PERSONAL NOTES FOR THE COIMBRA MINICOURSE
PROBABILITY AND UNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. This are personal notes, they contain mistakes and the notation is inconsistent. Read at your own risk.

1. FIRST LECTURE

1.1. **Expanding maps.** Let us start to investigate a particular, but very important, type of dynamical systems: piecewise smooth expanding maps.

More precisely, let $X := [0,1]^d$ together with a (possibly countable) collection of disjoint open sets $\{\Delta_i\}_{i\in\mathcal{I}\subset\mathbb{N}}$ such that

- $\cup_{i\in\mathcal{I}}\overline{\Delta}_i = X$;
- For each orthogonal basis $E := \{e_i\}$ let$L_k(x,j,E)$ be the number of connected components of $\{x + te_k\}_{t\in[-1,1]} \cap \Delta_j$. Then we assume that $L_j = \inf_E \sup_{x\in\Delta_j} \sup_k L_k(x,j,E) < \infty$.

Next, let $T : X \to X$ be such that, for each $i \in \mathcal{I}$, $T|_{\Delta_j}$ is a $\mathcal{C}^2$ invertible map. Finally we ask that the map be expanding and not too singular

$$\begin{aligned} &\|(D_xT)^{-1}\| \leq \lambda_j^{-1} < 1 \quad \text{for all } x \in \Delta_j; \\ &|\nabla(D_xT)^{-1}|_{L^d} < \infty. \end{aligned} \tag{1.1}$$

Given such a system we ask ourselves the following questions

1. can we investigate the behavior of the Birkhoff sums in some detail?
2. are there invariant measures absolutely continuos with respect to the Lebesgue measure?
3. if such measures exist, do they describe the statistic of the orbits ?
4. which measures converge, under the dynamics, to such measures, how fast?
5. are such measures stable under perturbations (stochastic or deterministic)?
6. can such measure be computed?

By Birkhoff sums we mean, given a measurable function $f : X \to \mathbb{R}$,

$$f_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)). \tag{1.2}$$

The importance of such sums is due to the fact that very often a measurement on a real system (physical, biological, economic ...) described by the dynamical system (X,T) is of the form f_n for some large n.

Birkhoff theorem assert that $\lim_{n\to\infty} f \circ T^n$ exist μ-almost surely for any invariant probability measure μ. Let us recall that, calling $\mathcal{M}(X)$ the set of Borel

Date: July 23, 2008.

measures on X, we can define the map $T' : \mathcal{M} \to \mathcal{M}$ by $T'\mu(f) := \mu(f \circ T)$ for each measurable bounded f. The a measure is invariant if $T'\mu = \mu$.

Invariant measures thus play a very important role, but they are too many (e.g. if T is continuos, then any invariant set supports at least an invariant measure by Krylov-Bogoliubov theorem). If we are interested in the behavior of (1.2) only for almost every point with respect to Lebesgue, then it seems natural and necessary to restrict our discussion to invariant measures absolutely continuos w.r.t. Lebesgue (if they exist). To do this some measure theory is needed.

1.2. **A bit of measure theory.** Let us define the following two norms on $\mathcal{M}(X)$:

$$\begin{aligned} |\mu| &:= \sup_{\varphi \in \mathcal{C}^0(X,\mathbb{R})} \frac{\mu(\varphi)}{|\varphi|_\infty} \\ \|\mu\| &:= \sup_{k\in\{1,\dots,d\}} \sup_{\varphi\in\mathcal{C}^1(X,\mathbb{R})} \frac{\mu(\partial_{x_k}\varphi)}{|\varphi|_\infty}. \end{aligned} \tag{1.3}$$

Note that, for each $\varphi \in \mathcal{C}^0(X,\mathbb{R})$ and $\varepsilon > 0$ one can find $\varphi_\varepsilon \in \mathcal{C}^1(X,\mathbb{R})$ such that $|\varphi - \varphi_\varepsilon| \le \varepsilon|\varphi|_\infty$, hence

$$\mu(\varphi) \le |\mu|\varepsilon|\varphi|_\infty + \mu(\varphi_\varepsilon) = |\mu|\varepsilon|\varphi|_\infty + \mu(\partial_{x_1}\int_0^{x_1}\varphi_\varepsilon) \le (|\mu|\varepsilon + \|\mu\|(1+\varepsilon))|\varphi|_\infty.$$

Taking the sup on φ and by the arbitrariness of ε, follows

$$|\mu| \le \|\mu\|. \tag{1.4}$$

Lemma 1.1. *Let $\mathcal{B} := \{\mu \in \mathcal{M}(X) \ : \ \|\mu\| < \infty\}$. If $\mu \in \mathcal{B}$ then it is absolutely continuos with respect to the Lebesgue measure m. Moreover*

$$\frac{d\mu}{dm} \in L^p(X,m) \quad \textit{for all } p < \frac{d}{d-1}.$$

Proof. Let $\varphi \in \mathcal{C}^0(X,\mathbb{R})$, then for each $\varepsilon \in (0,1)$ there exists $\varphi_\varepsilon \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$, supported in $[-\varepsilon, 1+\varepsilon]^d$, such that $|\varphi - \varphi_\varepsilon|_{\mathcal{C}^0(X,\mathbb{R})} \le \varepsilon$, $|\varphi_\varepsilon|_\infty \le |\varphi|_\infty(1+\varepsilon)$. In addition, if we define

$$\Gamma(\xi) := \begin{cases} -\frac{1}{2}\|\xi\| & \text{if } d = 1 \\ -\frac{1}{2\pi}\ln\|\xi\| & \text{if } d = 2 \\ \frac{1}{d(d-2)\alpha_d\|\xi\|^{d-2}} & \text{if } d \ge 3, \end{cases} \tag{1.5}$$

where α_d is the d-dimensional volume of the unit ball in $\mathbb{R}^d$, we can define the Newtonian potential $w_\varepsilon(x) = \int_{\mathbb{R}^d}\Gamma(x-z)\varphi_\varepsilon(z)dz$. It is then well know from potential theory that $\Delta w_\varepsilon = \varphi_\varepsilon$, thus

$$\begin{aligned} \mu(\varphi) \le \mu(\varphi_\varepsilon) + |\mu|\varepsilon &= \sum_{k=1}^d \mu(\partial_{x_k}\partial_{x_k}w_\varepsilon) + |\mu|\varepsilon \\ &\le \sum_{k=1}^d \|\mu\| \sup_{x\in X}\int |\partial_{x_k}\Gamma(x-z)\varphi_\varepsilon(z)dz| + |\mu|\varepsilon \\ &\le C\sum_{k=1}^d \|\mu\|\,|\varphi_\varepsilon|_{L^q}\left[\int_{[-1,2]^d}\frac{|x_k - z_k|^p}{\|x-z\|^{dp}}dz\right]^{\frac{1}{p}} + |\mu|\varepsilon, \end{aligned}$$

where $q^{-1}+p^{-1}=1$. Since the integral in square brackets is finite for $p<\frac{d}{d-1}$, we have, be the arbitrariness of ε,

$$\mu(\varphi)\leq C(\|\mu\|+|\mu|)|\varphi|_{L^q}.$$

This means that the linear functional $\mu:\mathcal{C}^0\to\mathbb{R}$ can be extended to a bounded functional on L^q. Since the dual of L^q is L^p it follows that there exists $h\in L^p$ such that $\mu(\varphi)=\int_X h(x)\varphi(x)dx$. □

Remark 1.2. *In fact it follows from the Gagliardo-Nirenberg-Sobolev inequality that the above Lemma holds also for* $p=\frac{d}{d-1}$.

Exercize 1. *Show that, for all* $\mu\in\mathcal{B}$, *setting* $h=\frac{d\mu}{dm}$, *holds* $|\mu|=|h|_{L^1}$ *and* $\|\mu\|=|h|_{BV}$.

The following characterization will be useful in the following: given $h\in L^1(X,m)$ we define

$$\operatorname{Var}^k(h)(x)=\sup_{\varphi\in\mathcal{C}^1([0,1],\mathbb{R})}\frac{\int_0^1 h(x_1,\dots,x_{k-1},z,x_{k+1},\dots,x_d)\varphi'(z)dz}{|\varphi|_\infty}.$$

Lemma 1.3. *For each* $\mu\in\mathcal{B}$, *setting* $h=\frac{d\mu}{dm}$,

$$\|\mu\|=\sup_{k\in\{1,\dots,n\}}|\operatorname{Var}^k(h)|_{L^1}.$$

Proof. First,

$$\|\mu\|\leq\sup_k\sup_{|\varphi|_\infty\leq1}\int h\partial_{x_k}\varphi=\sup_k\sup_{|\varphi|_\infty\leq1}\int\operatorname{Var}^k h\sup_{x_k}|\varphi|\leq\sup_k|\operatorname{Var}^k(h)|_{L^1}.$$

For the opposite inequality one need a bit of preparation.

For each $n\in\mathbb{N}$ and function $\eta\in\mathcal{C}_0^2([-1,1]^n,\mathbb{R}_+)$, $\int\eta=1$, let us define $\eta_\varepsilon(x)=\varepsilon^{-n}\eta(\varepsilon^{-1}x)$. Then, for each $h\in L^1([0,1]^n,m)$ and $\varphi\in\mathcal{C}_0^1(\mathbb{R}^n,\mathbb{R})$ let $h_\varepsilon(x)=\int dz\; h(z)\,\eta_\varepsilon(x-z)$,

$$\begin{aligned}\int\partial_{x_k}h_\varepsilon(x)\cdot\varphi(x)&=\int h(z)\partial_{x_k}\eta_\varepsilon(x-z)\cdot\varphi(x)\\&=-\int h(z)\partial_{z_k}\eta_\varepsilon(x-z)\cdot\varphi(x)\leq|h|_{BV}|\varphi|_\infty.\end{aligned}\tag{1.6}$$

That is $\sup_k|\partial_{x_k}h_\varepsilon|_{L^1}\leq|h|_{BV}$. On the other hand, for each $\delta>0$ and $k\in\{1,\dots,d\}$ there exists $\phi\in\mathcal{C}^1$, $|\phi|_\infty=1$, such that $|h|_{BV}\leq\int h\partial_{x_k}\phi+\delta$. Next, consider a compact support extension $\tilde\phi\in\mathcal{C}_0^1$ of ϕ on all $\mathbb{R}^n$ such that $|\tilde\phi|_\infty\leq1+\delta$ and choose $\varepsilon_0>0$ such that, for all $\varepsilon<\varepsilon_0$,

$$\sup_{x\in[0,1]^n}\left|\partial_{x_k}\phi(x)-\int_{\mathbb{R}^n}\eta_\varepsilon(x-z)\partial_{z_k}\tilde\phi(z)dz\right|\leq\delta|\mu|^{-1}.$$

Hence,

$$|h|_{BV}\leq\int h_\varepsilon\partial_{x_k}\tilde\phi+2\delta=-\int\partial_{x_k}h_\varepsilon\tilde\phi+2\delta\leq|\partial_{x_k}h_\varepsilon|_{L^1}(1+\delta)+2\delta.$$

Thus, by the arbitrariness of δ,

$$\liminf_{\varepsilon\to0}\sup_k|\partial_{x_k}h_\varepsilon|_{L^1}=|h|_{BV}.\tag{1.7}$$

Finally, let $\tilde\eta : \mathbb{R} \to \mathbb{R}_+$ and $\eta_\varepsilon(x) = \varepsilon^{-1}\tilde\eta(\varepsilon^{-1}x_k)$, using first (1.7) for $n = 1$, then Fatu and finally arguing as in (1.6),

$$\begin{aligned}|\operatorname{Var}^k(h)|_{L^1} &= \int dx_1 \cdots dx_{k-1}dx_{k+1}\cdots dx_d \operatorname{Var}^k h(x)\\ &= \int dx_1 \cdots dx_{k-1}dx_{k+1}\cdots dx_n \liminf_{\varepsilon\to 0}\int dx_k|\partial_{x_k}h_\varepsilon(x)|\\ &\le \liminf_{\varepsilon\to 0}|\partial_{x_k}h_\varepsilon|_{L^1} \le \liminf_{\varepsilon\to 0}\sup_{\substack{\varphi\in\mathcal{C}^1\\ |\varphi|_\infty\le 1}}\int h(x)\partial_{x_k}\varphi_\varepsilon(x) \le |h|_{BV}.\end{aligned}$$

□

2. Second Lecture

Lemma 2.1. *The ball $B = \{\mu \in \mathcal{B} \;:\; \|\mu\| \le 1\}$ is relatively compact in $(\mathcal{M}(X), |\cdot|)$.*

Proof. For each $t \in \mathbb{N}$, let us consider a partition $\{A_j\}$ of $[0, 1]$ in intervals of size t^{-1} and, for each $k \in \{1, \dots, d\}$, define

$$\begin{aligned}&P_{t,k}\varphi(x) = t\sum_j \mathbb{1}_{A_j}(x_k)\int_{A_j} dz\varphi(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_d)\\ &P_t\varphi = P_{t,1}\cdots P_{t,d}\varphi.\end{aligned} \tag{2.1}$$

First of all note that

$$P'_{t,k}\mu(\varphi) = \mu(P_{t,k}\varphi) = \int hP_{t,k}\varphi = \int P_{t,k}h\cdot\varphi.$$

Next, if $j \ne k$

$$P'_{t,k}\mu(\partial_{x_j}\varphi) = \int hP_{t,k}\partial_{x_j}\varphi = \int h\partial_{x_j}P_{t,k}\varphi \le \|\mu\|.$$

and

$$P'_{t,k}\mu(\partial_{x_k}\varphi) = \int hP_{t,k}\partial_{x_k}\varphi = \|\mu\|\left|\int_0^{x_k} dx_kP_{t,k}\partial_{x_k}\varphi\right|_\infty \le 4\|\mu\|.$$

In addition,

$$\mu(P_{i,k}\varphi - \varphi) = \|\mu\|\left|\int_0^{x_k} dx_k(P_{t,k}\varphi - \varphi)\right|_\infty.$$

If $x_k \in A_j = [jt^{-1}, (j + 1)t^{-1}]$, then

$$\int_0^{x_k} dx_k(P_{t,k}\varphi - \varphi) = \int_{jt^{-1}}^{x_k}\varphi \le |\varphi|_\infty t^{-1}.$$

Accordingly, $\|P'_t\mu\| \le 4^d\|\mu\|$ and $|P'_t\mu - \mu| \le 4^{d+1}t^{-1}$. In addition, notice that $P'_t\mu = t^d\sum_{i_1,\dots,i_d}\mu(\mathbb{1}_{A_{i_1}}\cdots\mathbb{1}_{A_{i_d}})m_{A_1\times\cdots\times A_{i_d}}$, where $t^{-d}m_{A_1\times\cdots\times A_{i_d}}$ is the Lebesgue measure restricted to the set $A_1\times\cdots\times A_{i_d}$. In other words the range of P'_t is a finite dimensional space. This implies that if $\{\mu_j\} \subset B$, then $\{P'_t\mu_j\}$ lives in a finite dimensional bounded set, hence it is compact. Thus there exists μ_t and n_j such that $\lim_{j\to\infty}\|P'_t\mu_{n_j} - \mu_t\| = 0$. In addition, for $t' \ge t$,

$$|\mu_t - \mu_{t'}| \le |\mu_t - P'_t\mu_{n_j}| + |\mu_t - P'_{t'}\mu_{n_j}| + |P'_t\mu_{n_j} - P'_{t'}\mu_{n_j}| \le Ct^{-1}$$

provided one choses j large enough. It follows that there exists a sequence t_j and a measure μ such that $\lim_{j\to\infty}|\mu - P_{t_j}\mu_{n_j}| = 0$. □

2.1. **Dynamical inequalities (Lasota-Yorke).** There exists $C > 0$ such that for each $\alpha \in (0,1)$, $\varepsilon > 0$ and $i \in \mathcal{I}$, there are smooth functions ϕ_i^ε supported in a $\alpha^{-i}\lambda_i^{-1}L_i\varepsilon$-neighborhood of Δ_i and such that $|\phi_i^\varepsilon|_\infty = 1$, $|\phi_i^\varepsilon|_{\mathcal{C}^1} \leq C\alpha^i\varepsilon^{-1}\lambda^i L_i^{-1}$ and $\phi_i(x) = 1$ for all $x \in \Delta_i$. Let us define

$$\sigma' := \lim_{\varepsilon \to 0} \left| \sum_{i \in \mathcal{I}} \phi_i^\varepsilon \lambda_j L_j \right|_\infty .$$

Note that, in the simple case in which the partition $\{\Delta_i\}$ is finite and can be chosen (eventually by refining it), such that $L_j = 1$, and if $\lambda = \lambda_i$, then $\sigma' = C_\Delta \lambda^{-1}$ where C_Δ is the complexity of the partition:

$$C_\Delta := \sup_{x \in X} \#\{i \in \mathcal{I} \, : \, x \in \overline{\Delta}_i\}.$$

Lemma 2.2 (Lasota-Yorke inequality)**.** *For each $\sigma \in (\sigma', 1)$ there exists a constant $B > 0$ such that, for each $\mu \in \mathcal{B}$, holds*

$$\begin{aligned} &|T'\mu| \leq |\mu| \\ &\|T'\mu\| \leq \sigma\|\mu\| + B|\mu|. \end{aligned}$$

Proof. First of all notice that, if $\mu \in \mathcal{B}$, then (Remembering Lemma 1.1 and Exercise 1)

$$|T'\mu| = \sup_{|\varphi|_{\mathcal{C}^0} \leq 1} \mu(\varphi \circ T) \leq |\mu|.$$

Next, for all $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \leq 1$ and $k \in \{1, \dots, d\}$ we have

$$\begin{aligned} T'\mu(\partial_{x_k}\varphi) &= \sum_{i \in \mathcal{I}} \mu(\mathbb{1}_{\Delta_i}(\partial_{x_k}\varphi) \circ T) \\ &= \sum_{i \in \mathcal{I}} \sum_{j=1}^{d} \mu(\mathbb{1}_{\Delta_i}\partial_{x_j}((DT)^{-1}_{kj}\varphi \circ T)) - \sum_{i \in \mathcal{I}} \sum_{j=1}^{d} \mu(\mathbb{1}_{\Delta_i}\varphi \circ T\partial_{x_j}((DT)^{-1}_{kj})). \end{aligned}$$

Setting $h = \frac{d\mu}{dm}$ and $\psi_{kj} = (DT)^{-1}_{kj}\varphi \circ T$, note that $\sum_j |\psi_{kj}|_\infty \leq \lambda_i^{-1}$, moreover we can rotate the coordinates as is most convenient (by redefining ψ_{kj} as well)

$$\begin{aligned} \mu(\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) &= \mu(\phi_i^\varepsilon \mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) \\ &\leq \int h(x)\partial_{x_j}\left[\phi_i^\varepsilon \int_0^{x_j} [\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}](x_1, \dots, x_{j-1}, z, x_{j+1}, \dots, x_d)dz\right] \\ &\quad + \lambda_i^{-1}L_i|\mu||\phi_i|_{\mathcal{C}^1}. \end{aligned}$$

Hence, remembering the hypotheses on T,

$$\begin{aligned} T'\mu(\partial_{x_k}\varphi) &= \int \operatorname{Var}^k h \left|\sum_{i \in \mathcal{I}} \phi_i^\varepsilon \lambda_i^{-1} L_i\right|_\infty + \sum_{i \in \mathcal{I}} \lambda_i^{-1} L_i|\mu||\phi_i|_{\mathcal{C}^1} + C\mu(\|\nabla(DT)^{-1}\|) \\ &\leq \|\mu\|\sigma + B|\mu| + (\sigma - \sigma')\|\mu\|. \end{aligned}$$

□

2.2. **Spectral properties.** In this subsection we will study the spectral properties of the operator T' acting on $\mathcal{B}$ and relate them with the dynamical properties of the system.

Remark 2.3. From now on we will assume $\sigma < 1$. *Note that, in some cases, this can be achieved by considering a power of the map (e.g., in one dimension with a finite partition).*

Lemma 2.4. *The operator T' has spectral radius equal one and essential spectral radius smaller than σ.*

Proof. The first assertion follows directly from Lemma 2.2. For the second we need a well known result.

Theorem 2.5 (Analytic Fredholm theorem–finite rank[1])**.** *Let D be an open connected subset of $\mathbb{C}$. Let $F : \mathbb{C} \to L(\mathcal{B}, \mathcal{B})$ be an analytic operator-valued function such that $F(z)$ is finite rank for each $z \in D$. Then, one of the following two alternatives holds true*

- $(\mathbb{1} - F(z))^{-1}$ *exists for no* $z \in D$
- $(\mathbb{1} - F(z))^{-1}$ *exists for all* $z \in D \backslash S$ *where S is a discrete subset of D (i.e. S has no limit points in D). In addition, if $z \in S$, then 1 is an eigenvalue for $F(z)$ and the associated eigenspace has finite multiplicity.*

For completeness we well give later a fast proof of this result.

Let $T'_{n,t} := (T')^n P_t$, clearly such an operator is finite rank, in addition

$$\|(T')^n \mu - T'_{n,t}\mu\| \leq \sigma^n \|(\mathbb{1} - P_t)\mu\| + B|(\mathbb{1} - P_t)\mu| \leq (1 + 4)\sigma^n \lambda^{-n} \|\mu\| + Bt^{-1}\|\mu\|.$$

By choosing $t = \sigma^n$ we have that there exists $C_1 > 0$ such that

$$\|(T')^n - T'_{n,t}\| \leq C_1 \sigma^n.$$

For each $z \in \mathbb{C}$ we can now write

$$\mathbb{1} - z(T')^n = (\mathbb{1} - z((T')^n - T'_{n,t})) - zT'_{n,t}.$$

Since

$$\|z((T')^n - T'_{n,t})\| \leq |z| C_1 \sigma^n < \frac{1}{2},$$

provided that $|z| \leq \frac{1}{2C_1}\sigma^{-n}$. Given any z in the disk $D_n := \{|z| < \frac{1}{2C_1}\sigma^{-n}\}$ the operator $B(z) := \mathbb{1} - z((T')^n - T'_{n,t})$ is invertible.[2] Hence

$$\mathbb{1} - z(T')^n = \left(\mathbb{1} - zT'_{n,t}B(z)^{-1}\right) B(z) =: (\mathbb{1} - F(z))B(z).$$

By applying Theorem 2.5 to $F(z)$ we have that the operator is either never invertible or not invertible only in finitely many points in the disk D_n. Since for $|z| < 1$ we have $(\mathbb{1} - z(T')^n)^{-1} = \sum_{k=0}^{\infty} z^k (T')^{nk}$, the first alternative cannot hold hence the Theorem follows. $\square$

[1] The present proof is patterned after the proof of the Analytic Fredholm alternative for compact operators (in Hilbert spaces) given in [46, Theorem VI.14]. There it is used the fact that compact operators in Hilbert spaces can always be approximated by finite rank ones. In fact the theorem holds also for compact operators in Banach spaces but the proof is a bit more involved.

[2] Clearly $B(z)^{-1} = \sum_{k=0}^{\infty} \left[z((T')^n - T'_{n,t})\right]^k$.

Proof of Theorem 2.5. First of all notice that, for each $z_0 \in D$ there exists $r > 0$ such that $D_{r(z_0)}(z_0) := \{z \in \mathbb{C} : |z - z_0| < r(z_0)\} \subset D$, and

$$\sup_{z \in D_{r(z_0)}(z_0)} \|F(z) - F(z_0)\| \leq \frac{1}{2}.$$

Clearly if we can prove the theorem in each such disk we are done.[3] Note that

$$\mathbb{1} - F(z) = \left(\mathbb{1} - F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}\right)(\mathbb{1} - [F(z) - F(z_0)]).$$

Thus the invertibility of $\mathbb{1} - F(z)$ in $D_r(z_0)$ depends on the invertibility of $\mathbb{1} - F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}$. Let us set $F_0(z) := F(z_0)(\mathbb{1} - [F(z) - F(z_0)])^{-1}$.

Let us start by looking at the equation

$$(\mathbb{1} - F_0(z))h = 0. \tag{2.2}$$

Clearly if a solution exists, then $h \in \text{Range}(F_0(z)) = \text{Range}(F(z_0)) := \mathbb{V}_0$. Since $\mathbb{V}_0$ is finite dimensional there exists a basis $\{h_i\}_{i=1}^N$ such that $h = \sum_i \alpha_i h_i$. On the other hand there exists an analytic matrix $G(z)$ such that[4]

$$F_0(z)h = \sum_{ij} G(z)_{ij} \alpha_j h_i.$$

Thus (2.2) is equivalent to

$$(\mathbb{1} - G(z))\alpha = 0,$$

where $\alpha := (\alpha_i)$.

The above equation can be satisfied only if $\det(\mathbb{1} - G(z)) = 0$ but the determinant is analytic hence it is either always zero or zero only at isolated points.[5]

Suppose the determinant different from zero, and consider the equation

$$(\mathbb{1} - F_0(z))h = g.$$

Let us look for a solution of the type $h = \sum_i \alpha_i h_i + g$. Substituting yields

$$\alpha - G(z)\alpha = \beta$$

where $\beta := (\beta_i)$ with $F_0(z)g =: \sum_i \beta_i h_i$. Since the above equation admits a solution, we have $\text{Range}(\mathbb{1} - F_0(z)) = \mathcal{B}$, Thus we have an everywhere defined inverse, hence bounded by the open mapping theorem.

We are thus left with the analysis of the situation $z \in S$ in the second alternative. In such a case, there exists h such that $(\mathbb{1} - F(z))h = 0$, thus one is an eigenvalue. On the other hand, if we apply the above facts to the function $\Phi(\zeta) := \zeta^{-1}F(z)$ analytic in the domain $\{\zeta \neq 0\}$ we note that the first alternative cannot take place

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[3] In fact, consider any connected compact set K contained in D. Let us suppose that for each $z_0 \in K$ we have a disk $D_{r(z_0)}(z_0)$ in the theorem holds. Since the disks $D_{r(z_0)/2}(z_0)$ form a covering for K we can extract a finite cover. If the first alternative holds in one such disk then, by connectedness, it must hold on all K. Otherwise each $S \cap D_{r(z_0)/2}(z_0)$, and hence $K \cap S$, contains only finitely many points. The Theorem follows by the arbitrariness of K.

[4] To see the analyticity notice that we can construct linear functionals $\{\ell_i\}$ on $\mathbb{V}_0$ such that $\ell_i(h_j) = \delta_{ij}$ and then extend them to all $\mathcal{B}$ by the Hahn-Banach theorem. Accordingly, $G(z)_{ij} := \ell_j(F_0(z)h_i)$, which is obviously analytic.

[5] The attentive reader has certainly noticed that this is the turning point of the theorem: the discreteness of S is reduced to the discreteness of the zeroes of an appropriate analytic function: a determinant. A moment thought will immediately explain the effort made by many mathematicians to extend the notion of determinant (that is to define an analytic function whose zeroes coincide with the spectrum of the operator) beyond the realm of matrices (the so called Fredholm determinants).

since for $|\zeta|$ large enough $\mathbb{1} - \Phi(\zeta)$ is obviously invertible. Hence, the spectrum of $F(z)$ is discrete and can accumulate only at zero. This means that there is a small neighborhood around one in which $F(z)$ has no other eigenvalues, we can thus surround one with a small circle γ and consider the projector

$$\begin{aligned} P :=& \frac{1}{2\pi i}\int_\gamma (\zeta - F(z))^{-1} d\zeta = \frac{1}{2\pi i}\int_\gamma \left[(\zeta - F(z))^{-1} - \zeta^{-1}\right] d\zeta \\ =& \frac{1}{2\pi i}\int_\gamma F(z)\zeta^{-1}(\zeta - F(z))^{-1} d\zeta. \end{aligned}$$

By standard functional calculus it follows that P is a projector and it clearly projects on the eigenspace of the eigenvector one. But the last formula shows that P must project on a subspace of the range of $F(z)$, hence it must be finite dimensional. □

2.3. **Peripheral spectrum.** It is then natural to start looking at the eigenvalues of modulus one. By Lemma 2.4 and the usual fact about the spectral decomposition of the operators [28], follows that there exists a finite set $\Theta \subset [0, 2\pi)$ such that we can write[6]

$$T' = \sum_{\theta \in \Theta} e^{i\theta} \Pi_\theta + R$$

where Π_θ are finite rank operators and the spectral radius of R is strictly smaller than one. Moreover, $\Pi_\theta \Pi_{\theta'} = \delta_{\theta\theta'} \Pi_\theta$, $\Pi_\theta R = R \Pi_\theta = 0$. It follows that, for each $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-ik\theta} (T')^k = \begin{cases} \Pi_\theta & \text{if } \theta \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Also, by Lemma 2.2 follows $\|\Pi_\theta \mu\| \le C|\mu|$. Since Π_θ is a finite rank projector, there must exist $\mu_{\theta,l} \in \mathcal{B}$, $\ell_{\theta,l} \in \mathcal{B}'$ such that $\Pi_\theta = \sum_l \mu_{\theta,l} \otimes \ell_{\theta,l}$, moreover $T'\mu_{\theta,l} = e^{i\theta}\mu_{\theta,l}$ and $\ell_{\theta,l}(T'\mu) = e^{i\theta}\ell_{\theta,l}(\mu)$ for all $\mu \in \mathcal{B}$. Hence, it must be $|\ell_{\theta,l}(\mu)| \le C|\mu| = C\int |h_\mu| dm$. Since $L^\infty(X, m)$ is the dual of L^1, it follows that there exists $\bar{\ell}_{\theta,l} \in L^\infty(X, m)$ such that

$$\ell_{\theta,l}(\mu) = \int \bar{\ell}_{\theta,l} h_\mu = \mu(\bar{\ell}_{\theta,l}).$$

Hence, for each $\mu \in \mathcal{B}$,

$$\mu(\bar{\ell}_{\theta,l}) = \ell_{\theta,l}(\mu) = e^{-i\theta}\ell_{\theta,l}(T'\mu) = e^{-i\theta} T'\mu(\bar{\ell}_{\theta,l}) = e^{-i\theta}\mu(\bar{\ell}_{\theta,l} \circ T).$$

The above implies that $\bar{\ell}_{\theta,l} \circ T = e^{-i\theta}\bar{\ell}_{\theta,l}$ Lebesgue a.s.. Let us set $\mu_* := \Pi_0 m$.

Lemma 2.6. *For each $\ell \in L^\infty(X, m)$ such that $\ell \circ T = \ell$, m-a.s., if we define the measure $\mu(\varphi) := \mu_*(\ell\varphi)$, then μ is invariant and $\mu \in \mathcal{B}$.*

Proof. First of all notice that $T'\mu(\varphi) = \mu_*(\ell \cdot \varphi \circ T) = \mu_*((\ell\varphi) \circ T) = \mu_*(\ell\varphi) = \mu(\varphi)$, that is μ is an invariant measure. Next, for each $\varepsilon > 0$ there exists $\ell_\varepsilon \in L^\infty$ such that $|\ell_\varepsilon|_\infty \le 2|\ell|_\infty$ and $\mu_*(|\ell - \ell_\varepsilon|) + m(|\ell - \ell_\varepsilon|) \le \varepsilon$. Then, setting $\mu_\varepsilon(\varphi) := \mu_*(\ell_\varepsilon \varphi)$

$$|(T')^n \mu(\varphi) - (T')^n \mu_\varepsilon(\varphi)| \le \varepsilon |\varphi|_\infty$$

[6]Remark that there cannot be Jordan blocks with eigenvector of modulus one, since this would imply that $\|(T')^n\|$ grows polynomially, contrary to Lemma 2.2.

implies

$$|\Pi_0\mu_\varepsilon - \mu| \le \limsup_{n\to\infty} \left| \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}(T')^k(\mu_\varepsilon - \mu)\right| \le \varepsilon$$

Hence, for each $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \le 1$,

$$\mu(\partial_{x_k}\varphi) = \lim_{\varepsilon\to 0} \Pi_0\mu_\varepsilon(\partial_{x_k}\varphi) \le \lim_{\varepsilon\to 0} \|\Pi_0\mu_\varepsilon\| \le C \lim_{\varepsilon\to 0} |\mu_\varepsilon| \le C.$$

□

Thus, for each $p \in \mathbb{N}$ and $\theta \in \Theta$, the measure $\mu_{p,\theta}(\varphi) := \mu_*(\bar{\ell}^p_{\theta,i}\varphi)$ is in $\mathcal{B}$ and $T'\mu_{p,\theta} = e^{ip\theta}\mu_{p,\theta}$. But this implies that $\{p\theta\}_{p\in\mathbb{N}} \subset \sigma_{\mathcal{B}}(T') \cap \{|z| = 1\}$ and since the latter is finite it must be $\theta = 2\pi\frac{s}{t}$ for some $s, t \in \mathbb{N}$. We have just proven the following

Lemma 2.7. *The peripheral spectrum of T', $\sigma_{\mathcal{B}}(T') \cap \{|z| = 1\}$, is the fine union of cyclic groups.*

3. Third Lecture

3.1. Dynamical properties.

Lemma 3.1. *If the map T is topologically transitive then 1 is a simple eigenvalue for T'. If all the powers of T are topologically transitive, then $\{1\}$ is the all peripheral spectrum.*

Proof. We do the proof only for $d = 1$, as in higher dimension it is more complex (see footnote below). If one it is not simple, then there exists an invariant set A, $\mu_*(A) \notin \{0, 1\}$. But then $\mathbb{1}_A \in BV$ which implies that A contains an open set, the same applies to A^c (this is true only for $d = 1$).[7] But then, by topological transitivity, there is an orbit that visits such opens sets, hence the sets are not invariant. The same argument applied to T^n concludes the Lemma. □

In conclusion, we have obtained conditions under which the system has a unique invariant measure μ_* absolutely continuos w.r.t. Lebesgue. In addition, there exists $\rho > 0$ such that for each $\mu \in \mathcal{B}$ we have

$$\|(T')^n\mu - \mu_*\| \le C\|\mu\|e^{-\rho n}.$$

3.2. Birkhoff averages.

From now on we assume that one is simple and is the only eigenvalue of modulus one. Let $f \in L^\infty(X, m)$, and let $\hat{f} = f - \mu_*(f)$, then

$$m(\hat{f}_n^2) = \frac{1}{n^2}\left[\sum_{k=0}^{n-1} m(\hat{f}^2 \circ T^k) + 2\sum_{j>k=0}^{n-1} m(\hat{f} \circ T^j \hat{f} \circ T^k)\right] \le Cn^{-1}|f|_\infty.$$

By Chebyshev inequality, we have

$$m(\{x \,:\, |\hat{f}_n| \le L^{-1}\}) \le C\frac{L^2}{n}.$$

[7] In higher dimensions one can have a Cantor like set with characteristic function in BV. Hence one must either use a different functional space (a convenient one in this respect has been introduced in [48]) or use explicitly the dynamics: for example note the one can easily bound the ε neighborhood of the boundary of the partition and that, by a commonly used argument, implies that there is a large measure of point with an open neighborhood whose preimages are all away from singularities. One can then proceed to prove that on such open sets the density must be continuos showing that any invariant set must contain an open set.

The above, by Borel-Cantelli, implies[8]

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x)=\mu_*(f)\quad m\text{-almost surely.}$$

That is μ_* is a physical measure (also SRB) and the unique one. In fact one can obtain much sharper results on the behavior of the $\hat f_n$.

3.3. **Open problems.** All the above has been obtained under some conditions on the partition $\{\Delta_j\}$. Many results of this type can be found in [48, 11, 12]. The necessity of some condition it is know ([52, 10]) yet no condition is necessary in the analytic or piecewise linear case ([50, 51, 9]) and some progress has been made in the general case [13]. Is it possible to weaken the known conditions? Almost no results for partitions with countably many elements or with singular maps (while we have just seen that some results are possible).

3.4. **Limit Theorems.** A first question may be the following: given $f\in\mathcal{C}^0$, $n\in\mathbb{N}$ and $a\in\mathbb{R}_+$ let

$$A_{a,n}(f):=\left\{x\in\mathbb{T}^1\;:\;\left|\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x)-\mu(f)\right|\geq a\mu(|f|)\right\}. \tag{3.1}$$

Question 1. *How large is $\mu(A_{a,n})$?*

Note that we can write $\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k(x)-\mu(f)=\frac{1}{n}\sum_{k=0}^{n-1}\hat f\circ T^k(x)$ where $\hat f:=f-\mu(f)$. So we can reduce the question to the study of zero average function. A more refined question could be.

Question 2. *Does it exists a sequence $\{c_n\}$ such that*

$$\frac{1}{c_n}\sum_{k=0}^{n-1}\hat f\circ T^k(x)$$

converges in some sense to a non zero object?

In the following we will use, for convenience, the operator $\mathcal{L}: BV\to BV$ defined by $\mathcal{L}h=\frac{dT'\mu}{dm}$, where $\frac{d\mu}{dm}=h$.

---

[8] Actually one must apply Borel-Cantelli with some care (but this is a quite standard an general strategy):

Consider the set $\mathcal{N}:=\{4^k+j2^k\;:\;k\in\mathbb{N}\;j<3\cdot 2^k\}$, then

$$\sum_{l\in\mathcal{N}} m(\{x\;:\;|\hat f_l|\leq L^{-1}\})\leq CL^2\sum_{k=0}^{\infty}\sum_{j=0}^{3\cdot 2^k}4^{-k}\leq CL^2\sum_{k=0}^{\infty}3\cdot 2^{-k}<\infty.$$

Hence Borel-Cantelli imply that every infinite sequence in $\mathcal{N}$ converges. Next notice that

$$|\hat f_n-\hat f_{n+m}|\leq|f|_\infty\frac{m}{n}$$

which readily imply the wanted result.

3.5. **Large deviations–well, half of it.** Note that it suffices to study the set

$$A^+_{a,n}(f) := \left\{ x \in \mathbb{T}^1 \ : \ \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) - \mu(f + a|f|) \geq 0 \right\}.$$

since $A_{a,n}(f) = A^+_{a,n}(f) \cap A^+_{a,n}(-f)$. On the other hand, setting $f_a := f - \mu(f + a|f|)$ holds

$m(A^+_{a,n}(f)) = \mu(\{x \ : \ e^{\lambda \sum_{k=0}^{n-1} f_a \circ T^k(x)} \geq 1\}) \leq \mu(e^{\lambda \sum_{k=0}^{n-1} f_a \circ T^k}) = m(he^{\lambda \sum_{k=0}^{n-1} f_a \circ T^k})$.

Then

$$\mu(A^+_{a,n}(f)) \leq m(\mathcal{L}^n_\lambda h) \tag{3.2}$$

where we have defined the operator $\mathcal{L}_\lambda g := \mathcal{L}(e^{\lambda f_a} g)$, $\mathcal{L}$ being the Transfer operator of the map T.

Since f_a is bounded it is easy to see that $\mathcal{L}_\lambda$ is a well defined operator on BV.

The basic idea to accomplish the wanted estimate is to show that the spectral radius of $\mathcal{L}_\lambda$ is strictly smaller than one. Since we are interested in λ small we will try to apply perturbation theory viewing $\mathcal{L}_\lambda$ as a perturbation of $\mathcal{L}$. Since $\mathcal{L}_\lambda g = \sum_{n=0}^{\infty} \mathcal{L}(\frac{\lambda^n f_a^n g}{n!})$ it is clear that $\mathcal{L}_\lambda$ depends analytically from λ and we can thus apply the usual perturbation theory for operators (see [28]). Accordingly, for λ small enough, the maximal eigenvalue is close to one and the associated eigenspace is one dimensional. Hence, there exists $h_\lambda \in BV$ and $\ell_\lambda \in BV'$, both analytic in λ, such that the project on the maximal eigenvalue of $\mathcal{L}_\lambda$ reads $\Pi_\lambda(h) = h_\lambda \ell_\lambda(h)$. Obviously

$$\mathcal{L}_\lambda h_\lambda = \alpha_\lambda h_\lambda, \tag{3.3}$$

and $\alpha_0 = 1$, $h_0 = h$ and $\ell_0 = m$. Notice that h_λ and ℓ_λ are not uniquely defined: by $\Pi_\lambda^2 = \Pi_\lambda$ follows $\ell_\lambda(h_\lambda) = 1$ but one normalization can be chosen freely, let us choose $m(h_\lambda) = 1$. All the above discussion is summarize by the following Lemma.

Lemma 3.2. *There exists constants $C_1, C_2 > 0$ and $\rho > 0$ such that, for $\lambda \leq C_1 |f_a|_\infty^{-1}$, $\mathcal{L}_\lambda = \alpha_\lambda \Pi_\lambda + Q_\lambda$, $\Pi_\lambda Q_\lambda = Q_\lambda \Pi_\lambda = 0$, $\|Q^n_\lambda\|_{BV} \leq C_2 \rho^n$. Moreover everything is analytic in λ.*

In view of the above fact we can differentiate (3.3) obtaining

$$\mathcal{L}'_\lambda h_\lambda + \mathcal{L}_\lambda h'_\lambda = \alpha'_\lambda h_\lambda + \alpha_\lambda h'_\lambda \ ; \quad m(h'_\lambda) = 0. \tag{3.4}$$

Integrating with respect to m yields

$$\alpha'_\lambda = m(\mathcal{L}_\lambda(f_a h_\lambda)) + m(\mathcal{L}_\lambda h'_\lambda).$$

Thus $\alpha'_0 = \mu(f_a) = -a\mu(|f|) < 0$. This means that we can choose λ such that the norm of $\mathcal{L}^n_\lambda$ is strictly smaller than one, yet to know how small we can take it, it is necessary to investigate the second derivative of α_λ. Taking the derivative of (3.4), integrating with respect to m and setting $\lambda = 0$ yields

$$\alpha''_0 = m(h\, f_a^2) + 2m(f_a\, h'_0).$$

On the other hand, setting $\Pi g := \Pi_0 g = h\, m(g)$, (3.4) implies

$$(\mathbb{1} - \mathcal{L})h'_0 = \mathcal{L}(f_a h - \mu(f_a)h) = \mathcal{L}(\mathbb{1} - \Pi)(f_a h).$$

On the other hand, setting $\hat{\mathcal{L}} := \mathcal{L}(\mathbb{1} - \Pi)$, Lemma 3.2 implies that the spectral radius of $\hat{\mathcal{L}}$ is smaller than ρ, hence

$$h'_0 = (\mathbb{1} - \hat{\mathcal{L}})^{-1} \hat{\mathcal{L}}(f_a h), \tag{3.5}$$

and

$$\alpha_0'' = \mu(f_a^2) + 2m(f_a(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(f_a h)). \tag{3.6}$$

Since $\alpha_\lambda = 1 - \alpha_0'\lambda + \frac{1}{2}\alpha_0''\lambda^2 + \mathcal{O}(\lambda^3)$, it follows that the situation is drastically different if α_0'' is positive or negative. Looking at (3.6) the sign is far from evident, yet a careful analysis shows that the sign is often positive.

Lemma 3.3. *Either $\alpha_0'' \geq C > 0$, with C independent on a, or there exists a $\phi \in BV$ such that $\hat{f} = \phi - \phi \circ T$.*

Proof. First of all $(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}g = (\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}\hat{f} \in \{g \in BV \ : \ m(g) = 0\}$, thus

$$\alpha_0'' = \mu(\hat{f}^2) + a^2\mu(|f|)^2 + 2m(\hat{f}(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(\hat{f}\,h)) \geq \mu(\hat{f}^2) + 2m(\hat{f}(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(\hat{f}\,h)).$$

Next consider the following

$$\begin{aligned}
0 \leq \mu\left(\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\hat{f}\circ T^k\right]^2\right) &= \frac{1}{n}\sum_{k,j=0}^{n-1}\mu(\hat{f}\circ T^k \hat{f}\circ T^j) \\
&= \mu(\hat{f}^2) + \frac{2}{n}\sum_{j=0}^{n-1}\sum_{k=j+1}^{n-1}\mu(\hat{f}\hat{f}\circ T^{k-j}) = \mu(\hat{f}^2) + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{j=1}^{k} m(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h)) \\
&= \mu(\hat{f}^2) + 2\sum_{j=1}^{n-1}\frac{n-j}{n}m(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h)).
\end{aligned}$$

Accordingly,

$$\begin{aligned}
0 \leq \sigma^2 := \lim_{n\to\infty}\mu\left(\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\hat{f}\circ T^k\right]^2\right) &= \mu(\hat{f}^2) + 2\sum_{j=1}^{\infty}m(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h)) \\
&= \mu(\hat{f}^2) + 2m(\hat{f}(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(\hat{f}\,h)).
\end{aligned}$$

Clearly, if $\sigma > 0$ the lemma is proven, thus we need only to analyze the case $\sigma = 0$.

If $\sigma = 0$, we have

$$\begin{aligned}
\mu\left(\left[\sum_{k=0}^{n-1}\hat{f}\circ T^k\right]^2\right) &= n\left[\mu(\hat{f}^2) + 2\sum_{j=1}^{n-1}\frac{n-j}{n}m(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h))\right] \\
&= -2n\sum_{j=n}^{\infty}m(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h)) - 2\sum_{j=1}^{n-1}jm(\hat{f}\hat{\mathcal{L}}^j(\hat{f}\,h)) \\
&\leq C_3\left[n\rho^n + \sum_{j=0}^{\infty}j\rho^j\right]|\hat{f}|_{L^1}\|\hat{f}\|_{BV} \leq C_4|\hat{f}|_{L^1}\|\hat{f}\|_{BV}
\end{aligned}$$

Accordingly, the sequence $\sum_{k=0}^{n-1}\hat{f}\circ T^k$ is bounded in L^2 and hence weakly compact. Let $\sum_{k=0}^{n_j-1}\hat{f}\circ T^k$ a weakly convergent subsequence, that is there exists $\phi \in L^2$ such that for each $\varphi \in L^2$ holds

$$\lim_{j\to\infty}\int\varphi\sum_{k=0}^{n_j-1}\hat{f}\circ T^k = \int\varphi\phi.$$

It follows that, for each $\varphi \in \mathcal{C}^1$,

$$\int \varphi[\hat f - \phi + \phi\circ T] = m(\varphi \hat f) + \lim_{j\to\infty}\sum_{k=0}^{n_j-1}\int \varphi \hat f\circ T^{k+1} - \int \varphi \hat f \circ T^k$$
$$= \lim_{j\to\infty}\int \varphi \hat f\circ T^{n_j} = m(\varphi)\mu(\hat f) = 0$$

And, since $\mathcal{C}^1$ is dense in L^2, it follows

$$\hat f = \phi - \phi\circ T. \tag{3.7}$$

A function with the above property is called a *coboundary*, in this case an L^2 coboundary since we know only that $\phi \in L^2$. In fact, this it is not not enough to conclude the Lemma: we need to show, at least, that $\phi \in \mathcal{C}^0$.

First of all notice that, since for each $\beta \in \mathbb{R}$ we have $\hat f = \phi + \beta - (\phi+\beta)\circ T$, we can assume without loss of generality $\mu(\phi) = 0$. But them

$$\hat{\mathcal{L}}\hat f\, h = \mathcal{L}\hat f\, h = \mathcal{L}\phi\, h - \phi\, h = \hat{\mathcal{L}}\phi\, h - \phi\, h = -(\mathbb{1} - \hat{\mathcal{L}})\phi\, h.$$

Hence[9]

$$\phi = h^{-1}(\mathbb{1} - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(\hat f\, h) \in BV$$

□

Accordingly, $\alpha_0'' = \sigma^2 + a^2\beta^2$, $\beta := \mu(|f|)$. The minimum (of the quadratic part) is thus achieved by choosing $\lambda_- = \frac{a\beta}{\sigma^2+a^2\beta^2}$ which, remembering (3.2), allows to conclude

$$m(A^+_{a,n}(f)) \le m(\mathcal{L}^n_{\lambda_-}h) \le Ce^{-\frac{a^2\beta^2}{2\sigma^2}n + \mathcal{O}(a^3 n\|f\|_{BV})}.$$

Since similar arguments hold for the set $A^+_{a,n}(-f)$, it follows that we have an exponentially small probability to observe a deviation from the average.

3.6. **The Central Limit Theorem.** We can now address the second question we have posed. From the above discussion is clear that we must chose $c_n = \sqrt{n}$.

Let $f \in BV$ and set $\hat f := f - \mu(f)$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\hat f\circ T^k(x) = 0 \quad m-\text{a.e.}$$

Let us set $\Psi_n := \frac{1}{\sqrt n}\sum_{k=0}^{n-1}\hat f\circ T^k$. We can consider Ψ_n a random variable with distribution $F_n(t) := mu(\{x \ :\ \Psi_n(x) \le t\})$. It is well know that, for each

[9]Indeed, for each $\varphi \in L^1$,

$$\int \varphi(\mathbb{1} - \hat{\mathcal{L}})^{-1}(\mathbb{1} - \hat{\mathcal{L}})(h\phi) = \int \varphi\phi h - \lim_{n\to\infty}\int \varphi\mathcal{L}^n h\phi$$
$$= \int \varphi\phi h - \lim_{n\to\infty}\int \varphi\circ T^n h\phi = \int \varphi\phi$$

where we have using the mixing of the system.

continuous function g holds[10]

$$\mu(g(\Psi_n)) = \int_{\mathbb{R}} g(t) dF_n(t)$$

where the integral is a Riemann-Stieltjes integral. It is thus clear that if we can control the distribution F_n, we have a very sharp understanding of the probability to have small deviations (of order $\sqrt{n}$) from the limit. From the work in the previous section it follows that there exists $\delta > 0$ such that, for each $|\lambda| \leq \delta\sqrt{n}$,

$$\begin{aligned}\varphi_n(\lambda) := \mu(e^{i\lambda\Psi_n}) = \mu(\mathcal{L}^n_{i\lambda/\sqrt{n}} h) &= (1 - \frac{\sigma^2\lambda^2}{2n} + \mathcal{O}(\lambda^3 n^{-\frac{3}{2}} + \rho^n)\|f\|_{BV})^n \\ &= e^{-\frac{\sigma^2\lambda^2}{2}}(1 + \mathcal{O}(\lambda^3 n^{-\frac{1}{2}} + n\rho^n)\|f\|_{BV}).\end{aligned} \tag{3.8}$$

The above quantity is called *characteristic function* of the random variable and determine the distribution via the formula

$$F_n(b) - F_n(a) = \lim_{\Lambda\to\infty} \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{e^{-ia\lambda} - e^{-ibt}}{i\lambda} \varphi_n(\lambda) d\lambda,$$

as can be seen in any basic book of probability theory.[11]

Formula (3.8) means in particular that

$$\lim_{n\to\infty} m(e^{\lambda\Psi_n}) = e^{-\frac{\sigma^2\lambda^2}{2}} =: \varphi(\lambda).$$

What can we infer out of the above facts? First of all a simple computation shows that

$$g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\lambda} \varphi(\lambda) d\lambda = \frac{1}{\sqrt{\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}$$

a random variable with such a density is called a Gaussian random variable with zero average and variance σ. Accordingly, formula (3.8) can be interpreted by saying that there exists a Gaussian random variable G such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \hat{f} \circ T^k \sim \frac{1}{\sqrt{n}} G(1 + \mathcal{O}(n^{-\frac{1}{2}}))$$

in distribution. But what does this means concretely. Actual estimates are made difficult by the fact that the distribution under study no not necessarily have a density, thus we are Fourier transforming function that behave quite badly at infinity. To overcome such a problem we can smoothen the quantities involved.

---

[10]If $g \in \mathcal{C}^1_0$, then

$$\int_{\mathbb{R}} g dF_n = -\int_{\mathbb{R}} F_n(t) g'(t) dt = -\int_{\mathbb{R}} dt \int_{\mathbb{T}^1} dx \chi_{\{z\,:\,\Psi_n(z)\leq t\}}(x) g'(t).$$

Applying Fubini yields

$$\int_{\mathbb{R}} g dF_n = -\int_{\mathbb{T}^1} dx \int_{\mathbb{R}} dt \chi_{\{z\,:\,\Psi_n(z)\leq t\}}(x) g'(t) = -\int_{\mathbb{T}^1} dx \int_{\Psi_n(x)}^{\infty} g'(t) dt = \int_{\mathbb{T}^1} dx g(\Psi_n(x)).$$

[11]In the case when there exists a density, that is an L^1 function f_n such that $F_n(b) - F_n(a) = \int_a^b f_n(t) dt$, then the formula above becomes simply

$$f_n(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\lambda} \varphi_n(\lambda) d\lambda,$$

and follows trivially by the inversion of the Fourier transform.

Let $j \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $\int_\mathbb{R} j(t)dt = 1$, $j(t) = j(-t)$, and $j(t) = 0$ for all $|t| > 1$, for each $\varepsilon > 0$ defined then $j_\varepsilon(t) := \varepsilon^{-1} j(\varepsilon^{-1}t)$ and

$$F_{n,\varepsilon}(t) := \int_\mathbb{R} j_\varepsilon(t-s)F_n(s)ds. \tag{3.9}$$

A simple computation shows that, for each $a, b \in \mathbb{R}$, holds

$$F_n(b+\varepsilon) - F_n(a-\varepsilon) \geq F_{n,\varepsilon}(b) - F_{n,\varepsilon}(a) \geq F_n(b-\varepsilon) - F_n(a+\varepsilon)$$

that is: if the measurements have a precision worst than 2ε, then $F_{n,\varepsilon}$ is as good as F_n to describe the resulting statistics. On the other hand calling $\varphi_{n,\varepsilon}$ the characteristic function associated to $F_{n,\varepsilon}$, holds $\varphi_{n,\varepsilon}(\lambda) = \varphi_n(\lambda)\hat{j}(\varepsilon\lambda)$, where $\hat{j}$ is the Fourier transform of j. Since now $F_{n,\varepsilon}$ is the law of a smooth random variable it has a density $f_{n,\varepsilon}$ and

$$f_{n,\varepsilon}(t) = \frac{1}{2\pi}\int_\mathbb{R} e^{-i\lambda t}\varphi_n(\lambda)\hat{j}(\varepsilon\lambda)d\lambda$$

since j is smooth it follows that there exists $C > 0$ such that $|\hat{j}(\lambda)| \leq C(1+\lambda^2)^{-2}$. We can finally use formula (3.8) to obtain a quantitative estimate

$$\begin{aligned}
f_{n,\varepsilon}(t) &= \frac{1}{2\pi}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} e^{-i\lambda t}\varphi_n(\lambda)\hat{j}(\varepsilon\lambda)d\lambda + \mathcal{O}(\varepsilon^{-5}n^{-\frac{3}{2}}) \\
&= \frac{1}{2\pi}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} e^{-i\lambda t}\varphi(\lambda)\hat{j}(\varepsilon\lambda)d\lambda + \mathcal{O}(\varepsilon^{-5}n^{-\frac{3}{2}} + n^{-\frac{1}{2}}) \\
&= g(t) + \mathcal{O}(\varepsilon + \varepsilon^{-5}n^{-\frac{3}{2}} + n^{-\frac{1}{2}}) = g(t) + \mathcal{O}(n^{-\frac{1}{2}})
\end{aligned}$$

provided we choose $n^{-\frac{1}{2}} \geq \varepsilon \geq n^{-5}$. Which, as announced, means that, if the precision of the instrument is compatible with the statistics, the typical fluctuations in measurements are of order $\frac{1}{\sqrt{n}}$ and Gaussian. This is well known by sperimentalists who routinely assume that the result of a measurement is distributed according to a Gaussian.[12]

3.7. **Perturbation theory.** To answer the questions posed at the beginning we need some perturbation theorems. Few such results are available (e.g., see [39], [53] for a review and [3] for some more recent results), here we will follow mainly the theory developed in [34] adapted to the special cases at hand.

For simplicity let us work directly with the densities and in the case $d = 1$. Then $\mathcal{L}$ is the transfer operator for the densities. We will start by considering an abstract family of operators $\mathcal{L}_\varepsilon$ satisfying the following properties.

Condition 1. *Consider a family of operators $\mathcal{L}_\varepsilon$ with the following properties*

(1) A uniform Lasota-Yorke inequality:

$$\|\mathcal{L}_\varepsilon^n h\|_{BV} \leq A\lambda^{-n}\|h\|_{BV} + B|h|_{L^1}, \quad |\mathcal{L}_\varepsilon^n h|_{L^1} \leq C|h|_{L^1}\ ;$$

(2) $\int \mathcal{L}h(x)dx = \int h(x)dx$;

[12] Note however that our proof holds in a very special case that has little to do with a real experimental setting. To prove the analogous statement in for a realistic experiment is a completely different ball game.

(3) For $L : BV \to BV$ define the norm

$$|||L||| := \sup_{\|h\|_{BV}\leq 1} |Lf|_{L^1},$$

that is the norm of L as an operator from $BV \to L^1$. Then we require that there exists $D > 0$ such that

$$|||\mathcal{L} - \mathcal{L}_\varepsilon||| \leq D\varepsilon.$$

Condition 1-(3) specifies in which sense the family $\mathcal{L}_\varepsilon$ can be considered an approximation of the unperturbed operator $\mathcal{L}$. Notice that the condition is rather weak, in particular the distance between $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ as operators on BV can be always larger than 1. Such a notion of closeness is completely inadequate to apply standard perturbation theory, to get some perturbations results it is then necessary to drastically restrict the type of perturbations allowed, this is done by Conditions 1-(1,2) which state that all the approximating operators enjoys properties very similar to the limiting one.[13]

To state a precise result consider, for each operator L, the set

$$V_{\delta,r}(L) := \{z \in \mathbb{C} \mid |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(L)) \leq \delta\}.$$

Since the complement of $V_{\delta,r}(L)$ belongs to the resolvent of L it follows that

$$H_{\delta,r}(L) := \sup\left\{\|(z-L)^{-1}\|_{BV} \mid z \in \mathbb{C} \setminus V_{\delta,r}(L)\right\} < \infty.$$

By $R(z)$ and $R_\varepsilon(z)$ we will mean respectively $(z-\mathcal{L})^{-1}$ and $(z-\mathcal{L}_\varepsilon)^{-1}$.

Theorem 3.4 ([34]). *Consider a family of operators $\mathcal{L}_\varepsilon : BV \to BV$ satisfying Conditions 1. Let $H_{\delta,r} := H_{\delta,r}(\mathcal{L})$; $V_{\delta,r} := V_{\delta,r}(\mathcal{L})$, $r > \lambda^{-1}$, $\delta > 0$, then, if $\varepsilon \leq \varepsilon_1(\mathcal{L}, r, \delta)$, $\sigma(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$. In addition, if $\varepsilon \leq \varepsilon_0(\mathcal{L}, r, \delta)$, there exists $a > 0$ such that, for each $z \notin V_{\delta,r}$, holds true*

$$|||R(z) - R_\varepsilon(z)||| \leq C\varepsilon^a.$$

Proof.[14] To start with we collect some trivial, but very useful algebraic identities.

For each operator $L : BV \to BV$ and $n \in \mathbb{Z}$ holds

$$\frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}L)^i(z-L) + (z^{-1}L)^n = \mathbb{1} \tag{3.10}$$

$$R(z)(z-\mathcal{L}_\varepsilon) + \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L}) + R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L}) = \mathbb{1} \tag{3.11}$$

$$(z-\mathcal{L}_\varepsilon)\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right] = \mathbb{1} - (z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z) \tag{3.12}$$

$$\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right](z-\mathcal{L}_\varepsilon) = \mathbb{1} - (z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L}), \tag{3.13}$$

where we have set $G_{n,\varepsilon} := \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L}_\varepsilon)^i$.

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[13] Actually only Condition 1-(1) is needed in the following. Condition 1-(2) simply implies that the eigenvalue one is common to all the operators. If 1-(2) is not assumed, then the operator $\mathcal{L}_\varepsilon$ will always have one eigenvalue close to one, but the spectral radius could vary slightly, see [42] for such a situation.

[14] This proof is simpler than the one in [34], yet it gives worst bounds, although sufficient for the present purposes.

Let us start applying the above formulae. For each $h \in BV$ and $z \notin V_{r,\delta}$ holds

$$\begin{aligned}\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\|_{BV} &\leq (r\lambda)^{-n}A\|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\|_{BV} + \frac{B}{r^n}|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h|_{L^1} \\ &\leq [(r\lambda)^{-n}A2C_1 + Br^{-n}D\varepsilon]H_{r,\delta}\|h\|_{BV} < \|h\|_{BV}\end{aligned}$$

Thus $\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)\|_{BV} < 1$ and the operator on the right hand side of (3.12) can be inverted by the usual Neumann series. Accordingly, $(z - \mathcal{L}_\varepsilon)$ has a well defined right inverse. Analogously,

$$\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{BV} \leq (r\lambda)^{-n}A\|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{BV} + Br^{-n}|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h|_{L^1}.$$

This time to continue we need some informations on the L^1 norm of the resolvent. Let $g \in BV$, then equation (3.10) yields

$$\begin{aligned}|R(z)g|_{L^1} &\leq \frac{1}{r}\sum_{i=0}^{n-1}|(z^{-1}\mathcal{L})^i g|_{L^1} + \|R(z)(z^{-1}\mathcal{L})^n g\|_{BV} \\ &\leq \frac{1}{r^n(1-r)}|g|_{L^1} + H_{\delta,r}A(r\lambda)^{-n}\|g\|_{BV} + H_{\delta,r}Br^{-n}|g|_{L^1} \\ &\leq r^{-n}(H_{\delta,r}B + (1-r)^{-1})|g|_{L^1} + H_{\delta,r}A(r\lambda)^{-n}\|g\|_{BV}\end{aligned}$$

Substituting, we have

$$\begin{aligned}&\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{BV} \leq \{(r\lambda)^{-n}AH_{\delta,r}2C_1[1 + Br^{-n}] \\ &+ Br^{-2n}[H_{\delta,r}B + (1-r)^{-1}]D\varepsilon\}\|h\|_{BV} < 1,\end{aligned}$$

again, provided ε is small enough and choosing n appropriately. Hence the operator on the right hand side of (3.13) can be inverted, thereby providing a left inverse for $(z - \mathcal{L}_\varepsilon)$. This implies that z does not belong to the spectrum of $\mathcal{L}_\varepsilon$.

To investigate the second statement note that (3.11) implies

$$R(z) - R_\varepsilon(z) = \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z) - R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z).$$

Accordingly, for each $\varphi \in BV$ holds

$$|R(z)\varphi - R_\varepsilon(z)\varphi|_{L^1} \leq \{r^{-n}(1-r)^{-1}\varepsilon + H_{\delta,r}(\lambda r)^{-n}2AC_1 + H_{\delta,r}B\varepsilon\}\|R_\varepsilon(z)\varphi\|_{BV}.$$

□

3.8. **Deterministic stability.** The $\mathcal{L}_\varepsilon$ are Perron-Frobenius (Transfer) operators of maps T_ε which are $\mathcal{C}^1$–close to T, that is $d_{\mathcal{C}^1}(T_\varepsilon, T) = \varepsilon$ and such that $d_{\mathcal{C}^2}(T_\varepsilon, T) \leq M$, for some fixed $M > 0$. In this case the uniform Lasota-Yorke inequality is trivial. On the other hand, for all $\varphi \in \mathcal{C}^1$ holds

$$\int(\mathcal{L}_\varepsilon f - \mathcal{L}f)\varphi = \int f(\varphi \circ T_\varepsilon - \varphi \circ T).$$

Now let $\Phi(x) := (D_xT)^{-1}\int_{Tx}^{T_\varepsilon x}\varphi(z)dz$, since

$$\Phi'(x) = -(D_xT)^{-1}D_x^2T\Phi(x) + D_xT_\varepsilon(D_xT)^{-1}\varphi(T_\varepsilon x) - \varphi(Tx)$$

follows

$$\int(\mathcal{L}_\varepsilon f - \mathcal{L}f)\varphi = \int f\Phi' + \int f(x)[(D_xT)^{-1}D_x^2T\Phi(x) + (1 - D_xT_\varepsilon(D_xT)^{-1})\varphi(T_\varepsilon x)].$$

Given that $|\Phi|_\infty \leq \lambda^{-1}\varepsilon|\varphi|_\infty$ and $|1 - D_xT_\varepsilon(D_xT)^{-1}|_\infty \leq \lambda^{-1}\varepsilon$, we have

$$\int(\mathcal{L}_\varepsilon f - \mathcal{L}f)\varphi \leq \|f\|_{BV}\lambda^{-1}|\varphi|_\infty\varepsilon + |f|_{L^1}\lambda^{-1}(B+1)\varepsilon|\varphi|_\infty \leq D\|f\|_{BV}\varepsilon|\varphi|_\infty.$$

By Lebesgue dominate convergence theorem we obtain the above inequality for each $\varphi \in L^\infty$, and taking the sup on such φ yields the wanted inequality.

$$|\mathcal{L}_\varepsilon f - \mathcal{L}f|_{L^1} \leq D\|f\|_{BV}\varepsilon.$$

We have thus seen that all the requirements in Condition 1 are satisfied. See [29] for a more general setting including piecewise smooth maps.

3.9. **Stochastic stability.** Next consider a set of maps $\{T_\omega\}$ depending on a parameter $\omega \in \Omega$. In addition assume that Ω is a probability space and consider a measure P on Ω. Consider the process $x_n = T_{\omega_n} \circ \cdots \circ T_{\omega_1}x_0$ where the ω are i.i.d. random variables distributed accordingly to P and let E_μ be the expectation of such process when x_0 is distributed according to μ. Then, calling $\mathcal{L}_\omega$ the transfer operator associated to T_ω, we have

$$E(f(x_{n+1}) \mid x_n) = \mathcal{L}_P f(x_n) := \int_\Omega \mathcal{L}_\omega f(x_n)P(d\omega).$$

Then if

$$|\mathcal{L}_\omega h|_{BV} \leq \lambda_\omega^{-1}|h|_{BV} + B_\omega|h|_{L^1}$$

integrating yields

$$|\mathcal{L}_P h|_{BV} \leq E(\lambda_\omega^{-1})|h|_{BV} + E(B_\omega)|h|_{L^1}$$

And the operator $\mathcal{L}_P$ satisfy a Lasota-Yorke inequality provided that $E(\lambda^{-1}) < 1$ and $E(B) < \infty$.

In addition, if for some map T and associated transfer operator $\mathcal{L}$,

$$E(|\mathcal{L}_\omega h - \mathcal{L}h|) \leq \varepsilon|h|_{BV}$$

then we can apply perturbation theory and obtain stochastic stability.

3.10. **Computability.** If we want to compute the invariant measure and the rate of decay of correlations, we can use the operator P_t defined in (2.1) and define $\mathcal{L}_{t,m} = P_t\mathcal{L}^m$. By the estimates in Lemma 2.1 it follows

$$|\mathcal{L}_{t,m}h|_{BV} \leq 4^d\sigma^m|h|_{BV} + B|h|_{L^1}.$$

We can then chose the smallest m so that $4^d\sigma^m = \sigma_1 < 1$. Moreover, we also saw that

$$|\mathcal{L}_{t,m}h - \mathcal{L}h| \leq t^{-1}|h|_{BV}.$$

So we are again in the realm of our perturbation theory and we have that the finite dimensional operator $\mathcal{L}_{t,m}$ has spectrum close to the one of the transfer operator. We can then obtain all the info we want by diagonalizing a matrix.

3.11. **Linear response.** Linear response is a theory widely used by physicists. In essence it says the follow: consider a one parameter family of systems T_s and the associated (e.g.) invariant measures μ_s, then, for a given observable f one want to study the response of the system to a small change in s, and, not surprisingly, one expects $\mu_s(f) = \mu_0(f) + s\nu(f) + o(s)$. That is one expects differentiability in s. Yet differentiability is is not ensured by Theorem 3.4. Is it possible to ensure conditions under which linear response holds? The answer is yes (for example if holds if the maps are sufficiently smooth and the dependence on the parameter is also smooth in an appropriate sense). To prove it one need a sophistication of Theorem 3.4 that can be found in [41].

3.12. **The hyperbolic case.** One can wonder is the previous approach can be applied to uniformly hyperbolic systems and partially hyperbolic system. The answer is yes although the work in this direction is still in progress and the price to pay is the need to consider rather unusual functional spaces (space of anysotropic distributions). Just to give a vague idea let us look at a totally trivial example: toral automorphisms.

Then one can consider the norms:

$$\|f\|_{p,q} := \sum_{k\in\mathbb{Z}^{2d}\setminus\{0\}} |f_k|\frac{|k|^p}{1+|\langle v^s, k\rangle|^{p+q}} + |f_0|,$$

where f_k are the Fourier coefficients of f and v^s is the unit vector in the stable direction. Then

$$\begin{aligned} \|[\mathcal{L}f\|_{p,q} &\leq C_1\|f\|_{p,q}, \\ \|[\mathcal{L}^n f\|_{p,q} &\leq C_3\mu^n\|f\|_{p,q} + B\|f\|_{p-1,q+1}. \end{aligned} \tag{3.14}$$

we have thus the Lasota-Yorke inequality. Moreover on can easily check the relative compactness of $\{\|f\|_{p,q} \leq 1\}$ with respect to the topology induced by the norm $\|\cdot\|_{p-1,q+1}$, hence our previous theory applies almost verbatim.

To have a more precise idea of what can be done, see [41, 4].

3.13. **Open problems.** The stochastic stability is reasonably well understood (Cowienson) but what about the smooth dependence from a parameter (linear response)? Counterexamples in $d = 1$ but unknown in higher dimensions. The uniformly hyperbolic case is well understood but not much is know on how to apply the present ideas to the partially hyperbolic case and to the case of systems with discontinuities, although a concentrated effort is taking place to extend the theory in such directions.

4. Fourth Lecture

4.1. **The problem.** We have seen during the first workshop many attempts to bring the theory of dynamical systems to bear on the issue of non-equilibrium statistical mechanics.

In Dolgopyat's lectures we have seen some techniques allowing to show how the complicate behavior of the nonlinearities can give rise to effective noise at the *macroscopic* level.

The main gap to close the circle is to learn how to treat system with *many* (say 10^{25}) components. This is very hard and, at the moment, can be done only in the very simple case of *coupled map lattices*.

4.2. **CML.** A couple map lattice is constructed a follows: given a dynamical system (X,T) we consider the space $\Omega := X^{\mathbb{Z}^d}$ (but more general sets than $\mathbb{Z}^d$ can be also considered) and the product map $F_0(x)_i = T(x_i)$. Next we consider a map $\Phi_\varepsilon : \Omega \to \Omega$ that is ε-close to the identity in a sense to be made precise. The CML that we will consider are then given by $F_\varepsilon := \Phi_\varepsilon \circ F_0$. Interesting cases are:

- T expanding map (either smooth or not)
- T uniformly hyperbolic (either smooth or not)
- T partially hyperbolic (either smooth or not)

The typical approach, going back to Bunimovich-Sinai, is to conjugate F_ε to F_0 and use Markov partitions (see the papers in the references for more details).

A more direct approach, and more dynamical in nature, is desirable (also because in the non-smooth case conjugation fails).

4.3. **Super-brief history of the transfer operator approach.** The possibility to investigate directly the transfer operator for a CML was first investigated by Keller and Künzle [33]. They were able to prove spectral gap in finitely many dimensions and existence of a measure with absolutely continuos marginals in infinite dimensions. Then Fischer, Rugh [15] and Rugh [47] managed to prove space-time decay of correlations in infinite dimensions in the *analytic* case. Then in Baladi, Degli Esposti, Järvenpää, Kupiainen [1] and Baladi, Rugh [2] the spectrum in the analytic case is precisely investigated. Finally, in [37] it was proved the spectral gap for piecewise expanding CML. The latter paper is what I will explain in the following.

4.4. **Expanding CML.** Consider the case in which $X = [0,1]$ and the map is piecewise $\mathcal{C}^2$ and $|DT| \geq \lambda > 2$. While

$$\Phi_\varepsilon(x)_i = x_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i x)(x_{i+z} - x_i),$$

with $\tau^i(x)_j = x_{i+j}$ and $\alpha_z \in \mathcal{C}^1$ with $\partial_{x_j}\alpha_z = 0$ if $|j| \geq 1$. Moreover, we assume

- $\alpha_z \geq 0$. Which, for ε small, insures $x_i \geq 0 \Longrightarrow \Phi_\varepsilon(x)_i \geq 0$.
- $\sum_i \alpha_i = 1$. Which for ε small, insures $x_i \leq 1 \Longrightarrow \Phi_\varepsilon(x)_i \leq 1$.

The goal is show existence and uniqueness of the SRB measure for small ε. For large, but still less than one, ε uniqueness may fail [5].

4.5. **Transfer operator and Lasota-Yorke inequality.** As we want to deal with infinite systems, it is convenient to first define the transfer operator on the set of Borel measures $\mathcal{M}(\Omega)$: for each measurable set A, let $\mathcal{L}\mu(A) := \mu(F_\varepsilon^{-1}(A))$.

Obviously $\mathcal{M}(\Omega)$ is too big to be useful, to restrict it we define two norms:

$$|\mu| := \sup_{|\varphi|_{\mathcal{C}^0} \leq 1} \mu(\varphi)$$

$$\|\mu\| := \sup_{i \in \mathbb{Z}^d} \sup_{\substack{\|\varphi\|_{\mathcal{C}^0} \leq 1 \\ \varphi \in \mathcal{C}^1}} \mu(\partial_{x_i}\varphi).$$

Clearly $|\mu| \leq \|\mu\|$. Let $\mathcal{B} := \{\mu \in \mathcal{M}(\omega) \,:\, \|\mu\| < \infty\}$.

Theorem 4.1 (Keller et al.)**.** *For ε small enough there exists $\theta \in (0,1)$ such that, for all $n \in \mathbb{N}$,*

$$\|\mathcal{L}^n \mu\| \leq A\theta^n \|\mu\| + B|\mu|.$$

That is nice but **compactness** is missing. In fact, compactness does not hold, thus we need a way to establish directly the existence of a gap.

4.6. **spectral gap.** To deal with this fix $a \in [0,1]$ and given $x \in \Omega$ let $(x^p)_q = x_q$ for $q \neq p$ and $(x^p)_p = a$. Then define $\Phi_{\varepsilon,p}$ to be the map

$$\Phi_{\varepsilon,p}(x)_q = \begin{cases} \Phi_\varepsilon(x^q)_q & \text{if } q \neq p \\ x_p & \text{if } q = p\,. \end{cases}$$

One can easily verify that

$$|(\mathcal{L} - \mathcal{L}_p)\mu| \leq C\varepsilon\|\mu\|,$$

where $\mathcal{L}_p$ is the operator associated to the coupling $\Phi_{\varepsilon,p}$. Indeed, letting $\Phi_t := (1-t)\Phi_\varepsilon - t\Phi_{\varepsilon,p}$, holds

$$\begin{aligned}\mu(\varphi\circ\Phi_\varepsilon - \varphi\circ\Phi_{\varepsilon,p}) &= \int_0^1 \mu(\frac{d}{dt}\varphi\circ\Phi_t) = \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}\varphi\cdot[\Phi_\varepsilon - \Phi_{\varepsilon,p}]_i) \\ &= \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}[\varphi(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i]) - \mu(\varphi\partial_{x_i}(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i]) \\ &\leq C\varepsilon\|\mu\|\cdot|\varphi|_\infty.\end{aligned}$$

Hence

$$|(\mathcal{L}^n - \mathcal{L}_p^n)\mu| \leq \sum_{k=0}^{n-1} |\mathcal{L}^{n-k-1}(\mathcal{L} - \mathcal{L}_p)\mathcal{L}_p^k\mu| \leq C\varepsilon n\|\mu\|.$$

Next, suppose that $\mu(\varphi) = 0$ for each function φ that does not depend on x_p, then

$$\|\mathcal{L}^{n+m}\mu\| \leq A\theta^n\|\mathcal{L}^m\mu\| + B|\mathcal{L}^m\mu| \leq C(\theta^n + m\varepsilon)\|\mu\| + B|\mathcal{L}_p^m\mu|.$$

Then, if h is the invariant density of the single site map,

$$\begin{aligned}\mathcal{L}_p^m\mu(\varphi) &= \mu(\varphi\circ(\Phi_{\varepsilon,p}\circ F_0)^m) \\ &= \int_\Omega \left[\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x))\right]\mu(dx) \\ &= \int_\Omega \partial_{x_p}\int_0^{x_p} dx_p \left[\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x))\right]\mu(dx) \\ &\leq \|\mu\|\sup_{x_{\neq p}}\int_0^1 dy \mathbb{1}_{[0,x_p]}(y)\left[\varphi(x_{\neq p}, T^m y) - \int_0^1 dz h(z)\varphi(x_{\neq p}, z)\right] \\ &\leq C\nu^n\|\mu\|\cdot|\varphi|_\infty,\end{aligned}$$

where ν is the rate of decay for the single site map. Putting the above estimates together yields

$$\|\mathcal{L}^{n+m}\mu\| \leq C(\theta^n + m\varepsilon + \nu^m)\|\mu\| \leq \sigma^{n+m}\|\mu\|,$$

for some $\sigma \in (0,1)$, provided we choose n, m, ε appropriately.

So, let $\mathcal{B}_p = \{\mu \in \mathcal{B} \;:\; \mu(\varphi) = 0 \text{ for all } \varphi \text{ independent of } p\}$. The situation looks good but there are two problem

(1) in general $\mu \in \mathcal{B}$ does not belong to $\mathcal{B}_p$ for any p.
(2) $\mu \in \mathcal{B}_p \not\Longrightarrow \mathcal{L}\mu \in \mathcal{B}_p$.

No problem: first show that each $\mu \in \mathcal{B}$ can be decomposed as

$$\mu = cm + \sum_{p \in \mathbb{Z}^d} \mu_p$$

where $m \in \mathcal{B}$ is a fixed probability measure and $\mu_p \in \mathcal{B}_p$. Then, for each $\mu_p \in \mathcal{B}_p$, write

$$\mathcal{L}\mu_p = \mathcal{L}_p\mu_p + \varepsilon \sum_{|q-p|\leq 1} \mathcal{L}_{q,p}\mu_p$$

where $\mathcal{L}_p\mathcal{B}_p \subset \mathcal{B}_p$ and $\mathcal{L}_{q,p}\mathcal{B}_p \subset \mathcal{B}_q$ and the operators have all uniformly bounded norm. Only a seemingly catastrophic problem is left: the decomposition sum does not converge in the $|\cdot|$ topology (let alone the $\|\cdot\|$ one).

No problem: let us associate to each measure μ the vector (c, μ_p) given by the terms of its decomposition (this means that one introduces the new super-abstract Banach space $\bar{\mathcal{B}} = \mathbb{C} \times (\times_{p\in\mathbb{Z}^d}\mathcal{B}_p)$ with norm $\|(c,\mu_p)\| := \max\{|c|, \sup_{p\in\mathbb{Z}^d} \|\mu\|_p\}$) and the operator

$$\overline{\mathcal{L}}(c, \mu_p) = (c, \mathcal{L}_p\mu_p + \varepsilon \sum_{|q-p|\leq 1} \mathcal{L}_{p,q}\mu_q + \zeta_p) =: (c, \mathcal{L}_*(\mu_p) + \bar{\zeta}),$$

where ζ_p is the decomposition of $\mathcal{L}m - m$. By applying the previous estimates one has that $\|\mathcal{L}_*\| < 1$. Is that good for something?

Well, $(1, \bar{\mu}) = (1, \mathcal{L}_*\bar{\mu} + \bar{\zeta})$ has the unique solution $\bar{\mu}^* := (\mathbb{1} - \mathcal{L}_*)^{-1}\bar{\zeta}$. Let φ be a local function that depends only the variables in the finite set $\Lambda \subset \mathbb{Z}^d$ and $\mu \in \mathcal{B}$ a probability measure with decomposition $(1, \bar{\mu})$, then

$$\mu(\varphi \circ F_\varepsilon^n) = m(\varphi) + \sum_{p\in\Lambda} \left(\mathcal{L}_*^n\bar{\mu} + \sum_{k=0}^{n-1} \mathcal{L}_*^k\bar{\zeta} \right)_p (\varphi) = \sum_{p\in\Lambda} \mu_p^*(\varphi) + \mathcal{O}(|\Lambda|\|\mathcal{L}_*\|^n).$$

By weak compactness and the Lasota-Yorke inequality we know that $\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}^k\mu$ has accumulation points in $\mathcal{B}$, let μ_* be one such accumulation point, then

$$\mu_*(\varphi) = \sum_{p\in\Lambda} \mu_p^*(\varphi)$$

Invariance, uniqueness and spatio-temporal decay of correlation for μ_* readily follow.

4.7. **Partially hyperbolic systems.** Let $X = [0,1]^2$, and T be a piecewise expanding map, then

$$F_\varepsilon(x, E)_i = \begin{pmatrix} Tx_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i(x,E))(x_{i+z} - x_i) \\ E_i + \varepsilon \sum_{|z|=1} \pi_z(\tau^i(x,E))(E_{i+z} - E_i) \end{pmatrix} \tag{4.1}$$

where α_z, π_z are smooth and $\alpha_z, \pi_z \geq 0$ and $\sum_z \alpha_z = \sum_z \pi_z = 1$. As explained by Dolgopyat this is a very hard case even if one has only two maps. So we need some simplifying assumptions. Let us start with a very drastic one that has recently been treated in [14].

4.8. **Cocycles and random walks.** Assume that $\alpha_z = 0$ (in fact, the case $\partial_E \alpha_z = 0$ can be treated in the same way) and $\partial_E \pi_z = 0$. Then the above system can be treated as a random walk in random environment. Indeed the if we take an initial condition such that $\sum_i E_i = 1$, such a condition will be preserved by the dynamics. But then we can interpret the E_i as the probability of an imaginary particle (a ghost particle) to be at site i. Looking at the dynamics we see that if we interpret the π_z as environment dependent transition probability, then the dynamics specify exactly the evolution of the probability distribution of the ghost particle if such a particle performs a random walk with transition probabilities π_z. Let $X_n \in \mathbb{Z}^d$ be the position of the ghost particle at time n.

4.9. **The egocentric point of view.** Consider the process $\boldsymbol{\omega} =: (\omega^n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ described by the action of the Markov operator $S : L^\infty(\Omega) \to L^\infty(\Omega)$ defined by

$$Sf(\omega) := \sum_{z\in\Lambda} \pi_z(\omega) f \circ F(\tau^z \omega) =: \sum_{z\in\Lambda} S_z f. \tag{4.2}$$

Remark 4.2. *It is easy to verify that the process $\boldsymbol{\omega}$, $\omega^0 = x$, has the same distribution as the process $(\tau^{X_n} T^n x)_{n\in\mathbb{N}}$.*

We can use the same techniques used to study CML to study the operator S and prove that it has a unique invariant measure $\mu_* \in \mathcal{B}$. We can then consider the measure $\mathbb{P}$ on $\Omega^{\mathbb{N}}$ of the associated Markov process started with a measure μ_*.

4.10. **Annealed statistical properties.**

Lemma 4.3. *There exists a vector $v \in \mathbb{R}^d$ and a matrix* Var ≥ 0 *such that, for each probability measure $\nu \in \mathcal{B}$ we have*

$$\frac{1}{N}\mathbb{E}(X_N) \to v$$
$$\frac{X_N - vN}{\sqrt{N}} \Rightarrow \mathcal{N}(0, \mathrm{Var}) \quad \textit{under } \mathbf{P}_\nu.$$

Note that if $v = 0$ (which can be insured by a symmetry assumption) and $\varphi \in \mathcal{C}^0(\mathbb{R}^d)$, then we have (essentially)

$$\lim_{N\to\infty} \sum_{q\in\mathbb{Z}^d} \mu_*(\varphi(N^{-\frac{1}{2}}q)E_q(Nt)) = \lim_{N\to\infty} \sum_{q\in\mathbb{Z}^d} \mathbb{E}(\varphi(N^{-\frac{1}{2}}X_{Nt})) = \int_{\mathbb{R}^d} \Psi(y,t)\varphi(x)dy$$

where

$$\partial_t \Psi = \sum_{ij} \mathrm{Var}^2_{ij}\, \partial_{y_i y_j} \Psi.$$

In other words, the *local average* of the E is described by the function Ψ which satisfy the heat equation. Since

$$\mathbb{E}\left(e^{\frac{i}{\sqrt{N}}\langle t, \Delta_k - v\rangle} \mid \mathcal{F}_k\right) = \sum_{z\in\Lambda} \pi_z(\tau^{X_k}\theta^k) e^{\frac{i}{\sqrt{N}}\langle t, z-v\rangle},$$

it is natural to introduce the operators, for all $t \in \mathbb{C}^d$,

$$\mathcal{M}_t h(\theta) := \sum_{z\in\Lambda} \pi_z(\theta) e^{\langle t, z-v\rangle} h(\tau^z F(\theta)) = \sum_{z\in\Lambda} e^{\langle t, z-v\rangle} S_z h. \tag{4.3}$$

Then,

$$\mathbb{E}\left(e^{\frac{i}{\sqrt{N}}\langle t, \tilde{X}_N\rangle}\right) = \nu(\mathcal{M}^N_{it/\sqrt{N}} 1). \tag{4.4}$$

The operator $\mathcal{M}'_t$ acting on the space $\mathcal{B}$ is an analytic perturbation of the operator $S' = \mathcal{M}'_0$. Unfortunately, S' does not have a nice spectrum on $\mathcal{B}$, but if we lift it to our covering space $\overline{\mathcal{B}}$ then it has a simple maximal eigenvalue and we can apply standard perturbation theory to prove that the maximal eigenvalue is of the form $1 - N^{-1}\langle t, \operatorname{Var}^2 t\rangle$ which implies the result.

4.11. **Kinetic limit.** The idea is to consider the system described by (4.1) and look at $E(\varepsilon^{-2}t)$ in the limit $\varepsilon \to 0$. This is very similar to the work of Gaspard-Gilbert and Bricmont-Kupiainen that we heard in the previous weeks. The goal is to show that, in the limit, we have a limiting stochastic process $e(t)$ that satisfy the SDE

$$de_i = \sum_{|i-k|=1} \alpha(e_i, e_k)dt + \sum_{|i-k|=1} \sigma\gamma(e_i, e_k)dB_{\{i,k\}} \tag{4.5}$$

with $B_{\{i,k\}} = -B_{\{k,i\}}$ independent standard Brownian motions. This is an open problem at the moment but similar results have been obtained for different model by Liverani-Olla (in preparation).

The above equation looks very similar to the non-gradient Ginzburg-Landau equation studied by Varadhan in [54]. So it is conceivable that it may be possible to take diffusive scaling limit (like in the previous example) and obtain a non-linear heat-equation.

This is the current research plan of several people.

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