A PIECE OF FUNCTIONAL ANALYSIS

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1. The setting

Consider two Banach space $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}_0, |\cdot|)$ such that $\mathcal{B} \subset \mathcal{B}_0$ and

i. $|h| \leq \|h\|$ for all $h \in \mathcal{B}$,
ii. if $h \in \mathcal{B}$ and $|h| = 0$, then $h = 0$.
iii. The unit ball $\{h \in \mathcal{B} \, : \, \|h\| \leq 1\}$ is relatively compact in $\mathcal{B}_0$.
iv. There exists $C > 0$ such that: for each $\varepsilon > 0$ there exists a finite rank operator $\mathbb{A}_\varepsilon \in L(\mathcal{B}, \mathcal{B})$ such that $\|\mathbb{A}_\varepsilon\| \leq C$ and $|h - \mathbb{A}_\varepsilon h| \leq \varepsilon \|h\|$ for all $h \in \mathcal{B}$.[1]

In addition consider a bounded operator $\mathcal{L} : \mathcal{B}_0 \to \mathcal{B}_0$, constants $A, B, C \in \mathbb{R}_+$, and $\lambda > 1$, such that

a. $|\mathcal{L}^n| \leq C$ for all $n \in \mathbb{N}$,
b. $\mathcal{L}(B) \subset B$
c. $\|\mathcal{L}^n h\| \leq A\lambda^{-n}\|h\| + B|h|$ for all $h \in \mathcal{B}$ and $n \in \mathbb{N}$.

In particular $\mathcal{L}$ can be seen as a bounded operator on $\mathcal{B}$.

Theorem 1.1. *The spectral radius of the operator $\mathcal{L} \in L(\mathcal{B}, \mathcal{B})$ is bounded by 1 while the essential spectral radius is bounded by λ^{-1}.*

The proof of the above theorem depends on an auxiliary fact which is of much interest in its own.

Theorem 1.2 (Analytic Fredholm theorem–finite rank[2])**.** *Let D be an open connected subset of $\mathbb{C}$. Let $F : \mathbb{C} \to L(\mathcal{B}, \mathcal{B})$ be an analytic operator-valued function such that $F(z)$ is finite rank for each $z \in D$. Then, one of the following two alternatives holds true*

- $(\mathbf{Id} - F(z))^{-1}$ *exists for no $z \in D$*
- $(\mathbf{Id} - F(z))^{-1}$ *exists for all $z \in D \backslash S$ where S is a discrete subset of D (i.e. S has no limit points in D). In addition, if $z \in S$, then 1 is an eigenvalue for $F(z)$ and the associated eigenspace has finite multiplicity.*

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[1]In fact, this last property is not needed. We require it since, on the one hand, it is true in all the applications we will be interested in and, on the other hand, drastically simplifies the argument. Note also that, if one uses the Fredholm alternative for compact operators rather than finite rank ones (Theorem 1.2), then one can ask the $\mathbb{A}_\varepsilon$ to be compact instead than finite rank making easier their construction in concrete cases.

[2]The present proof is patterned after the proof of the Analytic Fredholm alternative for compact operators (in Hilbert spaces) given in [1, Theorem VI.14]. There it is used the fact that compact operators in Hilbert spaces can always be approximated by finite rank ones. In fact the theorem holds also for compact operators in Banach spaces but the proof is a bit more involved.

Proof. First of all notice that, for each $z_0 \in D$ there exists $r > 0$ such that $D_{r(z_0)}(z_0) := \{z \in \mathbb{C} \,:\, |z - z_0| < r(z_0)\} \subset D$, and

$$\sup_{z \in D_{r(z_0)}(z_0)} \|F(z) - F(z_0)\| \leq \frac{1}{2}.$$

Clearly if we can prove the theorem in each such disk we are done.[3] Note that

$$\mathbf{Id} - F(z) = \left(\mathbf{Id} - F(z_0)(\mathbf{Id} - [F(z) - F(z_0)])^{-1}\right)(\mathbf{Id} - [F(z) - F(z_0)]).$$

Thus the invertibility of $\mathbf{Id} - F(z)$ in $D_r(z_0)$ depends on the invertibility of $\mathbf{Id} - F(z_0)(\mathbf{Id} - [F(z) - F(z_0)])^{-1}$. Let us set $F_0(z) := F(z_0)(\mathbf{Id} - [F(z) - F(z_0)])^{-1}$.

Let us start by looking at the equation

$$(\mathbf{Id} - F_0(z))h = 0. \tag{1.1}$$

Clearly if a solution exists, then $h \in \text{Range}(F_0(z)) = \text{Range}(F(z_0)) := \mathbb{V}_0$. Since $\mathbb{V}_0$ is finite dimensional there exists a basis $\{h_i\}_{i=1}^N$ such that $h = \sum_i \alpha_i h_i$. On the other hand there exists an analytic matrix $G(z)$ such that[4]

$$F_0(z)h = \sum_{ij} G(z)_{ij} \alpha_j h_i.$$

Thus (1.1) is equivalent to

$$(\mathbf{Id} - G(z))\alpha = 0,$$

where $\alpha := (\alpha_i)$.

The above equation can be satisfied only if $\det(\mathbf{Id} - G(z)) = 0$ but the determinant is analytic hence it is either always zero or zero only at isolated points.[5]

Suppose the determinant different from zero, and consider the equation

$$(\mathbf{Id} - F_0(z))h = g.$$

Let us look for a solution of the type $h = \sum_i \alpha_i h_i + g$. Substituting yields

$$\alpha - G(z)\alpha = \beta$$

where $\beta := (\beta_i)$ with $F_0(z)g =: \sum_i \beta_i h_i$. Since the above equation admits a solution, we have $\text{Range}(\mathbf{Id} - F_0(z)) = \mathcal{B}$, Thus we have an everywhere defined inverse, hence bounded by the open mapping theorem.

We are thus left with the analysis of the situation $z \in S$ in the second alternative. In such a case, there exists h such that $(\mathbf{Id} - F(z))h = 0$, thus one is an eigenvalue. On the other hand, if we apply the above facts to the function $\Phi(\zeta) := \zeta^{-1}F(z)$ analytic in the domain $\{\zeta \neq 0\}$ we note that the first alternative cannot take place

[3] In fact, consider any connected compact set K contained in D. Let us suppose that for each $z_0 \in K$ we have a disk $D_{r(z_0)}(z_0)$ in the theorem holds. Since the disks $D_{r(z_0)/2}(z_0)$ form a covering for K we can extract a finite cover. If the first alternative holds in one such disk then, by connectness, it must hold on all K. Otherwise each $S \cap D_{r(z_0)/2}(z_0)$, and hence $K \cap S$, contains only finitely many points. The Theorem follows by the arbitrariness of K.

[4] To see the analyticity notice that we can construct linear functionals $\{\ell_i\}$ on $\mathbb{V}_0$ such that $\ell_i(h_j) = \delta_{ij}$ and then extend them to all $\mathcal{B}$ by the Hahn-Banach theorem. Accordingly, $G(z)_{ij} := \ell_j(F_0(z)h_i)$, which is obviously analytic.

[5] The attentive reader has certainly noticed that this is the turning point of the theorem: the discreteness of S is reduced to the discreteness of the zeroes of an appropriate analytic function: a determinant. A moment thought will immediately explain the effort made by many mathematicians to extend the notion of determinant (that is to define an analytic function whose zeroes coincide with the spectrum of the operator) beyond the realm of matrices (the so called Fredholm determinants).

since for $|\zeta|$ large enough $\mathbf{Id} - \Phi(\zeta)$ is obviously invertible. Hence, the spectrum of $F(z)$ is discrete and can accumulate only at zero. This means that there is a small neighborhood around one in which $F(z)$ has no other eigenvalues, we can thus surround one with a small circle γ and consider the projector

$$
\begin{aligned}
P :=& \frac{1}{2\pi i}\int_\gamma (\zeta - F(z))^{-1} d\zeta = \frac{1}{2\pi i}\int_\gamma \left[(\zeta - F(z))^{-1} - \zeta^{-1}\right] d\zeta \\
=& \frac{1}{2\pi i}\int_\gamma F(z)\zeta^{-1}(\zeta - F(z))^{-1} d\zeta.
\end{aligned}
$$

By standard functional calculus it follows that P is a projector and it clearly projects on the eigenspace of the eigenvector one. But the last formula shows that P must project on a subspace of the rank of $F(z)$, hence it must be finite dimensional. $\square$

We can now prove our main result.

Proof of Theorem 1.1. The first assertion is a trivial consequence of (c), (a) and (i).

The second part is much deeper. Let $\mathcal{L}_{n,\varepsilon} := \mathcal{L}^n \mathbb{A}_\varepsilon$, clearly such an operator is finite rank, in addition

$$\|\mathcal{L}^n h - \mathcal{L}_{n,\varepsilon} h\| \leq A\lambda^{-n}\|(\mathbf{Id} - \mathbb{A}_\varepsilon)h\| + B|(\mathbf{Id} - \mathbb{A}_\varepsilon)h| \leq A(1+C)\lambda^{-n}\|h\| + B\varepsilon\|h\|.$$

By choosing $\varepsilon = \lambda^{-n}$ we have that there exists $C_1 > 0$ such that

$$\|\mathcal{L}^n - \mathcal{L}_{n,\varepsilon}\| \leq C_1\lambda^{-n}.$$

For each $z_0 \in \mathbb{C}$ we can now write

$$\mathbf{Id} - z\mathcal{L} = (\mathbf{Id} - z_0(\mathcal{L} - \mathcal{L}_{n,\varepsilon}) - (z - z_0)\mathcal{L}) - z_0\mathcal{L}_{n,\varepsilon}.$$

Since

$$\|z_0(\mathcal{L} - \mathcal{L}_{n,\varepsilon}) + (z - z_0)\mathcal{L}\| \leq |z_0|C_1\lambda^{-n} + C|z - z_0| < \frac{1}{2},$$

provided that $|z_0| \leq \frac{1}{4C_1}\lambda^n$ and $|z - z_0| \leq \frac{1}{4C}$, given any z_0 in the disk $D_n := \{|z| < \frac{1}{4C_1}\lambda^n - \frac{1}{4C}\}$ we can consider the domain $D_n(z_0) := \{|z - z_0| < \frac{1}{4C}\}$ and, in such a domain $B(z) := \mathbf{Id} - z_0(\mathcal{L} - \mathcal{L}_{n,\varepsilon}) - (z - z_0)\mathcal{L}$ is invertible. Hence

$$\mathbf{Id} - z\mathcal{L} = \left(\mathbf{Id} - z_0\mathcal{L}_{n,\varepsilon}B(z)^{-1}\right)B(z) =: (\mathbf{Id} - F(z))B(z).$$

By applying Theorem 1.2 to $F(z)$ we have that the operator is either never invertible or not invertible only in finitely many points in the disk $D_n'(z_0) := \{|z - z_0| < \frac{1}{8C}\}$. Since we can cover $D_n := \{|z| < \frac{1}{4C_1}\lambda^n - \frac{1}{4C}\}$ with finitely many such disk and since the operator is invertible for z_0 small enough (thus the first alternative cannot hold), the Theorem follows. $\square$

References

[1] M.Reed, B.Simon, *Methods of Modern Mathematical Physics. I-Functional Analysis*, Academic Press, New York (1980).

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