FAST-SLOW PARTIALLY HYPERBOLIC SYSTEMS VERSUS FRIEDLIN–WENTZELL RANDOM SYSTEMS.

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Abstract. We consider a simple class of fast-slow partially hyperbolic dynamical systems and show that the (properly rescaled) behaviour of the slow variable is very close to a Friedlin–Wentzell type random system for times that are rather long, but much shorter than the metastability scale. Also, we show the possibility of a "sink" with all the Lyapunov exponents positive, a phenomenon that turns out to be related to the lack of absolutely continuity of the central foliation.

1. Introduction

In [11, 12, 13] the first two authors studied the following class of partially hyperbolic systems of the fast-slow type on $\mathbb{T}^2$

$$F_\varepsilon(x,\theta) = (f(x,\theta), \theta + \varepsilon\omega(x,\theta)) \mod 1, \tag{1.1}$$

with $\varepsilon > 0$, small, $F_\varepsilon \in \mathcal{C}^5(\mathbb{T}^2,\mathbb{T}^2)$, and $\inf_{x,\theta} \partial_x f(x,\theta) \geq \lambda > 1$, $\|\omega\|_{\mathcal{C}^4} = 1$.[1] As usual it is important to specify the type of initial conditions under which we like to study the dynamical systems $(\mathbb{T}^2, F_\varepsilon)$. It is well known that, in order to be able to obtain meaningful results for long times, they must be random. More precisely, if we define $(x_n,\theta_n) = F_\varepsilon^n(x_0,\theta_0)$, then we would like to consider, at least, the initial condition $\theta_0 \in \mathbb{T}^1$ fixed, while $x_0 \in \mathbb{T}^1$ is distributed according to a probability measure with smooth density w.r.t. Lebesgue. Then (x_n,θ_n) can be viewed as a (Markov) random process.

We refer to the introductions of the above mentioned papers for a lengthy discussion of the relevance of such systems, the connection with averaging, homogenisation theory, metastability and statistical mechanics as well as for a discussion of the related literature.

Even though (1.1) is arguably the simplest possible model problem for a fast–slow partially hyperbolic system, its exact properties are not understood in full generality. If we want to develop a general theory for fast-slow partially hyperbolic systems, it is then important to see were do we stand and what are the open problems for the above basic model.

Probably the most striking fact concerning the dynamical systems $(\mathbb{T}^2, F_\varepsilon)$, and more generally fast-slow systems, is that they have many different relevant time scales. More precisely there exists some parameters $\alpha_0, c_0, c_1 > 0$:

- **Initial times:** If $n < c_1 \log \varepsilon^{-1}$, then the time is so short that the statistical property plays no significant role and one can, in principle, compute the

2000 *Mathematics Subject Classification.* 37A25, 37C30, 37D30, 37A50, 60F17.

Key words and phrases. Averaging, metastability, partially hyperbolic, decay of correlations.

This work has been supported by the European Advanced Grant Macroscopic Laws and Dynamical Systems (MALADY) (ERC AdG 246953). D.V. has been partially funded by the Russian Academic Excellence Project '5-100'.

[1] In fact in such papers it was assumed only $F_\varepsilon \in \mathcal{C}^4(\mathbb{T}^2,\mathbb{T}^2)$, here we need a bit more regularity.

trajectory numerically with arbitrary precision starting from a deterministic initial condition.

- **Short times:** If $c_1 \log \varepsilon^{-1} < n < \varepsilon^{-1+\alpha_0}$, then the x variable is, essentially, distributed according to the invariant measure of $f(\cdot, \theta_0)$ while $|\theta_n - \theta_0| \leq \varepsilon^{\alpha_0}$. Thus θ appears to be almost a constant of motion.
- **Long times:** If $\varepsilon^{-1+\alpha_0} < n < \varepsilon^{-1-\alpha_0}$, the evolution of the variable $\theta_{\varepsilon^{-1}t}$ is close (in a precise technical sense) to a random process described by a stochastic differential equation.
- **Very long times:** If $\varepsilon^{-1-\alpha_0} < n < e^{c_0 \varepsilon^{-1}}$, the system may behave like if several invariant SRB measures exist (*metastable state*).
- **Arbitrarily long times:** If $n > e^{c_0 \varepsilon^{-1}}$, finally the system exhibits its true statistical properties (SRB measures, decay of correlations, etc.).

The first regime poses interesting problems in the fields of numerical analysis, but we will not discuss them here. The second regime can be studied by applying standard results on the decay of correlations and we will not discuss it either, as it can be seen as a special case of the third. The third regime is the one on which most of this paper will focus. We will see that a precise understanding of this regime gives relevant informations also for longer times. The only other discussions involving the behaviour of the system for longer times will be our discussion of the central foliation, that obviously contains informations on the infinite time dynamics, although only of a very local nature. As for the last regime, we will only mention briefly the standing open problems.

Let us discuss the last three regimes in a bit more detail.

1.1. Long Times.

In this regime it is useful to rescale time, so we define

$$\theta_\varepsilon(t) = \theta_{\lfloor \varepsilon^{-1}t \rfloor} + (\varepsilon^{-1}t - \lfloor \varepsilon^{-1}t \rfloor)(\theta_{\lfloor \varepsilon^{-1}t \rfloor + 1} - \theta_{\lfloor \varepsilon^{-1}t \rfloor}). \tag{1.2}$$

Also it is convenient to see θ_ε as a random variable in $\mathcal{C}^0(\mathbb{R}_+, \mathbb{T})$. In the paper [12] it is shown that, in any interval $[0, T]$, θ_ε converges weakly as $\varepsilon \to 0$ to the solution of

$$\frac{\mathrm{d}\bar\theta}{\mathrm{d}t} = \bar\omega(\bar\theta) \qquad\qquad \bar\theta(0) = \theta_0, \tag{1.3}$$

where $\bar\omega(\theta) = \mu_\theta(\omega(\cdot, \theta))$, μ_θ being the unique SRB measure of $f(\cdot, \theta)$, see [4]. For future use, let us also define the function $\hat\omega(x, \theta) = \omega(x, \theta) - \bar\omega(\theta)$. Note that, by the differentiability of μ_θ with respect to θ (see [35, Section 8]) we have that $\bar\omega \in \mathcal{C}^{3-\alpha}$, for each $\alpha > 0$. Thus (1.3) is a well defined differential equation. Also [11] contains a results on the fluctuations: that is, if we define $\zeta_\varepsilon(t) = \varepsilon^{-1/2}(\theta_\varepsilon(t) - \bar\theta(t))$, then, in any time interval $[0, T]$, ζ_ε converges to ζ, defined by

$$\begin{aligned} d\zeta &= \bar\omega'(\bar\theta)\zeta dt + \hat{\boldsymbol\sigma}(\bar\theta) dB \\ \zeta(0) &= 0, \end{aligned} \tag{1.4}$$

where B is the standard 1-dimensional Brownian motion and the diffusion coefficient $\hat{\boldsymbol\sigma}$ is given by the Green-Kubo formula

$$\hat{\boldsymbol\sigma}(\theta)^2 = \mu_\theta\left(\hat\omega(\cdot, \theta)\hat\omega(\cdot, \theta)\right) + 2\sum_{m=1}^{\infty} \mu_\theta\left(\hat\omega(f_\theta^m(\cdot), \theta)\hat\omega(\cdot, \theta)\right), \tag{1.5}$$

where we have used the notation $f_\theta(x) = f(x, \theta)$. In addition, $\hat{\boldsymbol\sigma}(\theta)$ is differentiable (see [35, Section 8] again) and it is strictly positive, unless $\hat\omega(\theta, \cdot)$ is a coboundary for f_θ, see [36]. Thus, from now on we will assume:

(A1) for each $\theta \in \mathbb{T}$, the function $\omega(\cdot, \theta)$ is not cohomologous to a constant function with respect to f_θ.

Moreover, in [12] is proven a much sharper result: a local limit theorem with error, see Theorem 2.4 for details.

Here we go one step further and we prove that θ_ε is very close (in a technical sense to be specified later) to the following Friedlin-Wentzell type process for times of order $\varepsilon^{-\alpha}$, for some $\alpha > 0$,

$$\begin{aligned} &\mathrm{d}\boldsymbol{\eta}(t) = \bar\omega(\boldsymbol{\eta}(t))\mathrm{d}t + \sqrt{\varepsilon}\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}(t))\mathrm{d}B \\ &\boldsymbol{\eta}(0) = \theta_0. \end{aligned} \tag{1.6}$$

The above equation has been extensively studied, starting with [19], and is know to exhibit metastable states. Before considering longer times, it is convenient to understand very precisely the behaviour of (1.6) in the present regime. To our surprise, we were not able to locate the needed results in the literature, so we provide them here. This will allow us to obtain a very precise description of our system in this time scale.

Note that such a result can be used to considerably simplify various arguments in [13].

1.2. **Very Long Times.**
Up to now the only condition on $\bar\omega$ was that it is not a coboundary. It turns out that precise results in the very long time scale are known only in certain cases, namely when $\bar\omega$ has zeroes. This is due to the fact that, in such a case, equation (1.3) has, generically, attractive fixed points and then the dynamics tends to be localised. On the contrary, if $\bar\omega$ has no zeroes, then the average dynamics is, essentially, a rotation and its statistical properties need a longer time scale to manifest (see Lemma 4.1). Thus let us assume

(A2) $\bar\omega$ has a non-empty discrete set of non-degenerate zeros.

Note that the above is a generic condition, once the zeroes do exist. The previous mentioned results imply then that the dynamics spends most of the time in a $\sqrt{\varepsilon}$ neighbourhood of the attractive fixed points of (1.3). Indeed, the large deviation results of [12] imply that the probability of escaping one such sink, if possible at all, is exponentially small in ε^{-1}. This means that it will be necessary an exponentially long time for the distributions of the θ variable to change appreciably. That is, there are quasi stationary states (metastability). The occurrence of metastable states for pure deterministic systems was first found, in a different but similar context, by Kifer [29], in which it is shown that the system visits the different metastable states essentially following a Markov chain. However, the results there do not suffice to investigate the true invariant measures of the system.

On the contrary, our estimates show that, if we consider any accumulation point of $\frac{1}{n}\sum_{k=0}^{n-1} F^k_{\varepsilon,*}\nu_0$, where ν_0 is one of our initial probability distribution,[2] then such points must be very close to a convex combinations of the metastable states. As all the possible *physical* measures of the systems must belong to such accumulation set,[3] see [13, Lemma 9.8]; this provides detailed informations on the possible structure of the physical measures. In turn, this also allows to compute the Lyapunov exponents of the system.

Of course, the variable x undergoes uniform expansion, hence one Lyapunov exponent is trivially positive. The other is the Lyapunov exponent in the central direction (the direction associated to the slow variable θ). Indeed, in Section 7 we will see that the maps F_ε, and hence F_0, have an invariant center foliation. Let $(s_*(x,\theta),1)$ be the vectors defining the center distribution of F_0. Such a distribution

[2] F_* stands for the pushforward, namely $F_*\mu(\varphi) = \mu(\varphi\circ F)$.

[3] Recall that μ is a physical measure if, for all continuous g, $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g\circ F^k_\varepsilon(x) = \mu(g)$ for x belonging to a set of positive Lebesgue measure.

is known to be close to the one of F_ε. Let us denote with ψ_* the directional derivative of ω in the center direction, or, more precisely, with respect to the vector $(s_*(x,\theta),1)$, that is:

$$\psi_*(x,\theta) = \partial_x\omega(x,\theta)s_*(x,\theta) + \partial_\theta\omega(x,\theta). \tag{1.7}$$

It is convenient to define also the average $\bar\psi_*(\theta) = \mu_\theta(\psi_*(\cdot,\theta))$. Essentially, $1+\varepsilon\psi_*$ is the one step-contraction (or expansion) in the center direction. Then, for each invariant measure μ, $\mu(\log(1+\psi_*))$ is the central Lyapunov exponent associated to such a measure.

Naively, one could think that such an exponent is always negative, as it is driven by the contraction in the sinks in which the dynamics spends most of the time. Surprisingly, this is not always the case. This was already conjectured in [13] and is proven here. In addition, we show that the presence of a positive Lyapunov exponent in the central foliation is associated to the foliation non being absolutely continuous. Such a pathology of the central foliation was already discovered in other examples, e.g. volume preserving partially hyperbolic maps [39, 37], but here emerges in a totally natural and robust manner for systems whose invariant measure is not previously known and it is not constant.

1.3. **Arbitrarily Long Times.** In this regime, the system exhibits its true statistical properties, the first being the existence or not of physical measures. They are proven to exist in the case of mostly expanding central direction [2], and mostly contracting central direction [17, 10]. For central direction with zero Lyapunov exponents (or close to zero) physical measures are known to exist generically [46]. In the case of mostly expanding and mostly contracting central foliations the above papers contain also some information on uniqueness and mixing of the physical measure, but not of a quantitive nature.

Precise quantitive results beyond what already discussed are known only when the central Lyapunov exponent is negative. More precisely, calling $\{\theta_{k,-}\}_{k=1}^{\boldsymbol{z}}$ the zeroes of $\bar\omega$ such that $\bar\omega'(\theta_{k,-})<0$, the condition

(A2) $\max_{k\in\{1,\cdots,\boldsymbol{z}\}}\bar\psi_*(\theta_{k,-})<0$.

implies that the center Lyapunov exponent is Lebesgue-a.s. negative (see [13] or Section 6). Under such a condition in [13] is obtained a full classification of the SRB measures and sharp estimates of the decay of correlations.

Numerical simulations and the results in [1, 2, 34, 46] suggest that similar type of results should hold in much greater generality (in particular for positive Lyapunov or zero exponents). Yet, to obtain similar results in the case of non negative Lyapunov exponent, or for the case in which $\bar\omega$ has no zeroes, stands as an important open challenge.

Contents

2. Limit theorems: a recap

Standard pairs are a very convenient way to describe initial conditions which can be realised by interacting with a system in the past. The reader can find a basic introduction to standard pairs in [11] and a more precise description of the type of standard pairs needed in the present context in [12, 13]. Here we content ourselves with a super brief introduction, just to establish notations.

Let us fix a small $\delta > 0$, and $D_1, D_1', c_1 > 0$ large enough. Let us define the set of functions

$$\Sigma_{c_1} = \{G \in \mathcal{C}^3([a,b],\mathbb{T}) \,: a, b \in \mathbb{T}, b-a \in [\delta/2, \delta],\\ \|G'\| \leq \varepsilon c_1,\, \|G''\| \leq \varepsilon D_1 c_1,\, \|G'''\| \leq \varepsilon D_1' c_1\}.$$

We associate to any $G \in \Sigma_{c_1}$ the map $\mathbb{G}(x) = (x, G(x))$; the graph of any such G (i.e. the image of $\mathbb{G}$) will be called a *standard curve*.

Next, fix $D_2, c_2 > 0$ large enough. We define the set of c_2-*standard* probability densities on the standard curve G as

$$D_{c_2}(G) = \left\{\rho \in \mathcal{C}^2([a,b],\mathbb{R}_+) \,:\, \int_a^b \rho(x)\mathrm{d}x = 1,\, \left\|\frac{\rho'}{\rho}\right\| \leq c_2,\, \left\|\frac{\rho''}{\rho}\right\| \leq D_2 c_2\right\}.$$

A *standard pair* ℓ is given by $\ell = (\mathbb{G}, \rho)$, where $G \in \Sigma_{c_1}$ and $\rho \in D_{c_2}(G)$. To any standard pair $\ell = (\mathbb{G}, \rho)$ is uniquely associated a probability measure μ_ℓ on $\mathbb{T}^2$ defined as follows: for any Borel-measurable function g on $\mathbb{T}^2$ let

$$\mu_\ell(g) := \int_a^b g(\mathbb{G}(x))\rho(x)\mathrm{d}x.$$

Let L_{c_1,c_2} be the set of standard pairs. For each $\ell \in L_{c_1,c_2}$ we will use a_ℓ, b_ℓ, $\mathbb{G}_\ell$ and ρ_ℓ for the domain, graph and density associated to the standard pair. Moreover, for future reference, given $\ell \in L_{c_1,c_2}$, let us define

$$\theta_\ell^* = \int_{a_\ell}^{b_\ell} G_\ell(x)\rho_\ell(x)dx = \mu_\ell(\theta).$$

A standard family can be conveniently regarded as a *random standard pair*. More precisely: a *standard family* $\mathfrak{L}$ is given by the triplet $(\mathcal{A}, \mathcal{F}, p)$ where $\mathcal{A}$ is a countable set $\mathcal{F}$ is a map $\ell : \mathcal{A} \to L_{c_1,c_2}$ and p is a probability measure on $\mathcal{A}$.

A (c_1, c_2)-standard family $\mathfrak{L}$ identifies a unique probability measure $\mu_{\mathfrak{L}}$ on $\mathbb{T}^2$: for any Borel-measurable function g of $\mathbb{T}^2$, let

$$\mu_{\mathfrak{L}}(g) := \int_{\mathcal{A}} \mu_{\ell(\alpha)}(g)\mathrm{d}p.$$

We will use $\mathbb{L}_{(c_1,c_2)}$ to designate the set of standard families and $\overline{\mathbb{L}}_{(c_1,c_2)}$ to designate the standard measures, that is, the weak closure of the measures associated to a standard family.

The basic properties of standard families rest in the following fact.

Proposition 2.1 ([13, Proposition 5.2]). *There exist c_1, c_2 such that, if ε is sufficiently small and ℓ is a standard pair, $F_{\varepsilon *}\mu_\ell$ can be seen as the measure associated to a standard family.*

Accordingly, if our initial condition is expressed by a standard pair, then the pushforward of the initial measure will always consists of a convex combination of standard pairs. In particular all the physical measures will belong to $\overline{\mathbb{L}}_{(c_1,c_2)}$, [13, Lemma 9.8].

For a given, but arbitrary, smooth function ψ, let us define the function ζ_n as:

$$\zeta_n = \varepsilon \sum_{k=0}^{n-1} \psi \circ F_\varepsilon^k. \tag{2.1}$$

Next, let $z_n = (\theta_n, \zeta_n)$ and define the polygonal interpolation

$$z_\varepsilon(t) = z_{\lfloor t\varepsilon^{-1}\rfloor} + (t\varepsilon^{-1} - \lfloor t\varepsilon^{-1}\rfloor)(z_{\lfloor t\varepsilon^{-1}\rfloor+1} - z_{\lfloor t\varepsilon^{-1}\rfloor}).$$

For any $t \geq 0$ and $\theta_* \in \mathbb{T}^1$, we define the function $\bar{z}(t, \theta_*)$ to be the solution of the ODE

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bar{z}(t) = \left(\bar{\omega}(\bar{\theta}(t)), \bar{\psi}(\bar{\theta}(t))\right), \\ &\bar{z}(0) = (\theta_*, 0) \end{aligned} \tag{2.2}$$

where $\bar{\psi}(\theta) = \mu_\theta(\psi(\cdot,\theta)$, $\bar{\omega} = \mu_\theta(\omega(\cdot,\theta)$ and μ_θ is the unique SRB measure of the map $f_\theta(x) = f(x,\theta)$. We conveniently introduced functions θ_ε, ζ_ε and $\bar{\theta}$, $\bar{\zeta}$ so that $z_\varepsilon = (\theta_\varepsilon, \zeta_\varepsilon)$ and $\bar{z} = (\bar{\theta}, \bar{\zeta})$.

Then (see [12, Theorem 2.1]), as $\varepsilon \to 0$, provided that the initial conditions (x_0, θ_0) are distributed according to standard pairs such that $\theta_\ell^* = \theta_*$, the random variable $z_\varepsilon(t)$ converges in probability to $\bar{z}(t, \theta_*)$. It is then natural to attempt a description of the behavior of deviations from the averaged dynamics. For any $p = (x_0, \theta_0)$, let $\Delta z(t,p) = (\Delta\theta(t,p), \Delta\zeta(t,p)) := z_\varepsilon(t,p) - \bar{z}(t, \theta_0)$. In this respect, if ε is sufficiently small, we can obtain (see [12, Theorem 2.2, Corollaries 2.3-5]) the following

Theorem 2.2 ([12, Proposition 2.3]). *Fix $T > 0$ and define, for $R_\theta, R_\zeta > C_\# \sqrt{\varepsilon}$:*

$$Q(R_\theta, R_\zeta) = \{p \in \mathbb{T}^2 \, : \, \sup_{t\in[0,T]} |\Delta\theta(t,p)| \geq R_\theta \text{ or } \sup_{t\in[0,T]} |\Delta\zeta(t,p)| \geq R_\zeta\}.$$

Then, for any standard pair ℓ, we have $\mu_\ell(Q(R_\theta, R_\zeta)) < \exp(-c_\# \varepsilon^{-1} \min(R_\theta^2, R_\zeta^2))$, where $c_\#$ is a constant which does not depend on ℓ.

We say that a differentiable path h of length T is *admissible* if for any $s \in [0,T]$, $h'(s) \subset \operatorname{int}\Omega(h(s))$, where for any $\theta \in \mathbb{T}$, we define the (non-empty, convex and compact) set

$$\Omega(\theta) = \{\mu(\omega(\cdot,\theta)) \,|\, \mu \text{ is a } f_\theta\text{-invariant probability}\}. \tag{2.3}$$

Theorem 2.3 ([13, Theorem 6.3]). *Let $h \in \mathcal{C}^1([0,T], \mathbb{R})$ be an admissible path joining θ_0 to θ_1; then for any standard pair ℓ which intersects $\{\theta = \theta_0\}$ and ε small enough there exists a set $Q_h \subset \operatorname{supp}\ell$ so that $\mu_\ell(Q_h) > \exp(-c_\#\varepsilon^{-1}T)$ and $F_\varepsilon^{\lfloor T\varepsilon^{-1}\rfloor} Q_h \subset \mathbb{T}^1 \times B(\theta_1, C_\#\varepsilon^{5/12})$.*

In fact we can also obtain a Local Central Limit Theorem:

Theorem 2.4 ([12, Theorem 2.7]). *For any $T > 0$, there exists $\varepsilon_0 > 0$ so that the following holds. For any $\beta > 0$, compact interval $I \subset \mathbb{R}$, $|I| \le 1$, real numbers $\kappa > 0$, $\varepsilon \in (0, \varepsilon_0)$, $t \in [\varepsilon^{1/2000}, T]$, and standard pair $\ell \in L_{c_1,c_2}$, we have:*

$$\frac{\mu_\ell(\Delta\theta(t,\cdot) \in \varepsilon I + \kappa\varepsilon^{1/2})}{\sqrt{\varepsilon}} = \operatorname{Leb} I\left[\frac{e^{-\kappa^2/2\boldsymbol{\sigma}_t^2(\theta_\ell^*)}}{\boldsymbol{\sigma}_t(\theta_\ell^*)\sqrt{2\pi}}\right] + \mathcal{O}(\varepsilon^{1/2-\beta}). \tag{2.4}$$

where the variance $\boldsymbol{\sigma}_t^2(\theta)$ is given by

$$\boldsymbol{\sigma}_t^2(\theta) = \int_0^t e^{2\int_s^t \bar\omega'(\bar\theta(r,\theta))\mathrm{d}r}\,\hat{\boldsymbol{\sigma}}^2(\bar\theta(s,\theta))\mathrm{d}s. \tag{2.5}$$

and $\hat{\boldsymbol{\sigma}}^2 : \mathbb{T} \to \mathbb{R}_+$ is defined in (1.5).

Remark 2.5. *Essentially, the above theorem states that the distribution of $\theta_\varepsilon(t)$ looks like it has a regular density up to scale $\varepsilon^{\frac{3}{2}-\beta}$. To determine the properties of the distribution below such a scale, it requires further investigation (possibly in the spirit of [3]).*

Observe that, $\hat{\boldsymbol{\sigma}}$ defined above is bounded away from 0 by assumption $(A1)$, hence we conclude that

$$C_\# t \le \boldsymbol{\sigma}_t^2 \le C_\# \exp(c_\# t)t \tag{2.6}$$

The above results will be instrumental in the following section.

3. Deterministic versus random

The goal of this section is to show that the results of the previous section can be used to predict very precisely the behavior of the system in the long time regime.

Note that, for a fixed standard pair ℓ, the distribution on the left hand side of (2.4) is exactly the distribution, at time t, of the solution of the SDE

$$\begin{aligned} &\mathrm{d}\Delta(t,\theta_\ell^*) = \bar\omega'(\bar\theta(t,\theta_\ell^*))\Delta(t)\mathrm{d}t + \hat{\boldsymbol{\sigma}}(\bar\theta(t,\theta_\ell^*)\mathrm{d}B(t)\\ &\Delta(0,\theta_\ell^*) = 0, \end{aligned} \tag{3.1}$$

where $B(t)$ is a standard Brownian motion. Indeed the solution of (3.1) is given by

$$\Delta(t,\theta_\ell^*) = \int_0^t e^{\int_s^t \bar\omega'(\bar\theta(\tau,\theta_\ell^*))d\tau}\,\hat{\boldsymbol{\sigma}}(\bar\theta(s,\theta_\ell^*))\mathrm{d}B(s),$$

which is a zero mean Gaussian random variable with variance given by (2.5). We can then define the following process: Fix $T \in \mathbb{R}_+$, then for $t \in [0,T]$ define

$$\boldsymbol{\eta}_0(t,\theta_\ell^*) = \bar\theta(t,\theta_\ell^*) + \sqrt{\varepsilon}\Delta(t,\theta_\ell^*). \tag{3.2}$$

For times $t \in [kT,(k+1)T]$ we define the process as a Markov process, that is

$$\mathbb{E}(f(\boldsymbol{\eta}_0(s+kT,\theta_\ell^*))) = \mathbb{E}(f(\boldsymbol{\eta}_0(s,z)) \mid \boldsymbol{\eta}_0(kT,\theta_\ell^*) = z).$$

In the following we will suppress the dependence on the initial point, if it does not cause any confusion. From Theorem 2.4 it follows:

Lemma 3.1. *For any $\beta > 0$, $\alpha \in (0,\beta)$, $\varepsilon \in (0,\varepsilon_0)$, standard pair ℓ and $t \in [\varepsilon^{1/2000}, \varepsilon^{-\alpha}]$, there exists a coupling $\mathbb{P}_c$ between $\theta_\varepsilon(t)$, under μ_ℓ, and $\boldsymbol{\eta}_0(t,\theta_\ell^*)$:*

$$\mathbb{P}_c(|\theta_\varepsilon(t) - \boldsymbol{\eta}_0(t,\theta_\ell^*)| \ge \varepsilon) = \mathcal{O}(\varepsilon^{1/2-\beta}). \tag{3.3}$$

Proof. For brevity we call a coupling between two random variables X, Y such that $\mathbb{P}(|X-Y| \ge \varepsilon) \le \delta$ an (ε,δ)-coupling.

The proof is by induction, let us prove it first for $t \in [\varepsilon^{1/2000}, T]$. We partition $\mathbb{R}$ in bins I_n of size 1 centered around $b_n = n$. By Theorem 2.4 we have for any $\zeta > 0$,

$$\begin{aligned}\mu_\ell(\theta_\varepsilon(t,\cdot) \in \bar\theta(t,\theta^*_\ell) + \varepsilon I_n) &= \int_{\varepsilon I_n} \frac{e^{-y^2/(2\boldsymbol{\sigma}_t^2\varepsilon)}}{\boldsymbol{\sigma}_t\sqrt{2\pi\varepsilon}} dy + \mathcal{O}(\varepsilon^{1-\zeta/2}) \\ &= \mathbb{P}(\boldsymbol{\eta}_0(t) \in \bar\theta(t,\theta^*_\ell) + \varepsilon I_n) + \mathcal{O}(\varepsilon^{1-\zeta/2}).\end{aligned}$$

We can thus construct a coupling of the part of measure that belongs to the same bins, and if we do it for all $n \le \varepsilon^{-1/2-\zeta/2}$, then we have a mass of order $\mathcal{O}(\varepsilon^{1/2-\zeta})$ that cannot be coupled. On the other hand by Theorem 2.2 the total mass for both process that can belong to intervals with $n \ge \varepsilon^{-1/2-\zeta/2}$ is also smaller than $\varepsilon^{1/2-\zeta}$ (in fact much smaller). We have constructed an $(\varepsilon, C\varepsilon^{1/2-\zeta})$-coupling for some $C > 0$, and the Lemma is thus proven for $t \in [\varepsilon^{1/2000}, T]$.

Next, let us assume that each $t \le kT$ we can construct an $(\varepsilon, 2kC\varepsilon^{1/2-\zeta})$-coupling. In particular, the bound holds for $t \le T_k = kT - \varepsilon^{1/2000}$. Let $\{I_n\}$ be a partition of $\mathbb{T}$ in intervals of size ε. We know that the standard pair ℓ will give rise, at time T_k, to a standard family $\mathfrak{L}_k = (\mathcal{A}_k, \mathcal{F}_k, p_k)$ such that

$$\mu_\ell(\theta_\varepsilon(t,\cdot) \in I_n) = \sum_{\alpha\in\mathcal{A}_k} p_\alpha \mu_{\ell_\alpha}(\theta_\varepsilon(t - T_k,\cdot) \in I_n).$$

Then

$$\mu_\ell(\theta_\varepsilon(t,\cdot) \in I_n) = \sum_j \sum_{\{\alpha\,:\,\theta_{\ell_\alpha}\in I_j\}} p_\alpha \mu_{\ell_\alpha}(\theta_\varepsilon(t - T_k,\cdot) \in I_n).$$

By the inductive hypothesis we can make an $(\varepsilon, 2kC\varepsilon^{1/2-\zeta})$-coupling with $\boldsymbol{\eta}_0(T_k)$. On the other hand it is easy to check that, for each $I \subset \mathbb{R}$ and $\theta \in \mathbb{T}^1$,

$$e^{-C_\#\sqrt{\varepsilon}} \le \frac{\mathbb{P}(\boldsymbol{\eta}_0(t,\theta) \in I)}{\mathbb{P}(\boldsymbol{\eta}_0(t,\theta+\varepsilon) \in I)} \le e^{C_\#\sqrt{\varepsilon}}.$$

Thus, for each $\theta_{\ell_\alpha} \in I_j$ we can $(\varepsilon, C\varepsilon^{1/2-\zeta})$-couple $\theta_\varepsilon(t)$, under μ_{ℓ_α}, with $\boldsymbol{\eta}_0(t,\theta_{\ell_\alpha})$. This clearly, produces an $(\varepsilon, C(2k+1)\varepsilon^{1/2-\zeta})$-coupling up to time $(k+1)T - \varepsilon^{1/2000}$. Another step as before will allow to construct an $(\varepsilon, 2C(k+1)\varepsilon^{1/2-\zeta})$-coupling up to time $(k+1)T$.

We can iterate this procedure up to $k \le \varepsilon^{-\alpha}$ for $\alpha < \beta - \zeta$, and get an $(\varepsilon, \mathcal{O}(\varepsilon^{1/2-\beta}))$-coupling. The Lemma is thus proven, taking ζ small enough. □

In fact, it is possible to couple our process to the following, more interesting, Freidlin–Wentzell type equation (1.6)

$$\begin{aligned}&\mathrm{d}\boldsymbol{\eta}(t) = \bar\omega(\boldsymbol{\eta}(t))\mathrm{d}t + \sqrt{\varepsilon}\hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}(t))\mathrm{d}B \\ &\boldsymbol{\eta}(0) = \theta^*_\ell\end{aligned}$$

and B is the standard Brownian motion.

Since $F_\varepsilon \in \mathcal{C}^5$, we can apply [35, Theorem 8.1] with the Banach spaces $\{\mathcal{C}^i\}_{i=0}^s$, $s = 4$, and obtain that $\bar\omega, \hat{\boldsymbol{\sigma}} \in \mathcal{C}^{4-\alpha}$, for all $\alpha > 0$. Thus [19, Theorem 2.2] implies that $\boldsymbol{\eta}(t) = \boldsymbol{\eta}_0(t) + \varepsilon\boldsymbol{\eta}_2(t) + \varepsilon^{\frac{3}{2}}\boldsymbol{\eta}_r(t)$ where

$$\begin{aligned}&d\boldsymbol{\eta}_2(t) = \bar\omega'(\bar\theta(t))\boldsymbol{\eta}_2(t)dt + \frac{1}{2}\bar\omega''(\bar\theta(t))\Delta(t)^2dt + \hat{\boldsymbol{\sigma}}'(\theta(t))\Delta(t)dB(t) \\ &\boldsymbol{\eta}_2(0) = 0,\end{aligned}$$

and

$$\mathbb{E}(\boldsymbol{\eta}_r(t)^2) \le C_\#.$$

Hence, for each $\beta > 0$,

$$\mathbb{P}(|\boldsymbol{\eta}_r(t)| \ge \varepsilon^{-\beta}) \le C_\#\varepsilon^{2\beta}.$$

Let us denote by d_{TV} the total variation distance.

Lemma 3.2. *For any $\beta > 0$, $\alpha \in (0, \beta)$ and $t \in [0, \varepsilon^{-\alpha}]$ we have*

$$\mathrm{d}_{TV}(\boldsymbol{\eta}(t), \boldsymbol{\eta}_0(t)) = \mathcal{O}(\varepsilon^{1/2-\beta}).$$

Proof. Again we start by considering the time interval $[0, T]$ first. It should be possible to prove the Lemma by using the above estimates for $\boldsymbol{\eta}_2, \boldsymbol{\eta}_r$, yet, we find faster to use the following result from [27]:[4] for each $t \in [0, T]$, let $p_t^\varepsilon(\theta_\ell^*, \bar\theta(t, \theta_\ell^*)+y)$ be the distribution of the random variable $\boldsymbol{\eta}(t)$ determined by (1.6), then

$$\left|p_t^\varepsilon(\theta_\ell^*, \bar\theta(t, \theta_\ell^*) + y) - (2\pi\varepsilon)^{-\frac{1}{2}} e^{-V(t,y)/2\varepsilon} K_0(t, y)\right| \leq C_\# \varepsilon^{\frac{1}{2}} e^{-V(t,y)/2\varepsilon} \tag{3.4}$$

where, setting $D_{y,t} = \{\varphi \in \mathcal{C}^1 \, : \, \varphi(0) = \theta_\ell^*, \varphi(t) = \bar\theta(t, \theta_\ell^*) + y\}$,

$$V(t, y) = \inf_{\varphi \in D_{y,t}} \int_0^t \frac{(\dot\varphi(s) - \bar\omega(\varphi(s))^2}{\hat{\boldsymbol{\sigma}}(\varphi(s))^2} \mathrm{d}s \tag{3.5}$$

and $K_0 \in \mathcal{C}^1$. To compute V note that the minimum is attained on the solution φ of the Euler-Lagrange equations.

To the Lagrangian is associated the Hamiltonian $\mathcal{H}(\varphi, p) = \frac{\hat{\boldsymbol{\sigma}}(\varphi)^2}{4} p^2 + p\bar\omega(\varphi)$ where $p = 2\hat{\boldsymbol{\sigma}}(\varphi)^{-2}(\dot\varphi - \bar\omega(\varphi))$. The Hamiltonian equation of motions read

$$\begin{aligned} \dot\varphi &= \partial_p \mathcal{H} = \frac{\hat{\boldsymbol{\sigma}}(\varphi)^2}{2} p + \bar\omega(\varphi) \\ \dot p &= -\partial_\varphi \mathcal{H} = -\frac{\hat{\boldsymbol{\sigma}}(\varphi)\hat{\boldsymbol{\sigma}}'(\varphi)}{2} p^2 - \bar\omega'(\varphi) p. \end{aligned} \tag{3.6}$$

Note that $(\varphi(s), p(s)) = (\bar\theta(s, \theta_\ell^*), 0)$ is a solution of (3.6) with initial condition θ_ℓ^* and final condition $\bar\theta(t, \theta_\ell^*)$, which corresponds to $y = 0$.

We can linearise the equation of motion around the solution $(\bar\theta(s, \theta_\ell^*), 0)$. Let $(\varphi(t, p_0), p(t, p_0))$ be the solution of (3.6) with initial conditions (θ_ℓ^*, p_0) and set $(\xi(t), \eta(t)) = \partial_{p_0}(\varphi(t, p_0), p(t, p_0)|_{p_0=0}$. Then

$$\begin{aligned} \dot\xi &= \frac{\hat{\boldsymbol{\sigma}}(\theta)^2}{2} \eta + \bar\omega'(\bar\theta)\xi \\ \dot\eta &= -\bar\omega'(\bar\theta)\eta \\ \xi(0) &= 0\,, \quad \eta(0) = 1, \end{aligned}$$

It readily follows

$$\begin{aligned} \eta(s) &= e^{-\int_0^s \bar\omega'(\bar\theta(s_1), \theta_\ell^*) ds_1} \\ \xi(s) &= e^{\int_0^s \bar\omega'(\bar\theta(s_1), \theta_\ell^*) ds_1} \int_0^s e^{-2\int_0^{s_1} \bar\omega'(\bar\theta(s_2), \theta_\ell^*) ds_2} \frac{\hat{\boldsymbol{\sigma}}(\bar\theta(s_1, \theta_\ell^*))^2}{2} ds_1 \\ &= \frac{\boldsymbol{\sigma}_s(\theta_\ell^*)^2}{2} e^{-\int_0^s \bar\omega'(\bar\theta(s_1), \theta_\ell^*) ds_1}. \end{aligned}$$

If we want the solution belonging to $D_{y,t}$, then we have to solve the equation $F(p_0, y) = \varphi(t, p_0) - \bar\theta(t, \theta_\ell^*) - y = 0$. Since $\partial_{p_0} F = \xi(t) \neq 0$, we can apply the implicit function theorem in a neighborhood of $(0, 0)$ and conclude that the minimum in (3.5) is obtained for the solution of (3.6) with initial conditions

$$\varphi(0) = \theta_\ell^* \,; \quad p_0(y) = \xi(t)^{-1} y + \mathcal{O}(y^2).$$

Moreover, since we have

$$V(t, y) = \int_0^t \frac{p(s, p_0(y))^2 \hat{\boldsymbol{\sigma}}(\varphi(s, p_0(y))))^2}{2} \mathrm{d}s,$$

[4] Related results are present in [31], where they are investigated from the point of view of viscosity solutions.

this allows to compute

$$V(t,0)=0\,;\quad \partial_y V(t,0)=0$$
$$\partial_y^2 V(t,0)=\int_0^t \eta(s)^2\xi(t)^{-2}\hat{\boldsymbol{\sigma}}(\bar\theta(s,\theta_\ell^*))^2=2\boldsymbol{\sigma}_t(\theta_\ell^*)^{-2},$$

which yields

$$V(t,y)=\frac{y^2}{\boldsymbol{\sigma}_t^2}+\mathcal{O}(y^3). \tag{3.7}$$

This, together with the large deviation results in [19], shows in particular that

$$\int_{|y|\geq \varepsilon^{1/2-\zeta}} p_t^\varepsilon(\theta_\ell^*,\bar\theta(t,\theta_\ell^*)+y)\leq \varepsilon^{100}.$$

It suffices thus to consider $|y|\leq \varepsilon^{1/2-\zeta}$, for which we have

$$\left|p_t^\varepsilon(\theta_\ell^*,\bar\theta(t,\theta_\ell^*)+y)-(2\pi\varepsilon\boldsymbol{\sigma}_t^2)^{-\frac12}e^{-\frac{y^2}{2\varepsilon\boldsymbol{\sigma}_t^2}}\right|\leq C_\#\varepsilon^{1/2-3\zeta}e^{-\frac{y^2}{2\varepsilon\boldsymbol{\sigma}_t^2}}. \tag{3.8}$$

This proves the statement for $t\leq T$. To prove the result for longer times, we proceed by recurrence, relying on Markov Property: after each time interval of length T there is an additional amount of mass of order $\mathcal{O}(\varepsilon^{1/2-3\zeta})$ that cannot be coupled, which means that $\mathrm{d}_{TV}(\boldsymbol{\eta}(t),\boldsymbol{\eta}_0(t))=\mathcal{O}(k\varepsilon^{1/2-3\zeta})$ for $t\in[kT,(k+1)T]$. We deduce the result, taking $\alpha\leq\beta-3\zeta$ and ζ small.

□

By Lemmata 3.1 and 3.2 immediately follow

Corollary 3.3. *For any $\beta>0$, $\alpha\in(0,\beta)$, $\varepsilon\in(0,\varepsilon_0)$, standard pair ℓ and $t\in[\varepsilon^{1/2000},\varepsilon^{-\alpha}]$, there exists a coupling $\mathbb{P}_c$ between $\theta_\varepsilon(t)$, under μ_ℓ, and $\boldsymbol{\eta}(t)$, such that:*

$$\mathbb{P}_c(|\theta_\varepsilon-\boldsymbol{\eta}(t)|\geq\varepsilon)=\mathcal{O}(\varepsilon^{1/2-\beta}).$$

4. Long times

We have thus seen that our deterministic process remains very close, in a precise technical sense, to the Freidlin-Wentzell process (1.6) for a polynomially long time. To take advantage of this it is necessary to have a good understanding of the statistical properties of (1.6). To this issue is devoted the present section.

First of all, note that the generator associated to the process can be written as

$$\begin{aligned}L_\varepsilon\varphi&=\bar\omega\varphi'+\frac{\varepsilon}{2}\hat{\boldsymbol{\sigma}}^2\varphi''=\frac{\varepsilon}{2\rho_\varepsilon}(\hat{\boldsymbol{\sigma}}^2\rho_\varepsilon\varphi'-Z_\varepsilon v_\varepsilon\varphi)',\\ \rho_\varepsilon(\theta)&=Z_\varepsilon\hat{\boldsymbol{\sigma}}^{-2}e^{-\varepsilon^{-1}\Omega(\theta)}\left[1+v_\varepsilon\int_0^\theta e^{\varepsilon^{-1}\Omega(s)}ds\right],\\ \Omega(\theta)&=-2\int_0^\theta\frac{\bar\omega(s)}{\hat{\boldsymbol{\sigma}}^2(s)}ds\ \text{ for all }\theta\in\mathbb{R},\end{aligned} \tag{4.1}$$

where $v_\varepsilon\in\mathbb{R}$ is determined by the relation $\rho_\varepsilon(0)=\lim_{\theta\to1}\rho_\varepsilon(\theta)$, which insures that ρ_ε is a smooth periodic function and hence the measure factors properly on $\mathbb{T}$, while the normalisation constant Z_ε is determined by $\int_{\mathbb{T}}\rho_\varepsilon=1$. Accordingly,

$$v_\varepsilon=\frac{e^{\varepsilon^{-1}\Omega(1)}-1}{\int_0^1 e^{\varepsilon^{-1}\Omega(s)}ds}, \tag{4.2}$$

which shows that $v_\varepsilon = 0$ if and only if $\int_{\mathbb{T}} \frac{\bar\omega}{\hat{\boldsymbol{\sigma}}^2} = 0$. Also, note that,

$$1 + v_\varepsilon \int_0^\theta e^{\varepsilon^{-1}\Omega(s)} ds = \frac{\int_0^1 e^{\varepsilon^{-1}\Omega(s)} ds + \left[e^{\varepsilon^{-1}\Omega(1)} - 1\right] \int_0^\theta e^{\varepsilon^{-1}\Omega(s)} ds}{\int_0^1 e^{\varepsilon^{-1}\Omega(s)} ds}$$
$$= \frac{\int_\theta^{\theta+1} e^{\varepsilon^{-1}\Omega(s)} ds}{\int_0^1 e^{\varepsilon^{-1}\Omega(s)} ds},$$

and thus ρ_ε can also be written as

$$\rho_\varepsilon(\theta) = \frac{\int_\theta^{\theta+1} \hat{\boldsymbol{\sigma}}^{-2}(\theta) e^{-\varepsilon^{-1}(\Omega(\theta)-\Omega(s))} ds}{\int_0^1 \int_\theta^{\theta+1} \hat{\boldsymbol{\sigma}}^{-2}(\theta) e^{-\varepsilon^{-1}(\Omega(\theta)-\Omega(s))} ds\, d\theta}$$
$$(4.3) \qquad = \tilde Z_\varepsilon \int_\theta^{\theta+1} \hat{\boldsymbol{\sigma}}^{-2}(\theta) e^{-\varepsilon^{-1}(\Omega(\theta)-\Omega(s))} ds.$$

One can easily check that $L'_\varepsilon \rho_\varepsilon = 0$. That is, ρ_ε is the density of the invariant measure ν_ε of (1.6). In addition, if $v_\varepsilon = 0$, then the process is reversible.

Let us first consider the case when $\bar\omega$ has no zero. We suppose $\bar\omega > 0$ (the negative case is obtained by symmetry). Then for a constant c big enough, since $\Omega(\theta) - \Omega(s) \geq c_\#(s-\theta)$ for $s > \theta$,

$$\int_\theta^{\theta+1} \hat{\boldsymbol{\sigma}}^{-2}(\theta) e^{-\varepsilon^{-1}(\Omega(\theta)-\Omega(s))} ds = \int_\theta^{\theta + c\varepsilon|\log\varepsilon|} \hat{\boldsymbol{\sigma}}(\theta)^{-2} e^{-\varepsilon^{-1}(\Omega(\theta)-\Omega(s))} ds + O(\varepsilon^2)$$
$$= \hat{\boldsymbol{\sigma}}(\theta)^{-2} \int_0^{c\varepsilon|\log\varepsilon|} e^{\varepsilon^{-1}\Omega'(\theta)s} ds + O(\varepsilon^2|\log\varepsilon|^2)$$
$$= -\varepsilon\hat{\boldsymbol{\sigma}}(\theta)^{-2}(\Omega')(\theta)^{-1} + O(\varepsilon^2|\log\varepsilon|^2)$$
$$= \frac{\varepsilon}{2}\bar\omega(\theta)^{-1} + O(\varepsilon^2|\log\varepsilon|^2).$$

We deduce, after normalization, that $\rho_\varepsilon(\theta) = Z\bar\omega(\theta)^{-1} + O(\varepsilon|\log\varepsilon|^2)$, for some $Z \in \mathbb{R}_+$. We have the following result of convergence toward ν_ε.

Lemma 4.1. *Suppose that $\bar\omega$ has no zeros. There exists $c_\#$ and $C_\#$ such that for all $\theta \in \mathbb{T}$,*

$$\mathrm{d}_{TV}(p_t^\varepsilon(\theta,\cdot), \nu_\varepsilon(\cdot)) \leq C_\# e^{-c_\#\varepsilon t}.$$

Proof. By standard coupling arguments, it is sufficient to prove that two solutions of (1.6) with respect to two independent Brownian motions and starting from x and x' meet before a time of order ε^{-1} with a probability bounded from zero independently from ε and uniformly in x and x'. We follow here arguments developed in [20] in a more general context. We denote by h the isochron map associated to the the periodic solution $\bar\theta(t,\theta_0)$ for a $\theta_0 \in \mathbb{T}^1$, that is, denote T its period, the mapping from $\mathbb{T}^1$ to $\mathbb{R}/T\mathbb{Z}$ satisfying $h(\theta_0) = 0$ and $h'(\theta) = \bar\omega^{-1}(\theta)$. We have in particular $h(\bar\theta(t,\theta_0)) = t \bmod T$.

We study then the process $(\Psi_t)_{t\geq 0}$ defined as the lift of $h(\boldsymbol{\eta}_t)$ (i.e. the unique $\mathbb{R}$-valued trajectory satisfying $\Psi_0 \in [0,T)$ and $h(\boldsymbol{\eta}_t) = \Psi_t \bmod T$ for all $t \geq 0$). Ψ_t satisfies

$$(4.4) \qquad \mathrm{d}\Psi_t = \mathrm{d}t - \varepsilon\frac{\bar\omega'\hat{\boldsymbol{\sigma}}^2}{2\bar\omega^2} \circ h^{-1}(\Psi_t)\mathrm{d}t + \sqrt{\varepsilon}\frac{\hat{\boldsymbol{\sigma}}}{\bar\omega} \circ h^{-1}(\Psi_t)\mathrm{d}B_t,$$

where, for $u \in \mathbb{R}$, $h^{-1}(u)$ is to be understood as $h^{-1}(u \bmod T)$. Now if T denotes the period of the deterministic dynamics $\bar\theta$ defined by (1.3), we have with probability converging to 1 when ε goes to 0 that for $\zeta > 0$ small,

$$\max_{n \leq \varepsilon^{-1}} \sup_{t \in [nT,(n+1)T]} |\Psi_t - (\Psi_{nT} + t - nT)| \leq \varepsilon^{\frac12 - \zeta}.$$

Indeed from the Burkholder-Davis-Gundy inequality we get for $m \geq 1$

$$\mathbb{P}\left(\sup_{t\in[nT,(n+1)T]}\left|\int_{nT}^{t}\frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}(\Psi_t)\mathrm{d}B_t\right| \geq \frac{1}{2}\varepsilon^{-\zeta}\right)$$
$$\leq 2^m\varepsilon^{m\zeta}\mathbb{E}\left[\sup_{t\in[nT,(n+1)T]}\left|\int_{nT}^{(n+1)T}\frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}(\Psi_t)\mathrm{d}B_t\right|^m\right] \leq C_m\varepsilon^{m\zeta},$$

and we can simply choose $m\zeta > 1$. So with probability converging to one at each step of size T the third term in (4.4) gives a contribution of order $\frac{1}{2}\varepsilon^{\frac{1}{2}-\zeta}$, and the second term gives an even smaller contribution (of order ε).

Remark that the functions $u \mapsto \frac{\bar{\omega}'\hat{\boldsymbol{\sigma}}^2}{2\bar{\omega}^2}\circ h^{-1}(u)$ and $u \mapsto \frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}(u)$ are T periodic. We denote $v = \int_0^T \frac{\bar{\omega}'\hat{\boldsymbol{\sigma}}^2}{2\bar{\omega}^2}\circ h^{-1}(u)du$ and $\kappa = \int_0^T\left(\frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}\right)^2(u)du$. From the above estimates, we deduce that

$$\Psi_{nT} - nT - \Psi_0 = \varepsilon nv + \varepsilon^{\frac{1}{2}}G_{nT} + \varepsilon\int_0^{nT} b_t^\varepsilon \mathrm{d}t + \sqrt{\varepsilon}\int_0^{nT}\gamma_t^\varepsilon \mathrm{d}B_t, \tag{4.5}$$

where G_{nT} is a random variable of normal distribution, centered and with variance $n\kappa$, and for $kT \leq t \leq (k+1)T$ with $k \geq n-1$,

$$b_t^\varepsilon = \frac{\bar{\omega}'\hat{\boldsymbol{\sigma}}^2}{2\bar{\omega}^2}\circ h^{-1}(\Psi_t) - \frac{\bar{\omega}'\hat{\boldsymbol{\sigma}}^2}{2\bar{\omega}^2}\circ h^{-1}(\Psi_{kT}+t) = \mathcal{O}(\varepsilon^{\frac{1}{2}-\zeta}),$$
$$\gamma_t^\varepsilon = \frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}(\Psi_t) - \frac{\hat{\boldsymbol{\sigma}}}{\bar{\omega}}\circ h^{-1}(\Psi_{kT}+t) = \mathcal{O}(\varepsilon^{\frac{1}{2}-\zeta}).$$

Using similar estimates as above, we can prove that the two last terms of (4.5) are of order $\mathcal{O}(\varepsilon^{\frac{1}{2}-\zeta})$ with probability converging to 1, and thus $\Psi_{\lfloor\varepsilon^{-1}\rfloor T} - \lfloor\varepsilon^{-1}\rfloor T - \Psi_0$ converges in distribution to a Gaussian with mean v and variance κ. This implies indeed that two independent solutions of (1.6) meet before $\varepsilon^{-1}T$ with a positive probability independent from ε.

□

The above result shows that the convergence to equilibrium takes place on a rather long time scale. As we will see in Section 5.1, with the available technology, this allows only a partial understanding of the properties of the physical measures.

Next we consider the case when $\bar{\omega}$ admits $2\boldsymbol{z}$ non-degenerates zeroes, $\boldsymbol{z} > 0$. In such a case the true convergence to equilibrium takes place to an even longer time scale, yet the convergence to a metastable situation takes place much faster.

Let us denote $\theta_{i,-}$ the stable zeroes, for $i = 1, \dots, \boldsymbol{z}$, and $\theta_{i,+}$ the unstable ones. We aim at proving the following Proposition.

Proposition 4.2. *Suppose that the dynamics defined by* (1.3) *admits* $2\boldsymbol{z}$ *non-degenerate fixed points. Then there exists a constant* $c_\#$ *such that for all* x *and all* $\alpha > 0$ *there exists non-negative real numbers* $c_1(x), \dots, c_{\boldsymbol{z}}(x)$ *satisfying* $c_1(x) + \dots + c_{\boldsymbol{z}}(x) = 1$ *such that*

$$\sup_{t\in[c_\#|\log\varepsilon|,\varepsilon^{-\alpha}]} \mathrm{d}_{TV}\left(p_t^\varepsilon(x,\cdot), \sum_{i=1}^{\boldsymbol{z}} c_i(x)\mathcal{G}_i^\varepsilon(\cdot)\right) = \mathcal{O}\left(\varepsilon|\log\varepsilon|^{\frac{3}{2}}\right), \tag{4.6}$$

where $\mathcal{G}_i^\varepsilon$ *is a gaussian distribution with mean* $\theta_{i,-}$ *and variance* $\frac{\hat{\boldsymbol{\sigma}}^2(\theta_{i,-})}{2\bar{\omega}'(\theta_{i,-})\varepsilon}$.

We will not investigate the optimal constant $c_\#$ given in this Proposition, as we are interested in longer times. For results in this direction and related to the cut-off phenomena, see [5].

For $a > 0$ denote $I_i^a = [\theta_{i,-} - a, \theta_{i,-} + a]$. Since $\bar{\omega}$ is smooth, there exists a $a > 0$ such that, on each interval I_i, $\bar{\omega}$ is uniformly convex, with $\bar{\omega}''(\theta) \geq c_\#$ for $\theta \in I_i$.

We denote $I^a = \cup_{i=1}^n I_i^a$, and first give the following Lemma, which is a particular case of the more general result given in [28]. For the reader convenience we provide a short proof of this Lemma for our situation.

Lemma 4.3. *For all $\beta > 0$, there exist constants $c_\#$ and $C_\#$ such that*

$$\sup_{t\in[c_\#|\log\varepsilon|,\varepsilon^{-\alpha}]} \mathbb{P}\left(\boldsymbol{\eta}(t) \notin I^a\right) \le C_\# \varepsilon^\beta. \tag{4.7}$$

Proof. We divide our analysis in three cases, depending on the position of the starting point $\boldsymbol{\eta}(0)$.

Let us denote, for any $c > 0$ and $i \in \{1, \dots, n\}$, $J_i^c = \{x \in \mathbb{T}, |x - \theta_{i,+}| \le c\}$, $J^c = \cup_{i=1,\dots,n} J_i^c$ and $\bar{J}^c$ the complementary of J^c. From large deviations estimates (see [19]), we know that for any $b > 0$, with probability $O(e^{-c_\#\varepsilon^{-1}})$, the process $(\boldsymbol{\eta}(t))_{t\ge0}$ starting from any $x \in \bar{J}^b$ reaches $I^{\frac{a}{2}}$ before a time T_b independent from ε, and then does not leave I^a before $t = \varepsilon^{-\alpha}$.

Suppose now that $(\boldsymbol{\eta}(t))_{t\ge0}$ starts from a point $x \in J_i^b \cap \bar{J}^{\varepsilon^{1/2-\zeta}}$ for some $\zeta > 0$. Then for any $\delta > 0$, the solution $\bar\theta(\cdot, x)$ of (1.3) reaches $\bar{J}^b$ before $T_\varepsilon^\delta = \frac{\frac12-\zeta}{\bar\omega'(\theta_{i,+})-\delta}|\log\varepsilon|$ if b is taken small enough. Now we have

$$\boldsymbol{\eta}(t) - \bar\theta(t,x) = \int_0^t \big(\bar\omega(\boldsymbol{\eta}(s)) - \bar\omega(\theta(s,x))\big)ds + \sqrt{\varepsilon}\int_0^t \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}(s))dB_s,$$

and from Doob inequality and Burholder-Davis-Gundy inequality, denoting $Z_t^\varepsilon = \int_0^t \hat{\boldsymbol{\sigma}}(\boldsymbol{\eta}(s))dB_s$, we get for any $m \ge 1$:

$$\mathbb{P}\left(\sup_{t\in[0,T_\varepsilon^\delta]} |Z_t^\varepsilon| > \varepsilon^{-\xi}\right) \le \varepsilon^{m\xi}\mathbb{E}\left[\sup_{t\in[0,T_\varepsilon^\delta]} \left|Z_{T_\varepsilon^b}^\varepsilon\right|^m\right] \le C_\#\varepsilon^{m\zeta}|\log\varepsilon|.$$

Denote $t_* = \inf\{t \in [0, T_\varepsilon^\delta], |\boldsymbol{\eta}(t) - \bar\theta(t,x)| \ge b/2\}$. On the event $A^\varepsilon = \{\sup_{t\in[0,T_\varepsilon^\delta]} |Z_t^\varepsilon| \le \varepsilon^{-\xi}\}$ and if b is small enough we obtain

$$|\boldsymbol{\eta}(t) - \bar\theta(t,x)| \le \big(\bar\omega'(\theta_{i,+}) + \delta\big)\int_0^t |\boldsymbol{\eta}(s) - \bar\theta(s,x)|ds + \varepsilon^{\frac12-\xi},$$

and thus by Grönwall inequality as long as $t \le t_*$ we have the following upper bound:

$$|\boldsymbol{\eta}(t) - \bar\theta(t,x)| \le \varepsilon^{\frac12-\xi-\frac{\bar\omega'(\theta_{i,+})+\delta}{\bar\omega'(\theta_{i,+})-\delta}\left(\frac12-\zeta\right)}.$$

If we choose ξ and δ sufficiently small with respect to ζ, this right hand side term goes to 0 when ε tends to 0, which means that on the event A^ε (which satisfies $1 - \mathbb{P}(A^\varepsilon) = O(\varepsilon^\beta)$ for any $\beta > 0$), if ε is small enough, $t_* = T_\varepsilon^\delta$ and thus $(\boldsymbol{\eta}(t))_{t\ge0}$ reaches $\bar{J}^{b/2}$ before $t = T_\varepsilon^\delta$.

We suppose now that $(\boldsymbol{\eta}(t))_{t\ge0}$ starts from a point $x \in J_i^{\varepsilon^{1/2-\zeta}}$, and consider the process $(y(t))_{t\ge0}$ starting from x and satisfying

$$\mathrm{d}y(t) = \bar\omega'(\theta_{i,+})\big(y(t) - \theta_{i,+}\big)\mathrm{d}t + \sqrt{\varepsilon}\hat{\boldsymbol{\sigma}}(\theta_{i,+})\mathrm{d}B_t.$$

$y(t) - \theta_{i,+}$ has in fact a Gaussian distribution (projected on the torus) of mean $e^{\bar\omega'(\theta_{i,+})t}(x - \theta_{i,+})$ and variance $\varepsilon\hat{\boldsymbol{\sigma}}^2(\theta_{i,+})\left(\frac{e^{2\bar\omega(\theta_{i,+})t}-1}{2\bar\omega(\theta_{i,+})}\right)$. So considering a time $t_\varepsilon = \frac{\gamma}{\bar\omega'(\theta_{i,+})}|\log\varepsilon|$ for a $\gamma > 0$, we have $\mathbb{P}(y(t_\varepsilon) \in J_i^{2\varepsilon^{1/2-\zeta}}) = O(\varepsilon^{\gamma-\zeta})$. On the other hand, comparing the two processes, we obtain

$$\mathrm{d}\big(\boldsymbol{\eta}(t) - y(t)\big) = \bar\omega'(\theta_{i,+})\big(\boldsymbol{\eta}(t) - y(t)\big)\mathrm{d}t + h_1(\boldsymbol{\eta}(t) - \theta_{i,+})\mathrm{d}t + h_2(\boldsymbol{\eta}(t) - \theta_{i,+})\mathrm{d}B_t,$$

where $h_1(u) = O(u^2)$ and $h_2(u) = O(u)$. So if we denote τ^ε the exit time from $J^{\varepsilon^{1/2-\zeta}}$ for the process $(\boldsymbol{\eta}(t))_{t\geq 0}$, then

$$\mathbb{P}\left(\sup_{t\in[0,t_\varepsilon]}\left|\int_0^{t\wedge\tau^\varepsilon} h_2(\boldsymbol{\eta}(t)-\theta_{i,+})\mathrm{d}B_t\right| \geq \varepsilon^{1/2-2\zeta}\right) \leq C_\# \varepsilon^{m\zeta}|\log\varepsilon|,,$$

which implies that for $t \in [0, t_\varepsilon]$

$$|\boldsymbol{\eta}(t\wedge\tau^\varepsilon) - y(t\wedge\tau^\varepsilon)| \leq C_\#\varepsilon^{1-2\zeta-\gamma}.$$

This proves, for ζ and γ small enough, that $(\boldsymbol{\eta}(t))_{t\geq 0}$ reaches $\bar{J}^{\varepsilon^{1/2-\zeta}}$ before t_ε with a probability $1 - O(\varepsilon^{\gamma-\zeta})$.

We have thus proved, considering these different cases, that after a time of order $c|\log\varepsilon|$ the process $(\boldsymbol{\eta}(t))_{t\geq 0}$ is trapped in I^a until $t = \varepsilon^{-\alpha}$ with a probability of order $1 - O(\varepsilon^\iota)$ for some $\iota > 0$. Proceeding by recurrence, considering the trajectories that are not trapped yet, we prove that the process is in I^a with probability $1 - O(\varepsilon^\beta)$ for any $\beta > 0$, taking the constant c large enough.

□

Lemma 4.3 shows that after a time of order $|\log\varepsilon|$ (that allows the process to escape from the neighbourhoods of the unstable fixed points), the process stays with high probability in one of the intervals I_i. We can thus, when $(\boldsymbol{\eta}(t))_{t\geq 0}$ is trapped in I_i, couple $(\boldsymbol{\eta}_t - \theta_{i,-})_{t\geq 0}$ with the process $(x(t))_{t\geq 0}$ defined on the real line by the equation

$$\mathrm{d}x(t) = -f_i'(x(t))\mathrm{d}t + \sqrt{\varepsilon}g_i(x(t))\mathrm{d}B_t, \tag{4.8}$$

with f_i being smooth, satisfying $f_i(0) = 0$, $c_\# \leq f_i'' \leq C_\#$ and such that $-f_i'(x) = \bar{\omega}(\theta_{i,-} + x)$ for $|x| \leq a$, and g_i being smooth, $c_\# \leq g_i \leq C_\#$ and such that $g_i(x) = \hat{\boldsymbol{\sigma}}(\theta_{i,-} + x)$ for $|x| \leq a$.

Remark that the process $(x(t))_{t\geq 0}$ admits the invariant measure π_i^ε with density $h_i^\varepsilon(x) = Z_i^\varepsilon g_i^{-2}(x)e^{-\varepsilon^{-1}W_i(x)}$, where $W_i(x) = 2\int_0^x \frac{f_i'(s)}{g_i^2(s)}ds$ and Z_i^ε is a normalization constant. Then, for any β, we get for c large enough,

$$\begin{aligned}\int_{\mathbb{R}}(Z_i^\varepsilon)^{-1}h_i^\varepsilon(x)dx &= \int_{-c|\log\varepsilon|^{\frac{1}{2}}}^{c|\log\varepsilon|^{\frac{1}{2}}}(Z_i^\varepsilon)^{-1}h_i^\varepsilon(x)dx + O(\varepsilon^\beta)\\ &= \int_{-c|\log\varepsilon|^{\frac{1}{2}}}^{c|\log\varepsilon|^{\frac{1}{2}}} g^{-2}(0)e^{-\varepsilon^{-1}W_i''(0)\frac{x^2}{2}}(1+O(\varepsilon^{\frac{1}{2}}|\log\varepsilon|^{\frac{3}{2}})) + O(\varepsilon^\beta),\end{aligned}$$

so $Z_i^\varepsilon = \left(\frac{W_i''(0)}{2\pi\varepsilon}\right)^{-\frac{1}{2}} + O(\varepsilon|\log\varepsilon|^{\frac{3}{2}})$, and a similar calculation shows that $\mathrm{d}_{TV}(\pi_i^\varepsilon, \mathcal{G}_i^\varepsilon) = O(\varepsilon|\log\varepsilon|^{\frac{3}{2}})$.

To conclude the proof of Proposition 4.2, it suffices now to prove that x_t converges in distribution to π_i^ε fast enough, as stated in the following Lemma. We denote by P_t^i the transition probabilities associated to the process $(x(t))_{t\geq 0}$.

Lemma 4.4. *There exist positive constants $c_\#$ and $C_\#$ such that for all $x \in [-a, a]$,*

$$\mathrm{d}_{TV}(P_t^i(x,\cdot), \pi_i^\varepsilon(\cdot)) \leq \frac{C_\#}{\varepsilon}e^{-c_\# t}; \tag{4.9}$$

To prove this Lemma, we rely on the Harris recurrence type result of M. Hairer and J. Mattingly [21], that we recall in the following Theorem.

Theorem 4.5 (Hairer, Mattingly)**.** *Consider a Markov kernel $\mathcal{P}$ satisfying $\mathcal{P}V(x) \leq \gamma V(x) + K$ where V is a Lyapunov function, and $K < 0$, $\gamma \in (0, 1)$ are constants* *such that* $\inf_{x\in\mathcal{C}} \mathcal{P}(x,\cdot) \geq \zeta\nu(\cdot)$ *for a constant* $\zeta \in (0,1)$*, a probability measure* ν*, and the set* $\mathcal{C} = \{V(x) \leq R\}$ *with* $R > \frac{2K}{1-\gamma}$*. Then we have*

$$\rho_\beta(\mathcal{P}\mu_1, \mathcal{P}\mu_2) \leq \bar{\zeta}\rho_\beta(\mu_1,\mu_2)\,, \tag{4.10}$$

where ρ_β *is the weighted variational distance*

$$\rho_\beta(\mu_1,\mu_2) = \int_X (1+\beta V(x))|\mu_1-\mu_2|(dx)\,, \tag{4.11}$$

and one can choose for any $\zeta_0 \in (0,\zeta)$ *and* $\gamma_0 \in (\gamma + \frac{2K}{R}, 1)$*,*

$$\beta = \frac{\zeta_0}{K}\,, \tag{4.12}$$

and

$$\bar{\zeta} = (1-(\zeta-\zeta_0)) \vee \frac{2+R\beta\gamma_0}{2+R\beta}\,. \tag{4.13}$$

Proof of Lemma 4.4. Denoting L_i the diffusion operator associated to (4.8), we get

$$L_i f_i = -(f_i')^2 + \frac{\varepsilon g_i^2}{2} f_i''. \tag{4.14}$$

Since $f_i'' \geq c_\#$, then $(f_i')^2 \geq 2c_\# f$, which leads to, recalling that both f_i'' and g_i are bounded,

$$L_i f_i \leq -c_\# f_i + C_\#\varepsilon. \tag{4.15}$$

We deduce the following inequality for the kernel:

$$P_t^i f_i \leq e^{-c_\# t} f_i + C_\#\varepsilon\,. \tag{4.16}$$

We can then consider the dynamics at integer times and we denote $\mathcal{P} = P_1^i$. It follows

$$\mathcal{P} f_i \leq \gamma f_i + K\varepsilon\,, \tag{4.17}$$

where $0 < \gamma < 1$, and γ and K do not depend on ε. Moreover, using the estimate given in [27] as in (3.4), this time for the process $(x_t)_{t\geq 0}$, and (3.8), we can find for any $b > 0$ a constant $\zeta > 0$ that does not depend on ε and a probability measure μ^ε such that

$$\inf_{|x|\leq b\sqrt{\varepsilon}} \mathcal{P}(x,\cdot) \geq \zeta\mu^\varepsilon. \tag{4.18}$$

Since the set $\mathcal{C} = \{x \,:\, -f_i'(x) \leq R\}$, where R satisfies $R > \frac{2K\varepsilon}{1-\gamma}$, is included in $\{x \,:\, |x| \leq b\sqrt{\varepsilon}\}$ when b is large enough, our system satisfies the conditions to apply Theorem 4.5: we can find constants $\bar{\zeta}$ and β that do not depend on ε such that

$$\rho_{\beta/\varepsilon}(P_n^i(x,\cdot), \pi_i^\varepsilon(\cdot)) \leq \bar{\zeta}^n \rho_{\beta/\varepsilon}(\delta_x, \pi_i^\varepsilon), \tag{4.19}$$

which implies (remark that $\int f_i \mathrm{d}\pi_i^\varepsilon \leq C_\#\sqrt{\varepsilon}$) that for $x \in [-a,a]$,

$$\mathrm{d}_{TV}(P_n^i(x,\cdot), \pi_i^\varepsilon(\cdot)) \leq \frac{C_\#}{\varepsilon}\bar{\zeta}^n\,. \tag{4.20}$$

The result for non-integers times follows directly, since $\mathrm{d}_{TV}(P_{t+\delta}^i(x,\cdot), \pi_i^\varepsilon(\cdot)) \leq \mathrm{d}_{TV}(P_t^i(x,\cdot), \pi_i^\varepsilon(\cdot))$ for $\delta > 0$. $\square$

5. On the structure of the SRB measures

First of all let us recall that the work of Tsujii [40] implies that for partially hyperbolic endomorphisms, generically, the physical measures are finitely many and absolutely continuous with respect to Lebesgue, this applies to maps of the form (1.1). Unfortunately, on the one hand it is not clear how to check if such a property holds for a specific system, on the other hand such result provides little information of how the physical measure looks like. On the contrary, quite a bit of informations can be obtained by the results established in the previous sections.

Note that, for each standard family $\mathfrak{L}$, the set of averages of the pushforward $\frac{1}{n}\sum_{k=0}^{n-1}(F_\varepsilon)_*\mu_{\mathfrak{L}}$ is weakly compact, hence it has accumulation points. Clearly such accumulation points are invariant measures. Let $\mathcal{M}_{\rm sp}(F_\varepsilon)$ be the closure of the set of such accumulation points, thus $\mathcal{M}_{\rm sp}(F_\varepsilon)$ is a subset of the invariant measures of the system. Such a class of measure is often called U-Gibbs. See [16] for a presentation of their properties that far exceeds our present needs.

Recall that any physical measure μ must belong to $\mathcal{M}_{\rm sp}(F_\varepsilon)$ (see [13, Lemma 9.8]). It is then natural to study the set $\mathcal{M}_{\rm sp}(F_\varepsilon)$. This is a set well behaved with respect to the ergodic properties as the next Lemma shows.

Lemma 5.1. *If $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$, then its ergodic decomposition consists of measures that also belong to $\mathcal{M}_{\rm sp}(F_\varepsilon)$.*

Proof. Note that, by definition, $\mathcal{M}_{\rm sp}(F_\varepsilon)$ is a convex compact set. Hence, by Krein-Milman theorem, each measure in $\mathcal{M}_{\rm sp}(F_\varepsilon)$ can be seen as the convex combination of its extremal points. On the other hand, if $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$ and $\tilde\mu$ is invariant and absolutely continuous with respect to μ, then $\tilde\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$. Indeed, let h be the Radon-Nikodym derivative of $\tilde\mu$ with respect to μ, then, for each $\varphi \in \mathcal{C}^0$ and $n \in \mathbb{N}$,

$$\tilde\mu(\varphi) = \tilde\mu(\varphi \circ F_\varepsilon^n) = \mu(h \cdot \varphi \circ F_\varepsilon^n).$$

By standard approximation arguments, for each $\delta > 0$ there exists $h_\delta \in \mathcal{C}^\infty$ such that $\|h - h_\delta\|_{L^1(\mu)} \le \delta$, hence

$$\tilde\mu(\varphi) = \mu(h_\delta \cdot \varphi \circ F_\varepsilon^n) + \mathcal{O}(\delta\|\varphi\|_{\mathcal{C}^0}).$$

In addition, by definition, for each $\boldsymbol{\epsilon} > 0$ there exists a standard family such that

$$\mu(\varphi) = \sum_{\alpha\in\mathcal{A}} p_\alpha \mu_{\ell_\alpha}(\varphi) + \mathcal{O}(\boldsymbol{\epsilon}\|\varphi\|_{\mathcal{C}^0}). \tag{5.1}$$

Accordingly,

$$\tilde\mu(\varphi) = \sum_{\alpha\in\mathcal{A}} p_\alpha \frac{1}{n}\sum_{k=0}^{n-1} \mu_{\ell_\alpha}(h_\delta \cdot \varphi \circ F_\varepsilon^k) + \mathcal{O}((\delta + \boldsymbol{\epsilon}\|h_\delta\|_{\mathcal{C}^0})\|\varphi\|_{\mathcal{C}^0}).$$

In general the measures $\mu_{\ell,\delta}(\varphi) = \mu_\ell(h_\delta\varphi)$ are not standard pairs because the derivative of the density might be too big, however their push-foward for large enough times will eventually be described by standard families [13, Proposition 5.2]. Thus, taking first the limit for $n \to \infty$, then $\boldsymbol{\epsilon} \to 0$ and, finally, $\delta \to 0$, we see that $\tilde\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$, as claimed.

The above imply that if μ is an extremal point of $\mathcal{M}_{\rm sp}(F_\varepsilon)$, then it must be ergodic. If not, then there exists an invariant set $A \subset \mathbb{T}^2$, $\mu(A) \notin \{0, 1\}$. We can then define the probability measures $\mu_1(\varphi) = \mu(A)^{-1}\mu(\mathbb{1}_A\varphi)$ and $\mu_2(\varphi) = [1 - \mu(A)]^{-1}\mu(\mathbb{1}_{A^c}\varphi)$. By the previous arguments $\mu_i \in \mathcal{M}_{\rm sp}(F_\varepsilon)$, but this is impossible since μ, being a convex combination of the μ_i, would not be an extremal point, contrary to the hypothesis. We have thus seen that the extremal points are ergodic, hence they provide the ergodic decomposition of the measures in $\mathcal{M}_{\rm sp}(F_\varepsilon)$, as claimed. □

Let $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$, then, for each $\boldsymbol{\epsilon} > 0$, (5.1) and [12, Proposition 9.7, Lemma 11.3] imply that, denoting $h(\cdot, \theta)$ the density of μ_θ,

$$\begin{aligned}\mu(\varphi) = \mu(\varphi \circ F_\varepsilon^n) = \sum_{\alpha\in\mathcal{A}} p_\alpha \int_{\mathbb{T}} h(x,\theta)\varphi(x,\theta)\\ + \mathcal{O}([\boldsymbol{\epsilon} + \varepsilon n]\,[\|\varphi\|_{\mathcal{C}^0} + \|\partial_\theta\varphi\|_{\mathcal{C}^0}] + e^{-c_\# n}\|\varphi\|_{\mathcal{C}^1}).\end{aligned}$$

Next, for all $I_m = [m\varepsilon, (m+1)\varepsilon)$, define $\mathcal{I}_m = \{\alpha \in \mathcal{A} \;:\; \mu_{\ell_\alpha}(G_{\ell_\alpha}) \in I_m\}$ and consider a measure ν on $\mathbb{T}^1$ such that $\nu(I_m) = \lim_{\boldsymbol{\epsilon}\to 0}\sum_{\alpha\in\mathcal{I}_m} p_\alpha$. We can then write

$$\mu(\varphi) = \int_{\mathbb{T}^2} \varphi(x,\theta)h(x,\theta)dx\,\nu(d\theta) + \mathcal{O}(\varepsilon\log\varepsilon^{-1}\|\varphi\|_{\mathcal{C}^1}), \tag{5.2}$$

where we have chosen $n = C_\#\log\varepsilon^{-1}$ and taken the limit $\boldsymbol{\epsilon} \to 0$. This reduces the problem of understanding the structure of the measure μ to the one of determining the measure ν.

To gain some control on ν we can repeat the same argument, but for the longer time $n = n_1 + n_2 = n_1 + C_\#\log\varepsilon^{-1}$, with $n_1 = \lfloor t\varepsilon^{-1-\gamma}\rfloor$, for some $\gamma \geq 0$ and fixed t. We use again (5.1) and call $\mathfrak{L}_\alpha$ the standard pair that describes the push-forward at time n_1 of the standard pair ℓ_α,

$$\mu(\varphi) = \mu(\varphi \circ F_\varepsilon^n) = \sum_{\alpha\in\mathcal{A}} p_\alpha \sum_{\ell\in\mathfrak{L}_\alpha} p_\ell\mu_\ell(\varphi\circ F_\varepsilon^{n_2}) + \mathcal{O}(\boldsymbol{\epsilon}\|\varphi\|_{\mathcal{C}^0}).$$

By Corollary 3.3 it follows that for $\alpha \in \mathcal{I}_m$, the standard family $\mathfrak{L}_\alpha$ is made of standard pairs distributed as the process $\boldsymbol{\eta}(n)$. More precisely, let $p(\theta, t, \boldsymbol{\eta}(0))$ the probability distribution of such a $\boldsymbol{\eta}(n)$, and note that $|\partial_\theta p| + |\partial_{\boldsymbol{\eta}(0)} p| \leq C_\#\varepsilon^{-\frac{1}{2}}p$, then [12, Lemma 11.3] and Corollary 3.3 imply, for each $\beta > 0$,

$$\mu(\varphi) = \int_{\mathbb{T}^3} p(\theta, t\varepsilon^{-\gamma}, \theta')\varphi(x,\theta)h(x,\theta)dx\,\nu(d\theta')d\theta + \mathcal{O}(\varepsilon^{1/2-\gamma-\beta}\|\varphi\|_{\mathcal{C}^0} + \varepsilon\log\varepsilon^{-1}\|\varphi\|_{\mathcal{C}^1}).$$

This, together with (5.2), yields, for all $\bar\varphi \in \mathcal{C}^1$,

$$\begin{aligned}\int_{\mathbb{T}} \bar\varphi(\theta)\nu(d\theta) = \int_{\mathbb{T}^2} p(\theta, t\varepsilon^{-\gamma}, \theta')\bar\varphi(\theta)\nu(d\theta')d\theta\\ + \mathcal{O}(\varepsilon^{1/2-\gamma-\beta}\|\bar\varphi\|_{\mathcal{C}^0} + \varepsilon\log\varepsilon^{-1}\|\bar\varphi\|_{\mathcal{C}^1}).\end{aligned} \tag{5.3}$$

To use effectively the above equation it is convenient to consider separately the two main cases.

5.1. **Rotations.** We consider first the case in which $\bar\omega$ has no zeroes. In such a case the averaged equation has also the unique invariant measure $\bar\omega(\theta)^{-1}d\theta$. So, if we let T be the period, we can use (5.3), with $t \leq T$ and $\gamma = 0$, as

$$\begin{aligned}\int_{\mathbb{T}} \bar\varphi(\theta)\nu(d\theta) =& \frac{1}{T}\int_0^T \int_{\mathbb{T}^2} p(\theta, t, \theta')\bar\varphi(\theta)\nu(d\theta')d\theta dt\\ &+ \mathcal{O}(\varepsilon^{1/2-\beta}\|\bar\varphi\|_{\mathcal{C}^0} + \varepsilon\log\varepsilon^{-1}\|\bar\varphi\|_{\mathcal{C}^1}).\end{aligned} \tag{5.4}$$

By Lemma 3.2 we have

$$\begin{aligned}\int_{\mathbb{T}^2} p(\theta, t, \theta')\bar\varphi(\theta)\nu(d\theta')d\theta &= \int_{\mathbb{T}^2} \frac{e^{-(\bar\theta(t,\theta')-\theta)^2/(2\boldsymbol{\sigma}_t^2\varepsilon)}}{\boldsymbol{\sigma}_t\sqrt{2\pi\varepsilon}}\bar\varphi(\theta)\nu(d\theta')d\theta + \mathcal{O}(\varepsilon^{\frac{1}{2}-\beta}\|\bar\varphi\|_{\mathcal{C}^0})\\ &= \int_{\mathbb{T}} \bar\varphi(\bar\theta(t,\theta'))\nu(d\theta') + \mathcal{O}(\varepsilon^{\frac{1}{2}-\beta}\|\bar\varphi\|_{\mathcal{C}^1}).\end{aligned}$$

Substituting the above in (5.4) yields

$$\begin{aligned}\int_{\mathbb{T}} \bar{\varphi}(\theta)\nu(d\theta) &= \int_{\mathbb{T}} \frac{1}{T}\int_0^T \bar{\varphi}(\bar{\theta}(t,\theta'))dt\nu(d\theta') + \mathcal{O}(\varepsilon^{\frac{1}{2}-\beta}\|\bar{\varphi}\|_{\mathcal{C}^1})\\ &= \int_{\mathbb{T}} \frac{\bar{\varphi}(\theta)}{\bar{\omega}(\theta)}d\theta + \mathcal{O}(\varepsilon^{\frac{1}{2}-\beta}\|\bar{\varphi}\|_{\mathcal{C}^1}).\end{aligned}$$

Thus all the elements of $\mathcal{M}_{\rm sp}(F_\varepsilon)$, and hence also the eventual physical measures, are very close to the invariant measure of the averaged system. More precisely, we have proven:

Proposition 5.2. *For each $\beta > 0$, if $\boldsymbol{z} = 0$, then for each $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$ and $\varphi \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R})$,*

$$\mu(\varphi) = \int_{\mathbb{T}^2} \varphi(x,\theta)\frac{h(x,\theta)}{\bar{\omega}(\theta)}dx\,d\theta + \mathcal{O}(\varepsilon^{1/2-\beta}\|\varphi\|_{\mathcal{C}^1}).$$

The above result is good enough to compute the leading contribution to the Lyapunov exponent by arguing similarly to what we do in section 6 for the case in which there are sinks. However it does not suffices to investigate the mixing properties of the physical measure. In fact, Lemma 4.1 tells us that in the time $\varepsilon^{-\gamma}$, $\gamma \le 1/2$, the process $\boldsymbol{\eta}$ is still far from equilibrium.

We conjecture that in this case there exists a unique physical measure which mixes with speed ε^2; but to prove such a result following the present strategy it would be necessary to improve the error in [12, Theorem 2.8]. More precisely we would need the to compute explicitly the first term in the Edgeworth expansion.

5.2. **Sinks.** Suppose that $\bar{\omega}$ has $2\boldsymbol{z}$ non degenerate zeroes. In such a case, Proposition 4.2 and equation (5.3), choosing $\gamma \in (0, 1/4)$, imply that there exists positive constants $\{c_i\}_{i=1}^n$ such that

$$\int_{\mathbb{T}} \bar{\varphi}(\theta)\nu(d\theta) = \sum_{i=1}^n c_i \int_{\mathbb{T}} \mathcal{G}_i^\varepsilon(\theta)\bar{\varphi}(\theta)d\theta + \mathcal{O}(\varepsilon^{1/2-2\gamma}\|\bar{\varphi}\|_{\mathcal{C}^0} + \varepsilon\log\varepsilon^{-1}\|\bar{\varphi}\|_{\mathcal{C}^1}).$$

To compute the constant c_i one must look at times longer than the metastability time scale. Indeed, by the large deviations results in [12] one can compute the probability to go from one sink to another. This analysis will show that, generically, there is a c_i that is exponentially larger than the other, hence the invariant measure will look like a gaussian centred on the winning sink. We will not pursue this issue further as it is not needed for our present discussions. For future reference, let us collect the result so far obtained.

Proposition 5.3. *If $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$, and $\boldsymbol{z} > 0$, then there exists $\bar{c} = \{c_k\}_{i=1}^n$, $c_k \ge 0$, $\sum_k c_k = 1$, such that, for each $\varphi \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R})$,*

$$\mu(\varphi) = \sum_{k=1}^n c_k \int_{\mathbb{T}^2} \varphi(x,\theta)h(x,\theta)g_k^\varepsilon(\theta)dx\,d\theta + \mathcal{O}(\varepsilon^{1/2-2\gamma}\|\varphi\|_{\mathcal{C}^0} + \varepsilon\log\varepsilon^{-1}\|\varphi\|_{\mathcal{C}^1}),$$

where g_k^ε is a Gaussian distribution with mean $\theta_{k,-}$ and variance $\frac{\hat{\boldsymbol{\sigma}}^2(\theta_{k,-})}{2\bar{\omega}'(\theta_{k,-})\varepsilon}$.

Remark that the above proposition essentially provides an effective formula for computing the Lyapunov exponent for measures in $\mathcal{M}_{\rm sp}(F_\varepsilon)$, as we will see in the next section.

In [13] it is proven that if the measures in $\mathcal{M}_{\rm sp}(F_\varepsilon)$ have negative Lyapunov exponents, then there exists finitely many physical measures, and a checkable criteria for uniqueness is provided. A similar result for non negative Lyapunov exponents is missing, although, as already mentioned, in [40] it is proven that the physical measures exist generically.

6. Lyapunov exponents

In the previous section we have see that all the invariant measures obtained by the pushforward of a standard pair, must be ε^{β_1} close to each other in the $(\mathcal{C}^1)'$ topology. Hence they may differ substantially only at a scale smaller than ε. It remains open the question if the SRB measure always exists and if it is unique or not. To discuss such issues it seems to be necessary to have some information on the Lyapunov exponent of the central foliation. Recall from [13, Section 3] that the n-step central direction is defined by

$$d_p F_\varepsilon^n(s_n(p), 1) = \mu_n(0, 1)$$

from which it follows

$$\begin{aligned} s_n(p) &= \Xi_p(s_{n-1}(F_\varepsilon(p))) = \frac{[1+\varepsilon\partial_\theta\omega(p)]s_{n-1}(F_\varepsilon(p)) - \partial_\theta f(p)}{\partial_x f(p) - \varepsilon\partial_x\omega(p)s_{n-1}(F_\varepsilon(p))} \\ \mu_n(p) &= \prod_{k=0}^{n} \left[1 + \varepsilon(\partial_x\omega(p_k)s_{n-k}(p_k) + \partial_\theta\omega(p_k))\right], \end{aligned} \tag{6.1}$$

where $p_k = F_\varepsilon^k(p)$. Note that, for ε small enough, there exists $K > 0$ and $\sigma \in (0, 1)$, such that, for each $p \in \mathbb{T}^2$, $\Xi_p([-K, K]) \subset [-K, K]$ and $\sup_{s\in[-K,K]} |\Xi_p'(s)| \le \sigma$. From this it follows that there exists $\hat{s}(p)$ such that

$$|\hat{s}(p) - s_n(p)| \le \sigma^n.$$

The 1-dimensional line field $(1, \hat{s}(p))$ is called the central distribution which we denote $E^c(p)$. It is known to be F_ε-invariant and continuous in p.

Then, setting

$$\hat{\mu}_n(p) = \prod_{k=0}^{n} \left[1 + \varepsilon(\partial_x\omega(p_k)\hat{s}(p_k) + \partial_\theta\omega(p_k))\right],$$

we have

$$|\mu_n(p) - \hat{\mu}_n(p)| \le C_\# \varepsilon \hat{\mu}_n(p). \tag{6.2}$$

We thus have that the central Lyapunov exponent is given by the ergodic average

$$\chi_c(p) = \lim_{n\to\infty} \frac{1}{n} \log \mu_n = \lim_{n\to\infty} \frac{1}{n} \log \hat{\mu}_n.$$

Next, by Lemma 5.1, we can restrict ourselves to considering only $\mu \in \mathcal{M}_{\rm sp}(F_\varepsilon)$ ergodic. Then, the Birkhoff ergodic theorem imply that μ almost surely

$$\chi_c(p) = \mu\left(\log\left[1 + \varepsilon(\partial_x\omega \cdot \hat{s} + \partial_\theta\omega)\right]\right). \tag{6.3}$$

Taking the limit $n \to \infty$ in (6.1) we have

$$\hat{s}(p) = -\sum_{k=0}^{\infty} \frac{\partial_\theta f(F_\varepsilon^k(p))}{\prod_{j=0}^{k} \partial_x f(F_\varepsilon^j(p))} + \mathcal{O}(\varepsilon) = s_0(p) + \mathcal{O}(\varepsilon).$$

Next, we obtain an even more explicit expression. Indeed, by [12, Lemma 4.1], for each $k \le n \le C_\#\sqrt{\varepsilon}$, we can write, for $p = (x_0, \theta_0)$,

$$\begin{aligned} p_k &= (f_{\theta_0}^k(Y_n(x_0)), \theta_0) + \mathcal{O}(\varepsilon k) \\ \|1 - Y_n'\| &\le C_\# \varepsilon n^2. \end{aligned}$$

We can then choose $n = C \log \varepsilon^{-1}$, for C large enough, hence

$$s_0(p) = -\sum_{k=0}^{C\log\varepsilon^{-1}} \frac{\partial_\theta f(f_{\theta_0}^k(Y_n(x_0)), \theta_0)}{(f_{\theta_0}^k)'(Y_n(x_0))} + \mathcal{O}(\varepsilon(\log\varepsilon^{-1})^3)$$

Then, by Proposition 5.3 and (6.3), we have

$$\begin{aligned}\varepsilon^{-1}\chi_c(p) &= \mu\left(\partial_x\omega\cdot s_0+\partial_\theta\omega\right)+\mathcal{O}(\varepsilon)\\ &= \sum_j c_j\mu_{\theta_{j,-}}\left(\partial_x\omega(\cdot,\theta_{j,-})\cdot s_0(\cdot,\theta_{j,-})+\partial_\theta\omega(\cdot,\theta_{j,-})\right)+\mathcal{O}(\sqrt{\varepsilon}),\end{aligned}$$

for some $c_j>0$, $\sum c_j=1$, to be determined. Note that $\mu_{\theta_{j,-}}$ is absolutely continuous with respect to Lebesgue and its density h_j is in $\mathcal{C}^1$. Thus we can write

$$\begin{aligned}&\mu_{\theta_{j,-}}\left(\partial_x\omega(\cdot,\theta_{j,-})\cdot s_0(\cdot,\theta_{j,-})+\partial_\theta\omega(\cdot,\theta_{j,-})\right)\\ &=\int h_j\circ Y_n^{-1}(x)\left[\partial_x\omega(Y_n^{-1}(x),\theta_{j,-})\cdot s_0(Y_n^{-1}(x),\theta_{j,-})+\partial_\theta\omega(Y_n^{-1}(x),\theta_{j,-})\right]\\ &\quad+\mathcal{O}(\varepsilon(\log\varepsilon^{-1})^2)\\ &=\int h_j(x)\left[\partial_x\omega(x,\theta_{j,-})\cdot s_0(Y_n^{-1}(x),\theta_{j,-})+\partial_\theta\omega(x,\theta_{j,-})\right]+\mathcal{O}(\varepsilon(\log\varepsilon^{-1})^2)\\ &=-\sum_{k=0}^{\infty}\int h_j(x)\left[\partial_x\omega(x,\theta_{j,-})\frac{\partial_\theta f(f^k_{\theta_{j,-}}(x),\theta_{j,-})}{(f^k_{\theta_{j,-}})'(x)}\right]+\mu_{\theta_{j,-}}\left(\partial_\theta\omega(\cdot,\theta_{j,-})\right)\\ &\quad+\mathcal{O}(\varepsilon(\log\varepsilon^{-1})^3).\end{aligned}$$

We arrive then to the rather explicit formula

$$\begin{aligned}\varepsilon^{-1}\chi_c(p)&=\sum_j c_j\mu_{\theta_{j,-}}\left(\partial_\theta\omega(\cdot,\theta_{j,-})\right)\\ &\quad-\sum_j c_j\sum_{k=0}^{\infty}\mu_{\theta_{j,-}}\left(\partial_x\omega(\cdot,\theta_{j,-})\frac{\partial_\theta f(f^k_{\theta_{j,-}}(\cdot),\theta_{j,-})}{(f^k_{\theta_{j,-}})'}\right)+\mathcal{O}(\sqrt{\varepsilon}).\end{aligned}\tag{6.4}$$

For a measure $\mu\in\mathcal{M}_{\rm sp}(F_\varepsilon)$ let $R_\mu(F_\varepsilon)$ be the set of points in the support of μ for which the Lyapunov exponent exists. Then our argument shows that (6.4) holds for each $p\in\cup_{\mu\in\mathcal{M}_{\rm sp}}R_\mu(F_\varepsilon)=:R(F_\varepsilon)$.

6.1. **Skew product case.** In the special case when the map F is a skew product

$$F(x,\theta)=(f(x),\theta+\varepsilon\omega(x,\theta)),$$

all the terms $\partial_\theta f$ in (6.4) are zero and thus the formula (6.4) reduces to

$$\varepsilon^{-1}\chi_c(p)=\sum_j c_j\mu_{\theta_{j,-}}\left(\partial_\theta\omega(\cdot,\theta_{j,-})\right)+\mathcal{O}(\sqrt{\varepsilon}).$$

In addition, we have $c_j>0$ and $\partial_\theta\omega(x,\theta_{j,-})<0$ for all j and any x. Thus $\chi_c(p)<0$ for ε small enough.

6.2. **A counterintuitive example.** We discuss in detail an example introduced, but not conclusively studied, in [12]. Let $\ell\in\mathbb{N}$, $\ell>1$, $\alpha\in\mathbb{R}$, $\beta>0$ and consider the family

$$F_\varepsilon(x,\theta)=(\ell x+\sin(2\pi\theta)\left[\alpha\sin(2\pi x)+\beta\sin(2\ell\pi x)\right],\theta+\varepsilon\cos(2\pi x))\mod 1.\tag{6.5}$$

In the above examples $\omega(x,\theta)=\cos(2\pi x)$ does not depend on θ. In [12] is computed

$$\bar\omega'(\theta)=\sum_{k=1}^{\infty}\int_{\mathbb{T}}(\omega\circ f^k_\theta(x))'\frac{\partial_\theta f(x,\theta)}{f'_\theta(x)}\rho_\theta(x)\mathrm{d}x.$$

Observe that if $\theta=0$ or $\theta=1/2$ (so that $\sin(2\pi\theta)=0$), then $f_\theta(x)=\ell x$, thus $\mu_\theta=\text{Leb}$, $\bar\omega(\theta)=0$ and

$$\bar\omega'(0)=-(2\pi)^2\sum_{k=1}^{\infty}\ell^{k-1}\int_{\mathbb{T}}\sin(2\ell^k\pi x)[\alpha\sin(2\pi x)+\beta\sin(2\ell\pi x)]=-2\pi^2\beta.$$

Then $\theta = 0$ is a sink for the averaged dynamics and it turns out to be the only one.[5] Accordingly, (6.4) yields, for all $p \in R(F_\varepsilon)$,

$$\begin{aligned}\varepsilon^{-1}\chi_c(p) &= 4\pi^2 \sum_{k=0}^{\infty} \int_{\mathbb{T}} \sin(2\pi x) \frac{\alpha \sin(2\pi \ell^k x) + \beta \sin(2\pi \ell^{k+1} x)}{\ell^k} + \mathcal{O}(\sqrt{\varepsilon}) \\ &= 2\pi^2 \alpha + \mathcal{O}(\sqrt{\varepsilon}).\end{aligned}$$

As suggested in [12], we thus see that, for $\alpha > 0$, the central Lyapunov exponent is positive, although the average dynamics tends to concentrate the motion in a very small neighbourhood of zero. This seems to be counterintuitive and has an interesting implication that we are going to discuss in the next section.

In addition, $\chi_c > 0$ also holds for small perturbations of (6.5). Indeed, the locus where the most of mass of the physical measure sits is a $\sqrt{\varepsilon}$-band around a zero of $\bar{\omega}(\theta)$. This locus depends on F_ε continuously. The derivative $DF|_{E^c}$ in the central direction depends on F_ε continuously, too. Thus:

Proposition 6.1. *For any $l \in \mathbb{N}$, $l > 1$, and $\alpha, \beta > 0$ there exists $\varepsilon > 0$ and a C^1-open set $\mathcal{U}_\varepsilon$, that contains the maps $F_{\varepsilon'}$ defined in (6.5) with the given l, α, β for all $0 \leq \varepsilon' \leq \varepsilon$, such that for any $F \in \mathcal{U}_\varepsilon$ and $p \in R(F)$ we have $\chi_c(p) > 0$.*

In particular, if $F \in \mathcal{U}_\varepsilon$ has a physical measure, then it must have positive Lyapunov exponents.

7. The central foliation

In Section 6 we have seen that, contrary to naive intuition, it is possible that the central Lyapunov exponent χ_c is positive, despite having a statistical sink. To make things worse, we prove below that F_ε has an invariant foliation made of smooth compact leaves tangent to the central distribution. If $\chi_c > 0$, these leaves have to expand in average but at the same time their length is uniformly bounded.

The reason why this is not contradictory is that the center foliation fails to be absolutely continuous. This means that, despite each leaf being individually smooth, the foliation as a whole is very wild. This situation is strange but known to happen, see the papers of Ruelle, Shub and Wilkinson [39, 37] where they presented an open set of volume preserving partially hyperbolic systems with non absolutely continuous central foliation for a perturbation of the product of an Anosov map by an identity map on the circle. This behaviour was later observed in many other partially hyperbolic systems, see [42, 23, 43, 44, 45, 47].

In our class of dynamical systems, we find a similar phenomenon. Namely, let $\mathcal{U}_\varepsilon$ be given by Proposition 6.1. Then for every $F \in \mathcal{U}_\varepsilon$ we have $\chi_c > 0$.

Theorem 7.1. *For every map F from $\mathcal{U}_\varepsilon$*

(a) *the central distribution E^c is uniquely integrable to a C^1 foliation $\mathscr{W}^c$;*
(b) *if, in addition, F has j-pinching, $j \geq 1$, see Subsection 7.1, then $\mathscr{W}^c$ is C^j;*
(c) *every leaf $W \in \mathscr{W}^c$ is a diffeomorphic circle of uniformly bounded length;*
(d) *if F has a physical measure, then $\mathscr{W}^c$ is not absolutely continuous.*

Remark 7.2. *As already explained, according to Tsujii [40], the existence of the physical measure is generic and hence it holds generically for $F \in \mathcal{U}_\varepsilon$. Accordingly, the above Theorem implies that generically $\mathscr{W}^c$ is not absolutely continuous..*

For possible future use we will prove many of the above results in much larger generality than stated in Theorem 7.1. Let us start by describing such a more general setting.

[5] We refrain from proving it but it is not very hard to check it numerically.

By a k-dimensional C^r *foliation* $\mathscr{W}$ of M, $r \geq 1$, we mean a partition of M into k-dimensional, complete, connected C^1 submanifolds $W(z) \ni z$, called *leaves*, which depend continuously on z. Let D^n denote the open unit ball in $\mathbb{R}^n$. For each point $z \in M$ there is a *coordinate chart* (or *foliation box*) (U, ϕ) at z: a neighborhood $U \ni z$ and a homeomorphism $\phi\colon D^k \times D^{m-k} \to U$ such that for each $p \in D^{m-k}$, the set $W_U(\phi(0,p)) = \{\phi(z,p)\}_{z\in D^k}$, called the *local leaf*, is contained in $W(\phi(0,p))$ and $\phi(\cdot, p)\colon D^k \to W(\phi(0,p))$ is a C^r diffeomorphism which depends continuously on $p \in D^{m-k}$ in C^1 topology.

Given a foliation $\mathscr{W}$, denote by d_W the distance along the leaf $W \in \mathscr{W}$, by $W(z)$ the leaf passing through $z \in M$, and by $W_\delta(z)$ the ball of radius δ in $W(z)$ centered in z. A foliation of a simply connected Riemannian manifold $\tilde{M}$ is called *quasi-isometric* [30] if there are $a, b > 0$ such that for any $z_1, z_2 \in W(z_1)$, holds

$$d_W(z_1, z_2) \leq a \cdot d(z_1, z_2) + b.$$

A foliation $W(z)$ is tangent to a distribution $E(z)$ if for every $z \in M$ we have $T_z W(z) = E(z)$. A distribution E is called *integrable* if there exists a foliation tangent to E, and *uniquely integrable* if such foliation is unique. The existence theorem for solutions of ODEs implies that for every continuous distribution E, $\dim E = 1$, and for every $z \in M$ there exists a local curve $\gamma \ni z$ tangent to E. However, this γ may not be unique and thus there may be no way to construct a global foliation of these curves, see for instance [33]. To assert γ is unique the distribution must have greater regularity, such a Lipshitz. For $\dim E \geq 2$ the classic Frobenius Theorem indicates that even infinitely smooth distributions may fail to be integrable, let alone uniquely integrable. Thus the question of integrability of the central distribution is very important in the theory of partially hyperbolic dynamical systems.

The analogue of uniqueness of solutions of ODEs in the world of distributions is the following property. A continuous k-dimensional distribution $\mathscr{W}$ is called *locally uniquely integrable* if for each $z \in M$ there are k-dimensional C^1-submanifold $W_{loc}(z)$ and $\alpha(z) > 0$ such that every piecewise C^1 curve $\sigma\colon [0,1] \to M$ satisfying (i) $\sigma(0) = z$, (ii) $\dot\sigma(t) \in E(\sigma(t))$ for $t \in [0,1]$, and (iii) length$(\sigma) < \alpha(z)$, is contained in $W_{loc}(z)$. Obviously, if $\mathscr{W}$ is locally uniquely integrable, then it is integrable and the integral foliation is unique. In addition, we say a distribution is C^r *locally uniquely integrable* if every $W_{loc}(z)$ is a C^r submanifold, $r \geq 1$.

Now let $F\colon M \to M$ be a C^1 local diffeomorphism, not necessary 1-1 globally. We say that a foliation $\mathscr{W}$ is *invariant* under F if for every sufficiently small local leaf W of $\mathscr{W}$ its image $F(W)$ is also a local leaf of $\mathscr{W}$. Obviously, if $\mathscr{W}$ is invariant under F, then its tangent distribution $T\mathscr{W}$ is invariant under DF. The converse is also true if $T\mathscr{W}$ is uniquely integrable.

For every F-invariant measure μ and every F-invariant foliation $\mathscr{W}$ the *leafwise volume Lyapunov exponent* $\chi_{\mathscr{W}}$ is well-defined for μ-a.e. $z \in M$:

$$\chi_{\mathscr{W}}(z) = \lim_{n\to\infty} \frac{1}{n} \log |\det DF^n|_{T_z\mathscr{W}}(z)|\,.$$

If F is a partially hyperbolic endomorphism, see Subsection 7.1, and its central distribution E^c is uniquely integrable to $\mathscr{W}^c$, then $\chi_{\mathscr{W}} = \chi_c$, where χ_c is the central Lyapunov exponent in case $\dim E^c = 1$ and the sum of the central Lyapunov exponents in case $\dim E^c \geq 2$.

In the following subsections we prove some results which are not only sufficient to prove claims (a)–(d) but go well beyond.

7.1. **Claim (a)-(b): the central foliation exists and is unique.** Let $F\colon M \to M$ be a C^l local diffeomorphism, perhaps non-invertible globally. Let $\tilde{F}\colon \tilde{M} \to \tilde{M}$ be a lift of F to the universal cover $\tilde{M}$. The map $\tilde{F}$ is a 1-1 local diffeomorphism and thus a global diffeomorphism. In this paper, we say F is a *partially hyperbolic endomorphism* if $\tilde{F}$ has a uniform dominated splitting with a strong unstable bundle: there are constants $0 < \lambda_1 \leq \lambda_2 < \mu_1 \leq \mu_2$, $\mu_1 > 1$, and $C \geq 1$ and distributions $\tilde{E}^c(\tilde{z})$, $\tilde{E}^u(\tilde{z})$, called *center* and *unstable*, respectively, such that for every $\tilde{z} \in \tilde{M}$

- $T_{\tilde{z}}\tilde{M} = \tilde{E}^c(\tilde{z}) \oplus \tilde{E}^u(\tilde{z})$;
- the distributions $\tilde{E}^c$, $\tilde{E}^u$ are invariant under $D\tilde{F}$;
- $C^{-1}\lambda_1^n\|v^c\| \leq \|D\tilde{F}^n(\tilde{z})v^c\| \leq C\lambda_2^n\|v^c\|$ for each $v^c \in \tilde{E}^c(\tilde{z})$ and $n > 0$;
- $C^{-1}\mu_1^n\|v^u\| \leq \|D\tilde{F}^n(\tilde{z})v^u\| \leq C\mu_2^n\|v^u\|$ for each $v^u \in \tilde{E}^u(\tilde{z})$ and $n > 0$;

Denote $r = \max\{j \in \{1, ..., l\} \,|\, \lambda_2^j < \mu_1\}$, the latter inequality sometimes being called the *j-pinching* condition.

In [13, Section 3] is proved the existence of unstable and center invariant cone fields for F_ε for every $\varepsilon \leq \varepsilon_0$. This implies that the maps in $\mathcal{U}_\varepsilon$ are partially hyperbolic endomorphisms.

As opposed to the central distribution $\tilde{E}^c$, the unstable distribution $\tilde{E}^u$ of a C^l diffeomorphism is known to be C^l uniquely integrable.

The following theorem, a version of Brin's [9, Theorem 1] for endomorphisms, establishes a connection between the geometry of the unstable foliation and the integrability of the central distribution.

Theorem 7.3. *Let F be a partially hyperbolic endomorphism of a compact manifold M. Suppose the unstable foliation of the lift $\tilde{F}$ is quasi-isometric in the universal cover $\tilde{M}$. Then the distribution E^c is C^r locally uniquely integrable; in particular, F has a unique central foliation and it is C^r.*

We do not provide the proof explicitly as the proof of C^1 local unique integrability is literally identical to the proof of the unique integrability of E^{cs} there. The additional C^r regularity of the leaves follows from [22, Chapter 1, Theorem 4.10], applied to the inverse limit system for F. This proves claim (b).

The following lemma, a version of [9, Proposition 4] gives an elegant sufficient condition for a foliation to be quasi-isometric. Again, the proof follows Brin's word for word.

Lemma 7.4. *Let $\mathscr{W}$ be a k-dimensional foliation of the m-dimensional space $\mathbb{R}^m$. Suppose there is an $(m-k)$-dimensional plane A such that $T_{\tilde{z}}W(\tilde{z}) \cap A = \emptyset$ for each $\tilde{z} \in \mathbb{R}^m$. Then $\mathscr{W}$ is quasi-isometric.*

To prove claim (a) of Theorem 7.1 it is now sufficient to show

Proposition 7.5. *For any map $F \in \mathcal{U}_\varepsilon$ the unstable foliation $\mathscr{W}^u$ of the lift $\tilde{F}$ to the universal cover $\tilde{M} = \mathbb{R}^2$ satisfies the assumption of Lemma 7.4 with $k = 1$, $m = 2$.*

Proof. The metrics on $\mathbb{T}^2$ and $\mathbb{R}^2$ are flat and the connections are trivial. Thus we can trivially identify all the tangent spaces $T_z\mathbb{T}^2$, $z \in \mathbb{T}^2$, and $T_{\tilde{z}}\mathbb{R}^2$, $\tilde{z} \in \mathbb{R}^2$. The union of all possible $\tilde{E}^u(\tilde{z})$, $\tilde{z} \in \mathbb{R}^2$, is a subset of the unstable cone for F and thus avoids the central cone for F. Thus any direction within the central cone, including the vertical direction, works as A. □

Remark 7.6. *This straightforwardly generalizes to the maps of form (1.1) in any dimension. Thus all such maps have locally uniquely integrable central distributions.*

7.2. **Claim (c): central leaves are compact and have uniformly bounded volume.** To prove claim (c), we need more assumptions on M and F. Let $C_* > 0$ some arbitrary, but fixed, constant and M_1, M_2 be compact Riemannian manifolds.

Given $M = M_1 \times M_2$, let $\mathbb{F}_\varepsilon(M_1, M_2)$, $\varepsilon \le \varepsilon_0$, be the set of a partially hyperbolic endomorphisms $F\colon M \to M$, $\|F\|_{\mathcal{C}^2} \le C_*$, of the form

$$F\colon (x,\theta) \mapsto (f(x,\theta), \Omega(x,\theta)), \quad x \in M_1, \theta \in M_2, \quad \mathrm{dist}(\Omega(x,\theta), Id) \le \varepsilon, \tag{7.1}$$

and assume $f(\cdot,\theta)$ is strictly expanding in x for every $\theta \in M_2$. Note that we have $\mathcal{U}_\varepsilon \subset \mathbb{F}_\varepsilon(\mathbb{T}^1, \mathbb{T}^1)$.

Clearly, there is $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$ the set $\mathbb{F}_\varepsilon(M_1, M_2)$ is made of partially hyperbolic endomorphism in the sense of Subsection 7.1. Then by Theorem 7.3, every $F \in \mathbb{F}_\varepsilon(M_1, M_2)$ has a unique smooth central foliation $W^c(z)$.

Theorem 7.7. *For every $\varepsilon \le \varepsilon_0$ there is $V > 0$ such that for any $z \in M$*

- *$W^c(z)$ is homeomorphic to M_2;*
- *$\mathrm{vol}\,(W^c(z)) < V$.*

Proof. We prove it first for the special case $\varepsilon = 0$ where we have an explicit description of the central foliation of each $F_0 \in \mathbb{F}_0(M_1, M_2)$. Recall that every smooth expanding map is structurally stable. Thus all the maps $f(\cdot,\theta)$ are conjugated. Let $h(\cdot,\theta)$ be the map that conjugates $f(\cdot,0)$ with $f(\cdot,\theta)$. By definition, $h(x,0) = h(x,1) = x$. The graph of $h(x,\cdot)$ is a compact submanifold of M homeomorphic to M_2. Thus we have a continuous foliation of M by the graphs of $h(x,\cdot)$. This foliation is invariant under F_0 and must coincide with the central foliation $\mathscr{W}^c$ for F_0. In particular, the leaves of $\mathscr{W}^c$ are compact smooth submanifolds of M homeomorphic to M_2.

To prove Theorem 7.7 for $\varepsilon > 0$, we will use the structural stability of the foliations. Following [7], we say the central foliation $\mathscr{W}^c_F$ of the map F is *structurally stable* if, given any nearby C^1 map G,

(a) the central distribution of G uniquely integrates to the central foliation $\mathscr{W}^c_G$;
(b) there exists a globally defined homeomorphism h_G sending leaves of $\mathscr{W}^c_F$ to leaves of $\mathscr{W}^c_G$;
(c) $h_G \circ F \circ h_G^{-1}$ is isotopic to G along the leaves.

Proposition 7.8. *For any $\varepsilon \le \varepsilon_0$ and $F \in \mathbb{F}_\varepsilon(M_1, M_2)$ the foliation $(F, \mathscr{W}^c_F)$ is structurally stable.*

Proof. We lift F and $\mathscr{W}^c_F$ to the universal cover of M and use Theorem (7.1) from [24]. □

Since $\overline{\mathbb{F}}(M_1, M_2)$, the $\mathcal{C}^1$ closure of $\mathbb{F}(M_1, M_2)$, is compact in the $\mathcal{C}^1$ topology, we can use Proposition 7.8 to cover it with finitely many balls of structural stability. We conclude that there exists a globally defined homeomorphism h sending leaves of $\mathscr{W}^c_{F_0}$ to leaves of $\mathscr{W}^c_F$ and $h \circ F_0 \circ h^{-1}$ is isotopic to F along the leaves. In particular, the leaves of $\mathscr{W}^c_F$ are compact smooth submanifolds of M homeomorphic to M_2.

Moreover, because at every point $z \in M$ the central space $E^c(z) \subset T_zM$ belongs to the same cone $K^c = \{(\xi, \boldsymbol{\eta}) \in T_zM_1 \oplus T_zM_2 \mid |\xi| \le \gamma^c|\boldsymbol{\eta}|\}$, this can be proven as in [13, Section 3], by Pythagoras Theorem for every $z \in M$ we have

$$\mathrm{vol}\,(W(z)) \le \sqrt{1+\gamma^2} \cdot \mathrm{vol}\,(M_2).$$

□

The above proof yields an interesting by-product consequence for families of expanding maps which, although folklore, we couldn't find stated explicitly in the literature.

Remark 7.9. *Let $f_\theta\colon M \to M$, $\theta \in (-\theta_0, \theta_0)$, be a smooth family of expanding maps, C^r jointly in x and θ. Let $h(x,\theta)$ be the conjugacy map as above. Then $h(x,\theta)$ is C^r smooth in θ.*

Proof. Apply the above argument to the endomorphism $F(x,\theta) = (f_\theta(x), \theta)$. Note that F has r-pinching because $\lambda_2 = 1$. This implies that the graphs of $h(x,\cdot)$ are C^r-smooth, see [24]. □

Of course, as we will see in the Subsections 7.3 and 7.4, one cannot expect $h(x,\theta)$ to be smooth in x, or even absolutely continuous.

7.3. **Claim (d): the central foliation is not absolutely continuous.** Let M be a Riemannian manifold equipped with a continuous foliation $\mathscr{W}$. Denote by Leb the Lebesgue measure on M coming from the Riemannian volume. It follows from the classic works of Rokhlin that for any foliation box[6] $\mathcal{B}$ there exists a disintegration of $\mathrm{Leb}|_{\mathcal{B}}$ into the transversal measure $\tilde{\mu}_{\mathcal{B}}$ and leafwise conditional measures $\nu_{W,\mathcal{B}}$ defined for $\tilde{\mu}$-almost every leaf disk $W_{\mathcal{B}}$ within the box. The measures coming from different boxes are equivalent on their common domain so we drop the index $\mathcal{B}$ for brevity.

Since every leaf $W \in \mathscr{W}$ is a smooth submanifold of M, it has the induced Riemannian volume and the Lebesgue measure Leb_W coming from it. Regularity of ν_W with respect to Leb_W is a good indicator of how nicely the foliation box around the leaf W is immersed in M. We say a foliation is *absolutely continuous* if for Lebesgue almost every $z \in M$ the conditional measure $\nu_{W(z)}$ is absolutely continuous with respect to $\mathrm{Leb}_{W(z)}$. There are other definitions of absolute continuous foliations, see for instance [23], but they are beyond the scope of this paper.

Theorem 7.10. *Let F be a C^2 partially hyperbolic endomorphism of a compact smooth Riemannian manifold M. Assume that*

(a) *F has a physical measure μ;*
(b) *F has a C^1 invariant foliation $\mathscr{W}$ with leaves of uniformly bounded volume;*
(c) *for each leaf $W \in \mathscr{W}$ the restriction $F_W\colon W \to F(W)$ is a 1-1 map;*
(d) *the leafwise volume Lyapunov exponent $\chi_{\mathscr{W}}$ w.r.t. μ is strictly positive;*

Then the foliation $\mathscr{W}$ is not absolutely continuous.

Proof. Assume that the foliation $\mathscr{W}$ is absolutely continuous. Let Λ be the set of Lyapunov regular points for μ. Because μ is physical, we have $\mathrm{Leb}(\Lambda) > 0$. Thus there exists a set A, $\mathrm{Leb}(A) > 0$, such that for every $z \in A$ we have $\mathrm{Leb}_z(W(z) \cap \Lambda) > 0$.

Fix any $z \in A$ and denote by $\mathrm{Jac}_{\mathscr{W}} F^n$ the determinant of the restriction of DF^n to E^c. Then for any $n \geq 0$ for the volume of the leaf $W(F^n(z))$, remembering assumption (c), we can write

$$\begin{aligned}\operatorname{vol} W(F^n(z)) = \int\limits_{W(F^n(z))} dm_{F^n(z)} = \int\limits_{W(z)} |\mathrm{Jac}_{\mathscr{W}} F^n|\, dm_z \geq \int\limits_{W(z)\cap\Lambda} |\mathrm{Jac}_{\mathscr{W}} F^n|\, dm_z \\ = \int\limits_{W(z)\cap\Lambda} e^{n\cdot\frac{1}{n}\cdot\log|\mathrm{Jac}_{\mathscr{W}} F^n|}\, dm_z.\end{aligned}$$

Then, by Jensen's inequality for e^x,

$$\operatorname{vol} W(F^n(z)) \geq e^{n\cdot \int\limits_{W(z)\cap\Lambda} \frac{1}{n}\log|\mathrm{Jac}_{\mathscr{W}} F^n|\, dm_z}.$$

Note that for every $z' \in W(z)\cap\Lambda$ we have $\frac{1}{n}\log|\mathrm{Jac}_{\mathscr{W}} F^n(z')| \to \chi_{\mathscr{W}}$. Thus by Fatou's lemma

$$\lim_{n\to\infty} \int\limits_{W(z)\cap\Lambda} \frac{1}{n}\log|\mathrm{Jac}_{\mathscr{W}} F^n|\, dm_z \geq \int\limits_{W(z)\cap\Lambda} \chi_{\mathscr{W}}\, dm_z.$$

[6] As defined after Theorem 7.1

By assumption (b), the volume of the leaves is uniformly bounded, i.e, there exists $C \in \mathbb{R}$ such that for any $z \in M$ we have $C \geq \operatorname{vol} W(z)$. Thus we can write

$$C \geq \lim_{n\to\infty} \operatorname{vol} W(F^n(z)) \geq \lim_{n\to\infty} e^{n\cdot \int_{W(z)\cap\Lambda} \chi_{\mathscr{W}}\, dm_z} \geq \lim_{n\to\infty} e^{n\cdot\chi_{\mathscr{W}}\cdot m_z(W(z)\cap\Lambda)} = +\infty,$$

because $\chi_{\mathscr{W}} > 0$ by assumption (d) of the theorem. This contradiction proves the theorem. □

Proof of Theorem 7.1. The idea is to apply Theorem 7.10, with $\mathscr{W} = \mathscr{W}^c_F$, to the maps $F \in \mathcal{U}_\varepsilon$. We have thus to check the hypotheses of Theorem 7.10. The existence of a central foliation satisfying (b) is established in Subsection 7.2. We obviously have (c) for F_0 because of the special structure of $\mathscr{W}^c_{F_0}$, recall Subsection 7.2. Then that for any $F \in \mathcal{U}_\varepsilon$, $\varepsilon \leq \varepsilon_0$, and $W \in \mathscr{W}^c_F$ we know that $F|_W \colon W \to F(W)$ is a local diffeomorphism (thus, a covering) and, by Proposition 7.8, is isotopic to a map, topologically conjugated to some $F_0|_{W'}$ which is 1-1, where $W' \in \mathscr{W}^c_{F_0}$. Thus $F|_W$ is itself 1-1. Finally, (d) follows from Proposition 6.1. □

Let us conclude the paper stating few interesting related facts.

Corollary 7.11. *Suppose a C^2 partially hyperbolic endomorphism $F \colon M_1 \times M_2 \to M_1 \times M_2$ is a skew product*

$$F(x, \theta) = (f(x), \theta + \varepsilon\omega(x, \theta))$$

and has an absolutely continuous ergodic invariant measure μ. Then its central Lyapunov exponent with respect to μ is non-positive.

Proof. Assumptions (a)–(c) follow from the skew product structure. The central foliation in this case is the collection of all $\{x\} \times M_2$, $x \in M_1$, which is obviously absolutely continuous. But if we assume $\chi_c > 0$ this would imply that central foliation is not absolutely continuous. Thus $\chi_c \leq 0$. □

This fits well within general knowledge in the area. In different settings, it known [25] that a generic partially hyperbolic skew product with a non-invertible base dynamics has negative central volume Lyapunov exponent, which can only become zero in some degenerate cases but never above zero. Kleptsyn, Nalskii [26] used a similar approach to prove that a generic random dynamical systems on the circle contracts the orbits. Both results are based on the fundamental Baxendale's [6] theorem for stochastic flows.

Remark 7.12. *Note that the situation is different for diffeomorphisms, see [23]. In that setting it is sufficient to ask $\chi_c \neq 0$ instead of $\chi_c > 0$ to prove that the central foliation is not absolutely continuous.*

A final comment on the case $\chi_c = 0$. Consider on the one hand a rigid rotation skew product

$$F_\varepsilon(x, \theta) = (f(x), \theta + \varepsilon\omega(x))$$

which has the vertical circles as the absolutely continuous central foliation. On the other hand a system of the form

$$F(x, \theta) = (f(x, \theta), \theta)$$

with a generic $f(x, \theta)$, $\partial_x f > \lambda > 1$, which has a non-absolutely continuous central foliation (we prove it shortly in Subsection 7.4). Thus, both possibilities can happen. Clearly, there is the need for further investigation if we want to understand the absolutely continuity of the foliation in this case.

7.4. **Non-absolute continuity for $\varepsilon = 0$.** For the special case $\varepsilon = 0$ the central Lyapunov exponent $\chi_c = 0$ and thus the Theorems 7.1 and 7.10 do not apply. However, recall the classic result by Shub, Sullivan [38]:

Theorem 7.13. *Let $2 \le r \le \omega$. If two orientation preserving expanding C^r endomorphisms f and g of $\mathbb{T}^1$ are absolutely continuously conjugate, then they are conjugate by a C^r diffeomorphism.*

In particular, the multipliers of all the according periodic points of f and g must be the same. This is a degeneracy of codimension infinity. In the concrete family (6.5) the multiplier of the fixed point $(0, \theta)$ non-trivially changes with θ. Thus, for a generic F_0 the conjugacy $h(x, \theta)$ is not absolutely continuous in x. This implies that

Proposition 7.14. *The map F_0 generically has a non absolutely continuous central foliation $\mathscr{W}_0^c$.*

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