HOPF BIFURCATION

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We have seen that for an open set it may happen that, for a family of vector fields $V \in \mathcal{C}^2(\mathbb{R}^2 \times [-1,1], \mathbb{R}^2)$, $V(0,0) = 0$, and $\partial_x V(0,0)$ is invertible but has purely immaginary eigenvectros.

Such a vector field, which can be assumed $\mathcal{C}^\infty$ without loss of generality, yields the family of differential equations

$$\dot{x} = Ax + \lambda b + G(x, \lambda) \tag{0.1}$$

where $G(x,\lambda) = \mathcal{O}(\|x\|^2 + \lambda^2)$ where $A = \partial_x V(0,0)$, $b = \partial_\lambda V(0,0)$.

Exercize 1. *Show that with a change of variables $z = Bx - \alpha(\lambda)$ one can put* (0.1) *in the form*

$$\dot{z} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} z + G_1(z, \lambda) \tag{0.2}$$

where $\omega > 0$ and $G_1(0,0) = 0$, $\partial_x G_1(0,0) = G_1(0,\lambda) = 0$. (Hint: first prove that the trace of A must be zero, hence it must be

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

and $\det A = -a^2 + bc =: -\omega^2$ *must be negative. Try then with*

$$B = \begin{pmatrix} -\omega^{-1} & -a\omega^{-1} \\ 0 & c \end{pmatrix}.$$

It remains to determine α. Note that setting $x = z + \alpha$ one wants $G_1(0,\lambda) = A\alpha(\lambda) + b\lambda + G(\alpha(\lambda),\lambda) = 0$ that is $\alpha(\lambda) = -A^{-1}b\lambda - A^{-1}G(\alpha(\lambda),\lambda)$ which can be easily shown to have a unique $\mathcal{C}^2$ solution.)

We already know that, generically, $\partial_{xx} G_1(0,0)$ is not degenerate.

Exercize 2. *Show that, generically, the vector fields of the form* (0.1) *have the trace t of $\partial_{x\lambda} G_1(0,0)$ differrent from zero. Verify that the eigenvalues of the matrix $A + \lambda \partial_{x\lambda} G_1(0,0)$ are given by $\lambda t/2 \pm i\omega(1 + \mathcal{O}(\lambda))$. Finally show that, by a linear change of coordinate, and a linear reparametrization of the time* (0.2) *can be put in the form*

$$\dot{x} = \begin{pmatrix} \lambda & \omega + a\lambda \\ -\omega - a\lambda & \lambda \end{pmatrix} z + V_1(z, \lambda) \tag{0.3}$$

with $\omega, a \neq 0$, $V_1(0,\lambda) = \partial_x V_1(0,0) = \partial_\lambda V_1(0,0) = \partial_{x\lambda} V_1(0,0) = 0$ and $\partial_{xx} V_1(0,0)$ non degenerate.

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Since (0.3) is the generic case, it is now time to study it to conclude our understanding of the generic situation for generic two dimensional vector fields.

Given the fact that the trajectories rotate around zero almost in circles it may occur the idea to treat the problem in polar coordinates. In fact this point of view is quite advantageous and we will carry it out in order to show how a problem may simplify if viewed in different coordinates.

The polar coordinates can be written as $x = \rho v(\theta)$, where $\rho \in \mathbb{R}_+$, $\theta \in \mathbb{R}$ and $v(\theta) := (\cos\theta, \sin\theta)$.

Remark 0.1. *Note that such a change of coordinates is singular for $\rho = 0$ and it is not globally one-one. Yet to consider θ in the universal cover of S^1 rather than in S^1 will be very usefull in the following.*

If we substitute such coordinates in (0.3), we obtain

$$\dot\rho v(\theta) + \rho n(\theta)\dot\theta = \lambda\rho v(\theta) - \omega_\lambda \rho n(\theta) + V(\rho v(\theta)),$$

where $n(\theta) := (-\sin\theta, \cos\theta)$ and $\omega_\lambda = \omega - \lambda a$. That is

$$\begin{aligned} \dot\rho &= \lambda\rho + \langle v(\theta), V(\rho v(\theta))\rangle =: \lambda\rho + a(\theta,\rho)\rho^2 \\ \dot\theta &= -\omega + \rho^{-1}\langle n(\theta), V(\rho v(\theta))\rangle =: -\omega + b(\theta,\rho)\rho. \end{aligned} \tag{0.4}$$

Note that the equation (0.4) is well defined also for $\rho = 0$ but in such a case, instead of a fixed point, it has the periodic orbit $(\rho(t), \theta(t)) = (0, -\omega t)$. In some sense the polar coordinates have automatically regularized the behaviour at zero saving us the trouble to do it by hand as one should do in Cartesian coordinates. Since for small ρ we have $\dot\theta < 0$, it is convenient to use θ rather than t to parameterize the motion (here is now evident the advantage of using the universal cover of S^1). Calling again ρ the distance from the origin as a function of θ we have

$$\frac{d\rho}{d\theta} = \frac{\lambda\rho + a(\theta,\rho)\rho^2}{-\omega + b(\theta,\rho)\rho} =: -\frac{\lambda}{\omega}\rho - \beta(\theta,\lambda)\rho^2 - \gamma(\theta,\rho,\lambda)\rho^3, \tag{0.5}$$

where

$$\begin{aligned} &\beta(\theta,\lambda) = \omega^{-1}a(\theta,0) + \lambda\omega^{-2}b(\theta,0) \\ &\gamma(\theta,0,\lambda) = \lambda b(\theta,0)^2\omega^{-3} + a(\theta,0)b(\theta,0)\omega^{-2} + \lambda\partial_\rho b(\theta,0)\omega^{-1} + \partial_\rho a(\theta,0)\omega^{-1}. \end{aligned}$$

Note, for later use, that β is a trigonometric polynomial of third degree while γ has terms of forth and sixth order.

At this point the Poincarè section corresponds simply at looking at the trajectory for multiple of 2π. We wish then to define the map $S_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $S_\lambda\xi := \rho(2\pi, \xi, \lambda)$, where $\rho(2\pi, \xi, \lambda)$ is the solution of (0.5) with initial condition ξ seen at $\theta = 2\pi$.[1]

Remark that (0.5) can be written in integral form as

$$\rho(\theta,\xi,\lambda) = e^{-\frac{\lambda}{\omega}\theta}\xi - \int_0^\theta e^{-\frac{\lambda}{\omega}(\theta-\varphi)}\left[\beta(\varphi,\lambda)\rho(\varphi,\xi,\lambda)^2 - \gamma(\varphi,\rho(\varphi,\xi,\lambda),\lambda)\rho(\varphi,\xi,\lambda)^3\right]d\varphi \tag{0.6}$$

Exercize 3. *Prove that, for fixed λ, setting*

$$K_\xi(\rho)(\theta) = e^{-\frac{\lambda}{\omega}\theta}\xi - \int_0^\theta e^{-\frac{\lambda}{\omega}(\theta-\varphi)}\left[\beta(\varphi)\rho(\varphi,\xi)^2 - \gamma(\varphi,\rho(\varphi,\xi),\lambda)\rho(\varphi,\xi)^3\right]d\varphi$$

[1] Since $\dot\theta < 0$, here we are going back in time with respect to the previous section, hence S_λ is essentially what before was T_λ^{-1}.

for each $\delta > 0$ there exists $L, M > 0$ such that the set $\{|\rho(\theta) - e^{-\frac{\lambda}{\omega}\theta}\xi| \leq M|\xi| \;\; \forall |\theta| \leq L\}$ is invariant for K_ξ provided $|\xi| \leq \delta$, and hence contains its fixed point.

Taking into account the above exercize and terating (0.6) yields

(0.7)

$$\begin{aligned}\rho(\theta,\xi,\lambda) =& e^{-\frac{\lambda\theta}{\omega}}\xi - \int_0^\theta e^{-\frac{\lambda(\theta-\varphi)}{\omega}}\beta(\varphi,\lambda)\left[e^{-\frac{2\lambda\varphi}{\omega}}\xi^2 - 2\int_0^\varphi e^{-\frac{\lambda(2\varphi-3x)}{\omega}}\beta(x,\lambda)\xi^3\right]d\varphi \\ &- \int_0^\theta e^{-\frac{\lambda(\theta-\varphi)}{\omega}}\gamma(\varphi,0,\lambda)e^{-\frac{3\lambda\varphi}{\omega}}\xi^3 d\varphi + \mathcal{O}(\xi^4).\end{aligned}$$

To compute the map S_λ we must set $\theta = 2\pi$ in (0.7). As already noted in the previous section some integral will be zero. Indeed, from (0.4) follows that $a(\theta,0), b(\theta,0)$ are homogeneous polynomials of third order while $\partial_\rho a(\theta,0), \partial_\rho b(\theta,0)$ are of forth order. Moreover, setting

$$\Gamma(\theta,0) := \int_0^\theta \beta(\varphi,0)d\varphi,$$

we have $\Gamma(2\pi) = 0$, hence

$$\int_0^{2\pi}\beta(\varphi,0)\int_0^\varphi \beta(x,0)dxd\varphi = \int_0^{2\pi}\Gamma'(\varphi,0)\Gamma(\varphi,0)d\varphi = \frac{\Gamma(2\pi,0)^2 - \Gamma(0,0)^2}{2} = 0.$$

Given the above considerations we can finally write

$$S_\lambda(\rho) = e^{-\frac{2\pi\lambda}{\omega}}\rho + B\rho^3 + \mathcal{O}(\lambda\rho^2 + \rho^4). \tag{0.8}$$

where

$$B = -\int_0^{2\pi}\left[a(\theta,0)b(\theta,0)\omega^{-2} + \partial_\rho a(\theta,0)\omega^{-1}\right]d\theta$$

Exercize 4. *Compute, in terms of the Tailor coefficients of V, what it means $B \neq 0$.*

As before the fixed points of S_λ are $\rho = 0$ and $\rho_p := \sqrt{\frac{2\pi\lambda}{\omega B}} + \mathcal{O}(\lambda)$ which exists only if $\lambda B > 0$. In the latter case $S'_\lambda(\rho_p) = 1 + \frac{4\pi\lambda}{\omega} + \mathcal{O}(\lambda^{\frac{3}{2}})$, while $S'_\lambda(0) = 1 - \frac{2\pi\lambda}{\omega} + \mathcal{O}(\lambda^2)$. Thus, if one is attracting the other must be repelling. If, for $\lambda B < 0$, we want only an attracting focus (as we asked in the previous section), then it must be $B < 0$. Hence, for $\lambda > 0$ we have an attracting focus, while for $\lambda > 0$ the focus becomes repelling and it appears an attracting periodic orbit.

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