ON CIRCLE MAPS

CARLANGELO LIVERANI

As we have seen a generic vector field in $\mathbb{R}^2$ can have a very limited choose of bounded invariant sets: either a fixed point and the associated stable and unstable manifolds, or (by Poincarè-Bendixon) a periodic orbit. Yet one can have a differential equation on different manifolds, notably the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Problem 1. *Consider the vector fields $V(x) = \omega \in \mathbb{R}^2$ on $\mathbb{T}^2$ and show that the orbit of the associated flow can be everywhere dense. (Hint: The equation $\dot{x} = \omega = (\omega_1, \omega_2)$ on $\mathbb{T}^2$ has the solution $x(t) = (x_1(t), x_2(t)) = x_0 + \omega t \mod 1$. If one looks at the flow only at the times $\tau_n = n\omega_1^{-1}$, then $x(n\tau) = x_0 + (0, \alpha n) \mod 1$ where $\alpha := \frac{\omega_2}{\omega_1}$. One can then consider the circle map $f : S^1 \to S^1$ defined by $f(z) = z + \alpha \mod 1$. Clearly, if the orbit of such a map are dense in S^1 the original flow will be dense in $\mathbb{T}^2$. The density follows in the case $\alpha \notin \mathbb{Q}$. In fact this implies that f has no periodic orbits. Then $\{f^n(0)\}$ is made of distinct points and contains a converging subsequence (by compactness) hence for each $\varepsilon > 0$ exists $\bar{n} \in \mathbb{N}$ such that $|z - f^{\bar{n}}(z)| \leq \varepsilon$, that is $f^{\bar{n}}$ is a rotation by less than ε.)*

The above problem shows that, on $\mathbb{T}^2$ is is possible to have a new ω-limit set: $\mathbb{T}^2$ itself? At this point it is natural if such a situation can take place for an open set of vector fields. To understand the situation is is useful to generalize the setting of Problem 1.

Definition 1. *A smooth generator of the first homotopy group such that V is always transversal to such a curve is called a* global section *for the flow associated to V.*

Lemma 0.1. *Let $V \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero vector field with a global section γ and let ϕ^t be the associated flow. Assume that the flow has no periodic orbits. Then, for $x \in \gamma$ there exists $t \in \mathbb{R}_+$ such that $\phi^t(x) \in \gamma$.*

Proof. Note that $\mathbb{T}^2 \setminus \gamma$ is topologically a cylinder. Since the close curves divide a cylinder into two disjoint set, the Poincarè-Bendixon theorem applies. Thus if $x \in \gamma$ and the forward trajectory never meets γ then $\omega(x)$ must be a periodic orbit, but this contradicts the assumptions, hence the Lemma. $\square$

The above lemma shows that the map $f : \gamma \to \gamma$ that to each point associates its first return to γ is well defined.

Problem 2. *Show that if $V \in \mathcal{C}^2(\mathbb{T}^2, \mathbb{R}^2)$ and the return map is well defined, then it is $\mathcal{C}^2$. (Hint: Smooth dependence from the initial conditions for an ODE.)*

Theorem 1. *Let $V \in \mathcal{C}^2(\mathbb{T}^2, \mathbb{R}^2)$ be a nowhere zero vector field with no periodic orbits an a global section γ. Let $g : \gamma \to \gamma$ be the return map. If $g'(x) \neq 0$ for all $x \in \gamma$, then for each point $y \in \mathbb{T}^2$, $\omega(y) = \mathbb{T}^2$.*

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Proof. First of all note that, by the same arguments used in Lemma 0.1, the forward orbit of each $x \in \mathbb{T}^2$ meets γ. Second, since γ is smooth we can find a C^2 diffeomorphism $h : S^1 \to \gamma$. If we set $f = h^{-1} \circ g \circ h$, we can consider the return map as $\mathcal{C}^2$ map on the unit circle such that $f' \neq 0$ at each point. Note that a periodic point for the map f corresponds to a periodic orbit for the flow, hence f cannot have periodic orbits. The clam follows then by the results of the following sections in which it is proven that a smooth circle map with no periodic orbits has dense orbits. □

Motivated by the above theorem we will now study orientation preserving circle maps. It turns out to be interesting and helpful to study their properties in relations to their increasing smoothness.

1. The continuous case

We start with some facts that follow from the simple hypothesis of continuity.

First of all note that one can lift the map f to the universal cover $\mathbb{R}$ of the circle, that is defining $\pi : \mathbb{R} \to S^1$ as $\pi(x) = x \mod 1$, it is possible to find $F \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ such that

$$f \circ \pi = \pi \circ F.$$

Problem 3. *Construct explicitly such an F. Show that $F(x + 1) = F(x) + 1$.*

Lemma 1.1. *Let $f : S^1 \to S^1$ an homeomorphism and $F\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ a lift of f. Then the limit*

$$\tau(f) := \lim_{|n|\to\infty} \frac{1}{n} F^n(x) \mod 1$$

exists and is independent both from the point and the lift.

Proof. See [1]. □

Problem 4. *If there exists $L > 0$ such that $-L \leq a_{m+n} \leq a_n + a_m + L$ for all $n, m \in \mathbb{N}$, then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists. (Hint: let $\liminf_{n\to\infty} \frac{a_n}{n} = a > -\infty$, then for each $\varepsilon, m > 0$ exists $\bar{n} \in \mathbb{N}$, $\bar{n} > m$, such that $a_{\bar{n}} \leq a\bar{n} + \varepsilon\bar{n}$. Let $l \in \mathbb{N}$, $l > \bar{n}$, and write $l = k\bar{n} + r$, $r < \bar{n}$, then*

$$a \leq \frac{a_l}{l} \leq \frac{ka_{\bar{n}} + kL + a_r}{l} \leq \frac{k\bar{n}(a + \varepsilon) + kL + a_r}{l} = a + \varepsilon + \frac{L}{m} + \frac{a_r}{l}.$$

From which the claim follows.)

Problem 5. *Show that $\tau(f) \in \mathbb{Q}$ if and only if f has a periodic orbit. (Hint: see* [1]*).*

Problem 6. *Given $f \in \mathcal{C}^0(S^1, S^1)$, for any interval $I \subset S^1$, if $f(I) \subset I$, then f has a fixed point in I. (Hint: Stetting $I = [a, b]$ note that $g(x) = f(x) - x$ has a zero in I.)*

Problem 7. *If $\tau(f) \notin \mathbb{Q}$, then for each $n \in \mathbb{N} \setminus \{0\}$ and $x, y \in S^1$, $\{f^k(y)\}_{k\in\mathbb{N}} \cap [x, f^n(x)] \neq \emptyset$. (Hint: this is the same than saying $\bigcup_{k\in\mathbb{N}} f^{-k}[x, f^n(x)] = S^1$. If not consider $f^{-kn}[x, f^n(x)]$, this are contiguous intervals. If they do not cover all S^1, then their length must go to zero and $f^{-kn}x$ must have an accumulation point, call it z. Then*

$$z = \lim_{k\to\infty} f^{-kn}(x) = \lim_{k\to\infty} f^{-kn}(f^n(x)) = f^n(z).$$

Hence f must have a fixed point contradicting $\tau(f) \notin \mathbb{Q}$.)

Lemma 1.2. *For any homomorpfism $f : S^1 \to S^1$ with $\tau(f) \notin \mathbb{Q}$ and any $x, y \in S^1$ holds $\omega(x) = \omega(y)$.*

Proof. If $z \in \omega(x)$, then there exists $\{n_j\}$ such that $\lim_{j\to\infty} f^{n_j}(x) = z$. But then for each $j \in \mathbb{N}$ there exists k_j such that $f^{k_j}(y) \in [f^{n_j}(x), f^{n_{j+1}}(x)]$. Clearly $\lim_{j\to\infty} f^{k_j}(y) = z$, thus $z \in \omega(y)$. Reversing the role of x and y the Lemma follows. □

2. The smooth case

In this section we assume $f \in \mathcal{C}^2(S^1, S^1)$ and $\ln f' \in \mathcal{C}^1(S^1, \mathbb{R})$.[1]

Lemma 2.1. *If $\tau(f) \notin \mathbb{Q}$ and $x_0 \notin \omega(x_0)$, then*

$$\sum_{n=0}^{\infty} (f^n)'(x_0) < \infty.$$

Proof. Let $U(x_0) \ni x_0$ be the largest open interval not intersecting $\omega(x_0)$, call $K(x_0)$ its closure. First of all we see that the invariance of the ω-limit set implies $\{f^n(\partial K(x_0))\}_{n=1}^{\infty} \subset \omega(x_0)$. This implies that either $f^n K(x_0) \cap K(x_0) = \emptyset$ or $f^n K(x_0) \supset K(x_0)$ but the latter would imply the existence of a fixed point for f^{-n}, which is impossible, hence all the sets $\{f^n K(x_0)\}_{n\in\mathbb{Z}}$ must be disjoint. We can now conclude thanks to a typical distortion estimate: let $K_n(x_0) := f^n(K(x_0))$, then, setting $D := \left|\frac{f''}{f'}\right|_\infty$,

$$\begin{aligned} 1 > \sum_{n\in\mathbb{N}} |K_n(x_0)| &= \sum_{n\in\mathbb{N}} \int_{K(x_0)} (f^n)'(x)dx = \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} \frac{(f^n)'(x)}{(f^n)'(x_0)} dx \\ &\geq \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} |\ln f'(f^k(x)) - \ln f'(f^k(x_0))|} dx \\ &\geq \sum_{n\in\mathbb{N}} (f^n)'(x_0) \int_{K(x_0)} e^{-\sum_{k=0}^{n-1} D|K_k(x_0)|} dx \geq |K(x_0)| e^{-D} \sum_{n\in\mathbb{N}} (f^n)'(x_0). \end{aligned}$$

□

Problem 8. *If $\tau(f) \notin \mathbb{Q}$, then for each $x \in S^1$ there exist infinitely many $n \in \mathbb{Z}$ such that $\{f^k x\}_{|k|<n} \cap [x, f^n x] = \emptyset$.*

Lemma 2.2. *If $\tau(f) \notin \mathbb{Q}$, then, for all $x \in S^1$, $\omega(x) = S^1$.*

Proof. We use the same notation as in Lemma 2.1. Note that if there exists $n \in \mathbb{N}$, $n \neq 0$, such that $f^n(x_0) \in K(x_0)$ then, by the invariance of $\omega(x_0)$, it must be $f^n(x_0) \neq \partial K(x_0) \subset \omega(x_0)$ and then Problem 7 implies that there are infinitely many k such that $f^k(x_0) \in [x_0, f^n(x_0)] \subset K(x_0)$, but this is impossible since such an interval does not contain accumulation points of the forward trajectory. Thus, for each $n \in \mathbb{Z}$, $n \neq 0$, $f^(x_0) \notin K(x_0)$, accordingly there exist $\delta > 0$ such that each interval $[x_0, f^n(x_0)]$ has length at least δ.

Next, choose $L > 0$, by Lemma 2.1 there exists $m \in \mathbb{N}$ such that $(f^n)'(x_0) < L^{-1}$, for all $n > m$. We can then apply Problem 8 to find an $|n| > m$ such that $\{f^k x\}_{|k|<n} \cap [x_0, f^n(x_0)] = \emptyset$. Suppose $n < 0$ and let $J_- = [x_0, f^n(x_0)]$, then for each $k \in \{1, \ldots, -n-1\}$, $f^k J_- = [f^k x_0, f^{n+k} x_0]$, since the extreme of such

[1]These hypotheses can be slightly weakened, see [1].

an interval do not belong to J it follows that $f^k J_- \cap J_- = \emptyset$ (otherwise the first would be contained in the second and there would be a fixed point). Thus, setting $J = [x_0, f^{|n|}(x_0)]$, for all $k \in \{1, \dots, -n-1\}$, holds $f^k J \cap J = \emptyset$. The same result follows, setting $J_- = [x_0, f^{-n}(x_0)]$, for $n > 0$. Finally we conclude with another distortion argument

$$\begin{aligned}|f^{-|n|}J| = \int_J (f^{-|n|})'(x)dx = \frac{1}{(f^{|n|})'(x_0)} \int_J \frac{(f^{|n|})'(f^{-|n|}(f^{|n|}(x_0))}{(f^{|n|})'(f^{-|n|}x)} dx \\ \geq \frac{1}{(f^{|n|})'(x_0)} \int_J e^{-\sum_{k=0}^{|n|-1} D|f^k J|} dx \geq Le^{-D}\delta.\end{aligned}$$

Then choosing $L > e^D \delta^{-1}$ leads a length of $|f^{-|n|}J|$ larger than one, which contradicts the fact that f is an homeomorphism. $\square$

3. The analytic case

We have seen that the qualitative behavior of smooth circle maps with irrational rotation number is similar to the behavior of the rigid rotation in problem 1. What it is not clear is if the two dynamics can be conjugated (i.e. in the spirit of the flow box theorem). This latter problem turn out to be extremely subtle and to require much finer number theoretical consideration than distinguishing between rational and irrationals.

We leave the study of this case to the reader, which should be easy after she/he has carefully studied the note on the Siegel problem.

References

[1] Katok, Anatole; Hasselblatt, Boris *Introduction to the modern theory of dynamical systems.* With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.

E-mail address: `liverani@mat.uniroma2.it`