Chernov Memorial Lectures

Hyperbolic Billiards, a personal outlook.

Lecture Three

Deterministic walks in random environment

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Beyond the Lorentz gas

As already mentioned even maintaining the assumption of independent particles, there are still many physically relevant situations one would like to study.
One is the presence of an external field.
To this situation Chernov gave fundamental contributions, e.g. see his (1993) paper with Eyink, Lebowitz and Sinai on electrical conduction and with Dolgopyat (2009) on the Galton board.

Beyond the Lorentz gas

Another, as already mentioned, is the non periodic (random) case. Here very few results are known [see Lenci (2007) and Lenci-Troubetzkoy (2011) in which recurrence is discussed], in particular the problem of establishing a CLT is wide open. It seems therefore necessary to investigate models of intermediate difficulty.
Of course, at the end of the game one would like to study the situation of random environment in presence of an external field ....

An abstract model

As mentioned, it is convenient to introduce an abstract class of systems (which includes the random Lorentz gas).
The goal is to have a more flexible set of problems in which the difficulties can be studied (and hopefully overcome) one by one.

Random environment

For simplicity we will consider a walk in $\mathbb{Z}^d$, but similar considerations hold for a walk on a graph $\mathcal{G}$. Let $\mathcal{A}$ be a finite set; X be a compact manifold, $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ a set of endomorphisms of X and $\{\mathcal{P}_\alpha\}_{\alpha\in\mathcal{A}}$ a set of partitions of X with $2d+1$ elements:
$\mathcal{P}_\alpha = \{p_{\alpha,1},\dots,p_{\alpha,2d+1}\}$
By random environment we mean a translation invariant probability measure $\mathbb{P}_e$ on $\Omega = \mathcal{A}^{\mathbb{Z}^d}$.
We are interested in statistical properties that hold for $\mathbb{P}_e$-almost all $\omega\in\Omega$.

Deterministic walk

By determinist walk in the environment $\omega \in \Omega$ we mean the map $\mathbb{F}_{\boldsymbol{\omega}} : X \times \mathbb{Z}^d \to X \times \mathbb{Z}^d$ defined by

$$\mathbb{F}_{\boldsymbol{\omega}}(x, z) = (f_{\boldsymbol{\omega}_{z+e(\boldsymbol{\omega}_z, x)}}(x), z + e(\boldsymbol{\omega}_z, x))$$

$$e(\omega, x) = \sum_{i=1}^{2d+1} \mathbb{1}_{p_{\omega,i}}(x) w_i.$$

$w_i \in \{\pm e_1, \ldots, \pm e_d, 0\}$.

The initial condition

Given a deterministic initial condition the deterministic walk will simple go somewhere.
If one wants to obtain some sensible results, then it is necessary to look at the statistics of the possible motions, that it, we have to consider an initial condition with some randomness.
The minimal possibility is: for each $\varphi : X \times \mathbb{Z}^d \to \mathbb{R}$

$$\mathbb{E}(\varphi) = \int_X h_0(x)\varphi(x,0)dx$$

with $h_0 \in \mathcal{C}^1(X, \mathbb{R})$.

Random Lorentz gas as an example

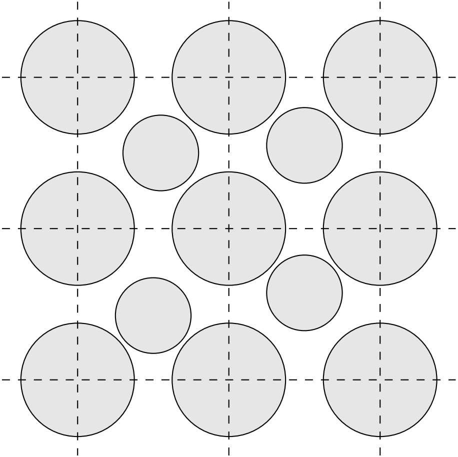

Figure: An obstacle configuration for the random Lorentz gas

A cell

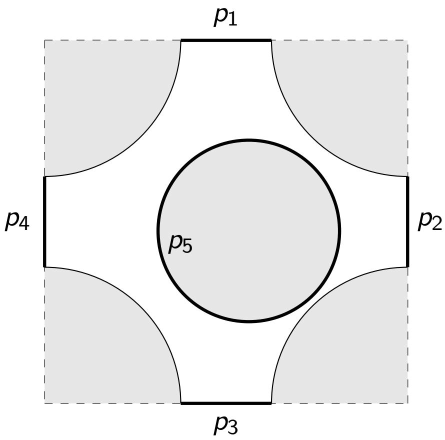

Figure: $X = \cup_{i=1}^{5} p_i \times [-\pi/2, \pi/2]$

Reduction to the abstract setting

Suppose that the obstacles can be only in K configurations and let $\mathcal{A} = \{1, \dots, K\}$. We have then associated the Billiard maps $\{f_i\}_{i \in \mathcal{A}}$, in this case the partition does not depend on $\mathcal{A}$. We can interpret X as the coordinates of the particle just after crossing the line defining the Poincarr̀ map. Thus, if the particle at time n is in the cell $z \in \mathbb{Z}^2$, with $x \in p_2$, we can consider it entering in the cell $z + e_1$ and then apply the Poincarè map $f_{\omega_{z+e_1}}$, thus the new coordinates will be

$$
\begin{aligned}
z(n+1) &= z + e_1 \\
x(n+1) &= f_{\omega_{z+e_1}}(x)
\end{aligned}
$$

that is a deterministic random walk.

Simpler examples 1

Let $d = 1$, $\mathcal{A} = \{-1, +1\}$, $\mathbb{P}_e$ is Bernoulli with probability $1/2$. The dynamics is defined by the map

$$f_\alpha(x) = f(x) = 4x \mod 1$$

with partitions $\mathcal{P}_{\pm 1}$: $p_{-1,-1} = [0, 1/4]$, $p_{-1,+1} = [1/4, 1]$ and $p_{+1,-1} = [0, 3/4]$, $p_{+1,+1} = [3/4, 1]$.
We consider the initial condition specified by the density $h_0 = 1$.

Simpler examples 1

As the map has a Markov structure, it is easy to check that at each site that can be reached the distribution of the x variable will always be uniform. Hence

$$\mathbb{P}(z(n+1) - z(n) = \pm 1 \mid \boldsymbol{\omega}, z(n), \dots, z(0)) = \frac{1}{2} \mp \frac{\boldsymbol{\omega}_z}{4}.$$

This is Sinai's walk, hence we do not have the classical CLT.

Simpler examples 2

Let $d = 1$, $\mathcal{A} = \{-1, +1\}$, $\mathbb{P}_e$ is Bernoulli with probability 1/2.
The partitions $\mathcal{P}_\alpha$ are given by $p_{\alpha,-1} = p_{-1} = [0, 1/2]$ and $p_{\alpha,+1} = p_{+1} = (1/2, 1]$.
The maps are defined by

$$f_{-1}(x) = \begin{cases} 2x & x \in [0, 1/4] \\ 4x \mod 1 & x > 1/4 \end{cases}$$

$$f_{+1}(x) = \begin{cases} 4x \mod 1 & x \in [0, 3/4] \\ 2x - 1 & x > 3/4. \end{cases}$$

The initial distribution is given by $h_0 = 1$.

Simpler examples 2

Denote by $\mathcal{L}_{\alpha,w}$, $w \in \{-1,+1\}$, the operator $\mathcal{L}_{\alpha,w}(\phi) = \mathcal{L}_{f_\alpha}(\mathbb{1}_{p_w}\phi)$. For any $\boldsymbol{\omega}$ and $z(1),\dots,z(n),z(n+1)$, denote by $\alpha_k = \boldsymbol{\omega}_{z(k+1)}$ and $w_k = w(k) = z(k+1) - z(k)$. Let $(x_n, z_n) = F^n_{\boldsymbol{\omega}}(x_0, 0)$. We have

$$\mathbb{P}(z(1),\dots,z(n) \mid \boldsymbol{\omega}) = \int_0^1 \mathbb{1}_{p_{w_0}}(x_0)\cdots\mathbb{1}_{p_{w_{n-1}}}(x_{n-1}),$$

where $x_k = f_{\alpha_{k-1}} \circ \cdots \circ f_{\alpha_0}(x_0)$. Hence

$$\mathbb{P}(z(1),\dots,z(n) \mid \boldsymbol{\omega}) = \int \mathcal{L}_{\alpha_{n-1},w_{n-1}} \dots \mathcal{L}_{\alpha_0,w_0}\mathbb{1}.$$

Simpler examples 2

By the Markov structure, the two dimensional vector space

$$\mathbb{V} = \{a_{-1}\mathbb{1}_{p_{-1}} + a_{+1}\mathbb{1}_{p_{+1}} \;:\; a_{-1}, a_{+1} \in \mathbb{R}\}$$

is left invariant by the operators $\{\mathcal{L}_{\alpha,w}\}_{\alpha,w}$.
If $\phi = a_{-1}\mathbb{1}_{p_{-1}} + a_{+1}\mathbb{1}_{p_{+1}}$, a direct computation shows that $\mathcal{L}_{\alpha,w}(\phi) = a_w \mathcal{L}_{\alpha,w}(\mathbb{1})$, and thus $\mathcal{L}_{\alpha',w'}\mathcal{L}_{\alpha,w}(\phi) = a_w \mathcal{L}_{\alpha',w'}\mathcal{L}_{\alpha,w}\mathbb{1}$.

Simpler examples 2

Let $\phi = \mathcal{L}_{\alpha_{n-2},w_{n-2}} \dots \mathcal{L}_{\alpha_0,w_0}\mathbb{1} = a_{-1}\mathbb{1}_{p_{-1}} + a_{+1}\mathbb{1}_{p_{+1}} \in \mathbb{V}$. Then

$$\mathbb{P}(z(1), \dots, z(n) \mid \boldsymbol{\omega}) = \int \mathcal{L}_{\alpha_{n-1},w_{n-1}}\phi = a_{w_{n-1}} \int \mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}$$

and

$$\mathbb{P}(z(1), \dots, z(n), z(n+1) \mid \boldsymbol{\omega}) = a_{w_{n-1}} \int \mathcal{L}_{\alpha_n,w_n}\mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}.$$

Simpler examples 2

Hence:

$$\mathbb{P}(w(n) \mid z(1), \dots, z(n), \omega) = \frac{\int \mathcal{L}_{\alpha_n, w_n} \mathcal{L}_{\alpha_{n-1}, w_{n-1}} \mathbb{1}}{\int \mathcal{L}_{\alpha_{n-1}, w_{n-1}} \mathbb{1}}$$

a function of $z(n-1)$, $z(n)$ and $w(n) = z(n+1) - z(n)$ only.
We have obtained a persistent random walk:
the transition probability depends not only on the current position of the particle but also on its previous position.

A much more challenging example

The first step to move toward the Lorentz gas situation is to give away with the Markov property.
$X = [0,1]$, all maps $f_\alpha : [0,1] \to [0,1]$ are piecewise C^2 and uniformly expanding (i.e. $|f'_\alpha| \geq \lambda > 1$), and the partitions $\mathcal{P}_\alpha = \{p_{\alpha,w}\}_{w\in\mathcal{W}}$ are made of subintervals of $[0,1]$.

A much more challenging example

Under some technical (explicitely computable) conditions there exists $\nu \in (0,1)$ such that for all ω

$$\mathbb{P}(z(n) \mid z(n-1), \dots, z(1), \omega) = \mathbb{P}(z(n) \mid z(n-1), \dots, z(m), \omega) + \mathcal{O}(\nu^{n-m} \|h_0\|_{\mathcal{C}^1}),$$

L.-Aimino work in progress.

A much more challenging example

The proof is base on the Hilbert metric technique introduced in the field of dynamical systems by Ferrero (1981), considerably extended in L. (1995) and adapted to the case of open dynamical systems in L.-Maume-Deschamps (2003).
I believe that this technique can be extended to billiards (Demers-L. work in progress).

Final comments

It should be noted that in considering the above examples we have forgotten two important properties of Billiards that instead I believe should play an important rôle in establishing the CLT:

- The Billiard flows is reversible: if $\boldsymbol{i}(x, v) = (x, -v)$, then $\phi_t \circ \boldsymbol{i} = \boldsymbol{i} \circ \phi_{-t}$.
- All the billiard maps have the same invariant measure, Liouville.

Final comments

In addition, in the simple examples discussed here, in which the Poincarè section is not random, $\mathbb{P}_e$ is also the invariant measure of the process as seen from the particle. This is a very helpful property that Lenci has used to establish recurrence for Lorentz tubes (one dimension).

So, to make further progresses, one must integrate the above facts into the proposed strategy.

Final comments

With this I reached the outer boundaries of my current knowledge,

thanks for listening