SPECTRAL PROPERTIES OF INTEGRAL OPERATORS IN BOUNDED, LARGE INTERVALS.

ENZA ORLANDI AND CARLANGELO LIVERANI

Abstract. We study the spectrum of one dimensional integral operators in bounded real intervals of length $2L$, for value of L large. The integral operators are obtained by linearizing a non local evolution equation for a non conserved order parameter describing the phases of a fluid. We prove a Perron-Frobenius theorem showing that there is an isolated, simple minimal eigenvalue strictly positive for L finite, going to zero exponentially fast in L. We lower bound, uniformly on L, the spectral gap by applying a generalization of the Cheeger's inequality. These results are needed for deriving spectral properties for non local Cahn-Hilliard type of equations in problems of interface dynamics, see [16].

1. Introduction

We study the spectrum of an integral operator acting on L^2 functions defined in intervals $[-L, L] \subset \mathbb{R}$, for value of L large. This problem arises when analyzing layered equilibria and front dynamics for the conservative, nonlocal, quasilinear evolution equation typified by

$$\partial_t m(t,x) = \nabla \cdot \big\{ \nabla m(t,x) - \beta(1 - m(t,x)^2)(J \star \nabla m)(t,x) \big\}, \tag{1.1}$$

where $\beta > 1$,

$$(J \star m)(x) = \int_{\mathbb{R}} J(x,y) m(y) \mathrm{d}y$$

and $J(\cdot, \cdot)$ is a regular, symmetric, translational invariant, non negative function with compact support and integral equal to one. This equation (1.1) first appeared in the literature in a paper [14] on the dynamics of Ising systems with a long–range interaction and so–called "Kawasaki" or "exchange" dynamics and later it was rigorously derived in [11]. In this physical context, $m(x, t) \in [-1, 1]$ is the spin magnetization density. It has been formally shown by Giacomin and Lebowitz [12], that in the sharp interface limit, i.e the limit in which the phase domain is very large with respect to the size of the interfacial region and time is suitably rescaled, the limit motion is given by Mullins Sekerka motion, a quasi-static free boundary problem in which the mean curvature of the interface plays a fundamental role. Equation (1.1) could be considered as a non local type of Cahn-Hilliard equation. Our intention is to provide basic spectral estimates useful for deriving higher dimensions spectral results in order to establish rigorously the relation between (1.1) and the singular limit motion described by the Mullins Sekerka equations, see [16]. We recall some previous results useful to better contextualize the problem. When $\beta > 1$

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Sadly, the first author died before finishing the revision of the paper. As the paper was almost complete and C.L. had some knowledge of its content, he decided to finish it in memory of a splendid person.

there is a phase transition in the underlying spin system, [15]. The pure phases correspond to the stationary spatially homogeneous solutions of (1.1) satisfying

$$m = \tanh \beta m.$$

For $\beta > 1$ there are three and only three different roots denoted

$$\pm m_\beta, 0.$$

The two phases $\pm m_\beta$ are thermodynamically stable while $m = 0$ is unstable. These statements, established in the context of the theory of Equilibrium Statistical Mechanics, see [15], are reflected by the corresponding stability properties of the space homogeneous solution of (1.1), see [12]. Equations (1.1) has also stationary solutions connecting the two coexisting phases: they are all identical modulo translations and reflection, see [12], to the "istanton" $\bar m(\cdot)$ which is $C^\infty(\mathbb{R})$, strictly increasing, antisymmetric function which identically verifies

$$\bar m(x) = \tanh \beta (J \star \bar m)(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

$\bar m(\cdot)$ is the stationary pattern that connects the minus and the plus phases as

$$\lim_{x \to \pm\infty} \bar m(x) = \pm m_\beta, \tag{1.3}$$

and it can be interpreted as a diffuse interface. The first results on these stationary patterns were obtained when analyzing the non conservative equation

$$\partial_t m(t,x) = -m(t,x) + \tanh \beta (J \star m)(t,x). \tag{1.4}$$

Equation (1.4) has been derived from the Glauber (non conservative) dynamic of an Ising spin system interacting via a Kac potential, see [6]. Since both the equations (1.1) and (1.4) have been derived from the same Ising spin systems, the first by a conservative dynamic while the latter by a non conservative one, both have as equilibrium solutions the homogeneous solution $\pm m_\beta$ and the stationary patterns connecting the two homogeneous phases. Stability properties of $\bar m$ have been derived both for the conservative evolution (1.1) (see [2], [3], [4]) and for the nonconservative evolution (1.4) (see [8]).

Let us recall few previous results which are needed in this paper. As proved in [8] the interface described by the istanton is "stable" for equation (1.4) and any initial datum "close to the instanton" is attracted and eventually converges exponentially to some translate of the instanton. Linearizing the evolution equation (1.4) at $\bar m$ one obtains the integral operator

$$\mathcal{L} v = v - \beta(1 - \bar m^2) J \star v \tag{1.5}$$

which is selfadjoint when $v \in L^2(\mathbb{R}, \frac{1}{\beta(1-\bar m^2)} dx)$. The spectrum of this operator has been studied in [7]. It has been proved that the spectrum of $\mathcal{L}$ is positive, the lower bound of the spectrum is 0 which is an eigenvalue of multiplicity one and the corresponding eigenfunction is $\bar m'(\cdot)$, the spatial derivative of the stationary solution $\bar m$, i.e

$$\mathcal{L} \bar m' = 0. \tag{1.6}$$

The remaining part of the spectrum is strictly bigger than some positive number. In this paper we consider operators of the type of the operator $\mathcal{L}$ defined in (1.5) but acting over functions in bounded intervals $[-L, L]$, L large. In fact, these are the operators naturally arising in the study of higher dimensional non local Cahn-Hilliard type equations.

Our goal is to derive qualitative and quantitative properties (in terms of L) of the principal eigenvalue together with the associated eigenfunctions and, most importantly, to show that the second eigenvalue can be bounded uniformly from below in terms of the length of the interval.

Remark that the above informations cannot be derived from the knowledge of the spectrum of $\mathcal{L}$ in $L^2(\mathbb{R}, \frac{1}{\beta(1-\bar{m}^2)}dx)$, obtained in [7]. Furthermore the method (Fourier analysis and Weyl's Theorem) used in [7] is not applicable to the present context, hence the need to develop a different, more flexible, approach.

It might be helpful to compare heuristically what we are doing with similar problems analysed previously in the context of reaction diffusion equations and Cahn-Hilliard equations. Dividing by $\beta(1-\bar{m}^2)$ the operator $\mathcal{L}$ we can define a new operator

$$\mathcal{G}v = \frac{v}{\beta(1-\bar{m}^2)} - J \star v = -\left[J \star v - v\right] + f''(\bar{m})v$$

where

$$f''(\bar{m}) = -1 + \frac{1}{\beta(1-\bar{m}^2)}$$

and

$$f(m) = -\frac{1}{2}m^2 + \frac{1}{\beta}\left[\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right)\right]$$

is a double equal well potential. The operator $\mathcal{G}$ on $L^2(\mathbb{R}, dx)$ and the operator $\mathcal{L}$ on $L^2(\mathbb{R}, \frac{1}{\beta(1-\bar{m}^2)}dx)$ have the same spectrum. Assume that v is smooth, taking into account that J is symmetric and therefore the first moment is null, we have that $J\star v - v \simeq \Delta v$. Heuristically $-[J\star v - v] + f''(\bar{m})v$ is equivalent to $-\Delta v + f''(\bar{m})v$. So the problem we are dealing with is in the same spirit of the problem dealt by De Mottoni and Schatzman, see [10, subsection 5.4]. They studied the spectrum of $-\Delta v + W(\bar{\theta})v$ in the finite interval $[-L, L]$ with Neumann boundary conditions. We denoted by $W(\bar{\theta})$ the corresponding of $f''(\bar{m})$ in [10]. This was a basic result used to obtain higher dimension spectral results for the Cahn-Hilliard equations, see for example [5] and [1]. In this paper we establish results for the spectrum of one dimensional integral operator in the finite interval $[-L, L]$. The main difficulty is to show that the spectral gap of our integral operator is bounded uniformly in L. This is achieved by applying a generalization of Cheeger's inequality, proven in [13] and lower bounding in our context the Cheeger's constant.

2. Notations and Results

Let $T_L = [-L, L]$ be a real interval, $L \geq 1$. We are actually interested in L large.

2.1. **The interaction.** Let $J \in C^\infty(\mathbb{R}, \mathbb{R})$, be a symmetric probability kernel, that is: $J(x) = J(-x)$, $J \geq 0$ and $\int J(x)\mathrm{d}x = 1$. We assume that $J(x) > 0$ for all $x \in (-1, 1)$ and $J(x) = 0$ for $|x| \geq 1$.[1] To define the interaction between x and y in $\mathbb{R}$ we set, by an abuse of notation, $J(x, y) = J(y - x)$. For a function v defined on T_L we set

$$(J \star_b v)(x) = \int_{T_L} J(x-y)v(y)\mathrm{d}y. \tag{2.1}$$

The suffix b is to reminds the reader that the integral is on the bounded interval T_L. Notice $\int_{T_L} J(x, y)dy = b(x)$ with $b(x) \in [\frac{1}{2}, 1]$ for $x \in T_L$. There are other ways to derive from J an integral kernel acting only on functions on the bounded interval T_L. One is the following

$$J^{neum}(x, y) = J(x, y) + J(x, 2L - y) + J(x, -2L - y), \tag{2.2}$$

[1] Note that the choice of $[-1, 1]$ as the support of J, i.e. the interaction length, is tantamount to a choice of the unit of measure. Indeed, the problem is scale invariant and the only quantity that really matters is the ratio between the range of the interaction (in this case 1) and the length of the interval (in this case L).

where $2L-y$ is the image of y under reflection on the right boundary $\{L\}$ and $-2L-y$ is the image of y under reflection on the left boundary $\{-L\}$. By the assumption on J, $J^{neum}(x,y) = J^{neum}(y,x)$ and $\int J^{neum}(x,y)\mathrm{d}y = 1$ for all $x \in T_L$. The choice to define by boundary reflections the interaction (2.2) has the advantage to keep J^{neum} a symmetric probability kernel. This definition first appeared in the paper [9, Section 2] and it was called there "Neumann" interaction. In [9] the authors studied spectral properties of operators closely related to the operator $\mathcal{L}$, see (1.5), defined on the space of the continuous symmetric functions on $\mathbb{R}$, $C^{\mathrm{sym}}(\mathbb{R})$.

We will consider in this paper operators with the integral kernel (2.1) acting on Hilbert spaces. We could denote (2.1) the Dirichelet interaction kernel. Our results can be, with minor modifications, immediately extended to the case when the integral kernel is J^{neum}.

2.2. **The istanton.** We call istanton the antisymmetric solution $\bar{m}$ of (1.2) with conditions at infinity given in (1.3). The function $\bar{m} \in C^\infty(\mathbb{R})$, it is strictly increasing, and there exist $a > 0$, $\alpha_0 > \alpha > 0$ and $c > 0$ so that

$$\begin{aligned} 0 < m_\beta^2 - \bar{m}^2(x) \le ce^{-\alpha|x|} \ , \\ |\bar{m}'(x) - a\alpha e^{-\alpha|x|}| \le ce^{-\alpha_0|x|}. \end{aligned} \tag{2.3}$$

A proof of these estimates and related results can be found in [17, Chapter 8, Section 8.2].

2.3. **The Operator.** For $\beta > 1$ set $p(x) = \beta(1 - \bar{m}^2(x))$ where $\bar{m}$ is the istanton. By the properties of $\bar{m}$ we have that

$$\lim_{|x|\to\infty} p(x) = \beta(1 - m_\beta^2) < 1, \tag{2.4}$$

and

$$\beta \ge p(x) \ge \beta(1 - m_\beta^2) > 0, \qquad x \in \mathbb{R}. \tag{2.5}$$

Denote

$$\mathcal{H} = L^2(T_L, \frac{1}{p(x)}\mathrm{d}x),$$

and for $v \in \mathcal{H}$ and $w \in \mathcal{H}$

$$\langle v, w\rangle = \int_{T_L} v(x)w(x)\frac{1}{p(x)}\mathrm{d}x,$$

$$\|v\|^2 = \int_{T_L} v^2(x)\frac{1}{p(x)}\mathrm{d}x. \tag{2.6}$$

To stress the dependence of $\mathcal{H}$ on L we will add, when needed, a suffix L, writing $\mathcal{H}_L$. We denote by

$$\|v\|_2, \qquad \|v\|_\infty,$$

respectively the $L^2(T_L, dx)$ and the $L^\infty(T_L, dx)$ norm of a function v. By (2.5) we have

$$\|v\|_2^2 \le \beta\|v\|^2. \tag{2.7}$$

We can now define precisely the operator we want to study: let $\mathcal{L}^0$ be the operator acting on $\mathcal{H}$ as

$$(\mathcal{L}^0 g)(x) = g(x) - p(x)(J \star_b g)(x). \tag{2.8}$$

2.4. **Results.** The following results for the operator $\mathcal{L}^0$ hold for any fixed value of L large enough.

Theorem 2.1. *For any $\beta > 1$ there exist $L_1(\beta)$ so that for $L \geq L_1(\beta)$ the following holds.*

(0)The operator $\mathcal{L}^0$ is a bounded, quasi compact, selfadjoint operator on $\mathcal{H}$.

(1) There exist $\mu_1^0 \in \mathbb{R}$ and $\psi_1^0 \in \mathcal{H}$, ψ_1^0 strictly positive in T_L so that

$$\mathcal{L}^0 \psi_1^0 = \mu_1^0 \psi_1^0.$$

The eigenvalue μ_1^0 has multiplicity one and any other point of the spectrum is strictly bigger than μ_1^0. There exist $c > 0$ independent on L so that

$$0 \leq \mu_1^0 \leq c e^{-2\alpha L}, \tag{2.9}$$

where $\alpha > 0$ is given in (2.3). Further $\psi_1^0 \in C^\infty(T_L)$, $\psi_1^0(z) = \psi_1^0(-z)$ for $z \in T_L$.

(2) Let μ_2^0 be the second eigenvalue of $\mathcal{L}^0$. We have that

$$\mu_2^0 = \inf_{\langle \psi, \psi_1^0 \rangle = 0; \|\psi\| = 1} \langle \psi, \mathcal{L}^0 \psi \rangle \geq D, \tag{2.10}$$

where $D > 0$ independent on L is given in (3.71).

(3) Let ψ_1^0 be the normalized eigenfunction corresponding to μ_1^0 we have

$$\left\| \psi_1^0 - \frac{\bar{m}'}{\|\bar{m}'\|} \right\| \leq C e^{-2\alpha L}, \tag{2.11}$$

where $C > 0$ is a constant independent on L.

3. Proof of the results

To prove Theorem 2.1 we introduce the following auxiliary operators. Denote by $\mathcal{A}$ the linear integral operator acting on functions $g \in \mathcal{H}$

$$\mathcal{A} g(x) = p(x)(J \star_b g)(x). \tag{3.1}$$

We denote by $\mathcal{B}$ the operator acting on $L^2(\mathbb{R}, \frac{1}{p(x)} dx)$:

$$\mathcal{B} g(x) = p(x)(J \star g)(x). \tag{3.2}$$

The operator $\mathcal{B}$ has been studied in [7] and we will use that, recall (1.6),

$$\bar{m}'(x) = (\mathcal{B} \bar{m}')(x). \tag{3.3}$$

We have the following result.

Theorem 3.1. *Take $L \geq 1$. The operator $\mathcal{A}$ is a compact, selfadjoint operator on $\mathcal{H}$, positivity improving. Further, there exist $\nu_0 > 0$ and $v_0 \in \mathcal{H}$, v_0 strictly positive even function, so that*

$$\mathcal{A} v_0(x) = \nu_0 v_0(x) \qquad x \in T_L. \tag{3.4}$$

The eigenvalue ν_0 has multiplicity one and any other point of the spectrum is strictly inside the ball of radius ν_0. The eigenfunction v_0 is in $C^\infty(T_L)$.

Proof. It is immediate to see that

$$\langle \mathcal{A} v, w \rangle = \langle v, \mathcal{A} w \rangle.$$

The compactness can be shown by proving that any bounded set of $\mathcal{H}$ is mapped by $\mathcal{A}$ in a relatively compact set. Namely since $J(\cdot, \cdot)$ is continuous in $T_L \times T_L$ and T_L is compact, then $J(\cdot, \cdot)$ is uniformly continuous. Thus given $\epsilon > 0$, we can find $\delta > 0$ so that $|x - y| \leq \delta$ implies $|J(x, z) - J(y, z)| \leq \epsilon$ for all $z \in T_L$. The same holds for $p(\cdot)J(\cdot, z)$. Let $B_M = \{v \in \mathcal{H} : \|v\|^2 \le M\}$. If $v \in B_M$ and $|x-y| \le \delta$ we have

$$|(\mathcal{A}v)(x) - (\mathcal{A}v)(y)| \le \epsilon c(\beta, J)\|v\| \le \epsilon c(\beta, J)M,$$

where $c(\beta, J) > 0$ depends only on β and J. Therefore the functions $\mathcal{A}[B_M] = \{w \in \mathcal{H} : w = \mathcal{A}v,\, v \in B_M\}$ are equicontinuous. Since they are also uniformly bounded by $c(\beta)\|J\|_2 M$, where $c(\beta) > 0$, we can use the Ascoli theorem to conclude that for every sequence $\{v_n\} \in B_M$, the sequence $\{\mathcal{A}v_n\}$ has a convergent subsequence (the limit might not be in $\mathcal{A}[B_M]$) in $C[T_L]$ and therefore in $\mathcal{H}$. To show the positivity improving we must show that there exists $n_L \in \mathbb{N}$ such that, for all $v(z) \ge 0$, $v \not\equiv 0$, we have $(\mathcal{A}^{n_L}v)(z) > 0$. To see it, note that $\mathcal{A}^n v(x) = \int_{T_L} K_n(x,y)v(y)\mathrm{d}y$ where

$$K_n(x,y) = p(x)\int \mathrm{d}x_1\mathrm{d}x_2....\mathrm{d}x_n p(x_1)J(x,x_1)p(x_2)J(x_1,x_2)...p(x_n)J(x_n,y).$$

On the other hand, by assumption, there exists $\omega > 0$ such that $\inf_{|x-y|\le 1/2} J(x,y) \ge \omega$. Next, we prove by induction that there exists $\zeta_* > 0$ such that

$$\inf_{|y-x|\le n/4} K(x,y) \ge \zeta_*^n.$$

This is obviously true for $n = 1$ and $\zeta_* \le \omega(1-m_\beta^2)$. Let us assume it true for n, then

$$K_{n+1}(x,y) = p(x)\int_{T_L} J(x,x_1)K(x_1,y) \ge \beta(1-m_\beta^2)\zeta_*^n \int_{T_L\cap\{|x_1-y|\le n/4\}} J(x,x_1).$$

If $|x-y| \le n/4$ then

$$K_{n+1}(x,y) \ge \frac{1}{2}\beta(1-m_\beta^2)\zeta_*^n$$

while, if $|x-y| \in [n/4, (n+1)/4]$, then

$$K_{n+1}(x,y) \ge \beta(1-m_\beta^2)\zeta_*^n \int_{T_L\cap\{|x_1-y|\in[(n-1)/4,n/4]\}} J(x,x_1) \ge \frac{\omega}{4}\beta(1-m_\beta^2)\zeta_*^n,$$

from which the statement follows with $\zeta_* = \frac{\omega}{4}\beta(1-m_\beta^2)$. Accordingly, setting $K(x,y) = K_{8L}(x,y)$ and $\zeta = \zeta_*^{8L}$ we have

$$K(x,y) > \zeta \qquad \text{for all } x, y \in T_L. \tag{3.5}$$

Thus the operator $\mathcal{A}$ is positivity improving and moreover one can apply the classical Perron Frobenius Theorem to the kernel $K(\cdot,\cdot)$. As a consequence we have that $\mathcal{A}^{8L}$ has a simple strictly positive maximal eigenvalue and a spectral gap. Accordingly, the maximum eigenvalue of the spectrum of $\mathcal{A}$, which we denote ν_0, has multiplicity one and any other point of the spectrum of $\mathcal{A}$ is strictly smaller than ν_0. Further the eigenfunction associated to ν_0 does not change sign. So we assume that it is strictly positive and we denote it v_0. Next we show that v_0 is even. Denote by $w(x) = v_0(-x)$. Since J and $p(\cdot)$ are even functions we have that

$$(\mathcal{A}w)(x) = (\mathcal{A}v_0)(-x) = \nu_0 v_0(-x) = \nu_0 w(x).$$

We then deduce that the function w is an eigenfunction associated to ν_0. Since ν_0 has multiplicity one we must have that $w(x) = v_0(x)$. Therefore v_0 is even. Next we show that $v_0 \in C^\infty(T_L)$. We start proving that it is $C^1(T_L)$. Since $p(\cdot)$ is $C^\infty(\mathbb{R})$ and $J \in C^\infty(\mathbb{R})$ differentiating we obtain

$$\nu_0 v_0'(z) = p'(z)(J \star_b v_0)(z) + p(z)(J \star_b v_0)'(z). \tag{3.6}$$

Therefore $v_0 \in C^1(T_L)$. Since $(J \star_b v_0)'(z) = (J' \star_b v_0)(z)$, differentiating again (3.6) we can show that $v_0 \in C^2(T_L)$. Repeating the argument yields $v_0 \in C^\infty(T_L)$. □

Lemma 3.2. (Lower bound on ν_0) *There exists positive constant $c > 0$ independent on L so that for any $L \geq 1$*

$$\nu_0 \geq 1 - ce^{-2\alpha L}, \tag{3.7}$$

where $\alpha > 0$ is the constant in (2.3).

Proof. Consider the following trial function

$$h(x) = \frac{\bar{m}'(x)}{\|\bar{m}'\|}, \qquad -L \leq x \leq L. \tag{3.8}$$

By the variational formula for spectral radius we have that

$$\nu_0 \geq \langle \mathcal{A}h, h\rangle,$$

hence

$$\begin{aligned}\langle \mathcal{A}h, h\rangle &= \int \mathrm{d}x \frac{1}{p(x)} h(x)p(x)(J \star_b h)(x) \\ &= \frac{1}{\|\bar{m}'\|^2} \int \mathrm{d}x \frac{1}{p(x)} \bar{m}'(x)(\mathcal{B}\bar{m}')(x) \\ &\quad + \int \mathrm{d}x \frac{1}{p(x)} h(x)p(x) \left[(J \star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J \star \bar{m}')(x)\right],\end{aligned} \tag{3.9}$$

where $\mathcal{B}$ is the operator defined in (3.2). By (3.3) and (2.6) we have that

$$\frac{1}{\|\bar{m}'\|^2} \int \mathrm{d}x \frac{1}{p(x)} \bar{m}'(x)(\mathcal{B}\bar{m}')(x) = 1.$$

Further, since $\bar{m}'$ is even, we have

$$\begin{aligned}&\int \mathrm{d}x h(x) \left[(J \star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J \star \bar{m}')(x)\right] \\ &= 2\int_{L-1}^{L} \mathrm{d}x h(x) \left[(J \star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J \star \bar{m}')(x)\right].\end{aligned} \tag{3.10}$$

For $x \in [L-1, L]$ we have

$$(J\star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J\star\bar{m}')(x) = \frac{1}{\|\bar{m}'\|}\left[\int_{x-1}^{L} \mathrm{d}y J(x,y)\bar{m}'(y) - \int_{x-1}^{x+1} \mathrm{d}y J(x,y)\bar{m}'(y)\right].$$

Let c be a positive constant independent on L, we obtain

$$\begin{aligned}\int_{x-1}^{L} \mathrm{d}y J(x,y)\bar{m}'(y) - \int_{x-1}^{x+1} \mathrm{d}y J(x,y)\bar{m}'(y) &= -\int_{L}^{x+1} \mathrm{d}y J(x,y)\bar{m}'(y) \\ &\geq -ce^{-\alpha L},\end{aligned} \tag{3.11}$$

since $\bar{m}'(\cdot)$ is strictly positive and exponentially decreasing, see (2.3). Inserting (3.11) in (3.10) we obtain the lower bound (3.7). $\square$

Lemma 3.3. (Upper bound on ν_0) *We have that for any $L \geq 1$*

$$\nu_0 < 1. \tag{3.12}$$

Proof. Multiply (3.4) in $L^2(T_L, \frac{1}{p(x)}dx)$ with the trial function h introduced in (3.8). We have

$$\nu_0\langle v_0, h\rangle = \langle \mathcal{A}v_0, h\rangle = \langle v_0, \mathcal{A}h\rangle.$$

By (3.3) write

$$\mathcal{A}h = \mathcal{B}\frac{\bar{m}'}{\|\bar{m}'\|} + p\left[J \star_b h - J \star h\right] = h + p\left[J \star_b h - J \star h\right].$$

Then

$$\begin{aligned}\langle v_0, p[J\star_b h - J\star h]\rangle &= \int_{-L}^{L} \frac{v_0(x)}{\|\bar m'\|}[(J\star_b \bar m')(x) - (J\star \bar m')(x)]\mathrm{d}x\\ &= \frac{2}{\|\bar m'\|}\int_{L-1}^{L} \mathrm{d}x v_0(x)\,[(J\star_b \bar m')(x) - (J\star \bar m')(x)] \\ &= -\frac{2}{\|\bar m'\|}\int_{L-1}^{L} \mathrm{d}x v_0(x)\int_{L}^{x+1}\mathrm{d}y J(x,y)\bar m'(y) < 0\end{aligned} \tag{3.13}$$

since v_0 is an even positive function, see Theorem 3.1. It follows

$$\nu_0\langle v_0, h\rangle < \langle v_0, h\rangle$$

Since both $\bar m'$ and v_0 are positive we have $\langle v_0, h\rangle > 0$, which proves the Lemma. □

Remark 3.4. *It is easy to verify that if $\mathcal{A}$ is defined replacing $J\star_b$ with the integral kernel $J^{neum}\star$ we will have $\nu_0 \ge 1 + ce^{-2\alpha L}$.*

Next we investigate the properties of the maximal eigenvector v_0.

Lemma 3.5. (Properties of v_0) *For any $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ there exists $r_0 = r_0(\epsilon_0)$ and $L_1 = L_1(\epsilon_0) > 0$ so that the following holds. Take $L \ge L_1$ and let v_0 be the strictly positive normalised eigenfunction of $\mathcal{A}$ on $\mathcal{H}_L$ corresponding to ν_0, see Theorem 3.1. We have that* [2]

$$v_0(x) \le e^{-\lfloor |x| - r_0\rfloor \alpha(\epsilon_0)} \sup_{|y|\in[r_0, r_0+1]} v_0(y) \qquad |x| \ge r_0 + 1 \tag{3.14}$$

where $\alpha(\epsilon_0)$ is given in (3.20)

$$|v_0'(x)| \le C, \qquad x \in T_L, \qquad \text{with } C \text{ independent on } L; \tag{3.15}$$

$$\gamma \ge \frac{v_0(x)}{v_0(y)} \ge \frac{1}{\gamma} \qquad |x-y| \le 1, \quad \text{for any} \quad x, y \in T_L \tag{3.16}$$

where $\gamma = \gamma(\epsilon_0) > 1$ is defined in (3.25). There exists $r_1 > 0$ and $\zeta_1 > 0$ independent of L so that

$$v_0(x) \ge \zeta_1 \qquad \text{for} \quad |x| \le r_1. \tag{3.17}$$

Proof. Fix $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and let $r_0 = r_0(\epsilon_0)$ be so that

$$p(x) < 1 - \epsilon_0, \qquad |x| \ge r_0. \tag{3.18}$$

Take $L_0(\epsilon_0)$ so that for $L \ge L_0(\epsilon_0)$, $r_0 \le \frac{L}{2}$. Furthermore, choose $L_2(\epsilon_0)$ so large so that for $L \ge L_2(\epsilon_0)$, see Lemma 3.2,

$$\nu_0 \ge 1 - ce^{-2\alpha L} > 1 - \frac{\epsilon_0}{2}. \tag{3.19}$$

Set $L_1(\epsilon_0) = \max\{L_0(\epsilon_0), L_2(\epsilon_0)\}$. Then for $L \ge L_1(\epsilon_0)$, and $|x| \ge r_0$,

$$\frac{p(x)}{\epsilon_0} \le e^{-\alpha(\epsilon_0)} < 1,$$

where

$$\alpha(\epsilon_0) = \ln\left(\frac{1 - \frac{\epsilon_0}{2}}{p(r_0)}\right) > 0. \tag{3.20}$$

Then, for each $|x| \ge r_0$,

$$v_0(x) \le e^{-\alpha(\epsilon_0)}\int_{x-1}^{x+1} J(x,y)v_0(y) \le e^{-\alpha(\epsilon_0)} \sup_{y\in[x-1,x+1]} v_0(y).$$

[2] By $\lfloor x\rfloor$ we mean the integer part of x.

Iterating the above, for $|x| \geq r_0 + 1$, yields

$$v_0(x) \leq e^{-\lfloor |x| - r_0 \rfloor \alpha(\epsilon_0)} \sup_{|y| \geq r_0} v_0(y).$$

Note that the sup on the right hand side of the above equation cannot be attained in $(-\infty, -r_0 - 1) \cup [r_0 + 1, \infty)$, otherwise, letting x_* be the place in which the sup is attained, one would get $v_0(x_*) \leq e^{-\alpha(\epsilon_0)} v_0(x_*)$ which is impossible since $v_0 > 0$. Equation (3.14) follows.

Next we show (3.15). By the eigenvalue equation

$$\nu_0 v_0'(x) = p'(x)(J \star_b v_0)(x) + p(x)(J' \star_b v_0)(x). \tag{3.21}$$

Therefore, by (2.3) and (2.5), (3.19), (2.7) and Theorem 3.1 we obtain

$$\begin{aligned} |v_0'(x)| &\leq \frac{1}{\nu_0} \left[2\beta m_\beta \|\bar{m}'\|_\infty \|J \star_b v_0\|_\infty + \|J' \star_b v_0\|_\infty \right] \\ &\leq \frac{1}{1 - \frac{\epsilon_0}{2}} \left[2\beta m_\beta \|\bar{m}'\|_\infty \|J\|_2 \|v_0\|_2 + \|J'\|_2 \|v_0\|_2 \right] \\ &= \frac{\beta^{\frac{1}{2}}}{1 - \frac{\epsilon_0}{2}} \left[2\beta m_\beta \|\bar{m}'\|_\infty \|J\|_2 + \|J'\|_2 \right] \equiv C. \end{aligned} \tag{3.22}$$

Now we show (3.16). Arguing as in (3.5) there exists n, independent on L, such that there exists $\zeta > 0$ so that for any $|x - x'| \leq 3$ in T_L, $(J)^n(x, x') \geq \zeta$. Then, taking into account that $\nu_0 \leq 1$, see Lemma 3.3, since $|x - y| \leq 1$, we have

$$\begin{aligned} x v_0(x) &= \frac{1}{\nu_0} p(x)(J \star_b v_0)(x) \\ &= \frac{1}{\nu_0^n} p(x) \int_{T_L \times_n T_L} \mathrm{d}x_1 \dots \mathrm{d}x_n p(x_1) \dots p(x_{n-1}) J(x, x_1) \dots J(x_{n-1}, x_n) v_0(x_n) \\ &\geq \beta^n (1 - m_\beta^2)^n \zeta \int_{\max\{y-1, -L\}}^{\min\{y+1, L\}} \mathrm{d}x' v_0(x'). \end{aligned} \tag{3.23}$$

On the other hand

$$\begin{aligned} v_0(y) &= \frac{1}{\nu_0} p(y)(J \star_b v_0)(y) \leq \frac{\beta}{\nu_0} \int_{T_L} \mathrm{d}x' J(y, x') v(x') \\ &\leq \frac{\beta}{\nu_0} \|J\|_\infty \int_{\max\{y-1, -L\}}^{\min\{y+1, L\}} \mathrm{d}x' v_0(x'). \end{aligned} \tag{3.24}$$

Therefore, by (3.19), for $L > L_1$, there exists $\gamma = \gamma(\epsilon_0) > 1$ so that

$$\frac{v_0(x)}{v_0(y)} \geq \frac{\beta^n (1 - m_\beta^2)^n \zeta \nu_0}{\beta \|J\|_\infty} \geq \frac{\beta^{n-1} (1 - m_\beta^2)^n \zeta (1 - \frac{\epsilon_0}{2})}{\|J\|_\infty} \equiv \frac{1}{\gamma}. \tag{3.25}$$

The other inequality in (3.16) is equivalent to the one proved. Next, we show (3.17). Since $v_0(x) > 0$ certainly $v_0(x) \geq c_L > 0$ for $x \in T_L$. We would like to show that there exists an interval independent on L so that for x in such an interval , $v_0(x) \geq \zeta > 0$ with $\zeta > 0$ independent on L. This is shown exploiting that v_0 is exponentially decreasing for $|x| \geq r_0$, see (3.14). Since

$$\|v_0\| = 1$$

we must have that there exists $r_1 > 0$, independent on L, so that

$$\int_{-r_1}^{r_1} \frac{1}{p(x)} (v_0(x))^2 \mathrm{d}x \geq \frac{1}{2}. \tag{3.26}$$

Since

$$\int_{-r_1}^{r_1} \frac{1}{p(x)}(v_0(x))^2 \mathrm{d}x \leq \frac{1}{\beta(1-m_\beta^2)} \int_{-r_1}^{r_1} v_0^2(x)\mathrm{d}x \tag{3.27}$$

we obtain, from (3.26) that

$$\int_{-r_1}^{r_1} v_0^2(x)\mathrm{d}x \geq \frac{1}{2}\beta(1-m_\beta^2) \equiv c_1^2. \tag{3.28}$$

Then, there exists $\bar{x} \in [-r_1, r_1]$ so that $v_0^2(\bar{x}) \geq \frac{1}{2r_1}c_1^2$. By (3.15) there exists $\epsilon \in (0,1)$ independent on L so that

$$v_0^2(x) \geq \frac{1}{4r_1}c_1^2, \quad x \in (\bar{x}-\epsilon, \bar{x}+\epsilon).$$

Similarly as proved in (3.16), one can show that given $\bar{r} > r_1$ there is $k_{\bar{r}}$, so that for any $k \geq k_{\bar{r}}$, there exists $\zeta^* > 0$ so that for any $x \in (-\bar{r}, \bar{r})$ and $y \in (-\bar{r}, \bar{r})$

$$J^k(x,y) \geq \zeta^*.$$

Then, take $x \in [-\bar{r}_1, \bar{r}_1]$, we get

$$\begin{aligned}
v_0(x) &= \frac{1}{\nu_0} p(x)(J \star_b v_0)(x) \\
&\geq \frac{1}{\nu_0^k} p(x) \int_{-r_1}^{r_1} \mathrm{d}x_1 \dots \int \mathrm{d}x_k p(x_1)\dots p(x_{k-1}) J(x,x_1)\dots J(x_{k-1},x_k) v_0(x_k) \\
&\geq \left(\frac{\beta(1-m_\beta^2)}{\nu_0}\right)^k \int_{-r_1}^{r_1} \mathrm{d}x_k v_0(x_k) J^k(x,x_k) \\
&\geq \left(\frac{\beta(1-m_\beta^2)}{\nu_0}\right)^k \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} \mathrm{d}x_k v_0(x_k) J^k(x,x_k) \\
&\geq \zeta^* \left(\frac{\beta(1-m_\beta^2)}{\nu_0}\right)^k \epsilon \frac{1}{\sqrt{r_1}} c_1 \equiv \zeta_1.
\end{aligned}$$

□

Theorem 3.1 shows that for any fixed L the operator $\mathcal{A}$ has a spectral gap, which might depend on L. We want to prove that the spectral gap can be lower bounded uniformly with respect to L. We achieve this following closely the paper of Gregory Lawler and Alan Sokal, [13]. We apply a generalisation of the Cheeger's inequality for positive recurrent continuous time jump processes and estimated the Cheeger's constant in our context. Denote by

$$Q(x,y) = \frac{p(x)}{\nu_0} J(x,y) \frac{v_0(y)}{v_0(x)} \qquad x,y \in T_L \tag{3.29}$$

and consider the operator

$$(Qf)(x) = \int_{-L}^{L} Q(x,y) f(y)\mathrm{d}y \tag{3.30}$$

for $f \in L^2(T_L, \pi(x)dx) = L^2(T_L, \pi)$,[3] where

$$\pi(x) = \frac{v_0^2(x)}{p(x)}. \tag{3.31}$$

[3] By a slight abuse of notation we will use π both to designate the measure and its density, as this does not create any ambiguity.

The operator Q is selfadjoint in $L^2(T_L,\pi)$, it is a positivity-preserving linear contraction on $L^2(T_L,\pi)$.[4] The constant function 1 is an eigenfunction of Q with eigenvalue 1. Denote by Ξ the map from $L^2(T_L,\frac{1}{p(x)}dx)$ to $L^2(T_L,\pi(x)dx)$, so that

$$\Xi f=\frac{f}{v_0}$$

The map Ξ is an isometry, $\|f\|_{L^2(T_L,\frac{1}{p(x)}dx)}=\|\Xi f\|_{L^2(T_L,\pi(x)dx)}$, and

$$\nu_0 Qf=\Xi\mathcal{A}\Xi^{-1}f.$$

Therefore the spectrum of $\widehat{\mathcal{A}}=\nu_0^{-1}\mathcal{A}$ is equal to the spectrum of Q.

Denote by $B=I-Q$ where I is the identity operator on $L^2(T_L,\pi)$. We have the following obvious result.

Lemma 3.6. *The spectrum of B is equal to the spectrum of $I-\widehat{\mathcal{A}}$, where I is the identity operator on $L^2(T_L,\frac{1}{p(x)}dx)$.*

Next we show that the spectrum of B restricted to functions orthogonal in $L^2(T_L,\pi)$ to the constant functions, so that $\int f(x)\pi(x)dx=0$, is strictly positive. The gap is bounded by a constant independent on L. To shorten notation we denote for functions f and g in $L^2(T_L,\pi)$

$$\int_{T_L} f(x)g(x)\pi(x)\mathrm{d}x=(f,g). \tag{3.32}$$

Denote by

$$\nu_1=\inf_{\{f:(f,1)=0\}}\frac{(f,Bf)}{(f,f)}. \tag{3.33}$$

We will show that there exists a constant D independent on L so that $\nu_1\geq D>0$. We obtain this by applying [13, Theorem 2.1] and estimating the Cheeger's constant. First notice that the linear bounded operator B defined on $L^2(T_L,\pi)$ can be written as

$$(Bg)(x)\equiv\int_{T_L}Q(x,y)[g(x)-g(y)]\mathrm{d}y, \tag{3.34}$$

i.e as the generator of a continuous time markovian jump process with transition rate kernel $Q(\cdot,\cdot)$ and invariant probability π. Define, see [13], the Cheeger's constant as

$$\boldsymbol{k}\equiv\inf_{A\in\mathcal{S},0<\pi(A)<1}\boldsymbol{k}(A) \tag{3.35}$$

where $\mathcal{S}$ denotes the $\pi-$ measurable sets of T_L and

$$\boldsymbol{k}(A)\equiv\frac{\int\pi(x)\mathrm{d}x1\!\!1_A(x)\left(\int Q(x,y)1\!\!1_{A^c}(y)\mathrm{d}y\right)}{\pi(A)\pi(A^c)}=\frac{(1\!\!1_A,Q1\!\!1_{A^c})}{\pi(A)\pi(A^c)}. \tag{3.36}$$

Taking into account that $(1\!\!1_A,1\!\!1_{A^c})=0$ we can write (3.36) as the following:

$$\boldsymbol{k}(A)=\frac{(1\!\!1_A,Q1\!\!1_{A^c})}{\pi(A)\pi(A^c)}=-\frac{(1\!\!1_A,B1\!\!1_{A^c})}{\pi(A)\pi(A^c)}=\frac{(1\!\!1_A,B1\!\!1_A)}{\pi(A)\pi(A^c)}. \tag{3.37}$$

Since

$$\int\pi(x)Q(x,y)\mathrm{d}x=\frac{1}{\nu_0}v_0(y)(J\star_b v_0)(y)=\pi(y) \tag{3.38}$$

$$\int\pi(x)Q(x,y)\mathrm{d}y=\frac{1}{\nu_0}v_0(x)(J\star_b v_0)(x)=\pi(x), \tag{3.39}$$

the constant M appearing in [13] in our case is simply $M=1$. Next, we recall [13, Theorem 2.1], which in the present context reads:

[4] In fact on all the spaces $L^p(T_L,\pi)$.

Theorem 3.7. *([13]) Let B be the bounded self-adjoint operator on $L^2(T_L, \pi)$ defined in (3.34) whose marginals, in term of the invariant measure, are given in (3.38) and (3.39). Then*

$$\kappa \frac{\boldsymbol{k}^2}{8} \le \nu_1 \le \boldsymbol{k} \tag{3.40}$$

where $\boldsymbol{k}$ is defined in (3.35), (3.36) and κ is a positive constant.

The interesting and deeper part of the previous theorem is the lower bound of ν_1. It states that if there does not exist a set A for which the flow from A to A^c is unduly small then the Markov chain must have rapid convergence to equilibrium, or more precisely that B restricted to function orthogonal to the constant must have spectrum strictly positive.

Theorem 3.8. *For $\beta > 1$ there exists $L_1(\beta)$ so that for $L \ge L_1(\beta)$ the Cheeger's constant associated to the operator B on $L^2(T_L, \pi)$, defined in (3.35) and (3.36) is bounded below by a positive constant D, given in (3.71), depending on β and on the interaction J, but independent on L so that*

$$\boldsymbol{k} \ge D. \tag{3.41}$$

Proof. For $\beta > 1$, fix any $\epsilon_0 = \epsilon_0(\beta)$, $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and take $L_1(\beta)$ so that Lemma 3.5 holds. For any $L \ge L_1(\beta)$ we estimate the Cheeger's constant, see (3.35). For each measurable set $A \subset T_L$, by definition (3.36) we have

$$\begin{aligned} \boldsymbol{k}(A) &\equiv \frac{\int \pi(x) \mathrm{d}x \mathbb{1}_A(x) \left(\int Q(x,y) \mathbb{1}_{A^c}(y) \mathrm{d}y\right)}{\pi(A)\pi(A^c)} \\ &= \frac{1}{\nu_0} \frac{\int \pi(x) \mathrm{d}x \mathbb{1}_A(x) \left(\int p(x) J(x,y) \frac{v_0(y)}{v_0(x)} \mathbb{1}_{A^c}(y) \mathrm{d}y\right)}{\pi(A)\pi(A^c)} \\ &\ge \frac{1}{\gamma} \beta(1-m_\beta^2) \frac{\int \pi(x) \mathrm{d}x \mathbb{1}_A(x) \left(\int J(x,y) \mathbb{1}_{A^c}(y) \mathrm{d}y\right)}{\pi(A)\pi(A^c)}, \end{aligned} \tag{3.42}$$

where $A^c = T_L \setminus A$ and we applied (2.5), (3.16) and Lemma 3.3. Define, for any $x \in \mathbb{R}$,[5]

$$\phi_A(x) = \begin{cases} \frac{\pi(A \cap [x, x+1))}{\pi([x,x+1))} & \text{if } \pi([x, x+1)) \ne 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.43}$$

The function ϕ_A defined in (3.43) is the conditional measure of the set A with respect to the interval $[x, x+1)$. When $x \notin \widetilde{T}_L = (-L-1, L)$ we have $\phi_A(x) = 0$. For the time being we are interested only in the case $\phi_A(0) \le \frac{1}{2}$. If $\phi_A(0) > \frac{1}{2}$, then we will see that the same argument can be applied to the set A^c (see equation (3.68)) for which $\phi_{A^c}(0) \le \frac{1}{2}$ again. Next we define two points x^*, x^{**} that roughly describe the distribution of A in T_L.

When $\phi_A(x) < \frac{1}{2}$ for all $x \ge 0$, we set $x^* = L$ and when and $\phi_A(x) < \frac{1}{2}$ for all $x \le 0$, then we set $x^{**} = -L-1$, otherwise

$$x^* = \min\left\{x \ge 0 : \phi_A(x) = \frac{1}{2}\right\}, \qquad x^{**} = \max\left\{x \le 0 : \phi_A(x) = \frac{1}{2}\right\}. \tag{3.44}$$

Our goal will be to lower bound the numerator of the last term in (3.42) in terms of $\pi(A)$. Upper bounding $\pi(A^c) < 1$ we immediately obtain a lower bound of $\boldsymbol{k}(A)$.

[5] Here we consider π as a measure on $\mathbb{R}$, but supported on T_L.

Let us start the lower bound. Recall that, for the moment, we are assuming $\phi_A(0) \le \frac{1}{2}$. We use the convention that $[x^{**}+1, x^*] = \emptyset$ if $x^{**}+1 > x^*$ and write[6]

$$
\begin{aligned}
&\int \pi(x)\mathrm{d}x \mathbb{1}_A(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&\ge \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[x^{**}+1,x^*]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&+ \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[x^*,x^*+1]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&+ \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[x^{**},x^{**}+1]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right).
\end{aligned} \tag{3.45}
$$

We partition the first integral in the right hand side of (3.45) as follows:[7]

$$
\begin{aligned}
&\int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[x^{**}+1,x^*]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&= \sum_{\lfloor x^{**}\rfloor+1 \le i \le \lfloor x^*\rfloor -1} \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[i,i+1)}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&+ \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[\lfloor x^*\rfloor,x^*]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&+ \int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap[x^{**}+1,\lfloor x^{**}\rfloor+1]}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right).
\end{aligned} \tag{3.46}
$$

Each addend of the right hand side of the previous equation can be estimated as follows:

$$
\begin{aligned}
&\int \pi(x)\mathrm{d}x \mathbb{1}_{A\cap B}(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right) \\
&\ge \pi(A\cap B) \inf_{x\in A\cap B} \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \right).
\end{aligned} \tag{3.47}
$$

To bound below (3.47) note that, for $B \in \{[x^{**}+1, \lfloor x^{**}\rfloor+1], \{[i,i+1)\}, [\lfloor x^*\rfloor, x^*]\}$, we have $\inf_{x\in A\cap B} \int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y \ge C > 0$ independent on L. This can be proved by exploiting that, for $x \in [x^{**}, x^*]$, $\phi_A(x) \le \frac{1}{2}$, which implies, by definition (3.43), that

$$
\frac{\pi(A^c \cap [x,x+1))}{\pi([x,x+1))} \ge \frac{1}{2}, \qquad x \in A\cap B. \tag{3.48}
$$

By the mean value Theorem we have that there exists $\bar{x} \in (x, x+1)$ so that

$$
\int_{[x,x+1)} \pi(z)\mathrm{d}z = \pi(\bar{x}). \tag{3.49}
$$

Hence, from (3.48), if $x \in A\cap B$,

$$
\frac{1}{2} \le \frac{\pi(A^c \cap [x,x+1))}{\pi([x,x+1))} = \int \frac{\pi(z)}{\pi(\bar{x})} \mathbb{1}_{A^c\cap[x,x+1)}(z)\mathrm{d}z. \tag{3.50}
$$

By (2.5) and (3.16), for $|z-\bar{x}| \le 1$, we obtain

$$
\frac{\pi(z)}{\pi(\bar{x})} = \frac{p(\bar{x})v_0^2(z)}{p(z)v_0^2(\bar{x})} \le \frac{\beta}{\beta(1-m_\beta^2)}\gamma^2. \tag{3.51}
$$

---

[6] The idea here is that the part of A that we ignore is dominated by the part that we keep. This is made precise by equation (3.65).

[7] Note that the summation on the right hand side of the equation is empty if $\lfloor x^{**}\rfloor + 1 > \lfloor x^*\rfloor - 1$.

Therefore
$$|A^c \cap [x, x+1)| \geq \frac{1}{2}(1-m_\beta^2)\frac{1}{\gamma^2} \equiv \omega. \tag{3.52}$$
Recalling that J is exactly supported by the unit interval (see the incipit of section 2.1), we have that there exists $J_\omega > 0$ so that
$$\inf_{|x-y|\leq 1-\frac{\omega}{2}} J(x,y) \geq J_\omega. \tag{3.53}$$
Moreover, from (3.52),
$$|A^c \cap [x, x+1-\omega/2]| \geq \frac{\omega}{2}. \tag{3.54}$$
Hence,
$$\begin{aligned}&\inf_{x\in A\cap[i,i+1)} \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right)\\ &\geq \inf_{x\in A\cap[i,i+1)} \left(\int J(x,y)\mathbb{1}_{A^c\cap[x,x+1-\frac{\omega}{2})}(y)\mathrm{d}y\right) \geq \frac{\omega}{2}J_\omega = D_1.\end{aligned} \tag{3.55}$$
Substituting (3.55) in (3.47) we obtain
$$\begin{aligned}\sum_{\lfloor x^{**}\rfloor+1\leq i\leq \lfloor x^*\rfloor -1} &\int \pi(x)\mathrm{d}x\mathbb{1}_{A\cap[i,i+1)}(x)\left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right)\\ &\geq D_1\pi([\lfloor x^{**}\rfloor+1, \lfloor x^*\rfloor]).\end{aligned} \tag{3.56}$$
The last two integrals in the formula (3.46) are estimated similarly. We therefore obtain the wanted lower bound for the first term in the right hand side of (3.45):
$$\int \pi(x)\mathrm{d}x\mathbb{1}_{A\cap[x^{**}+1,x^*]}(x)\left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right) \geq D_1\pi(A\cap[x^{**}+1,x^*]). \tag{3.57}$$
The last two terms in (3.45) are non zero only if $x^* < L$ and $x^{**} > -L-1$, respectively.[8] In such a case they can be estimated taking into account that, by the definition of x^* and x^{**}, $\phi_A(x^*) = \phi_A(x^{**}) = \frac{1}{2}$. This implies that $\pi(A\cap[x^*, x^*+1)) = \frac{1}{2}\pi([x^*, x^*+1))$, hence $\pi(A^c\cap[x^*, x^*+1)) = \frac{1}{2}\pi([x^*, x^*+1))$ and as we did before, we obtain
$$\begin{aligned}&\int \pi(x)\mathrm{d}x\mathbb{1}_{A\cap[x^*,x^*+1]}(x)\left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right)\\ &\geq \int \pi(x)\mathrm{d}x\mathbb{1}_{A\cap[x^*,x^*+1]}(x)\left(\int J(x,y)\mathbb{1}_{A^c\cap[x^*,x^*+1]}(y)\mathrm{d}y\right)\\ &\geq D_1\pi(A\cap[x^*,x^*+1]) = \frac{D_1}{2}\pi([x^*,x^*+1]).\end{aligned} \tag{3.58}$$
The same estimate holds for the last term of (3.45).
$$\int \pi(x)\mathrm{d}x\mathbb{1}_{A\cap[x^{**},x^{**}+1]}(x)\left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right) \geq \frac{D_1}{2}\pi([x^{**},x^{**}+1]). \tag{3.59}$$
Taking into account (3.45), (3.57), (3.58), (3.59) we obtain
$$\begin{aligned}&\int \pi(x)\mathrm{d}x\mathbb{1}_{A}(x)\left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right)\\ &\geq D_1\pi(A\cap[x^{**}+1,x^*]) + \frac{D_1}{2}\pi([x^*,x^*+1]) + \frac{D_1}{2}\pi([x^{**},x^{**}+1])\\ &\geq \frac{D_1}{2}\Big\{\pi(A\cap[x^{**}+1,x^*]) + \pi([x^*,x^*+1]) + \pi([x^{**},x^{**}+1])\Big\}.\end{aligned} \tag{3.60}$$

[8] Note that if $x^* = L$ and $x^{**} = -L-1$, then $\pi(A\cap[x^{**}+1,x^*]) = \pi(A)$ whereby yielding (3.66) without further ado.

To conclude the analysis of this case we must upper bound $\pi(A\cap[x^*,L])$ in terms of $\pi([x^*,x^*+1])$ and $\pi(A\cap[-L,x^{**}+1])$ in terms of $\pi([x^{**},x^{**}+1])$. We have

$$\begin{aligned}\pi(A\cap[x^*,L]) &= \int_{A\cap[x^*,L]}\frac{v_0^2(x)}{p(x)}\mathrm{d}x \le \frac{1}{\beta(1-m_\beta^2)}\int_{x^*}^{L} v_0^2(x)\mathrm{d}x\\ &\le \frac{1}{\beta(1-m_\beta^2)}\sum_{k\ge 0}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x\\ &= \frac{1}{\beta(1-m_\beta^2)}\sum_{\{k\ge 0\,:\,x^*+k\le r_0\}}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x\\ &\quad+\frac{1}{\beta(1-m_\beta^2)}\sum_{\{k\ge 0\,:\,x^*+k> r_0\}}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x.\end{aligned}\tag{3.61}$$

When $x^*+k\le r_0$ we can write

$$\begin{aligned}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x &= \int_{x^*+k-1}^{x^*+k} v_0^2(x+1)\mathrm{d}x = \int_{x^*+k-1}^{x^*+k} v_0^2(x)\frac{v_0^2(x+1)}{v_0^2(x)}\mathrm{d}x\\ &\le \gamma^2\int_{x^*+k-1}^{x^*+k} v_0^2(x) \le (\gamma^2)^k\int_{x^*}^{x^*+1} v_0^2(x)\mathrm{d}x,\end{aligned}\tag{3.62}$$

where $\gamma>1$, see (3.16). When $x^*+k>r_0$, by (3.14) and (3.16), we have

$$\begin{aligned}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x &\le \gamma^2 e^{-2(k-k(r_0))\alpha(\epsilon_0)}\int_{r_0}^{r_0+1} v_0^2(x)\mathrm{d}x\\ &=: \gamma^2 d_1^{2(k-k(r_0))}\int_{r_0}^{r_0+1} v_0^2(x)\mathrm{d}x\end{aligned}\tag{3.63}$$

where $k(r_0)$ is the first integer k so that $x^*+k(r_0)>r_0$, note that $k_0\le\lfloor r_0\rfloor+1$. Remembering (3.62), we obtain

$$\begin{aligned}\int_{x^*+k}^{x^*+k+1} v_0^2(x)\mathrm{d}x &\le \gamma^2(d_1^2)^{k-k(r_0)}\int_{x^*+k(r_0)}^{x^*+k(r_0)+1} v_0^2(x)\mathrm{d}x\\ &\le (\gamma^2)^{k(r_0)+2}(d_1^2)^{k-k(r_0)}\int_{x^*}^{x^*+1} v_0^2(x)\mathrm{d}x.\end{aligned}\tag{3.64}$$

Therefore, from (3.61), (3.62), (3.64) and denoting

$$D_2=\frac{1}{(1-m_\beta^2)}\left\{\sum_{k\le\lfloor r_0\rfloor+1}(\gamma^2)^k+(\gamma^2)^{\lfloor r_0\rfloor+3}\sum_{k\ge 0}(d_1^2)^k\right\},$$

we obtain

$$\pi(A\cap[x^*,L])\le D_2\pi([x^*,x^*+1]).\tag{3.65}$$

Similar computations hold for the interval $[x^{**},x^{**}+1]$ if $x^{**}>-L-1$. We can then continue the estimate left at (3.60) and write

$$\begin{aligned}&\frac{D_1}{2}\{\pi(A\cap[x^{**}+1,x^*])+\pi([x^*,x^*+1])+\pi([x^{**},x^{**}+1])\}\\ &\ge \frac{D_1}{2}\pi(A\cap[x^{**}+1,x^*])+\frac{D_1}{2D_2}\{\pi(A\cap[x^*,L])+\pi(A\cap[-L,x^{**}+1])\}\\ &\ge \min\left\{\frac{D_1}{2},\frac{D_1}{2D_2}\right\}\pi(A).\end{aligned}\tag{3.66}$$

Collecting the above facts and inserting them in (3.42) we obtain that, in the case $\phi_A(0) \le \frac{1}{2}$,

$$\boldsymbol{k}(A) \ge \frac{\beta(1-m_\beta^2)}{\gamma} \min\left\{\frac{D_1}{2}, \frac{D_1}{2D_2}\right\}. \tag{3.67}$$

To conclude we must lower bound the numerator of the last term in (3.42) when $\phi_A(0) \ge \frac{1}{2}$. Exploiting that the interaction J is symmetric and that, for $|x-y| \le 1$, $\frac{\pi(x)}{\pi(y)} \ge \frac{1}{\gamma^2}(1-m_\beta^2)$ we have

$$\begin{aligned} &\int \pi(x)\mathrm{d}x \mathbb{1}_A(x) \left(\int J(x,y)\mathbb{1}_{A^c}(y)\mathrm{d}y\right) \\ &= \int \mathrm{d}y \mathbb{1}_{A^c}(y) \left(\int J(y,x)\mathbb{1}_A(x)\pi(x)\mathrm{d}x\right) \\ &\ge \frac{1}{\gamma^2}(1-m_\beta^2) \int \mathrm{d}y \pi(y)\mathbb{1}_{A^c}(y) \left(\int J(y,x)\mathbb{1}_A(x)\mathrm{d}x\right). \end{aligned} \tag{3.68}$$

We have thus obtained the same expression present in (3.42) only with the roles of the sets A^c and A exchanged. Accordingly, we can proceed exactly as before, since $\phi_{A^c}(0) = 1 - \phi_A(0) \le \frac{1}{2}$, obtaining

$$\int \pi(x)\mathrm{d}x \mathbb{1}_{A^c}(x) \left(\int J(x,y)\mathbb{1}_A(y)\mathrm{d}y\right) \ge \min\left\{\frac{D_1}{2}, \frac{D_1}{2D_2}\right\} \pi(A^c). \tag{3.69}$$

Thus, by (3.42), (3.68) and (3.69) we have, when $\phi_A(0) \ge \frac{1}{2}$,

$$\boldsymbol{k}(A) \ge \frac{(1-m_\beta^2)}{\gamma^2} \frac{\beta(1-m_\beta^2)}{\gamma} \min\left\{\frac{D_1}{2}, \frac{D_1}{2D_2}\right\}. \tag{3.70}$$

Accordingly, denoting

$$D = \min\left\{\frac{\beta(1-m_\beta^2)}{\gamma}, \frac{\beta(1-m_\beta^2)^2}{\gamma^3}\right\} \min\left\{\frac{D_1}{2}, \frac{D_1}{2D_2}\right\} \tag{3.71}$$

and remembering (3.67) and (3.70), we obtain $\boldsymbol{k}(A) \ge D$. The thesis follows. $\square$

Proof of Theorem 2.1 For $\beta > 1$, fix any $\epsilon_0 = \epsilon_0(\beta)$, $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and take $L_1(\beta)$ so that Lemma 3.5 and Theorem 3.8 hold. Recall that $\mathcal{L}^0 = I - \mathcal{A}$, where I is the identity operator and $\mathcal{A}$ is the operator defined in (3.1). By Theorem 3.1 we have immediately that $\mathcal{L}^0$ is a bounded, selfadjoint, quasi compact operator. The smallest eigenvalue of $\mathcal{L}^0$ is $\mu_1^0 = 1 - \nu_0$ where ν_0 is the maximum eigenvalue of $\mathcal{A}$ and $\psi_1^0 = v_0$ is the corresponding eigenfunction. Equation (2.9) follows since Lemma 3.2 and Lemma 3.3 state $1 > \nu_0 \ge 1 - ce^{-2\alpha L}$. Point (2) is a direct consequence of Lemma 3.6, (3.40) and Theorem 3.8. Next we show point (3). Split

$$\frac{\bar{m}'}{\|\bar{m}'\|} = a\psi_1^0 + (\psi_1^0)^{ort}. \tag{3.72}$$

Then

$$a^2 + \|(\psi_1^0)^{ort}\|^2 = 1 \tag{3.73}$$

$$\frac{1}{\|\bar{m}'\|^2}\langle \mathcal{L}^0\bar{m}', \bar{m}'\rangle = a^2\mu_1^0 + \langle \mathcal{L}^0(\psi_1^0)^{ort}, (\psi_1^0)^{ort}\rangle \ge a^2\mu_1^0 + \mu_2^0\|(\psi_1^0)^{ort}\|^2. \tag{3.74}$$

By Lemma 3.2

$$\frac{1}{\|\bar{m}'\|^2}\langle \mathcal{L}^0\bar{m}', \bar{m}'\rangle \le ce^{-2\alpha L}$$

hence from (3.73) and (3.74)

$$ce^{-2\alpha L} \ge (1 - \|(\psi_1^0)^{ort}\|^2)\mu_1^0 + \mu_2^0\|(\psi_1^0)^{ort}\|^2.$$

By (2.9) and (2.10), that there exists a $C > 0$ independent on L so that

$$\|(\psi_1^0)^{ort}\|^2 \le Ce^{-2\alpha L}. \tag{3.75}$$

Then (2.11) follows by (3.72), (3.73) and (3.75). $\square$

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Enza Orlandi, Dipartimento di Matematica e Fisica, Università di Roma Tre, L.go S.Murialdo 1, 00156 Roma, Italy.

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it