A FEW ELEMENTARY FACTS FROM LOCAL BIFURCATION THEORY

CARLANGELO LIVERANI

1. Generic Vector fields

Let us consider a first order autonomous differential equation,

$$\dot{x} = V(x) \tag{1.1}$$

where $x \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^d)$ and $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$. We are interested in the *typical local behavior* of such systems. Unfortunately, before being able to even address such an issue, it is necessary to give a technical meaning to the three words *typical, local* and *behavior*. By *local* understanding in a region K we mean that for each point $x \in K$ we are able to consider some neighborhood of x in which we understand the solutions of (1.1).[1] To do so it would be nice to consider only neighborhoods U in which $V(x) \neq 0$ with, at most, the exception of one point. Of course, this is not always possible (think of the case $V \equiv 0$), our claim is that *typically* it is.

To define *typical* let us consider the following.

Definition 1. *Given a topological space Ω, we say that a set $A \subset \Omega$ is* generic *if it is open and dense.*

Now $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ is a Banach space[2] hence the topology is trivially determined by the norm. For now on *typical* will mean that it happens for a countable intersection of *generic* sets (this is also called a *residual* set).

Problem 1. *Prove that a residual set is dense.*

Problem 2. *Give an example of a typical set in $\mathbb{R}$ with zero Lebesgue measure.*

Next, let us define

$$A_K := \{V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n) \; : V(x) = 0 \text{ implies } D_x V \text{ hyperbolic } \forall\, x \in K\}$$

Problem 3. *Prove that, for each compact set $K \subset \mathbb{R}^d$, if $V \in A_K$, then V has only finitely many zeroes in K. (Hint: Let $\bar{x} \in K$ such that $V(\bar{x}) = 0$. Then, by assumption $D_{\bar{x}} V$ is invertible, so $V(\bar{x} + \xi) = 0$ can be written as*

$$D_{\bar{x}} V^{-1} (D_{\bar{x}} V \xi - V(\bar{x} + \xi)) = \xi.$$

Since $D_{\bar{x}} V \xi - V(\bar{x} + \xi) = o(\|\xi\|)$, it follows that the above equation has the unique solution $\xi = 0$ in a sufficiently small neighborhood of zero. Hence there exists a neighborhood of $\bar{x}$ in which there are no other zeroes. Next, for each point in K consider a neighborhood as follows: if the V is different from zero at such a point,

---

Date: November 28, 2007.

[1] Note that, if K is compact, then finitely many such neighborhoods will cover K. On the other hand if, for example, $K = \mathbb{R}^d$, then countably many neighborhoods will do the job.

[2] The norm being $\|V\| := |V|_\infty + |DV|_\infty$.

then consider a neighborhood for which the vector field is different from zero. If the vector field is zero at the point then consider the above neighborhood in which the point is the only zero. In such a way we have a covering of K, we can then extract a finite subcover hence proving the statement.)

Problem 4. *Prove that, for each compact set $K \subset \mathbb{R}^d$, A_K is generic. (Hint: Let $V \in A_K$ and $\{x_i\}_{i=1}^M$ be the zeroes of V. Then for each vector field $W \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\|W\| \leq 1$, consider the family $V(x,\mu) := V(x) + \mu W(x)$. For each $i \in \{1, \dots, M\}$, use the implicit function theorem to show that there exists $\varepsilon_i, \delta_i > 0$ and $X_i \in \mathcal{C}^1([-\varepsilon_i, \varepsilon_i], \mathbb{R}^n) \to \mathbb{R}^d$, $X_i(x_i) = 0$, such that $V(X_i(\mu), \mu) = 0$ and $V(x,\mu) = 0$, $\|x - x_i\| \leq \delta_i$, $|\mu| \leq \varepsilon_i$ implies that $x = X_i(\mu)$. Verify (using perturbation theory) that, for μ small enough $\partial_x V(X(\mu), \mu)$ is hyperbolic. Next, set $\delta = \min \delta_i$ and $\rho := \inf_{|x - x_i| \geq \delta} \|V(x)\|$. Clearly $V(x,\mu) \neq 0$ if $|x - x_i| \geq \delta$ and $\mu < \rho$. Hence a neighborhood of V of size $\min\{\varepsilon_i, \rho\}$ belongs to A_K, hence A_K is open. For the density, the first problem to rule out the possibility of infinitely many zeroes. A possible way is to consider the vector field $V_\varepsilon(x) = S_d \varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} V(x - z) e^{-\frac{\|x-z\|^2}{\varepsilon}} dx$, where S_d is determined by the normalization $S_d \int_{\mathbb{R}^d} e^{-\|x\|^2} dx = 1$. First verify that $\lim_{\varepsilon \to 0} \|V - V_\varepsilon\|_\infty + \|DV - DV_\varepsilon\|_\infty = 0$, then notice that V_ε is an analytic function. From this follows that the set of zeroes of each function is (generically) a finite union of codimension one manifolds.[3] One can than do small translations to ensure that all such manifolds have only transversal intersections. Next, if one has a field with finitely many zeroes, let $V(\bar{x}) = 0$ and $D_{\bar{x}} V$ is not hyperbolic, then show that one can find a matrix A such that, setting $V(x,\mu) := V(x) + \mu A(x - \bar{x})$ (in a ball containing K), holds $V(\cdot, \mu) \in A_K$ for all $\mu \neq 0$. This is better done in the coordinates in which $D_{\bar{x}} V$ is in normal Jordan form.)*

Definition 2. *We say that two vector fields V, W are equivalent in the open set U, if, for each $t > 0$, there exists a homeomorphism $F : U \to U$ such that, calling ϕ_t^V, ϕ_t^W the flows generated by the vector fields, holds $\phi_t^v \circ F = F \circ \phi_t^W$.*

Finally, we we say that two vector fields have the same local *behavior* at $\bar{x}$ if it is possible to find $U \ni \bar{x}$ such that they are equivalent in U.

It is now clear that we understand already the typical local behavior. In fact, either $V(\bar{x}) \neq 0$ and then the flow box Theorem tells us that the field has the same local behavior than a constant vector field; or, if $V(\bar{x}) = 0$, then Grobmann-Hartman Theorem tells us that the field has the same local behavior than its linear part.

2. Generic families of vector fields

Our next aim is to consider a situation in which the system has a control parameter. That is, it is described by the equations

$$\dot{x} = V(x, \lambda) \tag{2.1}$$

where $x \in \mathbb{R}^d$ and $\lambda \in [-2, 2]$ is the parameter that, in principle, can be varied. Now by *local* understanding in a region K we mean that for each point $(\bar{x}, \bar{\lambda}) \in K \times [-1, 1]$ we can find a neighborhood of the form $U \times (\lambda - \varepsilon, \lambda + \varepsilon)$ in which are able to

[3]This may seem obvious, but it is a subtle result. For example it follows from the Weierstrass preparation theorem. Do explicitly the cases $d = 1, 2$.

understand the behavior of the solutions of (2.1). Typicality will instead be (for simplicity) with respect to the space $\mathcal{C}^2(\mathbb{R}^d \times (-2,2), \mathbb{R}^d)$.

Let us now try to understand the local picture for typical families of vector fields. In analogy with the previous section, for compact $K \subset \mathbb{R}^d$, we can consider the sets

$$\bar{A}_K := \{V \in \mathcal{C}^1(\mathbb{R}^d \times (-2,2), \mathbb{R}^d) \;:\; V(x,\lambda) = 0 \text{ implies } \partial_x V(x,\lambda) \text{ hyperbolic} \\ \forall\, (x,\lambda) \in K \times [-1,1]\}$$

Problem 5. *Prove that if $V \in \bar{A}_K$, then for each $(\bar{x},\bar{\lambda}) \in K \times [-1,1]$ there exists an open set of the form $U \times (-\varepsilon + \bar{\lambda}, \varepsilon + \bar{\lambda}) =: U \times I$ such that either $V(x,\lambda) \neq 0$ or there exists $X \in \mathcal{C}^1(I, K)$ such that $V(X(\lambda),\lambda) = 0$ for each $\lambda \in K$ and there are no other zeroes in $U \times I$. Then, prove that $\bar{A}_K$ is open. (Hint: implicit function theorem.)*

Clearly the above situations can be treated exactly as we did in the previous section and are therefore locally understandable. Unfortunately, the above does not exhaust all the possibilities.

Lemma 2.1. *For each K with non empty interior $\bar{A}_K$ is not generic.*

Proof. Since $\bar{A}_K$ is open, the problem must be the density. To see this let us consider, for example, the case $d = 1$, a compact set K with interior containing zero and the family

$$V(x,\lambda) = \lambda a + \lambda x + bx^2.$$

Now let us consider any $W \in \mathcal{C}^1(\mathbb{R} \times [-1,1], \mathbb{R})$ and look at $\tilde{V}(x,\lambda,\mu) := V(x,\lambda) + \mu W(x,\lambda)$. The claim is that for each μ sufficiently small, then $\tilde{V}(x,\lambda,\mu) \notin \bar{A}_K$. In fact, there exists $(x(\mu),\lambda(\mu)) \in K$ such that both $\tilde{V}(x(\mu),\lambda(\mu),\mu) = 0$ and $\partial_x \tilde{V}(x(\mu),\lambda(\mu),\mu) = 0$. To see this we define the function $F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F(x,\lambda,\mu) := \begin{pmatrix} \lambda a + \lambda x + bx^2 + \mu W(x,\lambda) \\ \lambda + 2bx + \mu \partial_x W(x,\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{V} \\ \partial_x \tilde{V} \end{pmatrix},$$

clearly we are looking for $(x(\mu),\lambda(\mu))$ such that $F(x(\mu),\lambda(\mu)) = 0$. Since $F(0,0,0) = 0$ we can apply the implicit function theorem provided

$$\begin{pmatrix} 0 & a \\ 2b & 1 \end{pmatrix}$$

is invertible, that is if $ab \neq 0$. We have thus seen that the family has an open neighborhood disjoint from $\bar{A}_K$, hence the latter set cannot be dense. □

Thus, to have a generic situation we need to consider a larger set.

A natural possibility is given by the following. Before stating we need a bit of notation. Given a $d \times d$ matrix an a vector $w \in \mathbb{R}^d$ we consider the $d \times (d+1)$ matrix $(A\ w)$, by rank $(A\ w)$ we mean the number of linearly independent column.

$$B_K = \{V \in \mathcal{C}^2 \;:\; \forall\, (x,\lambda) \in K \times [-1,1]\ V(x,\lambda) = 0 \Longrightarrow \operatorname{rank}(\partial_x V\ \partial_\lambda V) = d; \\ \partial_{xx} V(w,w) \neq 0 \quad \forall w \in \mathbb{R}^d \text{ and } \partial_x V v = 0, \partial_x V^T w = 0 \Rightarrow \langle w, \partial_{xx} V(v,v)\rangle \neq 0\}.$$

Problem 6. *Show that, for each compact set K, B_K is dense.(Hint: show first that the generically the zeroes of V are contained in the finite union of dimension one varieties in $\mathbb{R}^{d+1}$.[4]. Next, show that the fact that $\partial_x V$ has a null space of dimension at most one is generic. If $\partial_x V$ is invertible, then the argument is the same as in*

[4] This can be done, as before, by reducing to the analytic case.

Problem 4. If, instead, the null space is not empty, there exists $w \in \mathbb{R}^d$ such that $\langle w, \partial_x V \xi\rangle = 0$ for all ξ. Next, check that $\langle w, \partial_\lambda V\rangle \neq 0$ is dense.The density of the second condition follows by similar considerations.)

The point of B_K is that it also open.

Lemma 2.2. *The set B_K is open. In addition, if $V \in B_K$ and $V(\bar{x}, \bar{\lambda}) = 0$, then there exists $\varepsilon > 0$ and a neighborhood $U \ni \bar{x}$ such the vector field $V(x, \lambda)$ either has no zeroes or the set of zeroes consists of a smooth curve in $U \times (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$. Moreover, on such a curve $\partial_x V$ is invertible everywhere with, at most, the exception of one point.*

Proof. Now the approach based on a simple application of the implicit function theorem suggested in Problem 4 does not work if $\partial_x V(\bar{x}, \bar{\lambda})$ is not invertible, yet one can salvage such a situation in the following way. First suppose, without loss of generality, that $(\bar{x}, \bar{\lambda}) = (0, 0)$. Next, the null space of $\partial_x V(0, 0)$ must have dimension one, otherwise rank $(\partial_x V(0,0)\ \ \partial_\lambda V(0,0)) < d$, let $v \in \mathbb{R}^d$, $\|v\| = 1$, be the unique vector such that $\partial_x V(0,0)v = 0$. Consider a vector $v \in \mathbb{R}^d$, $\|v\| = 1$, and the change of variables $(\lambda, x) = F_v(\xi, \tau)$ defined by

$$\begin{aligned} x &= \xi - \tau v \\ \lambda &= \langle \xi, v\rangle. \end{aligned}$$

It is easy to check that F^{-1} is defined by

$$\begin{aligned} \tau &= \lambda - \langle x, v\rangle \\ \xi &= \lambda v + x - \langle x, v\rangle v. \end{aligned}$$

Then

$$\det(DF^{-1}) = \det\left(\begin{array}{c|c} 1 & -v_i \\ \hline v_j & \delta_{ij} - v_i v_j \end{array}\right) = \det\left(\begin{array}{c|c} 1 & -v_i \\ \hline 0 & \delta_{ij} \end{array}\right) = 1.$$

Then define the field $\tilde{V} := V \circ F$. Since $F(0,0) = 0$, $\tilde{V}(0,0) = 0$. To apply the implicit function theorem in the new variable we need $\partial_\xi \tilde{V}$ to be invertible, but $\partial_\xi \tilde{V}(x, \lambda) = \partial_x V(x, \lambda) + \partial_\lambda V(x, \lambda) \otimes v$.[5] It follows that $\partial_\xi \tilde{V}(0, 0)$ must be invertible, otherwise there would exists $w \in \mathbb{R}^d$ such that, for all $\eta \in \mathbb{R}^d$, holds

$$0 = \langle w, \partial_\xi \tilde{V}(0,0)\eta\rangle = \langle w, \partial_x V(0,0)\eta\rangle + \langle w, \partial_\lambda V(0,0)\rangle\langle v, \eta\rangle.$$

Choosing $\eta = v$ follows $\langle w, \partial_\lambda V(0, 0)\rangle = 0$ and hence $\partial_x V(0, 0)^T w = 0$. But this would contradict rank $(\partial_x V(0,0)\ \ \partial_\lambda V(0,0)) = d$. So $V \in B_K$ implies invertibility in a neighborhood of zero. We have then a $\mathcal{C}^1$ function $\xi(\tau)$ such that $\tilde{V}(\xi(\tau), \tau) = 0$, with $\xi'(\tau) = (\partial_x V + \partial_\lambda V \otimes v)^{-1}\partial_x V v$. This means that we have a smooth curve of singular points for V in a neighborhood of $(0, 0)$.

To conclude, we note that $\partial_x V(x(\tau), \lambda(\tau))$ has a non empty null space iff $\frac{d\lambda}{d\tau} = 0$. Indeed,

$$\frac{d\lambda}{d\tau} = \langle \xi'(\tau), v\rangle = 1 - \langle (\partial_x V + \partial_\lambda V \otimes v)^{-1}\partial_\lambda V, v\rangle =: 1 - \langle \zeta, v\rangle,$$

where, by definition, $\partial_x V\zeta + \partial_\lambda V\langle v, \zeta\rangle = \partial_\lambda V$, but if $\frac{d\lambda}{d\tau} = 0$, then $\langle \zeta, v\rangle = 1$, hence $\partial_x V\zeta = 0$. This means in particular that $\zeta(0) = v$. Finally, either $\tau = 0$ is the only value of the parameter for which $\partial_x V\zeta = 0$ or such point accumulate to zero

[5] Given two vectors $v, w \in \mathbb{R}^d$, by $v \otimes w$ we mean the matrix with elements $(v \otimes w)_{ij} = v_i w_j$.

hence $\frac{d}{d\tau}\partial_x V\zeta(0) = \frac{d}{d\tau}\langle v, \zeta(0)\rangle = 0$. But differentiating $\partial_x V\zeta + \partial_\lambda V\langle v, \zeta\rangle = \partial_\lambda V$ and setting $\tau = 0$ yields

$$\partial_{xx}V(v,v) = \partial_\lambda V\langle v, \zeta'\rangle + \partial_x V\zeta',$$

which would imply $\partial_{xx}V(v,v) = \partial_x V\zeta'$ whereby contradicting the fact that $V \in B_K$. □

Problem 7. *Find a generic set $\tilde{B}_K$ such that if $V \in B_K$, then for each smooth curve $(x(\tau), \lambda(\tau))$ such that $V(x(\tau), \lambda(\tau)) = 0$ holds $V(x(\tau), \lambda(\tau)) \in A_K$ with, at most, one exception.*

Thus, typically, we have to worry only about families in which the vector field has, at most, one zero. In fact, due to the previous discussion, we need to consider only the case in which there is one zero and the derivative is not hyperbolic.

Problem 8. *Prove that generically if $V(0,0) = 0$ and $\partial_x V(0,0)$ is not hyperbolic then only the following possibilities can occur*

- *A has a zero eigenvalue*
- *A has two purely imaginary conjugated eigenvalues.*

(Hint: put the matrix in Jordan normal form).

3. One dimension

In the one dimensional case let K be a compact set containing a neighborhood of zero and $V \in B_K$, suppose, without loss of generality, that $V(0,0) = 0$. Then either $V(0,0) \in A_K$, and then we understand the local behavior of the solutions, or $\partial_x V(0,0) = 0$, but then it must be $\partial_\lambda V(0,0) \neq 0$ and $\partial_{xx}V(0,0) \neq 0$, hence

$$V(x,\lambda) = \lambda a + \lambda bx + cx^2 + g(x,\lambda), \tag{3.1}$$

where $g = o(x^2 + \lambda)$.

Then $V(x,\lambda)$ has no solutions if $ac > 0$, while for $ac < 0$ there are the two solutions $x = \pm\sqrt{\frac{\lambda a}{c}} + \mathcal{O}(\lambda)$. We have therefore the following generic picture: either two points collide and kill each other or there is a creation of two zeroes of the vector field.

Problem 9. *Prove that the two equilibrium pints of the vector field are one attractive and the other repulsive.*

The above scenario is called a *saddle-node* bifurcation.

A natural question is if there exists a simpler standard form of the above bifurcation. Indeed we can try to kill some of the terms in 3.1 by a change of variable.

Problem 10. *Show that with a change of variables of the type $z = \alpha\lambda + x$, $\mu = \rho\lambda$ one can change the vector field (4.1) to the from $\tilde{V}(z,\mu) = \mu + bz^2 + o(z^2 + \mu)$.*

The above is the *normal form* of the saddle node bifurcation. This type of reduction can be made for each bifurcation and give rise to the large field of normal form theory which, unfortunately, goes beyond the scopes of the present notes.

4. Two dimensions: a zero eigenvalue

In this case the vector field must have the form (possibly after a linear change of variable to put $\partial V_x(0,0)$ in normal form)

$$V(x,\lambda) = \bar{a}(\lambda) + A(\lambda)x + G(x,\lambda) \tag{4.1}$$

where $a(0) = 0$, $\|G(x,\lambda)\| \leq c\|x\|^2$ and

$$A(0) = \begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}$$

At this point is it easy to show that the scenario is exactly the same than in the one dimensional case. We leave the details to the reader.

5. Two dimensions: eigenvalues with zero real part

In this case we have the well know *Hopf* bifurcation. See related note.

6. The Hamiltonian case

It is important to note that non generic situations may appear due to symmetries or other type of constraints. To give an example of such a situation let us consider an Hamiltonian vector field, that is a vector field of the type $V(x,p) = (\partial_p H, -\partial_x H)$ for some function $H(x,p)$. In this case

$$DV = \begin{pmatrix} \partial_{xp}H & \partial_{pp}H \\ -\partial_{xx}H & -\partial_{xp}H \end{pmatrix}.$$

Note that the trace of DV is always zero, hence if one eigenvalue is null also the other must be. It is thus clear that in this case the situation with two complex conjugate eigenvalues is generic for a vector field while two, not one, zero eigenvalues is generic for a vector field family. We must thus consider, for example, a family of the type

$$V(x,\lambda) = (p, -\lambda x - x^2)$$

Problem 11. *Show that in the above family we have the collision of two fixed point (a center and a saddle) that collide and exchange type.*

Remark 6.1. *Among other things, in this case we have a new situation even for the local understanding of vector fields due to the possibility of two complex conjugate eigenvalues a* center*. It is not obvious what is the local picture under perturbation for such a situation. For the two dimensional case, it implies that the Hamiltonian has a minimum and hence we have persistence of the center under perturbations. In higher dimensional situation this issue is the subject of the so called KAM theory. See the note on Siegel Theorem to have a glimpse in such a theory.*

Problem 12. *Consider the case of an Hamiltonian with a degenerate minimum, e.g.* $H(x,p,\lambda) = \frac{1}{2}p^2 + x^4 + \lambda x$. *What can you say?*

Remark 6.2. *The reader wishing to get a bit deeper in the bifurcation theory may look at* [1, Chapter 6].

References

[1] V.I. Arnold, *Geometric Methods in the theory of Ordinary Differential Equations*, Springer-Verlang, New York (1988).

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it