The martingale approach after Varadhan and Dolgopyat

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Averaging and fast slow systems–a simple case

Suppose that $(x, y) \in M \times \mathbb{T}^1$ for some compact manifold M. Consider the ODE

$$\dot{x} = \omega(x, y)$$
$$\dot{y} = \varepsilon^{-1} v$$
$$x(0) = x_0, \; y(0) = y_0.$$

The second system has Lebesgue as the unique invariant measure (if $v \neq 0$). Averaging asserts that $x(t, \varepsilon)$ converges, as $\varepsilon \to 0$ to a trajectory $z(t)$ satisfying

$$\dot{z} = \bar{\omega}(z(t) := \int_{\mathbb{T}^1} \omega(z(t), y) dy$$
$$x(0) = x_0.$$

Averaging is a very rich theory. Consider this other example: $(x, y) \in M \times N$, N a Riemanian compact manifold,

$$\dot{x} = \varepsilon\omega(x, y)$$
$$\dot{y} = v(y)$$
$$x(0) = x_0,\ y(0) = y_0.$$

where the second equation, for each fixed x, gives rise to a *chaotic* system. Such systems have many invariant measures and the relevant one for averaging might depend on the initial condition.

To overcome this problem we must consider random initial conditions:

$$\mathbb{E}(\varphi(x(0), y(0))) = \int_N \varphi(x_0, y)\rho(y)dy$$

where dy is the Riemanian volume and $\rho \in \mathcal{C}^1(N, \mathbb{R}_{\geq})$ is a probability density. Then, letting $z_\varepsilon(t) = x(\varepsilon^{-1}t, \varepsilon)$, we have, under appropriate conditions, that z_ε converges as a random variable to $z(t)$ where

$$\dot{z}(t) = \bar{\omega}(z(t)) := \mu(\omega(z(t), \cdot)$$
$$z(0) = x_0.$$

μ being the SRB measure of the dynamics $\dot{y} = v(y)$.

First question

Averaging gives information on the motion of x for times of the order ε^{-1} but what about longer times (say ε^{-10})?

More: can we say something about the invariant measures of the systems.

Even more: can we treat an open set of systems: i.e. $v = v(x, y)$

Freidlin-Wentzell theory

Consider the SDE

$$dx = \omega(x)dt + \sqrt{\varepsilon}\sigma(x)dB$$
$$x(0) = x_0.$$

where B is the standard Brownian motion. Assume, for simplicity, that $\dot{x} = \omega(x)$ is a very simple dynamical systems: it has finitely many sinks and sources. Hence the only invariant ergodic measures are point mass measures either on a sink or a source.

Freidlin-Wentzell theory asserts that such a system has a unique invariant measure concentrated near the sinks with masses determined by an effective Markov chain whose transition rates can be computed by using large deviation theory.

Moreover the system will exhibit metastatibility behaviour on a scale of order $e^{c_{\#}\varepsilon^{-1}}$.

Second question

In some sense the above system is also of the fast-slow type being the Brownian motion infinitely fast.

A widely held point of view is that the small noise in a systems is just a way of modelling the weak interaction with the exterior made of other (deterministic but complex) systems. Such systems can be complex either in time (chaotic) or space (many degrees of freedom).

Is it possible to get the analogous of Freidlin-Wentzell theory in a completely deterministic system?

The answer to the above questions (and others coming from the theory of partially hyperbolic systems, and non-equilibrium statistical mechanics) lies at the end of a rather long (and only very partially walked) road. The aim of this course is to start along such a road form the beginning and proceed slowly but carefully till where the time will allow us.

The first part of the path (which is mostly what we will manage to discuss, hence the title of the course) is known for some time. I will follow the pedagogical exposition in

The martingale approach after Varadhan and Dolgopyat.
Jacopo De Simoi, Carlangelo Liverani.
preprint arXiv:1402.0090, pages 28.

The continuation is very recent and who is interested can find all the (rather ugly) details in the papers

Fast-slow partially hyperbolic systems: beyond averaging. Part I (Limit Theorems).
Jacopo De Simoi, Carlangelo Liverani;
preprint arXiv:1408.5453, pages 96.

Fast-slow partially hyperbolic systems: beyond averaging. Part II (Statistical Properties).
Jacopo De Simoi, Carlangelo Liverani;
preprint arXiv:1408.5454, pages 36.

To make the ideas as clear as possible we will consider the simplest possible model. Yet, the ideas that I will discuss should be applicable in much greater generality.

Let $F_\varepsilon \in \mathcal{C}^4(\mathbb{T}^2, \mathbb{T}^2)$ be defined by

$$F_\varepsilon(x, \theta) = (f(x, \theta), \theta + \varepsilon\omega(x, \theta)).$$

This is a family of fast-slow systems (in discrete time, for simplicity). The connection with Freidlin-Wentzell theory it is not immediately apparent but it will became clear as we proceed.

Since we want the fast dynamics to be *chaotic* we assume

$$\inf_{x,\theta} \partial_x f(x, \theta) = \lambda > 1.$$

So for each θ we have an expanding map of the circle.

Note that, for $x, z \in \mathbb{T}^1$ close enough, we have $|f^n(x) - f^n(z)| \geq \lambda^n |x - z|$, thus the fast dynamics has strong dependence from the initial conditions. Or, in more technical words, the map F_ε has a strictly positive Lyapunov exponent.

It is known that smooth expanding maps of the circle have a unique SRB measure. That is a unique invariant measure absolutely continuous with respect to Lebesgue. Let ρ_θ be the density of the SRB measure m_θ of the map $f(\cdot, \theta)$. This can be proved in many ways, let us see a probabilistic proof:

A dynamical systems interlude

Let $f \in \mathcal{C}^2(\mathbb{T}, \mathbb{T})$, $f' \geq \lambda > 1$. Consider the functions

$$D_a = \left\{ \rho \in \mathcal{C}^1(\mathbb{T}, \mathbb{R}_{\geq}) \ : \ \int \rho = 1, \frac{d}{dx} \ln \rho \leq a \right\}$$

and the set of probability measures

$M_a = \{ \nu \in \mathcal{M} \ : \ \frac{d\nu}{dx} = \rho \in D_a \}$.

Note that f has finitely many inverse branches $\{\varphi_i\}_{i=1}^d$.

then, for $\nu \in M_a$ and $\phi \in \mathcal{C}^0$

$$f_*\nu(\phi) = \int_{\mathbb{T}} \rho\phi \circ f = \sum_{i=1}^{d} \int_{\mathbb{T}} \rho \circ \varphi_i \varphi_i' \phi$$

Note that

$$\left| \frac{d}{dx} \rho \circ \varphi_i \varphi_i' \right| \leq \rho \circ \varphi_i \varphi_i' \left[a\varphi_i' + \|\varphi_i''(\varphi_i')^{-1}\|_{L^\infty} \right] \leq \rho \circ \varphi_i \varphi_i' \left[a\lambda^{-1} + C \right]$$

thus, setting $\rho_i = Z_i \rho \circ \varphi_i \varphi_i'$, $Z_i^{-1} = \int_{\mathbb{T}} \rho \circ \varphi_i \varphi_i'$, we have $\rho_i \in M_a$, for a large enough.

We have thus proven

$$f_*\nu = \sum_{i=1}^{d} Z_i \nu_i$$

Next, given $\nu, \mu \in M_a$ we have that their densities are $\geq e^{-a}$. We can then write

$$\nu(\phi) = \alpha \int \phi + (1-\alpha) \int \frac{\rho_\nu - \alpha}{1-\alpha} \phi$$

$$\mu(\phi) = \alpha \int \phi + (1-\alpha) \int \frac{\rho_\mu - \alpha}{1-\alpha} \phi$$

This means that we can couple α mass.

That is we can define a measure G on $\mathbb{T}^2$ by

$$G(\phi) = \alpha \int_{\mathbb{T}} \phi(x,x)dx + (1-\alpha) \int_{\mathbb{T}^2} \phi(x,y)\hat{\rho}_\nu(x)\hat{\rho}_\mu(y)$$

$$\hat{\rho}_\nu = \frac{\rho_\nu - \alpha}{1-\alpha} \; ; \quad \hat{\rho}_\mu = \frac{\rho_\mu - \alpha}{1-\alpha}$$

Note that, setting $F = f \times f$, we have that $F^n_* G$ is a coupling of $f^n_* \nu$ and $f^n_* \mu$. And, if α is small, letting $\hat{\nu}$ be the measure with density $\hat{\rho}_\nu$,

$$f_* \hat{\nu} = \sum_{i=1}^{d} Z_i \hat{\nu}_i$$

with $\hat{\nu}_i \in M_a$.

Hence, after m step, we can define a coupling G_m between $f_*^m\nu$ and $f_*^m\mu$ that couples $1-(1-\alpha)^m$ mass. Thus

$$\begin{aligned} |f_*^m\nu(\phi) - f_*^m\mu(\phi)| &\leq \int [\phi(x) - \phi(y)] G_m(dx, dy) \\ &\leq 2\|\phi\|_{L^\infty}(1-\alpha)^m. \end{aligned}$$

Which means that the equation $f_* m = m$ has a unique solution in M_a.

Lets go back to our story.

We are interested in the trajectories of such a dynamical system: let $(x_n, \theta_n) = F_\varepsilon^n(x_0, \theta_0)$.

As discussed before we will consider random initial conditions defined by the measure μ_0 on $\mathbb{T}^2$ by

$$\mathbb{E}_{\mu_0}(\varphi(x_0, \theta_0)) = \int_{\mathbb{T}^1} \varphi(x, \theta^*)\rho(x)dx$$

for some probability distribution $\rho \in \mathcal{C}^3(\mathbb{T}^1, \mathbb{R}_{\geq})$ and $\theta^* \in \mathbb{T}^1$.

Define the polygonalization (since we interpolate between close points the procedure is uniquely defined in $\mathbb{T}$.)

$$\Theta_\varepsilon(t) = \theta_{\lfloor \varepsilon^{-1} t \rfloor} + (t - \varepsilon \lfloor \varepsilon^{-1} t \rfloor)(\theta_{\lfloor \varepsilon^{-1} t \rfloor + 1} - \theta_{\lfloor \varepsilon^{-1} t \rfloor}),$$

with $t \in [0, T]$.

Note that Θ_ε is a random variable on $\mathbb{T}^2$ with values in $\mathcal{C}^0([0, T], \mathbb{T})$. Also, note the time rescaling done so that one expects regular but non trivial paths.

It is often convenient to consider random variables defined directly on the space $\mathcal{C}^0([0,T],\mathbb{T})$ rather than $\mathbb{T}$.

The space $\mathcal{C}^0([0,T],\mathbb{T})$ endowed with the uniform topology is a separable metric space. We can then view $\mathcal{C}^0([0,T],\mathbb{T})$ as a probability space equipped with the Borel σ-algebra.

Also, for each $t \in [0,T]$ let $\Theta(t) \in \mathcal{C}^0(\mathcal{C}^0([0,T],\mathbb{T}),\mathbb{T})$ be the random variable defined by $\Theta(t,\vartheta) = \vartheta(t)$, for each $\vartheta \in \mathcal{C}^0([0,T],\mathbb{T})$.

Since Θ_ε can be viewed as a continuous map from $\mathbb{T}$ to $\mathcal{C}^0([0,T],\mathbb{T})$, the measure μ_0 induces naturally a measure $\mathbb{P}^\varepsilon$ on $\mathcal{C}^0([0,T],\mathbb{T})$: $\mathbb{P}^\varepsilon = (\Theta_\varepsilon)_*\mu_0$.

Next, for each $\mathcal{A} \in \mathcal{C}^0(\mathcal{C}^0([0,T],\mathbb{T}),\mathbb{R})$, we will write $\mathbb{E}^\varepsilon(\mathcal{A})$ for the expectation with respect to $\mathbb{P}^\varepsilon$.

For $A \in \mathcal{C}^0(\mathbb{T},\mathbb{R})$ and $t \in [0,T]$,

$$\mathbb{E}^\varepsilon(A \circ \Theta(t)) = \mathbb{E}^\varepsilon(A(\Theta(t)))$$

is the expectation of the function $\mathcal{A}(\vartheta) = A(\vartheta(t))$, $\vartheta \in \mathcal{C}^0([0,T],\mathbb{T})$.

Here is the first result we want to establish in our journey:

Theorem 1 (Anosov, Kifer)**.** *The measures* $\{\mathbb{P}^\varepsilon\}$ *have a weak limit* $\mathbb{P}$*. Moreover,* $\mathbb{P}$ *is supported on the trajectory determined by the O.D.E.*

$$\dot{\overline{\Theta}} = \bar{\omega}(\overline{\Theta})$$

$$\overline{\Theta}(0) = \theta_0$$

where $\bar{\omega}(\theta) = \int_M \omega(x,\theta)\rho_\theta(x)dx$*.*

(Actually, working more it is possible to prove almost sure convergence, but we will not worry about this)