ABOUT MIXING

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1. The setting

Consider an expanding C^2 map of the circle. Then one can easily verify the Lasota-Yoke inequalities

$$|\mathcal{L}h|_{L^1} \leq |h|_{L^1}$$
$$|\mathcal{L}^n h|_{W_{1,1}} \leq \lambda^{-n}|h|_{W_{1,1}} + D|h|_{L^1}$$

where $|DT| > \lambda > 1$ and $D = \sup_x \frac{|D_x^2 T|}{|D_x T|^2}$. From abstract functional analytic facts it follows that the essential spectrum of $\mathcal{L}$ is contained in the disk of radius λ.

Lemma 1.1. *The map is T mixing.*

Proof. Let Π_θ be the projector on the eigenspace associated to the eigenvalue $e^{i\theta}$. Consider $\Pi_\theta\mathcal{L}$, this is a fine rank operator and must be a diagonal operator since it is norm bounded while Jordan blocks have norm that increases polynomially. Let $\{h_j\}$ be a basis of the rank and consider functionals ℓ_j such that $\ell_j(h_i) = \delta_{ij}$. Then $\Pi_\theta h = \sum_j \alpha_j h_j$, thus $\alpha_j = \ell_j(h)$, hence $\sum_j h_j \ell_j(h) = \Pi_\theta h = e^{-i\theta}\Pi_\theta\mathcal{L}h = e^{-i\theta}\sum_j h_j \ell_j(\mathcal{L}h)$. That is $\ell_j(h) = e^{-i\theta}\ell_j(\mathcal{L}h)$. But the above means

$$|\ell_j(h)| = |\ell_j(\mathcal{L}^n h)| \leq C\lambda^{-n}|h|_{W_{1,1}} + CD|h|_{L^1}$$

which, taking the limit for n to infinity, implies $|\ell_j(h)| \leq CD|h|_{L^1}$. Accordingly, ℓ_j belongs to the dual of L^1, that is can be identified with a function $\bar{\ell}_j \in L^\infty$ via the formula $\ell_j(h) = \int \bar{\ell}_j(x)h(x)dx$ and $\bar{\ell}_j \circ T = e^{i\theta}\ell_j$.

This means that $\mathcal{L}^n \bar{\ell}_j^* = e^{i\theta n}\mathcal{L}^n(\bar{\ell}_j^* \circ T^n) = e^{i\theta n}\bar{\ell}_j^*\mathcal{L}^n 1$. Or, better,

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}\mathcal{L}^k\bar{\ell}_j^* = \bar{\ell}_j^*\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}^k 1.$$

By compactness the last averages have a subsequence converging to $\Pi_\theta\ell_j^*$ and $\Pi_1 1$, respectively. Hence

$$\Pi_\theta\ell_j^* = \ell_j^*\Pi_1 1.$$

In addition, it is easy to verify that if the density is zero at same point, then it must be zero on all its preimages. Since such preimages are dense, then the density would be zero which is absurd since it must have integral one. So the density is strictly positive and thus, since it is continuous, uniformly so. But his implies that $\bar{\ell}_j$ has a version in $W_{1,1}$.

Now that means that all the sets $A_\alpha := \{|\bar{\ell}_j| \leq \alpha\}$ of positive measure must contain an interval I. But then $A_\alpha = T^n A_\alpha$ mod zero Lebesgue measure sets, but for n large enough $T^n I = \mathbb{T}^1$, thus $A = \mathbb{T}^1$. This means that $|\bar{\ell}_j|$ must be constant. We can write $\bar{\ell}_j = Ce^{i\varphi}$ for some real function φ everywhere continuous but at a

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point x_* where it has a jump that is an integer multiple of 2π and the point x_* can be chosen arbitrarily. Hence

$$e^{i\varphi\circ T} = \bar{\ell}_j \circ T = e^{i\theta}\bar{\ell}_j = e^{i(\varphi+\theta)}.$$

that is $\varphi \circ T = \varphi + \theta \mod 2\pi$. Finally, notice that the map T has at least one fixed point $\bar{x}$, thus, choosing $x_* \neq \bar{x}$, $\theta = 2\pi\Theta(\bar{x})$ which means that $e^{i\theta} = 1$. So non other eigenvalues can be present beside the eigenvalue one. Moreover, since $\ell_j = \ell_j \circ T$ implies ℓ_j constant, one must be a simple eigenvalue. This implies that, for all $h \in W_{1,1}$, $\int h = 1$, and $\varphi \in L^\infty$, holds

$$\lim_{n\to\infty} \int_{\mathbb{T}^1} \varphi \circ T^n h = \int \varphi h_* + \mathcal{O}(|\varphi|_{L^\infty} \nu^{-n} |h|_{W_{1,1}}),$$

where h_* is the eigenvector associated to one and ν is the radius of the spectrum of $\mathcal{L}$ once one is removed. By a standard approximation argument it follows that the system is mixing. □

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