$$r_n = e^{-6\sum_{k=0}^{n-1}\tau_k}r \geq e^{-Cr^{\frac{1}{p+3}}}r \geq \frac{r}{2},$$

provided r is chosen small enough. Hence, we have $r_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. The last thing we want to check is that there exists a limit conjugation

$$\Psi_n = \Psi_{n-1} + h_0 \circ \Psi_{n-1} = \Psi_0 + \sum_{k=0}^{n-1} h_k \circ \Psi_{n-k-1}$$

where we have set $\Psi_0(z) = z$. Hence

$$\begin{aligned}|\Psi_n - \Psi_0|_{r/2} &\leq \sum_{k=0}^{n-1}|h_k|_{e^{-\tau_k}r_k} \leq C\sum_{k=0}^{n-1}|f_k|_{r_k}\tau_k^{-p} \leq C\sum_{k=0}^{n-1} r^{\frac{p}{p+3}}|f_k|_{r_k}^{\frac{3}{p+3}} \\ &\leq Cr^{\frac{p}{p+3}}|f|_r\sum_{k=0}^{n-1}2^{\frac{3n}{p+3}} \leq Cr^{\frac{p}{p+3}}|f|_r.\end{aligned}$$

This means that the series is norm convergent, hence it defines a function H holomorpfhic on the disk or radius $r/2$ such that

$$H^{-1} \circ T \circ H = T_*$$

that is, in such a disk the map is conjugated with its linear part.

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