The martingale approach after Varadhan and Dolgopyat

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Let $F_\varepsilon \in \mathcal{C}^4(\mathbb{T}^2, \mathbb{T}^2)$ be defined by

$$F_\varepsilon(x, \theta) = (f(x, \theta), \theta + \varepsilon\omega(x, \theta)).$$

This is a family of fast-slow systems (in discrete time, for simplicity). We assume

$$\inf_{x,\theta} \partial_x f(x, \theta) = \lambda > 1.$$

A standard pair is a couple $\ell = (G, \rho)$. $G \in \mathcal{C}^r([a,b], \mathbb{T})$, $|b-a| \in [\delta, 2\delta]$, $|G'| \leq c\varepsilon$ and $\rho \in \mathcal{C}^r([a,b], \mathbb{R}_{\geq})$, $|\frac{d}{dx} \ln \rho(x)| \leq c$, $\int_a^b \rho = 1$. To each standard pair is associated the probability measure

$$\mu_\ell(\phi) = \int_a^b \rho(x)\phi(x, G(x))dx.$$

We assume that the initial condition is distributed according to a convex combination of standard pairs.

Define the polygonalization (since we interpolate between close points the procedure is uniquely defined in $\mathbb{T}$.)

$$\Theta_\varepsilon(t) = \theta_{\lfloor \varepsilon^{-1} t \rfloor} + (t - \varepsilon \lfloor \varepsilon^{-1} t \rfloor)(\theta_{\lfloor \varepsilon^{-1} t \rfloor + 1} - \theta_{\lfloor \varepsilon^{-1} t \rfloor}),$$

with $t \in [0, T]$.

Note that Θ_ε is a random variable on $\mathbb{T}^2$ with values in $\mathcal{C}^0([0,T],\mathbb{T})$. Let $\mathbb{P}^\varepsilon_\ell$ be the law of Θ_ε (with initial distribution μ_ℓ) as a random variable in $\mathcal{C}^0([0,T],\mathbb{T})$.

Here is the first result we want to establish in our journey:

Theorem 1 (Anosov, Kifer, Dolgopyat)**.** *For each standard pair* ℓ*, the measures* $\{\mathbb{P}_\ell^\varepsilon\}$ *have a weak limit* $\mathbb{P}$*. Moreover,* $\mathbb{P}$ *is supported on the trajectory determined by the O.D.E.*

$$\dot{\overline{\Theta}} = \bar{\omega}(\overline{\Theta})$$

$$\overline{\Theta}(0) = \theta_0$$

where $\bar{\omega}(\theta) = \int_M \omega(x,\theta)\rho_\theta(x)dx$*.*

The Key property of standard pairs is their invariance:

Given a standard pail ℓ consider the push forward $(F_\varepsilon)_*\mu_\ell$. Then there exists a finite collection of indices $\mathcal{A}$, a probability measure ν on $\mathcal{A}$ and standard pairs $\{\ell_\alpha\}_{\alpha\in\mathcal{A}}$ such that

$$(F_\varepsilon)_*\mu_\ell = \sum_{\alpha\in\mathcal{A}} \nu(\{\alpha\})\mu_{\ell_\alpha}.$$

Hence the weak closure of the coves combination of standard pairs is a compact convex set invariant under the pushforward. Thus it must contain at least an invariant measure of the system.

Now that we have identifies the class of measure we want to work with, it is natural to consider the situation in which our process starts with a measure μ_ℓ associated to a standard pair ℓ (hence we may have an ε uncertainty also in the slow variable). Again let $\mathbb{P}_\ell^\varepsilon$ be the associated measure in path space. Since

$$|\Theta_\varepsilon(t+h) - \Theta_\varepsilon(t)| \leq \sum_{i=\lfloor t\varepsilon^{-1}\rfloor - 1}^{\lceil (t+h)\varepsilon^{-1}\rceil + 1} \varepsilon|\omega(x_i, \theta_i)| \leq Ch.$$

It follows that $\mathbb{P}_\ell^\varepsilon$ is supported on the Lipschitz paths with Lipschitz constant less than C. Since such a set is compact in the uniform topology, it follows that $\{\mathbb{P}_\ell^\varepsilon\}$ is a tight family (i.e. is relatively compact in the weak topology).

Thus the first step in our strategy is totally trivial.

Next, a little computations with standard pairs.
Let $A \in \mathcal{C}^0(\mathbb{R}^m, \mathbb{R})$ and $\{t_i\}_{i=1}^m \subset [0, T]$, and ℓ be a standard pair

$$
\begin{aligned}
\mathbb{E}_\ell^\varepsilon(A(\Theta(t_1), \dots \Theta(t_m))) &= \mu_\ell(A(\Theta_\varepsilon(t_1), \dots, \Theta_\varepsilon(t_m))) \\
&= (F_\varepsilon)_* \mu_\ell(A(\Theta_\varepsilon(t_1 - \varepsilon), \dots, \Theta_\varepsilon(t_m - \varepsilon))) \\
&= \sum_{\alpha \in \mathcal{A}} \nu_\alpha \mu_{\ell_\alpha}(A(\Theta_\varepsilon(t_1 - \varepsilon), \dots, \Theta_\varepsilon(t_m - \varepsilon))) \\
&= \sum_{\ell_1 \in \mathfrak{L}_\ell} \nu_{\ell_1} \mu_{\ell_1}(A(\Theta_\varepsilon(0), \dots, \Theta_\varepsilon(t_m - t_1))) \\
&= \sum_{\ell_1 \in \mathfrak{L}_\ell} \nu_{\ell_1} \mathbb{E}_{\ell_1}^\varepsilon(A(\Theta(0), \dots, \Theta(t_m - t_1)))
\end{aligned}
$$

where $\mathfrak{L}_\ell$ is the family of standard pairs generated by ℓ at time $\varepsilon^{-1} t_1$.

Suppose

$$\lim_{\varepsilon \to 0} \sup_{\ell} \left| \mathbb{E}^{\varepsilon}_{\ell}(A(\Theta(0), \dots \Theta(t_m - t_1))) \right| = 0,$$

Then, for each set of functions $B_i \in \mathcal{C}^1$ and times $s_i \leq t_1$

$$\mathbb{E}^{\varepsilon}_{\ell} \left(\prod_j B_j(\Theta(s_j)) A(\Theta(t_1), \dots \Theta(t_m)) \right)$$

$$= \sum_{\ell_1 \in \mathfrak{L}_{1,\ell}} \cdots \sum_{\ell_{k+1} \in \mathfrak{L}_{k,\ell_k}} \left[\prod_{i=1}^{k} \nu_{\ell_i} B_i(\theta^*_{\ell_i}) \right]$$

$$\times \nu_{m+1} \mathbb{E}^{\varepsilon}_{\ell_{m+1}}(A(\Theta(0), \dots \Theta(t_m - t_1))) + \mathcal{O}(\varepsilon)$$

where $\theta^*_{\ell} = \mu_{\ell}(\theta)$.

Thus, if $\mathbb{P}$ is an accumulation point of $\mathbb{P}_\ell^\varepsilon$, it follows

$$\mathbb{E}\left(\prod_j B_j(\Theta(s_j))A(\Theta(t_1),\ldots\Theta(t_m))\right)=0$$

and, by the arbitrariness of the $\{B_j\}$, we have, for each $s \leq t_1$,

$$\mathbb{E}\left(A(\Theta(t_1),\ldots\Theta(t_m))\,|\mathcal{F}_s\right)=0.$$

In the limit we have recovered the possibility to condition with respect to the past.

To continue we need a computation: for each $n \leq \varepsilon^{-\frac{1}{2}}$

$$\mu_\ell \left(A(\theta_n)\right) = \int_a^b \rho(x) A\left(\theta_0 + \varepsilon \sum_{k=0}^{n-1} \omega(x_k, \theta_k)\right) dx$$

$$= \int_a^b \rho(x) A(\theta_\ell^*) dx + \varepsilon \sum_{k=0}^{n-1} \int_a^b \rho(x) A'(\theta_\ell^*) \omega(x_k, \theta_\ell^*) dx + \mathcal{O}(\varepsilon)$$

where $\theta_\ell^* = \mu_\ell(\theta_0)$. Since $|\theta_n - \theta_0| \leq Cn\varepsilon$.

$$
\begin{aligned}
\mu_\ell\left(A(\theta_n)\right) &= \int_a^b \rho(x) A(G_\ell(x))dx + \mathcal{O}(\varepsilon) \\
&\quad + \varepsilon \sum_{k=0}^{n-1} \int_a^b \rho(x) \langle \nabla A(\theta_\ell^*), \omega(f_*^k \circ Y_n(x), \theta_\ell^*)\rangle dx \\
&= \mu_\ell(A(\theta_0)) + \varepsilon \sum_{k=0}^{n-1} \int_{\mathbb{T}^1} \tilde{\rho}_n(x) \langle \nabla A(\theta_\ell^*), \omega(f_*^k(x), \theta_\ell^*)\rangle dx + \mathcal{O}(\varepsilon)
\end{aligned}
$$

where $f_*(x) = f(x, \theta_\ell^*)$ and $f_*^n(Y_n(x)) = x_n$. One can compute applying the implicit function theorem to show

$$
|f_*^k(Y_n(x)) - x_k| \leq C\varepsilon k \quad \forall k \leq n.
$$

while $\tilde{\rho}_n(x) = \left[\frac{\chi_{[a,b]}\rho}{Y_n'}\right] \circ Y_n^{-1}(x)$.

Note that $\int_{\mathbb{T}^1} \tilde{\rho}_n = 1$ but, unfortunately, $\|\tilde{\rho}\|_{\mathcal{C}^1}$ may be enormous; yet, $|Y_n' - 1| \leq C_\# \varepsilon n^2$. Moreover, $\bar{\rho} = (\chi_{[a,b]}\rho) \circ Y^{-1}$ has uniformly bounded variation. By the decay of correlations and the $\mathcal{C}^1$ dependence of the invariant measure on θ

$$
\begin{aligned}
&\int_{\mathbb{T}^1} \tilde{\rho}_n(x) \langle \nabla A(\theta_\ell^*), \omega(f_*^k(x), \theta_\ell^*)\rangle dx \\
&= \int_{\mathbb{T}^1} \bar{\rho}_n(x) \langle \nabla A(\theta_\ell^*), \omega(f_*^k(x), \theta_\ell^*)\rangle dx + \mathcal{O}(\varepsilon n^2) \\
&= m_{\text{Leb}}(\tilde{\rho}_n(x)) m_{\theta_\ell^*}\left(\langle \nabla A(\theta_\ell^*), \omega(\cdot, \theta_\ell^*)\rangle\right) + \mathcal{O}(\varepsilon n^2 + e^{-ck}) \\
&= \mu_\ell\left(\langle \nabla A(\theta_0), \overline{\omega}(\theta_0)\rangle\right) + \mathcal{O}(\varepsilon n^2 + e^{-ck}).
\end{aligned}
$$

Collecting all the above we have

$$\mu_\ell\left(A(\theta_n)\right) = \mu_\ell(A(\theta_0) + \varepsilon n \langle \nabla A(\theta_0), \bar{\omega}(\theta_0)\rangle) + \mathcal{O}(n^3\varepsilon^2 + \varepsilon).$$

Finally, we choose $n = \lceil \varepsilon^{-\frac{1}{3}} \rceil$ and set $h = \varepsilon n$. We define inductively standard families such that $\mathfrak{L}_{\ell_0} = \{\ell\}$ and for each standard pair $\ell_{i+1} \in \mathfrak{L}_{\ell_i}$ the family $\mathfrak{L}_{\ell_{i+1}}$ is a standard decomposition of the measure $(F_\varepsilon^n)^*\mu_{\ell_{i+1}}$. Thus, setting $m = \lceil t\varepsilon^{-\frac{2}{3}} \rceil - 1$,

$$
\begin{aligned}
&\mathbb{E}^{\varepsilon}_{\ell}(A(\Theta(t))) = \mu_{\ell}(A(\theta_{t\varepsilon^{-1}}))\\
&= \mu_{\ell}(A(\theta_0)) + \sum_{k=0}^{m-1} \mu_{\ell}\big(A\big(\theta_{\varepsilon^{-1}(k+1)h}\big)\big) - A\big(\theta_{\varepsilon^{-1}kh}\big)\big)\\
&= \mu_{\ell}(A(\theta_0))\\
&+ \sum_{k=0}^{m-1} \sum_{\ell_1 \in \mathfrak{L}_{\ell_0}} \cdots \sum_{\ell_{k-1} \in \mathfrak{L}_{\ell_{k-2}}} \prod_{j=1}^{k-1} \nu_{\ell_j} \left[\mu_{\ell_{k-1}}(\varepsilon^{\frac{2}{3}} \langle \nabla A(\theta_0), \bar{\omega}(\theta_0)\rangle) + \mathcal{O}(\varepsilon)\right]\\
&= \mathbb{E}^{\varepsilon}_{\ell}\left(A(\Theta(0)) + \sum_{k=0}^{m-1} \langle \nabla A(\Theta(kh)), \bar{\omega}(\Theta(kh))\rangle h \right) + \mathcal{O}(\varepsilon^{\frac{1}{3}} t)\\
&= \mathbb{E}^{\varepsilon}_{\ell}\left(A(\Theta(0)) + \int_0^t \langle \nabla A(\Theta(s)), \bar{\omega}(\Theta(s))\rangle ds \right) + \mathcal{O}(\varepsilon^{\frac{1}{3}} t).
\end{aligned}
$$

We have just proven

$$\lim_{\varepsilon\to 0}\sup_{\ell}\mathbb{E}_{\ell}^{\varepsilon}\left(A(\Theta(t)) - A(\Theta(0)) - \int_0^t \langle\nabla A(\Theta(s)), \bar{\omega}(\Theta(s))\rangle ds\right) = 0.$$

Which, setting

$$M_t = A(\Theta(t)) - A(\Theta(0)) - \int_0^t \langle\nabla A(\Theta(\tau)), \bar{\omega}(\Theta(\tau))\rangle d\tau$$

implies, for any accumulation point $\mathbb{P}$, and $s \leq t$,

$$\mathbb{E}\left(M_t \mid \mathcal{F}_s\right) = M_s.$$

It remains to prove the uniqueness of the martingale problem. This can now be done similarly to our first proof of the convergence of Friedlin-Wentzell to the deterministic part using the martingale property instead of Ito's formula.