HADAMARD-PERRON THEOREM

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1. Invariant manifold of a fixed point

He we will discuss the simplest possible case in which the existence of invariant manifolds arises: the Hadamard-Perron theorem.

Definition 1. *Given a smooth map $T : X \to X$, X being a Riemannian manifold, and a fixed point $p \in X$ (i.e. $Tp = p$) we call (local) stable manifold (of size δ) a manifold $W^s(p)$ such that*[1]

$$W^s(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^n x, p) = 0\}.$$

Analogously, we will call (local) unstable manifold (of size δ) a manifold $W^u(p)$ such that

$$W^u(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^{-n} x, p) = 0\}.$$

It is quite clear that $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$ (Problem 1). Less clear is that these sets deserve the name "manifold." Yet, if one thinks of the linear case it is obvious that the stable and unstable manifolds at zero are just segments in the stable and unstable direction, the next Theorem shows that this is a quite general situation.

Theorem 1.1 (Hadamard-Perron)**.** *Consider an invertible map $T : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $T \in \mathcal{C}^1(U, \mathbb{R}^2)$, such that $T(0) = 0$,*[2] *and*

$$D_0T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \tag{1.1}$$

where $0 < \mu < 1 < \lambda$.[3] *That is, the map T is hyperbolic at the fixed point 0. Then there exists stable and unstable manifolds at 0. They are $\mathcal{C}^1$ curves. Moreover, $\mathcal{T}_0W^{s(u)}(0) = E^{s(u)}(0)$ where $E^{s(u)}(0)$ are the expanding and contracting subspaces of D_0T.*

Proof. We will deal explicitly only with the unstable manifold since the stable one can be treated exactly in the same way by considering T^{-1} instead of T.

Since the map is continuously differentiable for each $\varepsilon > 0$ we can choose $\delta > 0$ so that, in a 2δ-neighborhood of zero, we can write

$$T(x) = D_0Tx + R(x) \tag{1.2}$$

where $\|R(x)\| \le \varepsilon\|x\|$, $\|D_xR\| \le \varepsilon$.

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[1] Sometime we will write $W^s_\delta(p)$ when the size really matters. By $B_\delta(x)$ we will always mean the open ball of radius δ centered at x.

[2] Note that is $Tp = p$ one can always translate the coordiantes as to have $p = 0$.

[3] Note that if D_0T has eigenvalues $0 < \mu < 1 < \lambda$ then one can always perform a change of variables such that (1.1) holds.

1.0.1. ***Existence–a fixed point argument.*** The first step is to decide how to represent manifolds. In the present case, since we deal only with curves, it seems very reasonable to consider the set of curves $\Gamma_{\delta,c}$ passing through zero and "close" to being horizontal, that is the differentiable functions $\gamma : [-\delta,\, \delta] \to \mathbb{R}^2$ of the form

$$\gamma(t) = \begin{pmatrix} t \\ u(t) \end{pmatrix}$$

and such that $\gamma(0) = 0$; $\|(1,0) - \gamma'\|_\infty \leq c$. It is immediately clear that any smooth curve passing through zero and with tangent vector, at each point, in the cone $\mathcal{C} := \{(a,\, b) \in \mathbb{R}^2 \mid |\frac{b}{a}| \leq c\}$, can be associated to a unique element of $\Gamma_{\delta,c}$, just consider the part of the curve contained in the strip $\{(x, y) \in \mathbb{R}^2 \mid |x| \leq \delta\}$. Moreover, if $\gamma \in \Gamma_{\delta,c}$ then $\gamma \subset B_{2\delta}(0)$, provided $c \leq 1/2$.

Notice that it suffices to specify the function u in order to identify uniquely an element in $\Gamma_{\delta,c}$. It is then natural to study the evolution of a curve through the change in the associated function.

To this end let us investigate how the image of a curve in $\Gamma_{\delta,c}$ under T looks like.

$$T\gamma(t) = \begin{pmatrix} \lambda t + R_1(t, u(t)) \\ \mu u(t) + R_2(t, u(t)) \end{pmatrix} := \begin{pmatrix} \alpha_u(t) \\ \beta_u(t) \end{pmatrix}.$$

At this point the problem is clearly that the image it is not expressed in the way we have chosen to represent curves, yet this is easily fixed. First of all, $\alpha_u(0) = \beta_u(0) = 0$. Second, by choosing $\varepsilon < \lambda$, we have $\alpha_u'(t) > 0$, that is, α_u is invertible. In addition, $\alpha_u([-\delta,\, \delta]) \supset [-\lambda\delta + \varepsilon\delta,\, \lambda\delta - \varepsilon\delta] \supset [-\delta,\, \delta]$, provided $\varepsilon \leq \lambda^{-1}$. Hence, α_u^{-1} is a well defined function from $[-\delta,\, \delta]$ to itself. Finally,

$$\left|\frac{d}{dt}\beta_u \circ \alpha_u^{-1}(t)\right| = \left|\frac{\beta_u'(\alpha_u^{-1}(t))}{\alpha_u'(\alpha_u^{-1}(t))}\right| \leq \frac{\mu c + \varepsilon}{\lambda - \varepsilon} \leq c$$

where, again, we have chosen $\varepsilon \leq \frac{c(\lambda-\mu)}{1+c}$.

We can then consider the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta,c}$ defined by

$$\tilde{T}\gamma(t) := \begin{pmatrix} t \\ \beta_u \circ \alpha_u^{-1}(t) \end{pmatrix} \tag{1.3}$$

which associates to a curve in $\Gamma_{\delta,c}$ its image under T written in the chosen representation. It is now natural to consider the set of functions $B_{\delta,c} = \{u \in \mathcal{C}^1([-\delta,\, \delta]) \mid u(0) = 0,\ |u'|_\infty \leq c\}$ in the vector space $\mathrm{L}ip([-\delta,\, \delta])$.[4] As we already noticed $B_{\delta,c}$ is in one-one correspondence with $\Gamma_{\delta,c}$, we can thus consider the operator $\hat{T} : \mathrm{L}ip([-\delta,\, \delta]) \to \mathrm{L}ip([-\delta,\, \delta])$ defined by

$$\hat{T}u = \beta_u \circ \alpha_u^{-1} \tag{1.4}$$

From the above analysis follows that $\hat{T}(B_{\delta,c}) \subset B_{\delta,c}$ and that $\hat{T}u$ determines uniquely the image curve.

The problem is then reduced to studying the map $\hat{T}$. The easiest, although probably not the most productive, point of view is to show that $\hat{T}$ is a contraction in the sup norm. Note that this creates a little problem since $\mathcal{C}^1$ it is not closed in the sup norm (and not even $\mathrm{L}ip([-\delta,\, \delta])$ is closed). Yet, the set $B^*_{\delta,c} = \{u \in$

[4] This are the Lipschitz functions on $[-\delta,\, \delta]$, that is the functions such that $\sup_{t,s\in[-\delta,\, \delta]} \frac{|u(s)-u(t)|}{|t-s|} < \infty$.

$\text{Lip}([-\delta,\,\delta]) \mid u(0)=0, \sup_{t,s\in[-\delta,\,\delta]} \frac{|u(s)-u(t)|}{|t-s|} < c\}$ is closed (see Problem 2). Thus $\overline{B_{\delta,c}} \subset B^*_{\delta,c}$. This means that, if we can prove that the sup norm is contracting, then the fixed point will belong to $B^*_{\delta,c}$ and we will obtain only a Lipschitz curve. We will need a separate argument to prove that the curve is indeed smooth.

Let us start to verify the contraction property. Notice that

$$\alpha_u^{-1}(t) = \lambda^{-1}t + \lambda^{-1}R_1(\alpha_u^{-1}(t), u(\alpha_u^{-1}(t))),$$

thus, given $u_1, u_2 \in B_{\delta,c}$, by Lagrange Theorem

$$\begin{aligned}|\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| &\leq \lambda^{-1}|\langle \nabla_\zeta R_1,\, (\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t), u_1(\alpha_{u_1}^{-1}(t)) - u_2(\alpha_{u_2}^{-1}(t)))\rangle| \\ &\leq \frac{\varepsilon}{\lambda}\left\{|\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| + |u_1(\alpha_{u_2}^{-1}(t)) - u_2(\alpha_{u_2}^{-1}(t))|\right\}.\end{aligned}$$

This implies immediately

$$|\alpha_{u_1}^{-1}(t) - \alpha_{u_2}^{-1}(t)| \leq \frac{\lambda^{-1}\varepsilon}{1-\lambda^{-1}\varepsilon}\|u_1 - u_2\|_\infty. \tag{1.5}$$

On the other hand

$$\begin{aligned}|\beta_{u_1}(t) - \beta_{u_2}(t)| &\leq \mu|u_1(t) - u_2(t)| + |\langle \nabla_\zeta R_2,\, (0, u_1(t) - u_2(t))\rangle| \\ &\leq (\mu+\varepsilon)\|u_1 - u_2\|_\infty.\end{aligned} \tag{1.6}$$

Moreover,

$$|\beta_u'(t)| \leq \mu + \varepsilon. \tag{1.7}$$

Collecting the estimates (1.5, 1.6, 1.7) readily yields

$$\begin{aligned}\|\hat{T}u_1 - \hat{T}u_2\|_\infty &\leq \|\beta_{u_1}\circ\alpha_{u_1}^{-1} - \beta_{u_1}\circ\alpha_{u_2}^{-1}\|_\infty + \|\beta_{u_1}\circ\alpha_{u_2}^{-1} - \beta_{u_2}\circ\alpha_{u_2}^{-1}\|_\infty \\ &\leq \left\{[\mu+\varepsilon]\frac{\lambda^{-1}\varepsilon}{1-\lambda^{-1}\varepsilon} + (\mu+\varepsilon)\right\}\|u_1 - u_2\|_\infty \\ &\leq \sigma\|u_1 - u_2\|_\infty,\end{aligned}$$

for some $\sigma \in (0,1)$, provided ε is chosen small enough.

Clearly, the above inequality immediately implies that there exists a unique element $\gamma_* \in \Gamma_{\gamma,c}$ such that $\tilde{T}\gamma_* = \gamma_*$, this is the *local* unstable manifold of 0.

1.0.2. ***Regularity–a cone field.*** As already mentioned, a separate argument is needed to prove that γ_* is indeed a $\mathcal{C}^1$ curve.

To prove this, one possibility could be to redo the previous fixed point argument trying to prove contraction in $\mathcal{C}^1_{Lip}$ (the $\mathcal{C}^1$ functions with Lipschitz derivative); yet this would require to increase the regularity requirements on T. A more geometrical, more instructive and more inspiring approach is the following.

Define the cone field $\mathcal{C}_{\theta,h}(x,u) := \{\xi \in B_h(x) \mid (a,b) = \xi - x;\ a \neq 0;\ |\frac{b}{a} - u| \leq \theta\}$, with $|u| \leq c\delta$, $\theta \leq c\delta$ and $h \leq \delta$. By construction $B_h(x)\cap\gamma_* \subset \mathcal{C}_{c\delta,h}$ for each $x \in \gamma_*$. We will study the evolution of such a cone field on γ_*.

For all $\xi \in \mathcal{C}_{\theta,h}(x,u)$, if $(a,b) = \xi - x$ and $(\alpha,\beta) = T\xi - Tx$, it holds

$$(\alpha,\beta) = D_xT(a,b) + \mathcal{O}(C\|(a,b)\|^2).$$

Thus, setting $(\alpha',\beta') = D_xT(a,b)$ and $u' = \frac{\beta'}{\alpha'}$, one can compute

$$\left|\frac{\beta}{\alpha} - u'\right| \leq \mu\lambda^{-1}[c_1h + \theta],$$

for some constant c_1 depending only on T and δ. Accordingly, if $h \le c_2\theta$, for same appropriate constant c_2, and δ is small enough, there exists $\sigma \in (0,1)$ such that

$$B_h(x) \cap T\mathcal{C}_{\theta,h}(x,u) \subset \mathcal{C}_{\sigma\theta,h}(Tx,u').$$

Hence, if there exists a vector field $(1,v) : [-\delta,\delta] \to \mathbb{R}^2$, and $\theta > 0$ such that, for all $h \le c_2\theta$, $x \in \gamma_*$, $\gamma_* \cap B_h(x) \subset \mathcal{C}_{\theta,h}(x,v(x))$, then

$$\begin{aligned}\gamma_* \cap B_{\sigma h}(x) &\subset T(\gamma_* \cap B_{\sigma h}(T^{-1}x)) \cap B_{\sigma h}(x) \subset B_{\sigma h}(x) \cap T\mathcal{C}_{\theta,\sigma h}(T^{-1}x, v(T^{-1}x)) \\ &\subset \mathcal{C}_{\sigma\theta,\sigma h}(x, v_1(x))\end{aligned}$$

where $a(1,v_1(x)) = D_{T^{-1}x}T(1,v(T^{-1}x))$ for the appropriate normalization constant a. Note that $\sigma h \le c_2\sigma\theta$ hence we can iterate the argument. Iterating the above inequality follows that, if $x \in \gamma_*$, since $\gamma_* \cap B_h(T^{-n}x) \subset \mathcal{C}_{c\delta,h}(T^{-n}x,0)$, then

$$\gamma_* \cap B_{\sigma^n h}(x) \subset \mathcal{C}_{c\sigma^n\delta,\sigma^n h}(x, v_n) \tag{1.8}$$

where $(a, av_n) = D_{T^{-n}x}T^n(1,0)$, for some $a \in \mathbb{R}_+$.

The estimate (1.8) clearly implies

$$\gamma_*'(x) = (1, \lim_{n\to\infty} v_n) \tag{1.9}$$

which indeed exists (see Problem 3). We have thus verified that γ_* is differentiable at each point. To verify the continuity of the derivative let $x, y \in \gamma_*$, and $\varepsilon > 0$, then

$$\|\gamma_*'(x) - \gamma_*'(y)\| \le \|D_{T^{-n}x}T^n(1,0) - D_{T^{-n}y}T^n(1,0)\| + C\sigma^n$$

so by choosing n large enough we can insure $C\sigma^n \le \varepsilon/2$, next, since $D_{T^{-n}\xi}T^n$ is a continuous function in ξ, we can choose y close enough to x so that $\|D_{T^{-n}x}T^n(1,0) - D_{T^{-n}y}T^n(1,0)\| \le \varepsilon/2$, hence the claim. □

There is an issue not completely addresses in our formulation of Hadamard-Perron theorem: the uniqueness of the manifolds.[5] It is not hard to prove that $W^{s(u)}(p)$ are indeed unique (see Problem 4).

There is another point of view that can be adopted in the study of stable and unstable manifolds: to "grow" the manifolds. This is done by starting with a very short curve in $\Gamma_{\delta,c}$, e.g. $\gamma_0(t) = (t,0)$ for $t \in [\lambda^{-n}\delta, \lambda^n\delta]$, and showing that the sequence $\gamma_n := T^n\gamma_0$ converges to a curve in the strip $[-\delta, \delta]$, independent of γ_0. From a mathematical point of view, in the present case, it corresponds to spell out explicitly the proof of the fixed point theorem. Nevertheless, it is a more suggestive point of view and it is more convenient when the hyperbolicity is non uniform. For example consider the map[6].

$$T\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}2x - \sin x + y\\ x - \sin x + y\end{pmatrix} \tag{1.10}$$

then 0 is a fixed point of the map but

$$D_0T = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

[5]Namely the doubt may remain that a less regular set satisfying Definition 1 exists.
[6]Some times this is called *Lewowicz map*

is not hyperbolic, yet, due to the higher order terms, there exist stable and unstable manifolds (see Problems 6, 7, 8).

Problems

1 Show that, if p is a fixed point, then $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$.

2 Prove that the set $B^*_{\delta,c}$ in section 1 is closed with respect to the sup norm $\|u\|_\infty = \sup_{t\in[-\delta,\delta]} |u(t)|$.

3 Prove that the limit in (1.9) is well defined.

4 Prove that, in the setting of Theorem 1.1, the unstable manifold is unique. (Hint: This amounts to show that the set of points that are attracted to zero are exactly the manifolds constructed in Theorem 1.1. Use the local hyperbolicity to show that.)

5 Show that Theorem 1.1 holds assuming only $T \in \mathcal{C}^1(U, U)$.

6 Consider the Lewowicz map (1.10), show that, given the set of curves $\Gamma_{\delta,c} := \{\gamma : [-\delta, \delta] \to \mathbb{R}^2 \mid \gamma(t) = (t, u(t));\ \gamma(0) = 0;\ |u'(t)| \in [c^{-1}t, ct]\}$, it is possible to construct the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta(1+c^{-1}\delta),\, c}$ in analogy with (1.3).

7 In the case of the previous problem show that for each $\gamma_i \in \Gamma_{\delta,c}$ holds $d(\tilde{T}\gamma_1, \tilde{T}\gamma_2) \le (1 - c\delta)d(\gamma_1, \gamma_2)$.

8 Show that for the Lewowicz map zero has a unique unstable manifold. (Hint: grow the manifolds, that is, for each $n > 1$ define $\delta_n := \frac{\rho}{n}$. Show that one can choose ρ such that $\delta_{n-1} \ge \delta_n(1+c^{-1}\delta_n)$. according to Problem 6 it follows that $\tilde{T} : \Gamma_{\delta_n,c} \to \Gamma_{\delta_{n-1},c}$. Moreover,

$$d(\tilde{T}^{n-1}\gamma_1, \tilde{T}^{n-1}\gamma_2) \le \prod_{i=1}^{n}(1 - c\delta_i)d(\gamma_1, \gamma_2).$$

Finally, show that, setting $\gamma_n(t) = (0, t) \in \Gamma_{\delta_n,c}$, the sequence $\tilde{T}^{n-1}\gamma_n$ is a Cauchy sequence that converges in $\mathcal{C}^0$ to a curve in $\Gamma_{1,c}$ invariant under $\tilde{T}$.)

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