PARABOLIC DYNAMICS AND ANISOTROPIC BANACH SPACES

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ABSTRACT. We investigate the relation between the distributions appearing in the study of ergodic averages of parabolic flows (e.g. in the work of Flaminio-Forni) and the ones appearing in the study of the statistical properties of hyperbolic dynamical systems (i.e. the eigendistributions of the transfer operator). In order to avoid, as much as possible, technical issues that would cloud the basic idea, we limit ourselves to a simple flow on the torus. Our main result is that, roughly, the growth of ergodic averages (and the characterization of coboundary regularity) of a parabolic flows is controlled by the eigenvalues of a suitable transfer operator associated to the renormalizing dynamics. The conceptual connection that we illustrate is expected to hold in considerable generality.

1. INTRODUCTION

In the last decade, *distributions* have become increasingly relevant both in parabolic and hyperbolic dynamics. On the *parabolic dynamics* side consider, for example, the work of Forni and Flaminio-Forni [25, 26, 21, 22, 23] on ergodic averages and cohomological equations for horocycle flows or of Bufetov [13] on translation flows; on the *hyperbolic dynamics* side it suffices to mention the study of the transfer operator through anisotropic spaces, started with [9].[1]

Since a frequent approach to the study of *parabolic dynamics* is the use of renormalization techniques,[2] where the renormalizing dynamics is often a *hyperbolic dynamics*, several people have been wondering on a possible relation between such two classes of distributions. Early examples of such line of thought can be found in Cosentino [16, Section 3] and Otal [47].

[1] But see, e.g., [49, 50, 43, 41] for earlier related results.

[2] Typical examples are circle rotations [52, 53], interval exchange maps via Teichmuller theory [55, 28], horocycle flow [26, 21].

In this paper we argue that the distributional obstructions discovered by Forni and the distributional eigenvectors of certain transfer operators are tightly related, to the point that, informally, one could say that they are exactly the same.

In order to present our argument in the simplest possible manner, instead of trying to develop it for the horocycle flow versus the geodesic flow (which would require a much more technical framework), we consider a very simple example that, while preserving the main ingredients of the horocycle-geodesic flow setting, allows to easily illustrate the argument. Yet, our example is not rigid (morally it corresponds to looking at manifolds of non constant negative curvature). So, notwithstanding its simplicity, it shows the flexibility of our approach, which has the potential of being greatly generalized. On the other hand, we cover only the case of periodic renormalization. Indeed, if the renormalizing dynamics are non linear, then it is not very clear how to define a good moduli space on which to act. The extension of our approach to the non periodic case remains an open problem.

Let us describe a bit more precisely our setting (see Section 2 for the exact, less discursive, description). As parabolic dynamics, we consider a flow ϕ_t, over $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, generated by a vector field $V \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R}^2)$, $\alpha \in \mathbb{R}_>$, such that, for all $x \in \mathbb{T}^2$, $V(x) \neq 0$. As hyperbolic dynamics, we consider a transitive Anosov map $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$, $r > 1 + \alpha$. By definition of Anosov map for all $x \in \mathbb{T}^2$ we have $T_x\mathbb{T}^2 = E^s(x) \oplus E^u(x)$, where we used the usual notation for the stable and unstable invariant distributions.[3] Since we want the latter system to act as a renormalizing dynamics for the former, we require,

$$(1.1) \qquad\qquad \forall x \in \mathbb{T}^2, \ V(x) \in E^s(x).$$

One might wonder which kind of flows admit the property (1.1) for some Anosov map $F \in \mathcal{C}^r$. Here is a partial answer whose proof can be found in Appendix A.

Lemma 1.1. *If a $\mathcal{C}^{1+\alpha}$, $\alpha > 0$, flow ϕ_t, without fixed points, satisfies (1.1) for some Anosov map $F \in \mathcal{C}^r$, $r \geq 1 + \alpha$, then it is topologically conjugated to a rigid rotation with rotation number ω such that*

$$(1.2) \qquad\qquad b\,\omega^2 + (a - d)\omega - c = 0$$

for some $a, b, c, d \in \mathbb{Z}$ such that $ad - cb = 1$.

Each $\mathcal{C}^{1+\alpha}$, $\alpha \geq 1$, flow ϕ_t without fixed points, or periodic orbits, is topologically conjugated to a rigid rotation. If the rotation number satisfies (1.2) and $\alpha \geq 2$, then ϕ_t satisfies (1.1) for some Anosov map $F \in \mathcal{C}^\beta$, for each $\beta < \alpha$.

Note that the condition (1.2) can be restated by saying that $\omega = r_1 + r_2\sqrt{D}$ where $r_1, r_2 \in \mathbb{Q}, D \in \mathbb{N}, r_2 \neq 0$. We stated the lemma in the above form because it connects better to the example worked-out in Section 5.

Remark 1.2. *Even though the above Lemma shows that it is always possible to reduce our setting to a linear model by a conjugation, such conjugation is typically of rather low regularity. We will see shortly that requiring F, and related objects, to be of high regularity is essential for the questions we are interested in. It is not obvious to us how to characterize the flows for which (1.1) holds for very smooth F. Yet, such a condition clearly singles out some smaller class of flows (compared*

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[3] Here "distribution" refers to a field of subspaces in the tangent bundle and has nothing to do with the meaning of "distribution" as generalized functions previously used. This is an unfortunate linguistic ambiguity for which we bear no responsibility.

to Lemma 1.1) to which our theory applies. Note however that there are plenty of examples, see Section 5.

Equation (1.1) implies that the trajectories $\{\phi_t(x)\}_{t \in (a,b)}$ are pieces of the stable manifolds for the map F. Thus, if we assume $r \geq 2 + \alpha$, we can define implicitly a function $\nu_n \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R})$ such that

$$(1.3) \qquad D_x F^n V(x) = \nu_n(x) V(F^n(x)),$$

where $|\nu_n| < C_\# \lambda^{-n}$ for some $\lambda > 1$. Without loss of generality we assume that F preserves the orientation of the invariant splitting, i.e. $\nu_n > 0$ (if not, use F^2).

Given the hypothesis (1.1) it is natural to ask, at least, that, for each $x \in \mathbb{R}^2$, the flow is regular with respect to the time coordinate i.e.

$$(1.4) \qquad \phi_{(\cdot)}(x) \in \mathcal{C}^r.$$

In fact, we will use slightly stronger hypotheses, see Definition 2.3 and Remark 2.4.

The reader may complain that the parabolic nature of the dynamics ϕ_t it is not very apparent. Indeed, a little argument is required to show that $\|D_x \phi_t\|$ can grow at most polynomially in t, see Section 4.1, and some more work is needed to show that there are cases in which it is truly unbounded, see Lemma 5.16.

Let us detail an easy consequence of (1.3). If, for each $n \in \mathbb{N}$, we define $\eta_n \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2 \times \mathbb{R}, \mathbb{T}^2)$ as $\eta_n(x,t) = F^n(\phi_t(x))$, then we have

$$\begin{cases} \frac{d}{dt} \eta_n(x,t) = D_{\phi_t(x)} F^n V(\phi_t(x)) = \nu_n(\phi_t(x)) V(\eta_n(x,t)) \\ \eta_n(0) = F^n(x). \end{cases}$$

It is then natural to define the time change[4]

$$(1.5) \qquad \tau_n(x,t) = \int_0^t ds \, \nu_n(\phi_s(x)),$$

and introduce the function $\zeta_n \in \mathcal{C}^{1+\alpha}$ by $\zeta_n(\tau_n(x,t), x) = \eta_n(x,t)$. Then

$$(1.6) \qquad \begin{cases} \frac{d}{dt} \zeta_n(x,t) = V(\zeta_n(x,t)) \\ \zeta_n(x,0) = F^n(x). \end{cases}$$

By the uniqueness of the solution of the above ODE, it follows, for all $t \in \mathbb{R}_>$

$$\phi_t(F^n(x)) = \zeta_n(x,t) = \eta_n(\tau_n^{-1}(x,t), x) = F^n(\phi_{\tau_n^{-1}(x,t)}(x)).$$

In other words, the image under F^n of a piece of trajectory, is the reparametrization of a (much shorter) piece of trajectory:

$$(1.7) \qquad F^n(\phi_t(x)) = \phi_{\tau_n(x,t)}(F^n(x)).$$

The above is the basic renormalization equation for the flow ϕ_t that we will use in the following.

Note that, by Lemma 1.1 and Furstenberg [27], the flow is uniquely ergodic, since its Poincaré map is uniquely ergodic. Let μ be the unique invariant measure. In addition, the flow is also minimal since it is topologically conjugated to a minimal flow (the linear one).

By unique ergodicity, given $g \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$, $\frac{1}{t} \int_0^t ds \, g \circ \phi_s(x)$ converges uniformly to $\mu(g)$. We have thus naturally arrived at our

First question: How fast is the convergence to the ergodic average?

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[4] By construction, for each $x \in \mathbb{T}^2$ and $n \in \mathbb{N}$, $\tau_n(x,t)$ is a strictly increasing function of t, and hence globally invertible. We will use the, slightly misleading, notation $\tau_n^{-1}(x, \cdot)$ for the inverse.

The question is equivalent to investigating the precise growth of the functionals $H_{x,t} : \mathcal{C}^r \to \mathbb{R}$ defined by

$$(1.8) \qquad H_{x,t}(g) := \int_0^t ds\, g \circ \phi_s(x).$$

Of course, if $\mu(g) \neq 0$, then $H_{x,t}(g) \sim \mu(g)t$, but if $\mu(g) = 0$, then we expect a slower growth.

Remark 1.3. *Note that the growth rate of an ergodic integral for functions of a given smoothness it is not a topological invariant, hence the fact that our systems can be topologically conjugated to a linear model, as stated in Lemma 1.1, is not of much help.*

In the work of Flaminio-Forni [21] is proven that the functionals (1.8), there defined for the horocycle flow on a surface of constant negative curvature, have a polynomial growth with exponent determined by a countable number of obstructions. That is, the growth is slower if the function g belongs to the kernel of certain set of functionals. The remarkable discovery of Forni (going back to [25, 26]) is that the possible power growths form a discrete set and that the associated obstructions cannot be expected, in general, to be measures: they are *distributions*.[5] See Remark 2.15 for further details.

In analogy with the above situation, one expects that in our simple model there exist a finite number[6] of functionals $\{O_i\}_{i=1,\dots,N_1} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$, and a corresponding set $\{\alpha_i\}_{i=1,\dots,N_1}$ of decreasing numbers $\alpha_i \in [0,1]$ such that if $O_j(g) = 0$ for all $j < i$ and $O_i(g) \neq 0$, then $H_{x,t}(g) = \mathcal{O}(t^{\alpha_i})$. As we mentioned just after equation (1.8), $O_1(g) = \mu(g)$ with $\alpha_1 = 1$.

Next, suppose that $O_i(g) = 0$ for all $i \leq N_1$ and $\alpha_{N_1} = 0$. That is, $H_{x,t}(g)$ remains bounded. By Gottschalk-Hedlund theorem [32] this implies that g is a continuous coboundary for the flow (since the flow is minimal). To investigate the regularity of the coboundary it is convenient to start with an alternative proof of Gottschalk-Hedlund theorem (limited to our context). For each $T \in \mathbb{R}_>$, consider the new functionals $\overline{H}_T : \mathcal{C}^r \to \mathcal{C}^{1+\alpha}$ defined by, for all $x \in \mathbb{T}^2$,

$$(1.9)\ \ \overline{H}_T(g)(x) := -\int_0^T dt\, \chi \circ \tau_{n_T}(x,t)g \circ \phi_t(x) = -\int_{\mathbb{R}_\geq} dt\, \chi \circ \tau_{n_T}(x,t)g \circ \phi_t(x),$$

where $n_T + 1 = \inf\{n \in \mathbb{N} \ : \ \inf_x \tau_n(x,T) \leq 1\}$ and $\chi \in \mathcal{C}^r(\mathbb{R}_\geq, [0,1])$ is a fixed function such that $\chi(s) = 1$ for all $s \leq 1/2$ and $\chi(s) = 0$ for all $s \geq 1$. Such a function χ can be thought as a "smoothing" of $\overline{\chi} \circ \tau_{n_T} := \max\{0, \frac{T-s}{T}\}$. Unfortunately, we cannot use $\overline{\chi}$ because such a choice would create serious difficulties later on (e.g., in the decomposition carried out in equation (4.8)), yet the reader can substitute $\overline{\chi}$ to χ to have an intuitive idea of what is going on.

The next Lemma restates Gottschalk-Hedlund and will be proved in Section 4.3.

Lemma 1.4. *For each $g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$, r large enough,[7] such that $\mathcal{O}_i(g) = 0$ for all $i \in \{1, \dots, N_1\}$, we have that $\overline{H}_T(g)$ converges uniformly, as $T \to \infty$, to some*

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[5] Apart, of course, for the first that, as above, is the invariant measure μ.

[6] For an explanation of "finitely many versus countably many" see, again, Remark 2.15.

[7] The condition on r is the same as in Theorem 2.8.

continuous function h such that

$$(1.10) \qquad h \circ \phi_t(x) - h(x) = \int_0^t ds\, g \circ \phi_s(x).$$

It follows that h is weakly differentiable in the flow direction and

$$(1.11) \qquad g(x) = \langle V(x), \nabla h(x) \rangle.$$

That is, g is a continuous coboundary.

The existence of a continuous coboundary is of some interest; but much more interesting is the existence of more regular solutions of (1.11) since this plays a role in establishing many relevant properties (see [39, Sections 2.9, 19.2]). Hence our

Second question: How regular are the solutions of the cohomological equation (1.11)?

Following Forni again, we expect that there exist finitely many distributional obstructions $\{O_i\}_{i=N_1+1,\dots,N_2} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$ and a set of increasing numbers $\{r_i\}_{i=N_1+1,\dots,N_2}$, $r_i \in (0, 1+\alpha)$ such that, if $O_j(g) = 0$ for all $j < i$ and $O_i(g) \neq 0$, then $h \in \mathcal{C}^{r_i}$.

Remark 1.5. *Note that in the present context, as the flow is only $\mathcal{C}^{1+\alpha}$, it is not clear if it makes any sense to look for coboundaries better than $\mathcal{C}^{1+\alpha}$. This reflects the fact that if one looks at the horocycle flows on manifolds of non constant negative curvature, then the associate vector field is, in general, not very regular. On the other hand rigidity makes not so interesting our simple example when both foliations are better than $\mathcal{C}^2$, [29, Corollary 3.3]. We will therefore limit ourself to finding distributions that are obstruction to Lipschitz coboundaries, i.e. if $O_i(g) = 0$ for all $i \leq N_2$, then h is Lipschitz (see Theorem 2.12). We believe this to be more than enough to illustrate the scope of the method.*

The goal of this paper is to prove the above facts by studying transfer operators associated to F, acting on appropriate spaces of distributions. In fact, we will show that the above mentioned obstructions $\{O_i\}$ can be obtained from the eigenvectors of appropriate transfer operators associated to F. As announced, this discloses the connection between the appearance of distributions in two seemingly different fields of dynamical systems.

Remark 1.6. *As already mentioned, in our model ϕ_t plays the role of the horocycle flow, while F the one of the geodesic flow. It is important to notice that most of the results obtained for the horocycle flows (and Flaminio-Forni's results in particular) rely on representation theory, thus requiring constant curvature of the space. In our context, this would correspond to the assumption that F is a toral automorphism and ϕ_t a rigid translation. One could then do all the needed computations via Fourier series (if needed, see Section 5.1 for details). It is then clear that extending our approach to more general parabolic flows, e.g. horocycle flows, (which should be quite possible using the results on flows by [31, 20, 19]) would allow to treat cases of variable negative curvature, and, more generally, cases where the tools of representation theory are not available or effective, whereby greatly extending the scope of the theory. To our knowledge the only other approach trying generalize the theory in such a direction is contained in the papers [11, 12]. However Bufetov's strategy relays on a coding of the system. Hence, it seems to suffer from the same limitations that affect the Markov partition approach to the study of hyperbolic systems.*

In particular, using coding techniques only a small portion of the transfer operator spectrum is accessible. These are exactly the limitations that the techniques used in this paper were designed to overcome. It would therefore be very interesting, and (we believe) possible, to extend the present approach to the setting described in [12].

The plan of the paper is as follow: in section 2 we state our exact assumptions, outline our reasoning and state precisely our results, assuming lemmata and constructions which are explained later on. Section 3 is devoted to our first question and proves our Theorem 2.8 concerning the distributions arising from the study of the ergodic integrals. Section 4 deals with our second question and proves our Theorem 2.12 dealing with the distributions arising from the study of the regularity of the cohomological equation. Section 5 is devoted to the discussion of examples. This is a rather long section, yet we think it is important as it explicitly shows how the abstract theory can be concretely applied and gives a clear idea of the type of work involved. In Section 5.1 we work out explicitly the simplest possible situation: a linear flow renormalized by a toral automorphism. Note that in this case all the obstructions generate by our scheme reduce, as it should be, to the Lebesgue measure. Yet, even in this simple situation, our scheme unveils much more structure than expected. In Section 5.2.1 we present a more general (non linear) one parameter family of systems and we prove that it behaves not so differently from the linear model: either the ergodic integral of a function g grows linearly in time or g is at least a $\mathcal{C}^{\frac{1}{2} - c_\# \gamma}$ coboundary, where γ is the perturbative parameter of the family of examples (see Lemmata 5.9, 5.11 and Corollary 5.12 for details). In sections 5.2.3, 5.2.5 we use perturbation theory to show that the flow is, generically, truly parabolic, that is the derivative has a polynomial growth (see Lemma 5.16). In the Appendix A we provide the details for some facts mentioned in the introduction without proof. In the Appendices B and C we recall the definition of the various functional spaces needed in the following (adapted to the present setting).

Notation. When convenient, we will use $C_\#, c_\#$ to designate a generic constant, depending only on F and ϕ_1, and $C_{a,b,...}$ for a generic constant depending also from $a, b, \dots$. Be advised that the actual value of such constants may change from one occurrence to the next.

2. Definitions and main Results

In this section we will introduce rigorously the model loosely described in the introduction and explicitly state our results. Unfortunately, this requires quite a bit of not so intuitive notations and constructions, which call for some explanation. The experienced reader can jump immediately to Theorems 2.8, 2.12 but we do not recommend it in general.

Let $\alpha, r \in \mathbb{R}_>$ with $r > 2 + \alpha$.

We start by recalling the definition of $\mathcal{C}^r$ Anosov map of the torus.

Definition 2.1. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The map is called Anosov if there exists two continuous closed nontrivial transversal cone fields $C^{u,s} : \mathbb{T}^2 \to \mathbb{R}^2$ which are strictly DF-invariant. That is, for each $x \in \mathbb{T}^2$,*

$$\text{(2.1)} \quad \begin{aligned} D_x F C^u(x) &\subset \text{Int } C^u(F(x)) \cup \{0\} \\ D_x F^{-1} C^s(x) &\subset \text{Int } C^s(F^{-1}(x)) \cup \{0\}. \end{aligned}$$

In addition, there exists $C > 0$ and $\lambda > 1$ such that, for all $n \in \mathbb{N}$,

$$
(2.2) \qquad
\begin{aligned}
\|D_x F^{-n} v\| &> C\lambda^n \|v\| && \text{if } v \in C^s(x); \\
\|D_x F^n v\| &> C\lambda^n \|v\| && \text{if } v \in C^u(x).
\end{aligned}
$$

It is well known that the above implies the following, seemingly stronger but in fact equivalent [39], definition

Definition 2.2. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$. The map is called* Anosov *if there exists a DF-invariant $\mathcal{C}^{1+\alpha}$, $r - 1 \geq \alpha > 0$, splitting $T_x M = E^s(x) \oplus E^u(x)$ and constants $C, \lambda > 1$ such that for $n \geq 0$*

$$
(2.3) \qquad
\begin{aligned}
\|DF^n v\| &< C\lambda^{-n} \|v\| && \text{if } v \in E^s; \\
\|DF^n v\| &< C\lambda^n \|v\| && \text{if } v \in E^u.
\end{aligned}
$$

As already mentioned we assume that the stable distribution E^s is orientable and that F preserves such an orientation. Further note that, since F is topologically conjugated to a toral automorphism [39, Theorem 18.6.1], F is topologically transitive.

Next, we consider a flow ϕ_t generated by a vector field V satisfying the following properties.

Definition 2.3. *Let the vector field V be such that*

   (i) $V \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R}^2)$;
  (ii) $\|V\| \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R}_>)$;
 (iii) for all $x \in \mathbb{T}^2$, $V(x) \neq 0$;
 (iv) for all $x \in \mathbb{T}^2$, $V(x) \in E^s(x)$.

Remark 2.4. *Note that Definitions 2.3-(ii) and (iv) imply condition (1.4) since, being $F \in \mathcal{C}^r$, so are the stable leaves [39]. In fact, Definition 2.3-(ii) essentially implies that we are just considering $\mathcal{C}^r$ time reparametrizations of the case $\|V\| = 1$. Hence, we are treating all the $\mathcal{C}^r$ reparametrizations on the same footing. This is rather convenient although not so deep in the present context. Yet, it could be of interest if the present point of view could be extended to the study of the mixing speed of the flow. Indeed, there is a scarcity of results on reparametrization of parabolic flows (see [24] for recent advances).*

Remembering (1.7), (1.5) and using the definition (1.8),

$$
(2.4) \qquad
\begin{aligned}
H_{x,t}(g) &= \int_0^t ds\, g \circ F^{-n} \circ \phi_{\tau_n(x,s)}(F^n(x)) \\
&= \int_0^{\tau_n(x,t)} ds_1 \left(\frac{g}{\nu_n} \right) \circ F^{-n} \circ \phi_{s_1} \circ F^n(x).
\end{aligned}
$$

It is then natural to introduce the *transfer operator* $\mathcal{L}_F \in L(\mathcal{C}^0, \mathcal{C}^0)$,[8]

$$
(2.5) \qquad
\begin{aligned}
\mathcal{L}_F(g) &:= (\nu_1 \circ F^{-1})^{-1} g \circ F^{-1} = g \circ F^{-1} \frac{\|V\|}{\|DFV\| \circ F^{-1}} \\
&= g \circ F^{-1} \frac{\|DF^{-1}V\|}{\|V\| \circ F^{-1}},
\end{aligned}
$$

where we have used (1.3).

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[8] Given a map F, in general a transfer operator associated to F has the form $\varphi \to \varphi \circ F^{-1} e^\phi$ for some function ϕ. Normally, the factor e^ϕ is called the *weight* while ϕ is the *potential*, [2].

We can now write

$$(2.6) \qquad H_{x,t}(g) = \int_0^{\tau_n(x,t)} ds_1 \, (\mathcal{L}_F^n g)(\phi_{s_1}(F^n(x))) = H_{F^n(x),\tau_n(x,t)}(\mathcal{L}_F^n g).$$

The above formula is quite suggestive: for each $x \in \mathbb{T}^2$ and $t \in \mathbb{R}_>$, if we fix $n = n_t(x)$ such that $\tau_{n_t}(x,t)$ is of order one, then $H_{x,t}(g)$ is expressed in terms of very similar functionals of $\mathcal{L}_F^{n_t} g$. Note that such functionals are uniformly bounded on $\mathcal{C}^0$ with respect to (x,t), that is: they can be seen as measures and, as such, they have uniform total variation. One can then naively imagine that to address the questions put forward in the introduction it suffices to understand the behavior of $\mathcal{L}_F^n$ for large n. This obviously is determined by the spectral properties of $\mathcal{L}_F$.

Unfortunately, it is well known that the spectrum of $\mathcal{L}_F$ depends strongly on the Banach space on which it acts. For example, in the trivial case when F is a toral automorphism and ϕ_t a rigid translation with unit speed, $e^{-h_{\text{top}}}\mathcal{L}_F$ acting on L^2 is an isometry,[9] hence the spectrum of $\mathcal{L}_F$ consists of the circle of radius $e^{h_{\text{top}}}$. The spectrum on $\mathcal{C}^0$ it is not much different. On the contrary, if we consider $\mathcal{L}_F$ acting on $\mathcal{C}^r$, then the spectral radius will be given by $e^{(r+1)h_{\text{top}}}$.

This seems to render completely hopeless the above line of thought.

Yet, as mentioned in the introduction, it is possible to define norms $\|\cdot\|_{p,q}$ and associated anisotropic Banach spaces $\mathcal{C}^{p+q} \subset \mathcal{B}^{p,q} \subset (\mathcal{C}^q)'$, $p \in \mathbb{N}^*, q \in \mathbb{R}_>$, $p + q \leq r$, such that each transfer operator with $\mathcal{C}^r$ weight can be continuously extended to $\mathcal{B}^{p,q} = \overline{\mathcal{C}^r}^{\|\cdot\|_{p,q}}$. The above are spaces of distributions (a fact that the reader might find annoying) but, under mild hypotheses on the weight used in the operators, several remarkable properties hold true[10]

i) a transfer operator (with $\mathcal{C}^r$ weight) extends by continuity from $\mathcal{C}^r$ to a bounded operator on $\mathcal{B}^{p,q}$;

ii) such a transfer operator is a quasi-compact operator with a simple maximal eigenvalue;

iii) the essential spectral radius of the transfer operator decreases exponentially with $\inf\{q, p\}$;

iv) the point spectrum is stable with respect to deterministic and random perturbations.

The possibility to make the essential spectrum arbitrarily small, by increasing p and q, will play a fundamental role in our subsequent analysis. Unfortunately, a further problem now arises: the weight of $\mathcal{L}_F$ contains the vector field V which, by hypothesis, is only $\mathcal{C}^{1+\alpha}$. Hence $\mathcal{L}_F$ leaves $\mathcal{C}^r$ invariant only for $r \leq 1 + \alpha$ (exactly the range in which we are not interested). Again it seems that we cannot use our strategy in any profitable manner.

Yet, such a problem has been overcame as well, e.g., in [34]. The basic idea is to extend the dynamics F to the oriented Grassmannian. Indeed, looking at (2.5), it is clear that the weight can be essentially interpreted as the expansion of a volume form. The simplest idea would then be to let the dynamics act on one forms on $\mathbb{T}^2$. Unfortunately, the length cannot be written exactly as a volume form on $\mathbb{T}^2$, hence the convenience of being a bit more sophisticated: the weight of $\mathcal{L}_F$ can be written

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[9] As usual h_{top} stands for the topological entropy of the map F we are considering.

[10] Before [9] it was unclear if spaces with such properties existed at all. Nowadays there exists a profusion of possibilities. We use the ones stemming from [33, 34] because they seem particularly well suited for the task at hand, but any other possibility (e.g. [4, 5]) should do.

as the expansion of a one dimensional volume form on the vector space containing V. As V is exactly the tangent vector to the curves along which we integrate, we are led, as in [34], to consider functions on the Grassmannian made by one dimensional subspaces. However, in the simple case at hand, the construction in [34] can be considerably simplified. Namely, we can limit ourselves to considering the compact set $\Omega_* = \{(x,v) \in \mathbb{T}^2 \times \mathbb{R}^2 \ : \ \|v\| = 1, \ v \in \overline{C^s(x)}\}$.[11] Moreover, since we have assumed that the stable distribution is orientable, then Ω_* is the disjoint union of two sets (corresponding to the two possible orientations). Let Ω be the connected component that contains the elements $(x, \widehat{V}(x))$, where $\widehat{V}(x) = \|V(x)\|^{-1}V(x)$. In addition, since we have also assumed that F preserves the orientation of the stable distribution, calling $\mathbb{F}$ the lift to the unitary tangent bundle of F we have $\Omega_0 = \mathbb{F}^{-1}(\Omega) \subset \Omega$.

Thus we have that $\mathbb{F} : \Omega_0 \subset \Omega \to \Omega$ is defined as

$$\mathbb{F}(x,v) = (F(x), \|D_x F v\|^{-1} D_x F v),$$
$$\mathbb{F}^{-1}(x,v) = (F^{-1}(x), \|D_x F^{-1} v\|^{-1} D_x F^{-1} v).$$

Also note that

(2.7) $$\mathbb{F}^{-1}(x, \widehat{V}(x)) = (F^{-1}(x), \widehat{V}(F^{-1}(x))).$$

Hence, if we define the natural extension $\phi_t(x,v) = (\phi_t(x), \|D_x \phi_t v\|^{-1} D_x \phi_t v)$, remembering (1.7), we have[12]

$$\mathbb{F}^n(\phi_s(x,v)) = (F^n(\phi_t(x)), \|D_x[F^n \circ \phi_t]v\|^{-1} D_x[F^n \circ \phi_t]v)$$
$$= \left(\phi_{\tau_n(x,t)}(F^n(x)), \frac{D_{F^n(x)}\phi_{\tau_n(x,t)} D_x F^n v + V(\phi_{\tau_n(x,t)}(F^n(x)))\langle \nabla \tau_n(x,t), v \rangle}{\|D_{F^n(x)}\phi_{\tau_n(x,t)} D_x F^n v + V(\phi_{\tau_n(x,t)}(F^n(x)))\langle \nabla \tau_n(x,t), v \rangle\|} \right).$$

The above formula does not look very nice, however we will be only interested in integrations along the flow direction. Accordingly, by (1.3) and

(2.8) $$D_x \phi_s(V(x)) = D_x \phi_s \left. \frac{d}{d\tau} \phi_\tau(x) \right|_{\tau=0} = \left. \frac{d}{d\tau}\phi_{s+\tau}(x) \right|_{\tau=0} = V(\phi_s(x)),$$

we have $D_{F^n(x)}\phi_{\tau_n(x,t)} D_x F^n V(x) = \nu_n(x) V(\phi_{\tau_n(x,t)}(F^n(x)))$. Hence, limited to $v = \widehat{V}(x)$, we recover an analogue of (1.7):

(2.9) $$\mathbb{F}^n(\phi_s(x, \widehat{V}(x))) = \phi_{\tau_n(x,s)}(\mathbb{F}^n(x, \widehat{V}(x))).$$

Remark 2.5. *Note that $\mathbb{F}$ is itself a uniformly hyperbolic map with the two dimensional repellor $\{(x,v) \in \Omega \ : \ v = \widehat{V}(x)\}$ and it has an invariant splitting of the tangent space with two dimensional unstable distribution and one dimensional stable.*

Next, we define the transfer operator associated to $\mathbb{F} : \mathcal{C}^0(\Omega_0, \mathbb{R}) \to \mathcal{C}^0(\Omega, \mathbb{R})$ as

(2.10) $$\mathcal{L}_{\mathbb{F}}\boldsymbol{g}(x,v) = \boldsymbol{g} \circ \mathbb{F}^{-1}(x,v) \frac{\|D_x F^{-1} v\| \, \|V(x)\|}{\|V \circ F^{-1}(x)\|}.$$

The key observation is that $\boldsymbol{\pi} \circ \mathbb{F}^{-1} = F^{-1} \circ \boldsymbol{\pi}$, where we have introduced the projection $\boldsymbol{\pi}(x,v) = x$. Hence, for each function $g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$, if we define $\boldsymbol{g} = \boldsymbol{\pi}^* g := g \circ \boldsymbol{\pi}$, then $\boldsymbol{g} \in \mathcal{C}^r(\Omega, \mathbb{R})$ and we have, for all $n \in \mathbb{N}$,

(2.11) $$\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g}(x, \widehat{V}(x)) = \mathcal{L}_F^n g(x).$$

---

[11] Note that Ω_* is a subset of the unitary tangent bundle of $\mathbb{T}^2$.

[12] See equation (4.2) if details are needed.

The above shows that understanding the properties of $\mathcal{L}_{\mathbb{F}}$ allows to control $\mathcal{L}_F$. In addition, from definition (2.10) it is apparent that $\mathcal{L}_{\mathbb{F}}(\mathcal{C}^{r-1}(\Omega, \mathbb{R})) \subset \mathcal{C}^{r-1}(\Omega, \mathbb{R})$.[13] We have thus completely eliminated the above mentioned regularity problem.

Accordingly, we define, for each $\boldsymbol{g} \in \mathcal{C}^0(\Omega, \mathbb{R})$, $t \in \mathbb{R}_>$ and $x \in \mathbb{T}^2$, the new functional

$$(2.12) \qquad\qquad \mathbb{H}_{x,t}(\boldsymbol{g}) = \int_0^t ds\, \boldsymbol{g}\left(\phi_s(x), \widehat{V}(\phi_s(x))\right)$$

and easily obtain an analogue of (2.6) for the operator $\mathcal{L}_{\mathbb{F}}$.

Lemma 2.6. *For each $\boldsymbol{g} \in \mathcal{C}^0(\Omega, \mathbb{R})$ and $g \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$, $n \in \mathbb{N}$, $t \in \mathbb{R}_>$ and $x \in \mathbb{T}^2$ we have*

$$\mathbb{H}_{x,t}(g \circ \boldsymbol{\pi}) = H_{x,t}(g)$$
$$\mathbb{H}_{x,t}(\boldsymbol{g}) = \mathbb{H}_{F^n x, \tau_n(x,t)}(\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g}).$$

Proof. The proof of the first formula is obvious by the definition, the second follows by direct computation using (2.9):

$$\mathbb{H}_{x,t}(\boldsymbol{g}) = \int_0^t ds\, \boldsymbol{g}\left(\phi_s(x), \widehat{V}(\phi_s(x))\right) = \int_0^t ds\, \boldsymbol{g} \circ \mathbb{F}^{-n} \circ \mathbb{F}^n \circ \boldsymbol{\phi}_s(x, \widehat{V}(x))$$

$$= \int_0^{\tau_n(x,t)} ds_1\, \nu_n(\boldsymbol{\pi} \circ \mathbb{F}^{-n} \circ \boldsymbol{\phi}_{s_1}(\mathbb{F}^n(x, \widehat{V}(x))))^{-1} \boldsymbol{g} \circ \mathbb{F}^{-n} \circ \boldsymbol{\phi}_{s_1} \circ \mathbb{F}^n(x, \widehat{V}(x))$$

$$= \int_0^{\tau_n(x,t)} ds_1\, (\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g})(\boldsymbol{\phi}_{s_1}(F^n(x), \widehat{V}(F^n(x)))) = \mathbb{H}_{F^n x, \tau_n(x,t)}(\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g}).$$

$\square$

As already mentioned, the basic fact about the operator $\mathcal{L}_{\mathbb{F}}$ is that there exists Banach spaces $\mathcal{B}^{p,q}$,[14] detailed in Appendix B, to which $\mathcal{L}_{\mathbb{F}}$ can be continuously extended.[15] Moreover, in Appendix B we prove the following result.

Proposition 2.7. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be an Anosov map. Let $p \in \mathbb{N}^*$ and $q \in \mathbb{R}$ such that $p + q \le r$ and $q > 0$. Let $\rho = \exp(h_{top})$ where h_{top} is the topological entropy of F. Then the spectral radius of $\mathcal{L}_{\mathbb{F}}$ on $\mathcal{B}^{p,q}$ is ρ and its essential spectral radius is at most $\rho \lambda^{-\min\{p,q\}}$. In addition, ρ is a simple eigenvalue of $\mathcal{L}_{\mathbb{F}}$ and all the other eigenvalues are strictly smaller in norm.*[16]

This last result has finally made precise our original naive idea: now the operator $\mathcal{L}_{\mathbb{F}}$ has a nice spectral picture and Lemma 2.6 shows that $H_{x,t}(g)$ can be written in terms of similar functionals acting on $\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g}$. Yet, a last difficulty appears: the $\mathbb{H}_{x,s}(\cdot)$, $s \le C_\#$, although uniformly bounded as functionals on $\mathcal{C}^0$, are not uniformly bounded on $\mathcal{B}^{p,q}$, in fact when acting on $\mathcal{B}^{p,q}$, for $p > 0$, they are not even continuous functionals![17] This last obstacle can be dealt with by a more sophisticated representation of $\mathbb{H}_{x,t}(\cdot)$ in terms of uniformly bounded elements of $(\mathcal{B}^{p,q})'$

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[13] Recall Definition 2.3-(ii).

[14] These are more general with respect to the previously mentioned ones. We use the same name to simplify notation and since no confusion can arise.

[15] By a slight abuse of notations we will still call $\mathcal{L}_{\mathbb{F}}$ such an extension.

[16] The conditions on p, q are not optimal. The lack of optimality begin due to the fact that we require $p \in \mathbb{N}$. See [6], and reference therein, for different approaches that remove such a constraint.

[17] This is due to the sharp cut-off of the test function at zero and s, see Lemma B.4.

plus a measure with total variation uniformly bounded in x, t. Such a representation is achieved in Lemma 3.1 which provides the last ingredient needed to close the argument.

Before being able to state precisely our first result we need another little bit of notation. Let $\{\boldsymbol{O}_{i,j}\}_{j=1}^{\boldsymbol{d}_i} \subset (\mathcal{B}^{p,q})'$ be the elements of a base of the eigenspaces associated to the discrete eigenvalues $\{\rho_i\}_{i \geq 1}$, $|\rho_i| > \exp(h_{\text{top}})\lambda^{-\min\{p,q\}}$, of $\mathcal{L}'_{\mathbb{F}}$ when acting on $(\mathcal{B}^{p,q})'$, $p + q \leq r - 1$.[18] Since $\mathcal{C}^{p+q} \subset \mathcal{B}^{p,q}$, we have $(\mathcal{B}^{p,q})' \subset (\mathcal{C}^r)'$. Hence $\{\boldsymbol{O}_{i,j}\} \subset \mathcal{C}^r(\Omega, \mathbb{C})'$. We then define $\tilde{O}_{i,j} = \boldsymbol{\pi}_* \boldsymbol{O}_{i,j}$, clearly $\tilde{O}_{i,j} \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C})'$. Note that $\boldsymbol{\pi}_*$ is far from being invertible, so many different distributions could be mapped to the same one. Thus the dimension d_j of the span of $\{\tilde{O}_{i,j}\}_{j=1}^{\boldsymbol{d}_i}$ will be, in general, smaller than $\boldsymbol{d}_j$ (see Section 5.1 for an explicit example). Let us relabel a subset of the $\tilde{O}_{i,j}$ so that the $\{\tilde{O}_{i,j}\}_{j=1}^{d_i}$ are all linearly independent and set $D_k = \sum_{i \leq k} d_i$. For convenience, let us relabel our distributions $\{O_i\}$, by $O_i = \tilde{O}_{k,l}$ for $i \in [D_k + 1, D_{k+1}]$ and $l = i - D_k$.

Theorem 2.8. *Provided r is large enough,[19] there exists N_1 such that the distributions (obstructions) $\{O_i\}_{i=1}^{N_1} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{C})'$ have the following properties. For each $i \leq N_1$, let $\mathbb{V}_i = \{g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C}) \; : \; O_j(g) = 0 \; \forall j < i \; ; O_i(g) \neq 0\}$. Then there exists $C, \delta > 0$ such that, for all $g \in \mathbb{V}_i$, there exists functions $\hat{\ell}_{k,j} \in L^\infty(\mathbb{T}^2 \times \mathbb{R}_>)$ such that, for all $t \in \mathbb{R}_>$ and $x \in \mathbb{T}^2$, we have*

$$\left| H_{x,t}(g) - t^{\alpha_k} \sum_{j=0}^{b_k - i + D_{k-1}} (\ln t)^j \hat{\ell}_{k,j}(x,t) \right| \leq \left\{ \begin{array}{l} C\, t^{\alpha_k - \delta} \|g\|_{\mathcal{C}^r} \; \text{if } \alpha_k > 0 \\ C \, \|g\|_{\mathcal{C}^r} \; \text{if } \alpha_k = 0, \end{array} \right.$$

where $i \in (D_{k-1}, D_k]$, $\alpha_k = \frac{\ln |\rho_k|}{h_{top}}$ and $b_k = d_k$ if $\alpha_k > 0$ and $b_k = d_k + 1$ if $\alpha_k = 0$. Also $\alpha_1 = 1$, $b_1 = 0$ and $\alpha_{N_1} = b_{N_1} = 0$.

The above Theorem will be proven in Section 3.

Remark 2.9. *Note that in Theorem 2.8 it could happen $N_1 = 1$, that is: either the integral grows like t or it is bounded. This is indeed the situation, for example, in the linear case (see Section 5.1) and hence for small smooth perturbations of the linear case as well (see Section 5.2). In such an event the result might seem less interesting, nonetheless it provides a relevant information.*

Remark 2.10. *A natural question is how to obtain a more explicit identification of the above mentioned distributions. In particular, the analogy with the situations studied by Flaminio-Forni would suggest $(\phi_t)_* O_j = O_j$, that is the distributions are invariant for the flow. However, note that in the present context, for $j > 1$, we know only that $O_j \in (\mathcal{C}^s)'$, for some $s \geq 2$, while in general $\phi_t \notin \mathcal{C}^2$, so $(\phi_t)_* O_j$ is, in principle, not even defined. Nevertheless, we expect some O_j to be invariant distribution for the flow, but the proof is not so obvious. We therefore limit ourselves, in the general case, to discussing O_1, that is known to be a measure (see however Lemma 5.11 for a more in depth discussion of a specific class of examples).*

Lemma 2.11. *The distribution O_1 is proportional to the unique invariant measure μ of ϕ_t.*

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[18] Remark that the compact pat of the spectrum of $\mathcal{L}_{\mathbb{F}}$ and $\mathcal{L}'_{\mathbb{F}}$ coincide (see [38, Remark 6.23]).

[19] For example, $e^{h_{\text{top}}}\lambda^{-r/2} < 1$ suffices. We refrain from giving a more precise characterization of the minimal r since, in the present context, it is not very relevant.

Proof. By Proposition 2.7 it follows that, for all $\boldsymbol{g} \geq 0$,

$$0 \leq \lim_{n \to \infty} \rho^{-n} \mathcal{L}_{\mathbb{F}}^n(\boldsymbol{g}) = \boldsymbol{h}_1 \boldsymbol{O}_1(\boldsymbol{g}),$$

where $\boldsymbol{h}_1$ and $\boldsymbol{O}_1$ are the right and left eigendistributions of $\mathcal{L}_{\mathbb{F}}$ associated to the eigenvalue ρ, respectively. Accordingly, $\boldsymbol{O}_1$ is a positive distribution, and hence a measure, thus also O_1 is a measure. By the ergodic theorem $H_{x,t}(g)$ grows proportional to t unless $g \in \mathbb{V}_0 = \{g \ : \ \mu(g) = 0\}$. By Theorem 2.8 it follows that $\mathrm{Ker}(O_1) \subset \mathbb{V}_0$. On the other hand the kernel of O_1 must be a codimension one closed subspace, hence $\mathrm{Ker}(O_1) = \mathbb{V}_0$. It follows that the two measures must be proportional.[20] $\square$

The next step is to study, in the case $O_i(g) = 0$ for all $i \in \{1, \ldots, N_1\}$, the regularity of the coboundary. As already mentioned (see Remark 1.5) it is natural to consider only $r_i \leq 1 + \alpha$. To study exactly the Hölder regularity would entail either to use a more complex Banach space or an interpolating argument. In the spirit of giving ideas rather than a complete theory, we content ourselves with considering Lipschitz regularity. To do so we have only to consider the derivative of $\overline{H}_T$ with respect to x. To study the growth of such a derivative, it is necessary to introduce new adapted transfer operators $\mathcal{L}_{\hat{A},\mathbb{F}}$ and $\widehat{\mathcal{L}}_{\mathbb{F}}$ the second of which is now defined on one forms (see equation (4.27) for the precise definitions) and acts on different Banach spaces $\widehat{\mathcal{B}}^{p,q}$ (see Appendix C). The Banach spaces $\widehat{\mathcal{B}}^{p,q}$ are a bit more complex than the $\mathcal{B}^{p,q}$ used in Theorem 2.8 insofar they are really spaces of currents rather then distributions (one has to think of dg, rather than g, as an element of the Banach space). Apart from this, the proof of our next result, to be found in Section 4, follows the same logic of the first proof.

Theorem 2.12. *Provided r is large enough,[21] there exist distributions (that we often call* obstructions*)* $\{O_i\}_{i=N_1+1}^{N_2} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$ *such that if $O_i(g) = 0$ for all $i \in \{1, \ldots, N_2\}$, then g is a Lipschitz coboundary. More precisely, for appropriate $p, q, p + q \leq r - 2$, there exists a potential $\widehat{A}$, an operator $\mathcal{L}_{\mathbb{F},\widehat{A}}$ acting on $\mathcal{B}^{p,q}$ and a Banach spaces $\widehat{\mathcal{B}}^{p,q}$ with a transfer operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ (depending on action of the map $\mathbb{F}$ on one forms) acting on it,[22] such that the distributions $\{O_i\}_{i=N_1+1}^{N_2}$ are described in terms of a base of the eigenspaces associated to the discrete eigenvalues of the operators $\mathcal{L}_{\mathbb{F}}, \mathcal{L}_{\mathbb{F},\widehat{A}}$ and $\widehat{\mathcal{L}}_{\mathbb{F}}$.*

Remark 2.13. *Note that, in principle, g could be a Lipschitz coboundary even if it is not in the kernel of the distributions $\{O_i\}_{i=1}^{N_2}$. Indeed, the Theorem provides only sufficient conditions. However, we believe the conditions to be generically also necessary, but to prove this quite some more work seems necessary. We limit ourselves to discussing such issue in a class of examples (see Lemma 5.9).*

The next sections of the paper are devoted to the proof of the above claims.
Last we would like to conclude this section with the following considerations.

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[20] Remark that this implies that O_1 is invariant for the flow ϕ_t. In fact, by using judiciously (2.9) one could have proven directly that $\boldsymbol{O}_1$ is invariant for $\boldsymbol{\phi}_t$. It is possible that such a proof would work also for eigendistributions with eigenvalues with modulus sufficiently close to one. Yet, for smaller eigenvalues the aforementioned regularity problems seem to kick in.

[21] Here r needs to be much larger than in the previous Theorem. A precise estimate is implicit in the proof, but the reader may be better off assuming $F \in \mathcal{C}^\infty$ and not worrying about this issue.

[22] See (4.27) for the exact definition of such operators.

Conjecture 2.14. *The natural analogues of Theorems 2.8 and 2.12 hold in the case of the horocycle flow on a surface of variable strictly negative curvature, where the renormalizing dynamics is the geodesic flow, with the only modification of having an infinite countable family of obstructions.*

Remark 2.15. *The difference between finitely many and countably many obstructions comes from the different spectrum of the transfer operators for maps and flows. In the former, the discrete spectrum is always finite. In the latter, one has a one parameter families of operators and it is then more natural to look at the spectrum of the generator. It turns out that such a spectrum is discrete on the right of a vertical line whose location depends on the flow regularity. Yet, it can have countably many eigenvalues (as the laplacian on hyperbolic surfaces), hence the countably many obstructions (see* [14, 15, 20] *for more details).*

3. Growth of the ergodic average

As already explained, there is one further, and luckily last, conceptual obstacle preventing the naive implementation our strategy: the functionals $\mathbb{H}_{x,t}$ are, in general, not continuous (let alone uniformly continuous with respect to (x,t)) on the spaces $\mathcal{B}^{p,q}$ that are detailed in Appendix B. That is, they do not belong to $(\mathcal{B}^{p,q})'$, for $p \neq 0$.[23] In fact, it is possible to introduce different Banach spaces on which the transfer operator is quasi-compact and the functionals $\mathbb{H}_{x,t}$ are continuous (this are spaces developed to handle piecewise smooth dynamics such as [17, 3, 18, 7]) but the essential spectral radius of our transfer operators on such spaces would always be rather large. Hence we would be able to obtain in this way, at best, only the very firsts among the relevant distributions we are seeking, whereby nullifying the appeal of our approach.

Before providing the proof of Theorem 2.8 we must thus circumvent such a problem. To this end we introduce, for each $x \in \mathbb{T}^2$ and $\varphi \in L^\infty(\mathbb{R}_>, \mathbb{R})$, the new "mollified" functional

$$(3.1) \qquad \mathbb{H}_{x,\varphi}(\boldsymbol{g}) = \int_{\mathbb{R}} \varphi(t) \cdot \boldsymbol{g} \circ \boldsymbol{\phi}_t(x, \widehat{V}(x))\, dt.$$

It is proven in Appendix B that $\mathbb{H}_{x,\varphi} \in (\mathcal{B}^{p,q})'$ provided $\varphi \in \mathcal{C}_0^{p+q}(\mathbb{R}, \mathbb{R})$.

3.1. **Proof of our first main result.**

Our key claim is that the functionals (3.1) suffice for our purposes. To be more precise let us fix $t > 0$ and define the sets $\mathcal{D}_{r,C}^s = \{\varphi \in \mathcal{C}^r([0,t], \mathbb{R}_\geq) \, : \, \|\varphi\|_{\mathcal{C}^r} \leq C\}$ and $\mathcal{D}_{r,C} = \{\varphi \in \mathcal{C}_0^r([0,t], \mathbb{R}_>) \, : \, \|\varphi\|_{\mathcal{C}^r} \leq C\}$, note that such sets are locally compact in $\mathcal{C}^{r-1}([0,t], \mathbb{R})$ and $\mathcal{C}_0^{r-1}([0,t], \mathbb{R})$, respectively.[24]

Lemma 3.1. *There exists $C_* > 0$ such that, for each $n \in \mathbb{N}$, $t \in \mathbb{R}_>$, $x \in \mathbb{T}^2$ and $\boldsymbol{g} \in \mathcal{C}^{r-1}(\Omega, \mathbb{R})$, there exists $K \in \mathbb{N}$, $\{n_i^\pm\}_{i=1}^K \subset \mathbb{N}$, $n_K^\pm = 0$, $C_\# > n_i^\pm - n_{i+1}^\pm \geq 0$,*

---

[23] The problem comes from the boundary in the domain of the integral defining them. There, in some sense, the integrand jumps to zero and cannot be considered smooth in any effective manner.
[24] Up to now the exact definition of the $\mathcal{C}^r$ norms was irrelevant, now instead it does matter. We make the choice $\|\varphi\|_{\mathcal{C}^r} = \sum_{k=0}^r 2^{r-k} \|\varphi^{(k)}\|_{L^\infty}$. It is well known that with such a norm $\mathcal{C}^r$ is a Banach algebra. Also, as usual, for a $\mathcal{C}^r$ function on a closed set, we mean that there exists an extension on some larger open set.

$n_i^- + n_i^+ > n_{i+1}^- + n_{i+1}^+$, and $\{\varphi_i^\pm\}_{i=1}^K \subset \mathcal{D}_{r,C_*}$, $\{\varphi^\pm\} \subset \mathcal{D}_{r,C_*}^s$ such that

$$\mathbb{H}_{x,t}(\boldsymbol{g}) = \sum_{\sigma \in \{+,-\}} \left(\sum_{i=1}^K \mathbb{H}_{F^{n_i^\sigma}(x),\varphi_i^\sigma}(\mathcal{L}_\mathbb{F}^{n_i^\sigma} \boldsymbol{g}) + \mathbb{H}_{x,\varphi^\sigma}(\boldsymbol{g}) \right).$$

Moreover, $\max\{|\operatorname{supp} \varphi^\pm|, |\operatorname{supp} \varphi_i^\pm|\} \le 1$.
Finally $\varphi_1^- = \varphi_1^+$ *and* $n_1^\pm = n_t$ *where* $n_t = \inf\{n \in \mathbb{N} \ : \ \tau_n(x,t) < 1\}$ *satisfies the bounds*

$$(3.2) \qquad\qquad \frac{\ln t}{h_{top}} - C_\# \le n_t \le \frac{\ln t}{h_{top}} + C_\#.$$

Before proving Lemma 3.1, let us use it and prove our first main result.

Proof of Theorem 2.8. By Proposition 2.7 we have

$$(3.3) \qquad\qquad \mathcal{L}_\mathbb{F} = \sum_{j=0}^m (\rho_j \Pi_j + Q_j) + R_{p,q}$$

where m is a finite number, ρ_j, $|\rho_{j+1}| \le |\rho_j| \le e^{h_{\text{top}}}$, are complex eigenvalues of $\mathcal{L}_\mathbb{F}$, Π_j are finite rank projectors, Q_j are nilpotent operators. That is, there exists $\{d_j\}_{j=1}^m$ such that $Q_j^{d_j} = 0$ and, if $d_j > 1$, then $Q^{d_j-1} \ne 0$. Finally, $R_{p,q}$ is a linear operator with spectral radius at most $e^{\beta_{\text{ess}}}$ where $e^{\beta_{\text{ess}}} = \lambda^{-\min(p,q)} e^{h_{\text{top}}}$. In addition, $\Pi_j R_{p,q} = R_{p,q} \Pi_j = Q_j R_{p,q} = R_{p,q} Q_j = 0$. Moreover, for each $i \ne j$, $[\Pi_i, \Pi_j] = [\Pi_i, Q_j] = [Q_i, Q_j] = 0$ and $\Pi_i^2 = \Pi_i$, $\Pi_i Q_i = Q_i \Pi_i = Q_i$. In other words the operator $\mathcal{L}_\mathbb{F}$ is quasi compact and it has a spectral decomposition in Jordan Block of size d_j plus a non compact part of small spectral radius. Note as well that $d_1 = 1$, $Q_1 = 0$ and Π_1 is a one dimensional projection corresponding to the eigenvalue $e^{h_{\text{top}}}$ which is the only eigenvalue of modulus $e^{h_{\text{top}}}$. Finally, set

$$\alpha_j = \frac{\ln |\rho_j|}{h_{\text{top}}} \ ; \quad \widetilde{N}_1 = \sum_{j=1}^m d_j.$$

If r is large enough, we can choose p, q, m and ε such that $|\rho_j| \ge 1$ for all $j \le m$, $\beta_{\text{ess}} < 0$ and $e^{\beta_{\text{ess}}} + \varepsilon < 1$, hence $\sup_n \|R_{p,q}^n\|_{p,q} \le C_\#$. Then, setting $\boldsymbol{g} = g \circ \boldsymbol{\pi}$, Lemmata 2.6, 3.1 and B.4, together with the spectral decomposition (3.3), imply

$$\left| H_{x,t}(g) - \sum_{j=0}^m \sum_{\sigma \in \{+,-\}} \sum_{i=1}^K \mathbb{H}_{F^{n_i^\sigma}(x),\varphi_i^\sigma}((\rho_j \Pi_j + Q_j)^{n_i^\sigma} \boldsymbol{g}) \right| \le C_\# \|g\|_{L^\infty} + C_\# \|R_{p,q}^n \boldsymbol{g}\|_{p,q}$$

$$\le C_\# \|\boldsymbol{g}\|_{p,q}.$$

On the other hand, setting

$$\ell_j(x,t,g) = \begin{cases} \rho_j^{-n_t} n_t^{-d_j+1} \sum_{\sigma \in \{+,-\}} \sum_{i=1}^K \mathbb{H}_{F^{n_i^\sigma}(x),\varphi_i^\sigma}((\rho_j \Pi_j + Q_j)^{n_i^\sigma} \boldsymbol{g}) & \text{if } |\rho_j| > 1 \\ n_t^{-d_j} \sum_{\sigma \in \{+,-\}} \sum_{i=1}^K \mathbb{H}_{F^{n_i^\sigma}(x),\varphi_i^\sigma}((\rho_j \Pi_j + Q_j)^{n_i^\sigma} \boldsymbol{g}) & \text{if } |\rho_j| = 1, \end{cases}$$

we have, in the first case,[25]

$$|\ell_j(x,t,g)| \leq C_\# \sum_{n=1}^{n_t} \rho_j^{-n_t} n_t^{-d_j} \|(\rho_j \Pi_j + Q_j)^n g\|_{p,q}$$

$$\leq C_\# \sum_{n=1}^{n_t} \rho_j^{-n_t} n_t^{-d_j} \rho_j^n n^{d_j} \|g\|_{p,q} \leq C_\# \|g\|_{p,q}$$

and the same estimate holds in the second case. Note that the function ℓ_j have a natural decomposition $\ell_j = \sum_{k=0}^{d_j-1} n_t^k \rho_j^{n_t} \ell_{j,k}$. Collecting the above yields

$$(3.4) \qquad \left| H_{x,t}(g) - \sum_{j=0}^{m} \rho_j^{n_t} \sum_{k=0}^{d_j-1} n_t^k \ell_{j,k}(x,t,g) \right| \leq C_\# \|g\|_{p,q}.$$

To conclude note that $\Pi_j = \sum_{i=1}^{d_j} h_{j,i} \otimes O_{j,i}$ with $h_{j,i} \in \mathcal{B}^{p,q}$ and $O_{j,i} \in (\mathcal{B}^{p,q})' \subset (\mathcal{C}^r(\Omega, \mathbb{R}))'$. Finally, since $\pi^* : \mathcal{C}^r(\mathbb{T}^2, \mathbb{R}) \subset \mathcal{C}^r(\Omega, \mathbb{R})$, we have that $\tilde{O}_{j,i} := \pi_* O_{i,j} \in (\mathcal{C}^r(\mathbb{T}^2, \mathbb{R}))'$, and $O_{j,i}(g) = \tilde{O}_{j,i}(g)$. Note that it might happen $\pi_* O_{j,i} = \pi_* O_{j',i'}$ or $\pi_* O_{j,i} = 0$ (see Section 5.1). Let N_1 be the cardinality of the set $\{\pi_* O_{j,i}\}$. Then, by construction, if $g \in \mathbb{V}_i$, then $\ell_j(x,t,g) \equiv 0$ for all j such that $i \leq D_{j-1}$. Hence the Theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.2. Decomposition in proper functionals.
This section is devoted to showing that the functionals $H_{x,t}$ can be written in terms of well behaved functionals plus a bounded error.

Proof of Lemma 3.1. Fix $x \in \mathbb{T}^2$ and $t \in \mathbb{R}_>$. By definition $\tau_{n_t}(x,t) \in (\Lambda^{-1}, 1)$ for some fixed $\Lambda > 1$.

Let $\delta \in (0, \Lambda^{-1}/4)$ small and $C_* > 0$ large enough to be fixed later. We can now fix $n_1 = n_t$. Note that the claimed bound on n_t follows directly by [31, Lemma C.3]. Next, chose $\psi \in \mathcal{D}_{r,C_*/2}$ such that $\text{supp}\,\psi \subset (\delta, \tau_{n_1} - \delta)$, $\psi|_{[2\delta, \tau_{n_1}-2\delta]} = 1$. Set $\psi^- = (1-\psi)\mathbb{1}_{[0,\tau_{n_1}/2]}$, $\psi^+ = (1-\psi)\mathbb{1}_{[\tau_{n_1}/2,\tau_{n_1}]}$. Then $\psi^\pm \in \mathcal{D}_{r,C_*}^s$ and we can use Lemma 2.6 to write

$$\mathbb{H}_{x,t}(g) = \mathbb{H}_{F^{n_1}(x),\tau_{n_1}(x,t)}(\mathcal{L}_{\mathbb{F}}^{n_1} g)$$

$$= \mathbb{H}_{F^{n_1}(x),\psi^-}(\mathcal{L}_{\mathbb{F}}^{n_1} g) + \mathbb{H}_{F^{n_1}(x),\psi}(\mathcal{L}_{\mathbb{F}}^{n_1} g) + \mathbb{H}_{F^{n_1}(x),\psi^+}(\mathcal{L}_{\mathbb{F}}^{n_1} g).$$

We are happy with the middle term which, by Lemma B.4, is a continuous functional of $\mathcal{L}_{\mathbb{F}}^{n_1} g$, not so the other two terms. We have thus to take care of them. A computation analogous to the one done in Lemma 2.6 yields, for each $n \in \mathbb{N}$,

$$(3.5) \qquad\qquad \mathbb{H}_{x,\varphi \circ \tau_n(x,\cdot)}(g) = \mathbb{H}_{F^n(x),\varphi}(\mathcal{L}_{\mathbb{F}}^n g).$$

We will use the above to prove inductively the formula

$$\mathbb{H}_{x,t}(g) = \mathbb{H}_{F^{n_k^-}(x),\psi_k^-}(\mathcal{L}_{\mathbb{F}}^{n_k^-} g) + \mathbb{H}_{F^{n_k^+}(x),\psi_k^+}(\mathcal{L}_{\mathbb{F}}^{n_k^+} g)$$

$$(3.6)$$

$$+ \sum_{\sigma \in \{+,-\}} \sum_{i=1}^{k} \mathbb{H}_{F^{n_i^\sigma}(x),\varphi_i^\sigma}(\mathcal{L}_{\mathbb{F}}^{n_i^\sigma} g)$$

---

[25] Note that the $n_i^\pm$ in Lemma 3.1 cannot be more than n_t, hence $K \leq n_t$.

where $\psi_1^{\pm} = \psi^{\pm}$, $\varphi_1^{\pm} = \frac{1}{2}\psi$, $n_1^{\pm} = n_1$, $\psi_k^{\pm} \in \mathcal{D}_{r,C_*}^s$, $\{\varphi_i^{\pm}\} \subset \mathcal{D}_{r,C_*}$, $\|\psi_k^{\pm}\|_{L^{\infty}} \leq 1$, $\|\varphi_k^{\pm}\|_{L^{\infty}} \leq 1$, $(b_k^{\pm}, b_k^{\pm} \mp \delta) \subset \operatorname{supp}\psi_k^{\pm} \subset (b_k^{\pm}, b_k^{\pm} \mp 2\delta)$, $\operatorname{supp}\varphi_k^{\pm} \subset (b_k^{\pm}, b_k^{\pm} \mp 1)$, $b_k^- = 0$ and $b_k^+ \in [0, \Lambda^{n_k^+}]$, $b_1^+ = t$.

Let us consider the first term on the right hand side of the first line of (3.6) (the second one can be treated in total analogy). Let $\operatorname{supp}(\psi_k^-) = [0, a_k]$ and define $m + 1 = \inf\{n \in \mathbb{N} \ : \ \tau_n^{-1}(F^{n_k^-}(x), a_k) \geq 1\}$. Note that, by construction, there exists a fixed $\overline{m} \in \mathbb{N}$ such that $m \geq \overline{m}$, also $\overline{m}$ can be made large by choosing δ small. Hence $\widehat{\psi_k^-}(s) = \psi_k^- \circ \tau_m(F^{n_k^-}(x), s)$ is supported in the interval $[0, 1)$ and the support contains $[0, \Lambda^{-1}]$.

Next, we need an estimate on the norm of $\widehat{\psi_k^-}$. We state it in a sub-lemma so the reader can easily choose to skip the, direct but rather tedious, proof.

Sub-Lemma 3.2. *Provided we choose δ small and C_* large enough, we have*

$$\widehat{\psi_k^-} \in \mathcal{D}_{r,C_*/2},$$

where $n_{k+1}^- = n_k^- - m$.

Proof. First of all $\|\psi_k^-\|_{L^{\infty}} \leq 1$, and[26]

$$
\begin{aligned}
\|\widehat{\psi_k^-}\|_{\mathcal{C}^r} &\leq \sum_{j=0}^{r} 2^{r-j} \|\psi_k^-\|_{\mathcal{C}^j} \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-1}} \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-2}} \cdots \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-j}} \\
&\leq 2^r + C_* \sum_{j=1}^{r} \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-1}} \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-2}} \cdots \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{r-j}}
\end{aligned}
$$

(3.7)

where $z_k = F^{n_k^-}(x)$ and, for each $j \in \mathbb{N}$ and $z \in \mathbb{T}^2$, $\tilde{\nu}_{z,m}(s) = \nu_m(\phi_s(z))$, where ν_m is defined in (1.3). Note that, although ν_m is, in general, only $\mathcal{C}^{1+\alpha}$, by (1.4) it follows that the map $s \to \tilde{\nu}_{z,m}(s) \in \mathcal{C}^{r-1}$ and hence, for all $z \in \mathbb{T}^2$, $\langle V, \nabla\tilde{\nu}_z \rangle \circ \phi_{(\cdot)} \in \mathcal{C}^{r-2}$. We can thus continue and compute

$$
\begin{aligned}
\frac{d}{ds} \tilde{\nu}_{z_k,m}(s) &= \frac{d}{ds} \prod_{j=0}^{m-1} \nu_1(F^j(\phi_s(z_k))) \\
&= \tilde{\nu}_{z_k,m}(s) \sum_{l=0}^{m-1} \frac{\langle \nabla\nu_1(F^l \circ \phi_s(z_k)), V(F^l \circ \phi_s(z_k)) \rangle}{\nu_1(F^l \circ \phi_s(z_k))} \tilde{\nu}_{z_k,l}(s) \\
&= \tilde{\nu}_{z_k,m}(s) \sum_{l=0}^{m-1} \left[\frac{\langle V, \nabla\nu_1 \rangle}{\nu_1}\right] \circ \phi_{\tau_l(z_k,s)}(F^l(z_k)) \tilde{\nu}_{z_k,l}(s).
\end{aligned}
$$

The above, by induction, implies that there exist increasing constants $A_q \geq 1$ such that $\|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^q} \leq A_q \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^0}$. Indeed, $\left[\frac{\langle V, \nabla\nu_1 \rangle}{\nu_1}\right] \circ \phi_{\cdot} \in \mathcal{C}^{r-2}$, and

$$
\begin{aligned}
\left\|\left[\frac{\langle V, \nabla\nu_1 \rangle}{\nu_1}\right] \circ \phi_{\tau_l(z_k,\cdot)}(F^l(z_k))\right\|_{\mathcal{C}^q} &\leq \sum_{i=0}^{q} 2^{q-i} \left\|\left[\frac{\langle V, \nabla\nu_1 \rangle}{\nu_1}\right] \circ \phi_{\cdot}(F^l(z))\right\|_{\mathcal{C}^q} \|\tilde{\nu}_{F^l(z_k),l}\|_{\mathcal{C}^{q-1}}^i \\
&\leq C_{\#} \sum_{i=0}^{q} A_{q-1}^i \lambda^{-il} \leq C_{\#} A_{q-1}^q.
\end{aligned}
$$

---

[26] Here we use the formula $\|f \circ g\|_{\mathcal{C}^r} \leq \sum_{k=0}^{r} 2^{r-k} \|f\|_{\mathcal{C}^k} \|Dg\|_{\mathcal{C}^{r-1}} \|Dg\|_{\mathcal{C}^{r-2}} \cdots \|Dg\|_{\mathcal{C}^{r-k}}$, that can be verified by induction.

Thus,

$$\|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^{q+1}} = 2^q \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^0} + \|\frac{d}{ds}\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^q}$$

$$\leq 2^q \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^0} + \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^q} \sum_{l=0}^{m-1} C_\# A_{q-1}^q A_{q-1} \lambda^{-l}$$

$$\leq \left[2^q + A_q C_\# A_{q-1}^{q+1} \right] \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^0} =: A_{q+1} \|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^0}.$$

We did not try to optimize the above computation since the only relevant point is that the A_q do not depend on m. Accordingly, if we choose δ small (and hence m large) enough, we have $\|\tilde{\nu}_{z_k,m}\|_{\mathcal{C}^q} \leq \frac{1}{4}$ for all $q \leq r-1$. Using the above fact in (3.7) yields

$$\|\widehat{\psi_k^-}\|_{\mathcal{C}^r} \leq 2^r + C_* \sum_{j=1}^r 4^{-j} = 2^r + \frac{1}{3}C_*$$

which implies the Lemma provided we choose C_* large. □

By (3.5), we have

$$\mathbb{H}_{F^{n_k^-}(x),\psi_k^-}(\mathcal{L}_\mathbb{F}^{n_k^-} \boldsymbol{g}) = \mathbb{H}_{F^{n_{k+1}^-}(x),\widehat{\psi_k^-}}(\mathcal{L}_\mathbb{F}^{n_{k+1}^-} \boldsymbol{g}).$$

Again we can write $\psi_{k+1}^- = (1-\psi)\widehat{\psi_k^-}\mathbb{1}_{[0,2\delta]}$ and $\varphi_{k+1}^- = \widehat{\psi_k^-} - \psi_{k+1}^-$. Then,

$$\sup\{\|\psi_{k+1}^-\|_{\mathcal{C}^r([0,1],\mathbb{R})}, \|\varphi_{k+1}^-\|_{\mathcal{C}_0^r(\mathbb{R},\mathbb{R})}\} \leq C_*$$

and $[0,\delta] \subset \operatorname{supp} \psi_{k+1}^- \subset [0,2\delta]$. Accordingly

$$\mathbb{H}_{F^{n_k^-}(x),\psi_k^-}(\mathcal{L}_\mathbb{F}^{n_k^-} \boldsymbol{g}) = \mathbb{H}_{F^{n_{k+1}^-}(x),\psi_{k+1}^-}(\mathcal{L}_\mathbb{F}^{n_{k+1}^-} \boldsymbol{g}) + \mathbb{H}_{F^{n_{k+1}^-}(x),\varphi_{k+1}^-}(\mathcal{L}_\mathbb{F}^{n_{k+1}^-} \boldsymbol{g}).$$

The Lemma is thus proven by taking $k = K$, so that $n_K^\pm = 0$.[27] □

4. Coboundary regularity

We first prove several claims stated in the introduction and set up some notation. Then we prove our main results concerning coboundary regularity.

4.1. **Parabolic.**
In the introduction we called the flow ϕ_t *parabolic*, but no evidence was provided for this name. It is now time to substantiate such an assertion.

Remark 4.1. *Note that the following Lemma shows only that the differential cannot grow more than polynomially, yet the possibility remains open that it does not grow at all, as in the linear model (or when the map is Lipschitz conjugated to the linear model). In such a case the flows should be more properly called* elliptic. *This is not always the case, as one can see in an explicit class of examples worked out in Section 5.2 (see Lemma 5.16).*

Lemma 4.2. *There exists $C, \beta > 0$ such that, for all $x \in \mathbb{T}^2$ and $t \in \mathbb{R}$, letting $\xi(s) = D_x\phi_s$, we have*

$$\|\xi\|_{\mathcal{C}^{r-1}((0,t),GL(2,\mathbb{R}))} \leq C|t|^\beta.$$

---

[27] If more steps are needed on one side, say the plus side, one can simply set $n_{k+1}^- = n_k^-$, $\psi_{k+1}^- = \psi_k^-$ and $\varphi_k^- = 0$ for all the extra steps.

Proof. Since $\phi_{-t} = \phi_t^-$, where ϕ_t^- is the flow generated by $-V$, and the following argument is insensitive to orientation, it suffices to consider the case $t \geq 0$. It turns out to be convenient to define $V^\perp(x)$ as the perpendicular vector to $V(x)$ such that $\|V^\perp(x)\| = \|V(x)\|^{-1}$. In this way we can use $\{V(x), V^\perp(x)\}$ as basis of the tangent space at x, and the changes of variable are uniformly bounded, with determinant one and $\mathcal{C}^r$ in the flow direction. In such coordinates we have

$$(4.1) \qquad D_x\phi_t = \begin{pmatrix} 1 & a(x,t) \\ 0 & b(x,t) \end{pmatrix}.$$

To have a more precise understanding of the above matrix elements, we have to use the knowledge that the dynamics is renormalizable. To start with we must differentiate (1.7):

$$(4.2) \begin{aligned} D_{\phi_t(x)}F^n \cdot D_x\phi_t &= D_{F^n(x)}\phi_{\tau_n(x,t)} \cdot D_xF^n + V(\phi_{\tau_n(x,t)}(F^n(x))) \otimes \nabla\tau_n(x,t) \\ &= D_{F^n(x)}\phi_{\tau_n(x,t)} \cdot D_xF^n \left[\mathbb{1} + \nu_n(x)^{-1}V(x) \otimes \nabla\tau_n(x,t) \right], \end{aligned}$$

where we have used (1.3) and (2.8). Hence, setting

$$(4.3) \qquad A_{x,t,n} = \mathbb{1} + \nu_n(x)^{-1}V(x) \otimes \nabla\tau_n(x, \tau_n^{-1}(x,t)),$$

we have

$$(4.4) \qquad D_x\phi_t = D_{\phi_{\tau_n(x,t)} \circ F^n(x)}F^{-n} \cdot D_{F^n(x)}\phi_{\tau_n(x,t)} \cdot D_xF^n \cdot A_{x,\tau_n(x,t),n}.$$

Thus, by equations (4.1) and (4.4) we have, for each $n \in \mathbb{N}$,

$$b(x,t) = \langle V^\perp(\phi_t(x)), D_{\phi_{\tau_n(x,t)} \circ F^n(x)}F^{-n} \cdot D_{F^n(x)}\phi_{\tau_n(x,t)} \cdot D_xF^n \cdot A_{x,\tau_n(x,t),n}V^\perp(x)\rangle$$

$$(4.5) \quad = \langle V^\perp(\phi_t(x)), D_{\phi_{\tau_n(x,t)} \circ F^n(x)}F^{-n} \cdot D_{F^n(x)}\phi_{\tau_n(x,t)} \cdot D_xF^nV^\perp(x)\rangle.$$

Choose n so that $\tau_n \in [\Lambda^{-1}, 1]$, hence n is proportional to $\ln t$. By compactness it follows that $\|D_{F^n(x)}\phi_{\tau_n(x,t)}\| \leq C_\#$. Hence, there exists $\beta_0 > 0$ such that

$$\sup_{x \in \mathbb{T}^2} |b(x,t)| \leq C_\# t^{\beta_0}.$$

On the other hand, by the semigroup property, for each $m \in \mathbb{N}$,

$$D_x\phi_m = \prod_{i=0}^{m-1} \begin{pmatrix} 1 & a(\phi_i(x),1) \\ 0 & b(\phi_i(x),1) \end{pmatrix} = \begin{pmatrix} 1 & \sum_{j=0}^{m-1} a(\phi_j(x),1)b(x,j) \\ 0 & b(x,m) \end{pmatrix}.$$

Since, again by compactness, $|a(x,1)| \leq C_\#$, it follows

$$|a(x,m)| \leq C_\# \sum_{j=0}^{m-1} |b(x,j)| \leq C_\# \sum_{j=0}^{m-1} j^{\beta_0} \leq C_\# m^{\beta_0+1}.$$

Hence

$$\|\xi\|_{\mathcal{C}^0((0,t),GL(2,\mathbb{R}))} \leq C_\# t^\beta$$

with $\beta = \beta_0 + 1$.

To estimate the derivatives notice that $\dot\xi(s) = D_{\phi_s(x)}V\xi(s)$. To understand the regularity of the above equation, recall that the stable foliation can be expressed in local coordinates by $(x_1, G(x_1, x_2))$, where $G(\cdot, x_2) \in \mathcal{C}^r$, $G(0, x_2) = x_2$, so that $\{(x_1, G(x_1, x_2))\}_{x_1 \in \mathbb{R}}$ is the leaf through the point $x = (x_1, x_2)$, and $(1, \partial_{x_1}G(x)) = V(x)$. It is known that, in such coordinates, $\partial_{x_2}G(\cdot, x_2) \in \mathcal{C}^{r-1}$ uniformly, see [35] and references therein. Then, by Schwarz Theorem [48], if follows

that $\partial_{x_2}\partial_{x_1}G(\cdot,x_2) \in \mathcal{C}^{r-2}$. Hence, $DV \circ \phi_t$ is a $\mathcal{C}^{r-2}$ function of t, with uniformly bounded norm. Accordingly, for each $k \in \{0, \ldots, r-2\}$,

$$\|\xi\|_{\mathcal{C}^{k+1}((0,t),GL(2,\mathbb{R}))} \leq \|\dot{\xi}\|_{\mathcal{C}^k((0,t),GL(2,\mathbb{R}))} + 2^{k+1}\|\xi\|_{\mathcal{C}^0((0,t),GL(2,\mathbb{R}))}$$
$$\leq C_k\|\xi\|_{\mathcal{C}^k((0,t),GL(2,\mathbb{R}))},$$

from which the Lemma readily follows. □

4.2. Some preliminary facts.

In this section we establish some facts and formulae needed in the following.

First of all recall that given a one form $\omega(x) = \sum_{i=1}^{2} a_i(x)dx_i$ and a diffeomorphism $G \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{T}^2)$ the pullback of the form is given by

$$(4.6) \qquad G^*\omega(x) = a_i(G(x))(D_xG)_{ij}dx_j,$$

where we have used the usual convention on the summation of repeated indexes; moreover recall that for a vector field v the pushforward is given by

$$G_*v(x) = D_{G^{-1}(x)}G \cdot v(G^{-1}(x)).$$

Next, we spell out the cocycle properties of τ_n.

Lemma 4.3. *For each $x \in \mathbb{T}^2$, $n, m \in \mathbb{N}$ we have*

$$\tau_m(F^n(x), \tau_n(x,s)) = \tau_{n+m}(x,s)$$

Proof. By definition

$$\tau_m(F^n(x), \tau_n(x,t)) = \int_0^{\tau_n(x,t)} \nu_m(\phi_s(F^n(x)))ds$$
$$= \int_0^t \nu_m(\phi_{\tau_n(x,s)}(F^n(x)))\nu_n(\phi_s(x))ds$$
$$= \int_0^t \nu_m(F^n \circ \phi_s(x))\nu_n(\phi_s(x))ds$$
$$= \int_0^t \nu_{n+m}(\phi_s(x))ds = \tau_{n+m}(x,t),$$

where we have used (1.7). □

By Lemma 4.3, and using (1.9), (1.7), we can write, for all $n \in \{0, \ldots, n_T\}$,[28]

$$(4.7) \qquad \overline{H}_T(g)(x) = -\int_{\mathbb{R}_{\geq}} \chi \circ \tau_{n_T-n}(F^n(x), s)\left(\frac{g}{\nu_n}\right) \circ F^{-n} \circ \phi_s \circ F^n(x)ds.$$

As we are now aware of the fact that the discontinuity of the test function χ at zero will create problems,[29] we take care of the problem right away. Given $\varpi \in (0, 1/4)$, small enough, let $T > 0$ and $n_* \in \mathbb{N}$ be large enough and such that $\sup_{z \in \mathbb{T}^2} \tau_{n_*}(z, 1) \leq \varpi$ and $n_T \geq n_*$. Then, we can chose $n = n_T - n_*$ and write

$$\chi \circ \tau_{n_*}(F^{n_T-n_*}(x), s) = \chi(s)\chi \circ \tau_{n_*}(F^{n_T-n_*}(x), s) + (1 - \chi(s))\chi \circ \tau_{n_*}(F^{n_T-n_*}(x), s)$$
$$= \chi(s) + (1 - \chi(s))\chi \circ \tau_{n_*}(F^{n_T-n_*}(x), s).$$

---

[28] We have used the fact that, by definition, $\tau_{n_T}(x, T) \geq 1$ while supp $\chi \subset [0, 1]$.

[29] The integral will not belong to the dual of the appropriate Banach space.

Thus, setting $\chi_*(z,s) = (1 - \chi(\tau_{n_*}^{-1}(F^{-n_*}(z),s)))\chi(s)$, we can write (4.7) as

$$
\begin{aligned}
\overline{H}_T(g)(x) = &- \int_{\mathbb{R}} \chi_*(F^{n_T}(x),s)\left(\frac{g}{\nu_{n_T}}\right) \circ F^{-n_T} \circ \phi_s \circ F^{n_T}(x)ds \\
(4.8) \qquad &- \int_{\mathbb{R}_{\geq}} \chi \circ \tau_{n_T-n_*}(x,s)g \circ \phi_s(x)ds \\
=: &\, \overline{H}_T^{\star}(g)(x) - \int_{\mathbb{R}_{\geq}} \chi \circ \tau_{n_T-n_*}(x,s)g \circ \phi_s(x)ds.
\end{aligned}
$$

Note that the first term contains now a smooth compactly supported test function, while the second term is exactly of the same initial form, that is (1.9), (apart from the fact that n_T is replaced by $n_T - n_*$). Hence, it suffices to study $\overline{H}_T^{\star}(g)(x)$. Before doing so let us show that $\overline{H}_T(g)$ is really the right quantity to consider.

4.3. Continuous coboundary.

In Section 3.1 we have seen that if g belongs to the kernel of enough distributions O_i, then the $H_{x,t}(g)$ are all uniformly bounded. In the introduction we claimed that this implies that g is a continuous coboundary, now is the time to prove it.

Proof of Lemma 1.4. Setting, as before, $\boldsymbol{g} = g \circ \boldsymbol{\pi}$, by equations (1.9), (3.1), (3.5) and using repeatedly formula (4.8) we have

$$
\begin{aligned}
\overline{H}_T(g)(x) &= -\mathbb{H}_{x,\chi\circ\tau_{n_T}(x,\cdot)}(\boldsymbol{g}) \\
&= -\sum_{l=0}^{K_T} \mathbb{H}_{F^{n_*l}(x),\chi_*(F^{n_*l}(x),\cdot)}(\mathcal{L}_{\mathbb{F}}^{n_*l}\boldsymbol{g}) - \int_{\mathbb{R}_{\geq}} \chi(s)g \circ \phi_s(x)ds,
\end{aligned}
$$

where $K_T n_* = n_T$. By hypothesis $\mathcal{O}_i(g) = 0$ for all $i \in \{1,\dots,N_1\}$, hence $\|\mathcal{L}_{\mathbb{F}}^n\boldsymbol{g}\|_{p,q} \leq C_g\theta^n$ for some $\theta \in (0,1)$ (see the proof of Theorem 2.8 for details). Thus, by Lemma B.4, it follows that also $|\mathbb{H}_{F^n(x),\chi_*(F^{n_*l}(x),\cdot)}(\mathcal{L}_{\mathbb{F}}^n\boldsymbol{g})| \leq C_g\theta^n$. Hence, for all $T' \geq T > 0$,

$$
(4.9) \qquad \|\overline{H}_T(g) - \overline{H}_{T'}(g)\|_{\mathcal{C}^0} \leq C_g\theta^{n_T},
$$

which, recalling (3.2), implies the existence of the limit.

To prove the second statement of the Lemma, observe that, recalling the properties of χ specified after (1.9),

$$
\begin{aligned}
\langle V(x), \nabla\overline{H}_T(g)(x)\rangle = &- \int_0^T dt\, \chi \circ \tau_{n_T}(x,t)\langle D_x\phi_t V(x), (\nabla g) \circ \phi_t(x)\rangle \\
&- \int_0^T dt\, \chi' \circ \tau_{n_T}(x,t)\left[\int_0^t \langle D_x\phi_s V(x), (\nabla\nu_{n_T}) \circ \phi_s(x)\rangle ds\right]g \circ \phi_t(x)dt \\
= &- \int_0^T dt\, \chi \circ \tau_{n_T}(x,t)\langle V(\phi_t(x)), (\nabla g) \circ \phi_t(x)\rangle \\
&- \int_0^T dt\, \chi' \circ \tau_{n_T}(x,t)\left[\int_0^t \langle V(\phi_s(x)), (\nabla\nu_{n_T}) \circ \phi_s(x)\rangle ds\right]g \circ \phi_t(x)dt
\end{aligned}
$$

where we have used (2.8) and the notation of the previous section. Hence,

(4.10)
$$\langle V(x), \nabla \overline{H}_T(g)(x) \rangle = -\int_0^T dt\, \chi \circ \tau_{n_T}(x,t) \left(\frac{d}{dt} g \circ \phi_t(x) \right)$$

$$- \int_0^T dt\, \chi' \circ \tau_{n_T}(x,t) \left[\nu_{n_T} \circ \phi_t(x) - \nu_{n_T}(x) \right] g \circ \phi_t(x) dt$$

$$= -\int_0^T dt\, \frac{d}{dt} \left(\chi \circ \tau_{n_T}(x,t) g \circ \phi_t(x) \right) + \nu_{n_T}(x) \int_0^T dt\, \chi' \circ \tau_{n_T}(x,t) g \circ \phi_t(x)$$

$$= g(x) + \nu_{n_T}(x) \int_0^T dt\, \chi' \circ \tau_{n_T}(x,t) g \circ \phi_t(x).$$

On the other hand

$$\int_0^T dt\, \chi' \circ \tau_{n_T}(x,t) g \circ \phi_t(x) = \mathbb{H}_{F^n(x), \chi'}(\mathcal{L}_{\mathbb{F}}^n \boldsymbol{g}),$$

which, by the same argument as before, is uniformly bounded. Integrating (4.10) along the flow, yields, for all $t \in \mathbb{R}$,

$$\overline{H}_T(g)(\phi_t(x)) - \overline{H}_T(g)(x) = \int_0^t \frac{d}{ds} \overline{H}_T(g)(\phi_s(x)) ds$$

$$= \int_0^t \langle V(\phi_s(x)), \nabla \overline{H}_T(g)(\phi_s(x)) \rangle ds = \int_0^t g \circ \phi_s(x) ds + \mathcal{O}(\lambda^{-n_T} t).$$

The Lemma follows remembering (3.2) and taking the limit $T \to \infty$. □

To study the coboundary regularity we will investigate the regularity of $\overline{H}_t(g)$. In reality, we will investigate only the first derivative, see Remark 1.5 for a discussion of this choice.

4.4. An explicit formula for the coboundary derivative.
Our goal here is to establish an explicit formula for the derivative of (4.8).

For each vector field $\boldsymbol{v} \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R}^2)$, noticing that[30]

$$\nabla_z \chi_*(z,s) = -\chi(s) \chi'(\tau_{n_*}^{-1}(F^{-n_*}(z), s)) \frac{(D_z F^{-n_*})^* \nabla_z \tau_{n_*}(F^{-n_*}(z), \tau_{n_*}^{-1}(F^{-n_*}(z), s))}{(\partial_t \tau_{n_*})(F^{-n_*}(z), \tau_{n_*}^{-1}(F^{-n_*}(z), s))},$$

and setting, $\boldsymbol{s}_*(x,s) = \tau_{n_*}^{-1}(F^{-n_*}(x), s)$,

(4.11)
$$\vartheta_j(x,s) = -\chi(s) \chi'(\boldsymbol{s}_*(x,s)) \frac{\langle \nabla_x \tau_{n_*}(F^{-n_*}(x), \boldsymbol{s}_*(x,s)), D_x F^{j-n_*} \boldsymbol{v} \rangle}{\nu_{n_*} \circ \phi_{\boldsymbol{s}_*(x,s)}(F^{-n_*}(x))}$$

we have that

$$\langle \boldsymbol{v}(x), \nabla \overline{H}_T^\star(g)(x) \rangle = -\int_{\mathbb{R}} ds\, \vartheta_{n_T}(F^{n_T}(x), s) \left(\frac{g}{\nu_{n_T}} \right) \circ F^{-n_T} \circ \phi_s \circ F^{n_T}(x)$$

$$+ \int_{\mathbb{R}} ds\, \chi_*(F^{n_T}(x), s) \langle D_x(F^{-n_T} \circ \phi_s \circ F^{n_T}) \boldsymbol{v}, \left[\frac{g}{\nu_{n_T}^2} \nabla \nu_{n_T} \right] \circ F^{-n_T} \circ \phi_s \circ F^{n_T}(x) \rangle$$

$$- \int_{\mathbb{R}} ds\, \chi_*(F^{n_T}(x), s) \langle D_x(F^{-n_T} \circ \phi_s \circ F^{n_T}) \boldsymbol{v}, \frac{\nabla g}{\nu_{n_T}} \circ F^{-n_T} \circ \phi_s \circ F^{n_T}(x) \rangle.$$

---

[30] Given a matrix A we use A^* to designate the transpose.

Recall that

$$(4.12) \qquad \nabla \nu_n = \nabla \prod_{j=0}^{n-1} \nu_1 \circ F^j = \sum_{j=0}^{n-1} \nu_n (DF^j)^* \left[\frac{\nabla \nu_1}{\nu_1} \circ F^j \right].$$

Setting (with a mild abuse of notation) $\chi_*(s) = \chi_*(F^{n_T}(x), s)$, we can write

$$
\begin{aligned}
(4.13) \quad \langle \boldsymbol{v}(x), \nabla \overline{H}_T^{\star}(g)(x) \rangle &= - \int_{\mathbb{R}} dt\, \vartheta_{n_T}(F^{n_T}(x), t) \left(\frac{g}{\nu_{n_T}} \right) \circ F^{-n_T} \circ \phi_t \circ F^{n_T}(x) \\
&\quad - \int_{\mathbb{R}} dt\, \chi_*(t) \frac{[(F^{-n_T} \circ \phi_t)^* dg]\,(F_*^{n_T} \boldsymbol{v})}{\nu_{n_T} \circ F^{-n_T} \circ \phi_t} \circ F^{n_T}(x) \\
&\quad + \sum_{j=0}^{n_T - 1} \int_{\mathbb{R}} dt\, \chi_*(t) \frac{\left[(F^{-n_T+j} \circ \phi_t)^* (\mathcal{L}_F^j g \cdot d \ln \nu_1) \right] (F_*^{n_T} \boldsymbol{v})}{\nu_{n_T - j} \circ F^{-n_T+j} \circ \phi_t} \circ F^{n_T}(x) dt.
\end{aligned}
$$

Next, we need an explicit formula for $d \ln \nu_1$. To this end notice that

$$(4.14) \qquad \partial_{x_k} V = p_k^* V + p_k \hat{V}^\perp,$$

where $(\hat{V}_1, \hat{V}_2) = \hat{V} = \|V\|^{-1} V$ and $\hat{V}^\perp = (-\hat{V}_2, \hat{V}_1)$. Then, differentiating $\|V\|^2$ and $DFV = \nu_1 V \circ F$, respectively, we have

$$
\begin{aligned}
(4.15) \quad p_k^* &= \partial_{x_k} \ln \|V\| \in C^{r-1} \\
(\partial_{x_k} DF) V + p_k DF \hat{V}^\perp &= -p_k^* \nu_1 V \circ F + \partial_{x_k} \nu_1 V \circ F \\
&\quad + \nu_1 \sum_j \partial_{x_k} F_j \left[p_j^* V + p_j \hat{V}^\perp \right] \circ F.
\end{aligned}
$$

Multiplying the latter by $\hat{V}^\perp \circ F$, since $DF^*(\hat{V}^\perp \circ F) = \frac{\|V\| \det DF}{\nu_1 \|V \circ F\|} \hat{V}^\perp$, yields

$$\langle \hat{V}^\perp \circ F, \partial_{x_k} DFV \rangle + p_k \frac{\|V\| \det DF}{\|V\| \circ F \nu_1} = \nu_1 \sum_j \partial_{x_k} F_j p_j \circ F.$$

Note that, due to the condition that F is uniformly hyperbolic we can assume (eventually using a power of F instead of F)

$$(4.16) \qquad \frac{\|V\| \det DF}{\|V\| \circ F \nu_1} > \lambda > 1 > \lambda^{-1} > \nu_1;$$

hence, setting

$$(4.17) \qquad \Gamma(x, v)_k = \langle (D_x Fv)^\perp, \partial_{x_k} D_x Fv \rangle \frac{\|V(x)\|}{\det D_x F},$$

we have[31]

$$(4.18) \qquad p = \frac{\nu_1^2 \|V\| \circ F}{\|V\| \det DF} DF^* p \circ F - \Gamma(\hat{V}).$$

It is then natural to set

$$(4.19) \qquad A = \frac{\nu_1^2 \|V\| \circ F}{\|V\| \det DF}$$

---

[31] By $\Gamma(\hat{V})$ we mean the function $\Gamma(\cdot, \hat{V}(\cdot))$ and $p = (p_1, p_2) \in \mathbb{R}^2$.

and write[32]

$$(4.20) \qquad p = -\sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} A \circ F^j \right] (DF^m)^* \Gamma(\hat{V}) \circ F^m.$$

In the same way, but multiplying the second of (4.15) by $\hat{V} \circ F$, we obtain

$$\partial_{x_k} \ln \nu_1 = p_k E + B_k$$

$$(4.21) \qquad B_k = \frac{\langle \hat{V} \circ F, (\partial_{x_k} DF) \hat{V} \rangle \|V\|}{\nu_1 \|V\| \circ F} + p_k^* - (DF^* p^* \circ F)_k$$

$$E = \frac{\langle \hat{V} \circ F, DF \, \hat{V}^{\perp} \rangle}{\nu_1 \|V\| \circ F}.$$

Note that, by equations (4.21) and (4.20),

$$(DF^{-n_T+j})^*(\nabla \ln \nu_1) \circ F^{-n_T+j} = (DF^{-n_T+j})^* B \circ F^{-n_T+j}$$

$$(4.22) \qquad - E \circ F^{-n_T+j} \sum_{m=0}^{n_T-j} \left[\prod_{l=0}^{m-1} A \circ F^{l-n_T+j} \right] (DF^{-n_T+j+m})^* \Gamma(\hat{V}) \circ F^{-n_T+j+m}$$

$$- E \circ F^{-n_T+j} \sum_{m=1}^{\infty} \left[\prod_{l=-n_T+j}^{m-1} A \circ F^l \right] (DF^m)^* \Gamma(\hat{V}) \circ F^m.$$

We can thus express the last line of (4.13) in terms of the transfer operators (acting on one forms $\omega = \langle \bar{\omega}(x), dx \rangle$ and functions g, respectively)

$$(4.23) \qquad \begin{aligned} (\widehat{\mathcal{L}}_F \omega)_x &= \langle (D_x F^{-1})^*(\nu_1^{-1} \bar{\omega}) \circ F^{-1}(x), dx \rangle \\ \mathcal{L}_{F,A} g &= \mathcal{L}_F(Ag). \end{aligned}$$

Indeed, using the above notation, equation (4.22), and setting $\omega_B = \langle B, dx \rangle$ and $\omega_\Gamma = \langle \Gamma(\hat{V}), dx \rangle$, allows to rewrite (4.13) as

$$\langle \boldsymbol{v}(F^{-n_T}(x)), \nabla \overline{H}_T^{\star}(g)(F^{-n_T}(x)) \rangle = -\int_{\mathbb{R}} dt\, \vartheta_{n_T}(x,t)(\mathcal{L}_F^{n_T} g) \circ \phi_t(x)$$

$$- \int_{\mathbb{R}} dt\, \chi_*(x,t) \{ [\phi_t^* \widehat{\mathcal{L}}_F^{n_T} dg](F_*^{n_T} \boldsymbol{v}) \}(x)$$

$$+ \sum_{j=0}^{n_T-1} \int_{\mathbb{R}} dt\, \chi_*(x,t) \{ [\phi_t^* \widehat{\mathcal{L}}_F^{n_T-j}((\mathcal{L}_F^j g) \cdot \omega_B)](F_*^{n_T} \boldsymbol{v}) \}(x)$$

$$(4.24) \qquad - \sum_{j=0}^{n_T-1} \sum_{m=0}^{n_T-j} \int_{\mathbb{R}} dt\, \chi_*(x,t) \{ [\phi_t^*(\widehat{\mathcal{L}}_F^{n_T-j-m}(\mathcal{L}_{F,A}^m E \mathcal{L}_F^j g) \cdot \omega_\Gamma)](F_*^{n_T} \boldsymbol{v}) \}(x)$$

$$- \sum_{j=0}^{n_T-1} \sum_{m=1}^{\infty} \int_{\mathbb{R}} dt\, \Psi_m(x,t)(\mathcal{L}_{F,A}^{n_T-j} E \mathcal{L}_F^j g) \circ \phi_t(x) \, ;$$

$$\Psi_m(x,t) = \chi_*(x,t) \prod_{l=0}^{m-1} A \circ F^l \circ \phi_t(x) \cdot [(F^m \circ \phi_t)^* \omega_\Gamma(F_*^{n_T} \boldsymbol{v})(x)].$$

---

[32] The series is convergent due to (4.16).

Next, we need bounds on ϑ_m and Ψ_m.[33] Let us call ν_n^u the maximal eigenvalue of DF^n, then $\|D_x F^n\| \leq C_\# \nu_n^u(x)$.

Lemma 4.4. *For each $m \in \mathbb{N}$ and $x \in \mathbb{T}^2$, we have,*

$$\|\vartheta_m(x, \cdot)\|_{\mathcal{C}^{r-1}(\mathbb{R}_>, \mathbb{R})} \leq C_{r, n_*} \nu_m^u(x) \|\boldsymbol{v}\|$$

$$\|\Psi_m(x, \cdot)\|_{\mathcal{C}^{r-1}(\mathbb{R}_>, \mathbb{R})} \leq C_{r, n_*} \nu_m(x) \nu_{n_T}^u(x) \|\boldsymbol{v}\|.$$

Proof. First of all note that, by the smoothness of the stable manifolds of an Anosov map (see [35] and references therein) it follows that $\nu_1 \circ \phi_{(\cdot)} \in \mathcal{C}^{r-1}(\mathbb{R}, \mathcal{C}^{1+\alpha}(\mathbb{T}^2))$. Hence, by using repeatedly Schwarz theorem [48], we have that, for each $p < r$, $\nabla(\partial_t^p \nu_1) \circ \phi_t(x) = \partial_t^p(\nabla \nu_1) \circ \phi_t(x)$. This implies $\sup_{x \in \mathbb{T}^2} \|\nabla \nu_1 \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^{r-1}} \leq C_r$. Also, since $\mathcal{C}^{r-1}$ is a Banach algebra and

$$\partial_t \nu_n \circ \phi_t(x) = \sum_{k=0}^{n-1} \nu_n \circ \phi_t(x) \nu_k \circ \phi_t(x) \langle V, \nabla(\ln \nu_1) \rangle \circ F^k \circ \phi_t(x)$$

$$\partial_t D_{\phi_t(x)} F^n = \sum_{k=0}^{n-1} \sum_{j=1}^{2} \left[D_{\phi_t(x)} F^{n-k-1} \partial_{x_j} D_{F^k \circ \phi_t(x)} F D_{\phi_t(x)} F^k \right] \nu_j \circ \phi_t(x) V_j \circ F^k \circ \phi_t(x)$$

we have, by induction on n and r, $\|\nu_n \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^{r-1}((0,1), \mathbb{R})} \leq C_r \|\nu_n(\phi_{(\cdot)} x)\|_{\mathcal{C}^0((0,1), \mathbb{R})}$ and $\|D_{\phi_{(\cdot)}(x)} F^n\|_{\mathcal{C}^{r-1}((0,1), \mathbb{R})} \leq C_r \|\nu_n^u \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^0((0,1), \mathbb{R})}$, thus, recalling (4.12), $\|\nabla \nu_n \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^{r-1}((0,1), \mathbb{R})} \leq C_r \nu_n(x) \nu_n^u(x)$.[34]

In addition, by (1.5) we have

$$\nabla \tau_n(x, t) = \int_0^t D_x \phi_s^* \nabla \nu_n \circ \phi_s(x) ds.$$

It follows, using Lemma 4.2 and since $\mathcal{C}^r$ is an algebra, that

$$(4.25) \qquad \|\nabla \tau_n \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^r((0,1), \mathbb{R}^2)} \leq C_r \nu_n(x) \nu_n^u(x).$$

The first inequality in the Lemma follows remembering the definition (4.11).

Let us prove the second. By equations (4.17), (4.19) and the smoothness of the stable manifolds of an Anosov map (see [35] and references therein) it follows that $\Gamma(\hat{V}) \circ \phi_t$ and $A \circ \phi_t$ are uniformly (in x) $\mathcal{C}^{r-1}$-bounded functions of t. For each $x \in \mathbb{T}^2$ we have

$$\partial_t \prod_{l=0}^{m-1} A \circ F^l \circ \phi_t(x) = \prod_{l=0}^{m-1} A \circ F^l \circ \phi_t(x) \cdot \sum_{l=0}^{m-1} \frac{\nu_l \circ \phi_t(\partial_s A \circ \phi_s|_{s=0}) \circ F^l \circ \phi_t(x)}{A \circ F^l \circ \phi_t(x)}.$$

Since each further derivative of a function composed with $F^l \circ \phi_t$ produces the multiplicative factor ν_l, it follows

$$\left\| \sum_{l=0}^{m-1} \frac{\nu_l \circ \phi_t(\partial_s A \circ \phi_s|_{s=0}) \circ F^l \circ \phi_t(x)}{A \circ F^l \circ \phi_t(x)} \right\|_{\mathcal{C}^{r-2}} \leq C_\#.$$

On the other hand, notice that $\det D_x F = \nu_1(x) \nu_1^u(x) \frac{\theta \circ F(x)}{\theta(x)}$, where $\theta(x)$ depends only on the angle between the stable and unstable direction at x and on $\|V(x)\|$.

---

[33] Remark that the point of the next Lemma is that the bounds do not depend on r, apart from an irrelevant multiplicative constant.

[34] Here we have used Grönwall's inequality to prove $\|\nu_n \circ \phi_{(\cdot)}(x)\|_{\mathcal{C}^0((0,1), \mathbb{R})} \leq C_\# \nu_n(x)$, and the same for ν_n^u.

Accordingly,

$$\left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^0} \leq C_\# \frac{\nu_m(x)}{\nu_m^u(x)}.$$

Thus, we have, by induction,

$$\left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^{r-1}} \leq 2^{r-1} \left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^0} + \left\| \partial_t \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^{r-2}}$$

$$\leq 2^{r-1} \left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^0} + C_\# \left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^{r-2}}$$

$$\leq C_r \left\| \prod_{l=0}^{m-1} A \circ F^l \circ \phi_{(\cdot)}(x) \right\|_{\mathcal{C}^0} \leq C_r \frac{\nu_m(x)}{\nu_m^u(x)}.$$

Analogously, $\|D_x F^m \circ \phi_{(\cdot)}\|_{\mathcal{C}^{r-1}} \leq C_r \nu_m^u(x)$, from which the Lemma follows. □

As in section 3 we are left with one last problem: the potentials may be non smooth. Such a problem can be solved in the same way as before: extending all the objects to a subset Ω of the unitary tangent bundle.

Recall that $(x, v) \in \Omega$ is a three dimensional subset of $\mathbb{T}^2 \times \mathbb{R}^2$, thus we can naturally write vectors in $T\Omega$ as (w, η), $w \in T\mathbb{T}^2$ and $\eta \in \mathbb{R}^2$. Accordingly, a one form $\mathbf{g}$ on Ω at a point (x, v) acts on a vector (w, η) as $\mathbf{g}_{(x,v)}((w, \eta))$.

Next, let us define

$$\widehat{A} \circ \mathbb{F}^{-1}(x, v) = \frac{\|V \circ F^{-1}(x)\| \det D_x F^{-1}}{\|D_x F^{-1} v\|^2 \|V(x)\|}$$

(4.26)
$$\widehat{B}_k(x, v) = \frac{\langle D_x F v, (\partial_{x_k} D_x F) v \rangle}{\|D_x F v\|^2} + p_k^*(x) - (D_x F^* p^* \circ F(x))_k$$

$$\widehat{E}(x, v) = \frac{\langle D_x F v, D_x F v^\perp \rangle}{\|D_x F v\|^2 \|V(x)\|}.$$

We can then define the operators, acting, respectively, on functions $\mathbf{g}$ and on one forms $\mathbf{g}$ defined on Ω by

$$\mathcal{L}_{\mathbb{F}, \widehat{A}} \mathbf{g} = \mathcal{L}_{\mathbb{F}}(\widehat{A} \mathbf{g})$$

(4.27)
$$\left[\widehat{\mathcal{L}}_{\mathbb{F}} \mathbf{g} \right]_{(x,v)} = \frac{\|D_x F^{-1} v\| \|V(x)\|}{\|V \circ F^{-1}(x)\|} \left[(\mathbb{F}^{-1})^* \mathbf{g} \right]_{(x,v)}.$$

The relation with the previously defined operators is given by the following Lemma.

Lemma 4.5. *For each $x \in \mathbb{T}^2$ and $w \in \mathbb{R}^2$ we have*[35]

$$[\mathcal{L}_{\mathbb{F}, \widehat{A}}(g \circ \boldsymbol{\pi})](x, \hat{V}(x)) = \mathcal{L}_{F,A}(g)(x) = (\mathcal{L}_{F,A}(g)) \circ \boldsymbol{\pi}(x, \hat{V}(x))$$

$$\left[\widehat{\mathcal{L}}_{\mathbb{F}} \boldsymbol{\pi}^* dg \right]_{(x, \hat{V}(x))}(w, 0) = \left[\widehat{\mathcal{L}}_F dg \right]_x(w) = \left[\boldsymbol{\pi}^*(\widehat{\mathcal{L}}_F dg) \right]_{(x, \hat{V}(x))}(w, 0).$$

Proof. By direct computation $\widehat{A}(x, \hat{V}(x)) = A(x)$ and the first statement of the Lemma follows. The second follows directly from the definition since $(\mathbb{F}^{-n})^* \boldsymbol{\pi}^* dg = \boldsymbol{\pi}^*(F^{-n})^* dg$. □

---

[35] See (4.23) for the definition of $\mathcal{L}_{F,A}$ and $\widehat{\mathcal{L}}_F$.

Recalling equation (2.11) and Lemma 4.5, and setting $\boldsymbol{x} = (x, \hat{V}(x))$, $\boldsymbol{g} = g \circ \boldsymbol{\pi}$, we can rewrite (4.24) as:

$$
\begin{aligned}
\langle \boldsymbol{v}(F^{-n_T}(x)), \nabla \overline{H}_T^\star(g)(F^{-n_T}(x)) \rangle = &-\int_\mathbb{R} dt\, \vartheta_{n_T}(x,t)(\mathcal{L}_\mathbb{F}^{n_T} \boldsymbol{g}) \circ \boldsymbol{\phi}_t(\boldsymbol{x}) \\
&- \int_\mathbb{R} dt\, \chi_*(x,t)\{[\boldsymbol{\phi}_t^* \widehat{\mathcal{L}}_\mathbb{F}^{n_T} \boldsymbol{\pi}^* dg](\mathbb{F}_*^{n_T}(\boldsymbol{v},0))\}(\boldsymbol{x}) \\
&+ \sum_{j=0}^{n_T-1} \int_\mathbb{R} dt\, \chi_*(x,t)\{[\boldsymbol{\phi}_t^* \widehat{\mathcal{L}}_\mathbb{F}^{n_T-j}((\mathcal{L}_\mathbb{F}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_B)](\mathbb{F}_*^{n_T}(\boldsymbol{v},0))\}(\boldsymbol{x}) \\
&- \sum_{j=0}^{n_T-1} \sum_{m=0}^{n_T-j} \int_\mathbb{R} dt\, \chi_*(x,t)\{[\boldsymbol{\phi}_t^*(\widehat{\mathcal{L}}_\mathbb{F}^{n_T-j-m}(\mathcal{L}_{\mathbb{F},\widehat{A}}^m \widehat{E} \mathcal{L}_\mathbb{F}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_\Gamma)](\mathbb{F}_*^{n_T}(\boldsymbol{v},0))\}(\boldsymbol{x}) \\
&- \sum_{j=0}^{n_T-1} \sum_{m=1}^\infty \int_\mathbb{R} dt\, \Psi_m(x,t)(\mathcal{L}_{\mathbb{F},\widehat{A}}^{n_T-j} \widehat{E} \mathcal{L}_\mathbb{F}^j \boldsymbol{g}) \circ \boldsymbol{\phi}_t(\boldsymbol{x}) \\
\hat{\boldsymbol{\omega}}_B = \langle (\widehat{B}, 0), (dx, dv) \rangle; \quad & \hat{\boldsymbol{\omega}}_\Gamma = \langle (\Gamma(v), 0), (dx, dv) \rangle.
\end{aligned}
$$

(4.28)

Last, for each time dependent function $\varphi \in L^\infty(\mathbb{R}^3, \mathbb{R})$, each form $\mathfrak{g}$ on Ω and time dependent vector field $w \in L^\infty(\mathbb{R}^3, \mathbb{R}^2)$, with compact support in $\mathbb{R}_>$, we define[36]

$$
\begin{aligned}
\mathbb{H}_{x,\varphi}(\boldsymbol{g}) &= \int_\mathbb{R} \varphi(x,s) \cdot \boldsymbol{g} \circ \boldsymbol{\phi}_s(x, \widehat{V}(x))\, ds \\
\mathbb{H}_{x,w}^1(\mathfrak{g}) &= \int_\mathbb{R} \mathfrak{g}_{\boldsymbol{\phi}_s(x,\hat{V}(x))}((D_x \phi_s w(x,s), 0), 0)) ds.
\end{aligned}
$$

(4.29)

With such a notation we can finally write (4.28) as

$$
\begin{aligned}
\langle \boldsymbol{v}(x), \nabla \overline{H}_T^\star(g)(x) \rangle = &-\mathbb{H}_{F^{n_T}(x), \vartheta_{n_T}}(\mathcal{L}_\mathbb{F}^{n_T} \boldsymbol{g}) \\
&- \mathbb{H}_{F^{n_T}(x), \chi_* \mathbb{F}_*^{n_T}(\boldsymbol{v},0)}^1(\widehat{\mathcal{L}}_\mathbb{F}^{n_T} \boldsymbol{\pi}^* dg) \\
&+ \sum_{j=0}^{n_T-1} \mathbb{H}_{F^{n_T}(x), \chi_* \mathbb{F}_*^{n_T}(\boldsymbol{v},0)}^1(\widehat{\mathcal{L}}_\mathbb{F}^{n_T-j}((\mathcal{L}_\mathbb{F}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_B)) \\
&- \sum_{j=0}^{n_T-1} \sum_{m=0}^{n_T-j} \mathbb{H}_{F^{n_T}(x), \chi_* \mathbb{F}_*^{n_T}(\boldsymbol{v},0)}^1(\widehat{\mathcal{L}}_\mathbb{F}^{n_T-j-m}(\mathcal{L}_{\mathbb{F},\widehat{A}}^m \widehat{E} \mathcal{L}_\mathbb{F}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_\Gamma) \\
&- \sum_{m=1}^\infty \sum_{j=0}^{n_T-1} \mathbb{H}_{F^{n_T}(x), \Psi_m}(\mathcal{L}_{\mathbb{F},\widehat{A}}^{n_T-j} \widehat{E} \mathcal{L}_\mathbb{F}^j \boldsymbol{g}) =: \mathcal{J}_{n_T}(F^{n_T}(x)).
\end{aligned}
$$

(4.30)

It follows that, if $n_T = n_* K_T$, the derivative of (4.8) takes the form

$$
\langle \boldsymbol{v}, \nabla \overline{H}_T(g) \rangle = \sum_{l=1}^{K_T} \mathcal{J}_{ln_*}(F^{ln_*}(x)) - \int_{\mathbb{R}_\geq} \chi(s)[\phi_s^* dg(\boldsymbol{v})](x) ds.
$$

(4.31)

The above corresponds to Lemma 3.1 in the present context.

---

[36] The first is just a slight generalization of (3.1).

4.5. **Lipschitz coboundary.**

Having shown that the problem can be cast in a setting completely analogous to the one already discussed in Section 3, we are now ready to conclude.

Proof of Theorem 2.12. This is the same argument carried out in the proof of Theorem 2.8, only now we also need the spectral picture for the operator $\mathcal{L}_{\mathbb{F},\widehat{A}}$ on $\mathcal{B}^{p,q}$ and of the operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ acting on an appropriate (new) space $\widehat{\mathcal{B}}^{p,q}$, $p+q \leq r-2$. Indeed, by the arguments in appendix B it follows that $\mathcal{L}_{\mathbb{F},\widehat{A}}$ is quasi compact on $\mathcal{B}^{p,q}$ and in appendix C we show that there exists a Banach space $\widehat{\mathcal{B}}^{p,q}$ on which $\widehat{\mathcal{L}}_{\mathbb{F}}$ has spectral radius $\rho > 0$ and essential spectral radius bounded by $\lambda^{-\min\{p,q\}}\rho$. Also the functionals $\mathbb{H}^1_{x,w}$, for $w \in \mathcal{C}^r_0$, are bounded by

$$|\mathbb{H}^1_{x,w}(\mathbf{g})| \leq C_\# \|w\|_{\mathcal{C}^{r-2}} \|\mathbf{g}\|_{\widehat{\mathcal{B}}^{p,q}}.$$

As before we notice that in (4.31) the last term is uniformly bounded, hence we have to worry only about the terms $\mathcal{J}_{ln_*}$. Looking at (4.30) we see that each $\mathcal{J}_{ln_*}$ consists of five terms.

By Lemma 4.4 we see that the first term is bounded only if p, q (and hence r) are large enough and g belongs to the kernel of enough eigenprojectors of $\mathcal{L}_{\mathbb{F}}$ so that the essential spectral radius of $\mathcal{L}_{\mathbb{F}}$, when restricted to the invariant subspace to which g belongs, is smaller than $\|\nu^u_1\|_\infty^{-n}$. Analogously, the second term is bounded if $\boldsymbol{\pi}^* dg$ belongs to the kernel of enough eigenprojectors of $\widehat{\mathcal{L}}_{\mathbb{F}}$ (again the spectral radius of the remainder must be smaller of $\|\nu^u_1\|_\infty^{-n}$). The third term has a bit more complex structure. First of all note that the multiplication by a smooth function is a bounded operator from $\mathcal{B}^{p,q}$ to itself, while the multiplication by a smooth one-form is a bounded operator from $\mathcal{B}^{p,q}$ to $\widehat{\mathcal{B}}^{p,q}$ (to verify it just use the norms definitions (B.3) and (C.1)). Next, assuming that g belongs to the above subspaces and using the spectral decomposition of $\widehat{\mathcal{L}}_{\mathbb{F}}$ we can write,[37] for some $m_* \in \mathbb{N}$, $\rho_* \in (0,1)$ and all $l \leq n_T$,

$$\sum_{k=0}^{m_*} \sum_{j=0}^{l-1} \mathbb{H}^1_{F^{n_T}(x),\chi_*\mathbb{F}^{n_T}_*(\boldsymbol{v},0)}([\rho_k\Pi_k + Q_k]^{l-j}((\mathcal{L}^j_{\mathbb{F}}\boldsymbol{g})\cdot\hat{\boldsymbol{\omega}}_B)) + \mathcal{O}(\rho^l_*)$$

$$= \sum_{k=0}^{m_*} \sum_{j=0}^{l-1} \sum_{p=0}^{\min\{l-j,d_k-1\}} \binom{l-j}{p} \mathbb{H}^1_{F^{n_T}(x),\chi_*\mathbb{F}^{n_T}_*(\boldsymbol{v},0)}(\rho_k^{l-j-p}Q^p_k((\mathcal{L}^j_{\mathbb{F}}\boldsymbol{g})\cdot\hat{\boldsymbol{\omega}}_B))$$

$$+ \mathcal{O}(\rho^l_*).$$

On the other hand, for $p \leq l - j$,

$$\sum_{j=0}^{l-1} \binom{l-j}{p} \mathbb{H}^1_{F^{n_T}(x),\chi_*\mathbb{F}^{n_T}_*(\boldsymbol{v},0)}(\rho_k^{l-j-p}Q^p_k((\mathcal{L}^j_{\mathbb{F}}\boldsymbol{g})\cdot\hat{\boldsymbol{\omega}}_B))$$

$$= \rho_k^{l-p} \sum_{s=0}^p c_{s,p}l^{p-s}\mathbb{H}^1_{F^{n_T}(x),\chi_*\mathbb{F}^{n_T}_*(\boldsymbol{v},0)}(Q^p_k(\mathcal{K}_s\boldsymbol{g})\cdot\hat{\boldsymbol{\omega}}_B)) + \mathcal{O}(\rho^l_*)$$

where $\mathcal{K}_s = \sum_{j=0}^\infty j^s(\rho_k^{-1}\mathcal{L}_{\mathbb{F}})^j$ and $\sum_{s=0}^p c_{s,p}l^{p-s}j^s = \binom{l-j}{p}$. By identifying the range of Π_k with $\mathbb{R}^{d_k}$ we can then define the functional $\ell_{k,s} : \mathcal{B}^{p,q} \to \mathbb{R}^{d_k}$ as

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[37] That is $\widehat{\mathcal{L}}^j_{\mathbb{F}} = \sum_{k=0}^{m_*}[\rho_k\Pi_k + Q_k]^j + \mathcal{O}(\|\nu^u_1\|_\infty^{-j})$, where $Q_k^{d_k} = 0$ and $\Pi_kQ_k = Q_k\Pi_k = Q_k$ and $\Pi_k^2 = \Pi_k$. Also we use the useful convention $Q_k^0 = \Pi_k$.

$\ell_{k,s}(\boldsymbol{g}) = \Pi_k(\mathcal{K}_s(\boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_B)$. Accordingly, if $\ell_{k,s}(\boldsymbol{g}) = 0$ for all $k \leq m_*$ and $s \leq d_k$, we have that also the third term is uniformly bounded.

Similar arguments hold for the fourth and fifth term. This implies that if g belongs to and appropriate finite codimensional subspace (determined by the above eigendistributions) then the $\overline{H}_T(g)$ are equicontinuous functions of x. Hence there exists $\{T_j\}$ such that $\overline{H}_{T_j}(g)$ converges uniformly to a Lipschitz function. We have thus shown that $\overline{H}_T(g)$ has a convergent subsequence to a Lipschitz function h which, for all $t \in \mathbb{R}$, satisfies

$$h \circ \phi_t(x) - h(x) = \int_0^t g \circ \phi_s(x) ds$$

Hence, h satisfies (1.11) and g is a Lipschitz coboundary.   □

5. EXAMPLES

Lemma 1.1 shows that the flows to which our theory applies must necessarily enjoy several properties, the reader might be left wondering if such flows exist at all (apart, of course, for the trivial one consisting of rigid rotations).

To construct examples the simplest strategy is to reverse the logic and start with a $\mathcal{C}^r$ Anosov map which is orientation preserving.[38] Given such a map, we have an associated stable distribution. If we choose any strictly positive function $\mathcal{N} \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$ there are only two fields V such that $V(x) \in E^s(x)$ for all $x \in \mathbb{T}^2$ and $\|V(x)\| = \mathcal{N}(x)$, they correspond to the two possible orientations. We can then choose any of the two and we have, at the same time, an example that satisfies all our assumptions and a justification of such assumptions. Indeed, in general the distribution E^s of a $\mathcal{C}^r$ Anosov map will be only $\mathcal{C}^{1+\alpha}$ with $\alpha \in (0,1)$, [39]. Notice however that it is possible to have situations in which $\alpha > 1$ and yet F is not $\mathcal{C}^1$ conjugated to a toral automorphism [39, Exercise 19.1.5]. Of course, in the latter case the unstable foliation will be irregular [29, Corollary 3.3].

The above partially clarifies the applicability of our work. Nevertheless, other reasons of unhappiness persist. In particular all our discussion, up to this point, has been a bit abstract as we did not discuss which type of concrete objects it really yields. To gain a better understanding we start by working out the linear case that, surprisingly, is not completely trivial.

5.1. A "trivial" example.
For the reader convenience we discuss here the case in which F is linear and ϕ_t is generated by a constant vector field. As already mentioned in the introduction this is an analogue of the case, for the geodesic-horocycle flow setting, of compact manifolds of constant negative curvature. Hence it can be dealt with directly by representation theory (i.e. Fourier transform, in the present setting), without using the strategy put forward in this paper.

Let $A \in SL(2, \mathbb{N})$; let $F_A : \mathbb{T}^2 \to \mathbb{T}^2$ be the Anosov map defined by $F_A(\xi) := A\xi$ mod 1. Since $\det(A) = 1$ the map is invertible, and has eigenvalues $\lambda_A, \lambda_A^{-1} \in \mathbb{R}$, with $\lambda_A > 1$. Let $V_A = (1, \omega)$ be the eigenvector associated to the eigenvalue λ_A^{-1}. Note that ω is a quadratic irrational, as in Lemma 1.1. Let $\phi_t(\xi) = \xi + tV_A \mod 1$.

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[38] After all, this is what is done for the horocycle flow: one starts from the geodesic flow.

In this case by applying Fourier transform to equation (1.11) we obtain, for $k \in \mathbb{Z}^2$, calling $\hat{f}_k$ the Fourier coefficients of f,

$$\sum_{k \in \mathbb{Z}^2} 2\pi i \langle V, k \rangle \hat{h}_k e^{2\pi i \langle k, \xi \rangle} = \sum_{k \in \mathbb{Z}^2} \hat{g}_k e^{2\pi i \langle k, \xi \rangle}.$$

Note that we have the trivial obstruction $\hat{g}_0 = 0$. If this is satisfied, note that $\langle V, k \rangle = k_1 + \omega k_2 \neq 0$ for all $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$, since ω is irrational. Thus, we can write

$$(5.1) \qquad\qquad \hat{h}(k) = -i \frac{\hat{g}(k)}{2\pi \langle V, k \rangle}.$$

Since ω is a quadratic irrational, it is well known (e.g. by using standard results on continuous fractions) that $|\langle V, k \rangle| \geq C_\# \|k\|^{-1}$. Hence, if $g \in W^{r,2}$ (the Sobolev space with the first r derivatives in L^2), then $h \in W^{r-1,2}$. In particular, if $g \in \mathcal{C}^\infty$, then $h \in \mathcal{C}^\infty$.

That is, in this example all the aforementioned distributions do not exist and the only obstruction is the trivial one: the one given by the invariant measure. If such an obstruction is satisfied (i.e. if $\mathrm{Leb}(g) = 0$), then the ergodic integrals are bounded and g is a coboundary with the maximal regularity one can expect.

Nevertheless, it is very instructive to apply to this example also our strategy. This will give us a feeling for what might happen in general.

5.1.1. *Ergodic integrals.*

Let us change coordinates in Ω. One convenient choice is $\theta(\xi, s) = (\xi, v(s))$ with $v(s) = (1, s)(1 + s^2)^{-\frac{1}{2}}$ and $s < 0$. In this co-ordinates we have that the set Ω, defined just before (2.7), reads $\mathbb{T}^2 \times [-\beta_1, -\beta_2]$ for some $0 < \beta_2 < \beta_1$. Also, calling $\widehat{\mathbb{F}} = \theta^{-1} \circ \mathbb{F} \circ \theta$, we have

$$\widehat{\mathbb{F}}(\xi, s) = (F(\xi), \psi(s))$$

$$\psi(s) = \frac{c + sd}{a + sb}; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In addition,

$$(5.2) \qquad\qquad \psi^{-1}(s) = \frac{as - c}{d - bs}.$$

The map ψ^{-1} is a contracting map with derivative $(\psi^{-1})'(s) = (d - bs)^{-2}$ and a unique fixed point $\bar{s}$ in $[-\beta_1, -\beta_2]$.

The smallest eigenvalue of A is given by $\bar{\nu} = (d - \bar{s}b)^{-1} < 1$ and the corresponding eigenvector is $\bar{v} = (1, \bar{s})$.

Setting, as usual, $\theta^* \boldsymbol{g} = \boldsymbol{g} \circ \theta$, for $\boldsymbol{g} \in \mathcal{C}^0(\Omega, \mathbb{C})$, and introducing the multiplication operator $\Xi \boldsymbol{g}(\xi, s) = \frac{\sqrt{1 + s^2}}{\|V(\xi)\|} \boldsymbol{g}(\xi, s)$, let us define, recalling that $\mathcal{L}_\mathbb{F}$ is defined in equation (2.10),

$$(5.3) \qquad\qquad \mathcal{L}_{\widehat{\mathbb{F}}} = \Xi \, \theta^* \mathcal{L}_\mathbb{F} (\theta^*)^{-1} \, \Xi^{-1}.$$

By direct computation it follows[39]

$$(5.4) \qquad\qquad \mathcal{L}_{\widehat{\mathbb{F}}} \boldsymbol{g}(\xi, s) = \boldsymbol{g} \circ \widehat{\mathbb{F}}^{-1}(\xi, s)(d - bs).$$

By Theorem 2.8 the obstruction are determined by the spectrum of $\mathcal{L}_\mathbb{F}$ on the Banach spaces $\mathcal{B}^{p,q}$ (see Appendix B). Note that (5.3) implies that the operator

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[39] Remember that $s < 0$, hence $d - sb > 0$.

$\mathcal{L}_{\mathbb{F}}$ is conjugated to the operator $\mathcal{L}_{\widehat{\mathbb{F}}}$ on the space $\Xi\, \theta^*(\mathcal{B}^{p,q})$ which, to simplify notation, we still call $\mathcal{B}^{p,q}$ since the identification is trivial; hence they have the same spectrum. Accordingly, it suffices to study the spectrum of $\mathcal{L}_{\widehat{\mathbb{F}}}$. We detail it in the next Lemma; see Remark 5.7 for the consequences of the Lemma on the existence of the obstructions.

Lemma 5.1. *For each $q \geq p \in \mathbb{N}$, setting $D_p = \{z \in \mathbb{C} \,:\, |z| > \bar{\nu}^{p-1}\}$,*

$$\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}_{\widehat{\mathbb{F}}}) \cap D_p = \{\bar{\nu}^{2k-1}\}_{k\in\mathbb{N}} \cap D_p,$$

and the eigenvalues have multiplicity one. In addition, there exist explicit formulae for the right and left eigenvectors associated to the point spectrum (see equations (5.6), (5.7) and (5.8)).

Proof. It is convenient to consider a larger Banach space $\mathcal{B}_*^{p,q}$ which is defined exactly as $\mathcal{B}^{p,q}$ with the only difference that the set Σ of admissible curves (see Definition B.1) is smaller, as it consists only of stable curves (that, in this case, are just segments in the direction $\hat{V}$). Indeed, since the sup is taken on a smaller set, it follows $\|\cdot\|_{\mathcal{B}_*^{p,q}} \leq \|\cdot\|_{\mathcal{B}^{p,q}}$ which implies $\mathcal{B}^{p,q} \subset \mathcal{B}_*^{p,q}$. In addition, note that if $\mathcal{L}$ is a bounded operator both on $\mathcal{B}^{p,q}$ and $\mathcal{B}_*^{p,q}$ and $R \subset \mathbb{C}$ is a region in which both $\sigma_{\mathcal{B}_*^{p,q}}(\mathcal{L})$ and $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L})$ consist only of point spectrum with finite multiplicity, then $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}) \cap R \subset \sigma_{\mathcal{B}_*^{p,q}}(\mathcal{L}) \cap R$. Indeed, if $h \in \mathcal{B}^{p,q}$ is an eigenvector of $\mathcal{L}$, then it is an eigenvector of $\mathcal{L}$ also when it is viewed as an operator from $\mathcal{B}_*^{p,q}$ to itself. In fact, [5, Lemma A.1] implies $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}) \cap R = \sigma_{\mathcal{B}_*^{p,q}}(\mathcal{L}) \cap R$.

The advantage in considering $\mathcal{B}_*^{p,q}$ rests in the fact that the norm (and all the theory) simplifies considerably since the stable and unstable distributions are constant.[40] This implies in particular that $\partial_\sigma g = \langle \nabla_{x,y} g, \hat{V} \rangle$, $\partial_u g = \langle \nabla_{x,y} g, \hat{V}^\perp \rangle$ and ∂_s all commute.[41]

Setting $B_{\beta,r} = \{\varphi \in \mathcal{C}_0^\beta([-r,r],\mathbb{C}) \,:\, \|\varphi\|_{\mathcal{C}^\beta([-r,r],\mathbb{C})} \leq 1\}$, the analogous of definition (B.3) is equivalent to

$$\|g\|_{\mathcal{B}_*^{p,q}} = \sup_{\substack{x\in\mathbb{T}^2 \\ r\in[\delta/2,\delta]}} \sup_{\substack{p_\sigma+p_u+p_s=p_* \\ p_* \leq p}} \sup_{\varphi\in B_{q+p_*,r}} \left| \int_{-r}^r [\partial_\sigma^{p_\sigma} \partial_u^{p_u} \partial_s^{p_s} g]\,(x+\hat{V}t,\bar{s})\varphi(t)dt \right|,$$

where, for further convenience, we make the choice $\delta \leq \bar{\nu}$. In addition, since φ is compactly supported in $[-r,r]$,

$$\int_{-r}^r [\partial_\sigma g]\,(x+\hat{V}t,\bar{s})\varphi(t)dt = \int_{-r}^r \frac{d}{dt}g(x+\hat{V}t,\bar{s})\varphi(t)dt = -\int_{-r}^r g(x+\hat{V}t,\bar{s})\varphi'(t)dt.$$

Hence, we can write

$$\|g\|_{\mathcal{B}_*^{p,q}} = \sup_{x\in\mathbb{T}^2} \sup_{r\in[\delta/2,\delta]} \sup_{p_u+p_s=p_* \leq p} \sup_{\varphi\in B_{q+p_*,r}} \left| \int_{-r}^r [\partial_u^{p_u} \partial_s^{p_s} g]\,(x+\hat{V}t,\bar{s})\varphi(t)dt \right|.$$

Next, we establish a Lasota-Yorke inequality by a super simplified version of the arguments used in [33, Lemma 2.2].

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[40] If the reader wonders why not always use such a space, he should consider that such a space is directly tied to the map and does not work for a small perturbation, as the invariant distributions will be different. Hence the spaces $\mathcal{B}_*^{p,q}$ would be useless in the following sections.

[41] Of course, ∂_σ is just the derivative in the stable direction and ∂_u the one in the unstable direction of F and ∂_s is the derivative with respect to s.

Sub-Lemma 5.2. *For all $p, q > 0$ there exist $C > 0$ such that, for all $n \in \mathbb{N}$,*

$$\|\mathcal{L}_{\widehat{\mathbb{F}}}^n h\|_{\mathcal{B}_*^{p,q}} \leq C\bar{\nu}^{-n} \|h\|_{\mathcal{B}_*^{p,q}}$$

$$\|\mathcal{L}_{\widehat{\mathbb{F}}}^n h\|_{\mathcal{B}_*^{p,q}} \leq C\bar{\nu}^{n(\min\{p,q\}-1)} \|h\|_{\mathcal{B}_*^{p,q}} + C\bar{\nu}^{-n}\|h\|_{\mathcal{B}_*^{p-1,q+1}}.$$

Proof. Note that $\partial_u \mathcal{L}_{\widehat{\mathbb{F}}} h = \bar{\nu} \mathcal{L}_{\widehat{\mathbb{F}}} \partial_u h$ and

$$(5.5) \qquad \partial_s \mathcal{L}_{\widehat{\mathbb{F}}} h = (d-bs)^{-2} \mathcal{L}_{\widehat{\mathbb{F}}} \partial_s h - b(d-bs)^{-1} \mathcal{L}_{\widehat{\mathbb{F}}} h.$$

Iterating the above equations yield, for each $p_u + p_s = p_* \leq p$, $\varphi \in B_{q+p_*, r}$,

$$\left| \int_{-r}^r \left[\partial_u^{p_u} \partial_s^{p_s} \mathcal{L}_{\widehat{\mathbb{F}}}^n g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right| \leq \bar{\nu}^{(2p_s + p_u)n} \left| \int_{-r}^r \left[\mathcal{L}_{\widehat{\mathbb{F}}}^n \partial_u^{p_u} \partial_s^{p_s} g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right|$$

$$+ C_p \sum_{0 \leq p_s' \leq p_* - 1 - p_u} \left| \int_{-r}^r \left[\mathcal{L}_{\widehat{\mathbb{F}}}^n \partial_u^{p_u} \partial_s^{p_s'} g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right|.$$

On the other hand

$$\int_{-r}^r \left[\mathcal{L}_{\widehat{\mathbb{F}}}^n h \right] (x + \hat{V}t, \bar{s})\varphi(t)dt = \int_{-\bar{\nu}^{-n}r}^{\bar{\nu}^{-n}r} h(x + \hat{V}t, \bar{s})\varphi(\bar{\nu}^n t)dt.$$

To compare the above with the integrals defining the norm we introduce a smooth partition of unity $\{\phi_k\}$ of the form $\phi_k(t) = \phi(t - k\delta/2)$, $\operatorname{supp} \phi \subset [-\delta, \delta]$, and write $\varphi(\bar{\nu}^n t) = \sum_k \varphi_{k,n}(t - k\delta/2)$ where $\varphi_{k,n}(t) = \phi(t)\varphi(\bar{\nu}^n(t + k\delta/2))$. Then, setting $x_k = x + k\delta/2\hat{V}$,

$$\int_{-r}^r \left[\mathcal{L}_{\widehat{\mathbb{F}}}^n h \right] (x + \hat{V}t, \bar{s})\varphi(t)dt = \sum_k \int_{-\delta}^{\delta} h(x_k + \hat{V}t, \bar{s})\varphi_{k,n}(t)dt.$$

Note that the sum consists of at most $C_\# \bar{\nu}^{-n}$ terms. This yields the first estimate of the Sub-Lemma since $\|\varphi_{k,n}\|_{\mathcal{C}^{q+p_*}} \leq C_\#$. Moreover we have

$$\left| \int_{-r}^r \left[\partial_u^{p_u} \partial_s^{p_s} \mathcal{L}_{\widehat{\mathbb{F}}}^n g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right| \leq \bar{\nu}^{p_* n} \sum_k \left| \int_{-\delta}^{\delta} [\partial_u^{p_u} \partial_s^{p_s} g] (x_k + \hat{V}t, \bar{s})\varphi_{k,n}(t)dt \right|$$

$$+ C_p \bar{\nu}^{-n} \|g\|_{\mathcal{B}_*^{p-1,q+1}}.$$

If $p_* = p$ then

$$\left| \int_{-r}^r \left[\partial_u^{p_u} \partial_s^{p_s} \mathcal{L}_{\widehat{\mathbb{F}}}^n g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right| \leq C_q \bar{\nu}^{(p-1)n} \|g\|_{\mathcal{B}_*^{p,q}} + C_p \bar{\nu}^{-n} \|g\|_{\mathcal{B}_*^{p-1,q+1}},$$

If $p_* < p$, recalling the choice $\delta \leq \bar{\nu}$,

$$\left\| \varphi_{k,n}(t) - \phi(t) \sum_{j=0}^{q-1} \frac{\varphi^{(j)}(\bar{\nu}^n k\delta/2)\bar{\nu}^j}{j!} t^j \right\|_{\mathcal{C}^{q+p_*}} \leq C_q \bar{\nu}^q$$

$$\left\| \phi(t) \sum_{j=0}^{q-1} \frac{\varphi^{(j)}(\bar{\nu}^n k\delta/2)\bar{\nu}^j}{j!} t^j \right\|_{\mathcal{C}^{q+p_*+1}} \leq C_q.$$

Thus

$$\left| \int_{-r}^r \left[\partial_u^{p_u} \partial_s^{p_s} \mathcal{L}_{\widehat{\mathbb{F}}}^n g \right] (x + \hat{V}t, \bar{s})\varphi(t)dt \right| \leq C_q \bar{\nu}^{(p_*-1)n} \bar{\nu}^{qn} \|g\|_{\mathcal{B}_*^{p^*,q}} + C_p \bar{\nu}^{-n} \|g\|_{\mathcal{B}_*^{p-1,q+1}}$$

$$\leq C_q \bar{\nu}^{(q-1)n} \|g\|_{\mathcal{B}_*^{p,q}} + C_p \bar{\nu}^{-n} \|g\|_{\mathcal{B}_*^{p-1,q+1}},$$

from which the second inequality in the statement of the Lemma follows by taking the sup on x, p_u, p_s and φ. $\qquad\square$

Sub-Lemma 5.3. *For each $q, p \in \mathbb{N}$ the essential spectral radius of $\mathcal{L}_{\widehat{\mathbb{F}}}$ acting on $\mathcal{B}_*^{p,q}$ is bounded by $\bar{\nu}^{\min\{p,q\}-1}$.*

Proof. One can check that the unit ball of $\mathcal{B}_*^{p,q}$ is weakly compact in $\mathcal{B}_*^{p-1,q+1}$ by a much simplified version of the proof of [44, Lemma 2.9]. Then the Lemma follows by the Lasota-Yorke inequality established in Sub-Lemma 5.2 and Hennion theorem [36]. $\qquad\square$

We can now study the point spectrum. We start by establishing a lower bound on its cardinality.

Sub-Lemma 5.4. *For each $p, q \in \mathbb{N}$, $q, p \in \mathbb{N}$, we have*

$$\sigma_{\mathcal{B}_*^{p,q}}(\mathcal{L}_{\widehat{\mathbb{F}}}) \supset \{\bar{\nu}^{2k-1}\}_{k \geq 0}.$$

Proof. We look for $\mathcal{L}_{\widehat{\mathbb{F}}} \boldsymbol{g} = \mu \boldsymbol{g}$ of the form $\boldsymbol{g}(\xi, s) = g(\xi)f(s)$. Then

$$\mu g(\xi)f(s) = g \circ F^{-1}(\xi)f(\psi^{-1}(s))(d - bs).$$

By the computation Fourier modes done at the beginning of section 5.1 it follows that $g(\xi) = 1$; hence

$$\mu f(s) = f(\psi^{-1}(s))(d - bs).$$

Iterating the above relation yields

$$f(s) = \mu^{-n} \prod_{k=0}^{n-1} (d - b\psi^{-k}(s))f(\psi^{-n}(s)).$$

Since $\psi^{-n}(s)$ converges to $\bar{s}$ there are two possibilities; either $f(\bar{s}) \neq 0$ or $f(\bar{s}) = 0$. In the first case we can assume, without loss of generality, that $f(\bar{s}) = 1$. Then

$$(5.6) \qquad f(s) = \prod_{k=0}^{\infty} \mu^{-1}(d - b\psi^{-k}(s)) = \prod_{k=0}^{\infty} \mu^{-1}(d - b\bar{s} + b[\psi^{-k}(\bar{s}) - \psi^{-k}(s)]),$$

provided the product converges. If we choose $\mu = d - b\bar{s} = \bar{\nu}^{-1}$, then, for any $\tau \in (\bar{\nu}^2, 1)$ we have

$$\prod_{k=0}^{\infty} \mu^{-1}(\mu + b[\psi^{-k}(\bar{s}) - \psi^{-k}(s)]) = e^{\sum_{k=0}^{\infty} \mathcal{O}(\tau^k)}$$

which shows the convergence. Since $f \in \mathcal{C}^{\infty}$, we have that $f \in \mathcal{B}_*^{p,q}$ for all $p, q \in \mathbb{N}$. We have thus an eigenvector.

Next, consider the case $f(\bar{s}) = 0$. It is natural to look for solutions of the form $f(s) = (s - \bar{s})^p M(s)$, $M(\bar{s}) = 1$, for $p \in \mathbb{N}$, then, recalling (5.2),

$$\mu(s - \bar{s})^p M(s) = (d - bs)(\psi^{-1}(s) - \psi^{-1}(\bar{s}))^p M \circ \psi^{-1}(s)$$
$$= (d - bs)^{-p+1}(d - b\bar{s})^{-p}(s - \bar{s})^p M \circ \psi^{-1}(s).$$

Iterating again we have

$$(5.7) \qquad M(s) = \prod_{k=0}^{\infty} \mu^{-1}(\bar{\nu}^{-1} + b(\psi^{-k}(\bar{s}) - \psi^{-k}(s)))^{-p+1}\bar{\nu}^p.$$

The above is convergent and non zero provided $\mu = \bar{\nu}^{2p-1}$. Such a choice yields $M(s) = f(s)^{-p+1}$ and since $1/f \in \mathcal{C}^{\infty}$ again we have an eigenvector in $\mathcal{B}_*^{p,q}$. $\qquad\square$

Next, we need a lower bound on the cardinality of the point spectrum. The idea to achieve it is to investigate the spectrum of $\mathcal{L}'_{\widehat{\mathbb{F}}}$ by considering the distributions

$$(5.8) \qquad \delta^n_*(h) = \int_{\mathbb{T}^2} [\partial^n_s h](x, \bar{s}) dx.$$

Note that, for $q \geq p \geq n$, $\delta^*_n \in (\mathcal{B}^{p,q}_*)'$. Indeed, $\mathbb{T}^2$ can be partitioned by pieces of stable curves, hence δ^n_* can be seen as an convex combination of the functionals that define the $\mathcal{B}^{p,q}_*$ norm, thus it is bounded.

Lemma 5.5. *The spaces* $\mathbb{W}_n = \{h \in \mathcal{B}^{p,q}_* : \delta^0_*(h) = \cdots = \delta^n_*(h) = 0\}$ *are invariant for* $\mathcal{L}_{\widehat{\mathbb{F}}}$.

Proof. Note that

$$\delta^0_*(\mathcal{L}_{\widehat{\mathbb{F}}} g) = \int_{\mathbb{T}^2} g(F^{-1}(x), \bar{s}) \bar{\nu}^{-1} dx = \bar{\nu}^{-1} \int_{\mathbb{T}^2} g(x, \bar{s}) dx = \bar{\nu}^{-1} \delta^0_*(g).$$

Moreover, iterating (5.5) it follows that there exist smooth functions a_k such that

$$\partial^n_s \mathcal{L}_{\widehat{\mathbb{F}}} g = \sum_{k=0}^n a_k(s) \mathcal{L}_{\widehat{\mathbb{F}}} \partial^k_s g.$$

Thus,

$$\delta^n_*(\mathcal{L}_{\widehat{\mathbb{F}}} g) = \sum_{k=0}^n a_k(\bar{s}) \int_{\mathbb{T}^2} [\partial^k_s g](F^{-1}(x), \bar{s}) \bar{\nu}^{-1} = \sum_{k=0}^n a_k(\bar{s}) \bar{\nu}^{-1} \delta^k_*(g).$$

$\square$

Note that the above implies that $\mathcal{L}'_{\widehat{\mathbb{F}}}$ has eigenvectors formed by linear combinations of δ^n_*.

Sub-Lemma 5.6. *For each* $m \geq 0$, *the spectral radius of* $\mathcal{L}_{\widehat{\mathbb{F}}}$, *when acting on* $\mathbb{W}_{m-1} \cap \mathcal{B}^{p,q}_*$, $p, q \geq 2m$, *is bounded by* $\bar{\nu}^{2m-1}$.

Proof. Setting $A(x, s) = (d - bs)^{-1}$ and $\mathcal{L}_{\widehat{\mathbb{F}}, k} g = A^{2k} \mathcal{L}_{\widehat{\mathbb{F}}} g$, by equation (5.5)

$$\partial_s \mathcal{L}^n_{\widehat{\mathbb{F}}, m} g = (d - bs)^{-2} \mathcal{L}_{\widehat{\mathbb{F}}, m} \partial_s \mathcal{L}^{n-1}_{\widehat{\mathbb{F}}, m} g + b(2m-1)(d - bs)^{-1} \mathcal{L}^n_{\widehat{\mathbb{F}}, m} g$$

$$= \mathcal{L}^n_{\widehat{\mathbb{F}}, m+1} \partial_s g + b(2m-1) \sum_{k=0}^{n-1} \mathcal{L}^k_{\widehat{\mathbb{F}}, m+1} A \mathcal{L}^{n-k}_{\widehat{\mathbb{F}}, m} g.$$

Iterating the above formula yields

$$\left| \int_{-r}^r \left[\partial^j_u \partial^k_s (\mathcal{L}^n_{\widehat{\mathbb{F}}} h) \right] (x + \hat{V}t) dt \right| \leq \bar{\nu}^{(2k-1)n} \left| \int_{-r}^r \left[\partial^j_u \mathcal{L}^n h_{\bar{s}, k} \right] (x + \hat{V}t) dt \right|$$

$$+ C_k \sum_{l=0}^{k-1} \bar{\nu}^{(2l-1)n} \left| \int_{-r}^r \left[\partial^j_u \mathcal{L}^n h_{\bar{s}, l} \right] (x + \hat{V}t) dt \right|$$

where $h_{\bar{s}, k}(x) = (\partial^k_s h)(x, \bar{s})$ and $\mathcal{L}g(x) = g \circ F^{-1}(x)$ is the transfer operator associated to the SRB measure for F (which, of course, it is just Lebesgue). It is then natural to consider the Banach spaces $\mathcal{B}^{p,q}_0$ obtained by closing $\mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$ with respect to the norms

$$\|g\|_{\mathcal{B}^{p,q}_0} = \sup_{\substack{x \in \mathbb{T}^2 \\ r \in [\delta/2, \delta]}} \sup_{p_u \leq p} \sup_{\|\varphi\|_{\mathcal{C}^{q+p_u}_0} \leq 1} \left| \int_{-r}^r \left[\partial^{p_u}_u g \right] (x + \hat{V}t) \varphi(t) dt \right|.$$

Note that $\|h_{\bar{s},k}\|_{\mathcal{B}_0^{p,q}} \leq \|h\|_{\mathcal{B}_*^{p,q}}$ if $k \leq p$. Then

$$(5.9) \qquad \|\mathcal{L}_{\widehat{\mathbb{F}}}^n h\|_{\mathcal{B}_*^{p,q}} \leq \bar{\nu}^{(2p-1)n}\|\mathcal{L}^n h_{\bar{s},p}\|_{\mathcal{B}_0^{p,q}} + C_p \sum_{j=0}^{p-1} \bar{\nu}^{(2j-1)n}\|\mathcal{L}^n h_{\bar{s},j}\|_{\mathcal{B}_0^{p,q}}.$$

It is well known that the point spectrum of the operator $\mathcal{L}$ on each $\mathcal{B}_0^{p,q}$ consists of just $\{1\}$, with the left eigenvector being Lebesgue measure, while the essential spectrum is bounded by $\bar{\nu}^{\min\{p,q\}}$.[42]

The above readily implies that if $h \in \mathbb{W}_0$, then, by (5.8), $h_{\bar{s},0}$ is of zero average and, for each $\beta > \bar{\nu}$ and $q \geq 2$,

$$\|\mathcal{L}_{\widehat{\mathbb{F}}}^n h\|_{\mathcal{B}_*^{2,q}} \leq C_\beta \bar{\nu}^n \beta^n \|h_{\bar{s},0}\|_{\mathcal{B}_0^{2,q}} \leq C_\beta \bar{\nu}^n \beta^n \|h\|_{\mathcal{B}_*^{2,q}}.$$

Thus the spectral radius of $\mathcal{L}_{\widehat{\mathbb{F}}}$ on $\mathbb{W}_0$ must be less than $\bar{\nu}$, and since $\mathbb{W}_0$ has codimension one it follows that $\mathcal{L}_{\widehat{\mathbb{F}}}$ can have at most one eigenvalue larger than $\bar{\nu}$. By our knowledge of the essential spectral radius and [5, Lemma A.1] it follows that on each $\mathcal{B}_*^{p,q}$, $p,q \geq 1$, the operator can have at most one eigenvalue larger that $\bar{\nu}$.

To bound the spectral radius of $\mathcal{L}_{\widehat{\mathbb{F}}}$, when acting on $\mathbb{W}_{m-1} \cap \mathcal{B}_*^{p,q}$, $p,q \geq 2m$, we proceed by induction. We already proved the statement for $m = 1$. Suppose it is true for m, then, for $q \geq 2m + 2$, equation (5.9) implies, for $h \in \mathbb{W}_m$ and $\beta > \bar{\nu}$,

$$\|\mathcal{L}_{\widehat{\mathbb{F}}}^n h\|_{\mathcal{B}_*^{2m+2,q}} \leq C_\beta \beta^{(2m+1)n} \sum_{j=0}^{2m+2} \|h_{\bar{s},j}\|_{\mathcal{B}_0^{2m+2,q}},$$

which proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are finally ready to conclude: by Sub-Lemma 5.3 and Proposition 2.7 we know that the essential spectrum of $\mathcal{L}_{\widehat{\mathbb{F}}}$ acting on $\mathcal{B}_*^{p,q}$ and on $\mathcal{B}^{p,q}$, $p \geq q$, are both bounded by $\bar{\nu}^{p-1}$. Hence [5, Lemma A.1] implies that $\sigma_{\mathcal{B}_*^{p,q}}(\mathcal{L}_{\widehat{\mathbb{F}}}) \cap D_p = \sigma_{\mathcal{B}^{p,q}}(\mathcal{L}_{\widehat{\mathbb{F}}}) \cap D_p$. On the other hand Sub-Lemma 5.6 implies that there can be at most m eigenvalues larger than $\bar{\nu}^{2m-1}$, provided $p > 2m$. Finally Sub-Lemma 5.4 shows that are at least m eigenvalues, implying the Lemma. $\qquad\qquad\square$

Remark 5.7. *Note that, as remarked just before Theorem 2.8, the projection $\boldsymbol{\pi}_*$ is not one-one. In the present case $O_k = \boldsymbol{\pi}_* \boldsymbol{\ell}_k$ are all zero apart when $k = 0$ in which case it is the invariant measure of ϕ_t (see equation (5.8)). Hence as we already saw, there are no non trivial obstructions. It is however interesting that this does not imply that the spectrum of $\mathcal{L}_{\mathbb{F}}'$ consists only of zero and $e^{h_{top}}$.*

5.1.2. *Cohomological equation.*

To study the regularity of the coboundary we consider separately the various contributions in equation (4.30). The first contribution is of the same type of the previous section, only now the test function is much bigger, however we have seen that if g is of zero average, then this term is exponentially small.

To analyze the second contribution we must analyze the operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ defined in (4.27). Doing the same type of conjugation than before we can reduce the problem

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[42] The first assertion follows by noticing that the decay of correlations is super exponential for analytic functions (just compute in Fourier transform), the second can be seen by the same arguments (actually, even simpler) we used for the spectrum of $\mathcal{L}_{\widehat{\mathbb{F}}}$.

to studying the operator $\widehat{\mathcal{L}}_{\widehat{\mathbb{F}}} = \Xi\,\theta^*\widehat{\mathcal{L}}_{\mathbb{F}}\,(\theta^*)^{-1}\Xi^{-1}$.[43] A direct computation shows that

(5.10) $$\left[\widehat{\mathcal{L}}_{\widehat{\mathbb{F}}}\mathbf{g}\right]_{(\xi,s)} = (d - bs)\left[(\widehat{\mathbb{F}}^{-1})^*\mathbf{g}\right]_{(\xi,s)}.$$

In this coordinates a one form reads $\mathbf{g} = \langle\boldsymbol{\eta}, d\xi\rangle + \boldsymbol{\sigma}ds$. Thus we can identify one forms with vector functions $(\boldsymbol{\eta}, \boldsymbol{\sigma}) : \mathbb{T}^2 \times [-\beta_1, -\beta_2] \to \mathbb{R}^3$. If we do such an identification, another direct computation shows that we are reduced to studying the operator

$$\left[\widehat{\mathcal{L}}(\boldsymbol{\eta}, \boldsymbol{\sigma})\right](\xi, s) = (d - bs)((DF^{-1})^*\boldsymbol{\eta} \circ \widehat{\mathbb{F}}^{-1}(\xi, s), (d - bs)^{-2}\boldsymbol{\sigma} \circ \widehat{\mathbb{F}}^{-1}(\xi, s)).$$

In addition, we are interested only in forms such that $\boldsymbol{\sigma} \equiv 0$ (see (4.28)). We are thus reduced to study the transfer operator

(5.11) $$[\mathcal{L}\boldsymbol{\eta}](\xi, s) = (d - bs)A^{-1}\boldsymbol{\eta} \circ \widehat{\mathbb{F}}^{-1}(\xi, s).$$

Since we can write $\boldsymbol{\eta}(\xi, s) = p_+(\xi, s)v^+ + p_-(\xi, s)v^-$, it follows that the eigenvalues must be of the form $\boldsymbol{\eta}(\xi, s) = p(\xi, s)v^{\pm}$ where $v^- = V$, $v^+ = V^{\perp}$ are the eigenvectors of A. Calling $\lambda_{\pm}$ the corresponding eigenvalues we have that the eigenvalues of $\mathcal{L}$ must be eigenvalues of

$$\mathcal{L}_{\pm}p = (d - bs)\lambda_{\pm}^{-1}p \circ \widehat{\mathbb{F}}^{-1}.$$

The above are simply multiples of the operator $\mathcal{L}_{\widehat{\mathbb{F}}}$ defined in (5.4). Note that $\lambda_+^{-1} = \lambda_- = \bar{\nu} < 1$. Accordingly, Lemma 5.1 implies that, outside the essential spectrum, we have $\sigma\left(\widehat{\mathcal{L}}_{\widehat{\mathbb{F}}}\right) = \{\bar{\nu}^{2k-2}\}_{k\in\mathbb{N}}$, where all the eigenvalues have multiplicity two, apart form the largest one which is simple. The eigenvalues that are possibly relevant are the three larger or equal to $\bar{\nu}$, however the projection of the corresponding eigendistributions that are not identically zero, when applied to ∇g, yield $\mathrm{Leb}(\partial_{v^-}(\|V\|^{-1}g)) = \mathrm{Leb}(\partial_{v^+}(\|V\|^{-1}g)) = 0$. Hence, also this term is uniformly bounded. To analyze the other terms note that $\partial_x DF = 0 = \partial_x\|V\|$, hence $\Gamma = p_0 = p_0^* = B = E = 0$ while $A = \bar{\nu}^2$, hence the remaining terms are identically zero.

It follows that no nontrivial obstructions exists and, if $\mathrm{Leb}(g) = 0$, then g is a Lipschitz coboundary (as we already knew from the simple Fourier transform computation).

5.2. Some considerations on the general case.

Given the previous discussion, a natural question is if there are or not cases in which non trivial obstructions exist. This is a difficult question to answer, here we content ourselves with the discussion of small perturbations of the linear case. See Remark 5.8 for considerations on the non perturbative case.

We will see that small perturbations of the linear case do not have obstructions to non trivial growth of ergodic integrals or to $\mathcal{C}^{\frac{1}{2}-\epsilon}$ coboundary. Hence, for a small enough perturbation, an ergodic integral either grows linearly in time or, if the function is of zero average, the function is, at least, a $\mathcal{C}^{\frac{1}{2}-\epsilon}$ coboundary (see Lemma 5.11 and Corollary 5.12).

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[43] Of course, now θ^* is not the composition operator but rather the pushforward of one forms while Ξ is a again a multiplication operator but now acting on forms.

5.2.1. *A one-parameter family of examples.*

Let us be more concrete: consider the symplectic maps studied in [46, 45]:

$$(5.12) \qquad F_\gamma(x, y) = (2x + y - \gamma\varphi(x), x + y - \gamma\varphi(x)),$$

where $\varphi \in \mathcal{C}^\infty(\mathbb{T}, \mathbb{R})$, $\int_\mathbb{T} \varphi = 0$, and $\gamma \geq 0$. For example one can choose $\varphi(x) = \frac{1}{2\pi}\sin 2\pi x$. Also choose $\|V\| = 1$ and call V_γ the vector field. Note that F_0 is the linear total automorphism discussed in the previous section.

Using the same co-ordinates as in the previous section we can reduce ourselves to the study of the map

$$\widehat{\mathbb{F}}_\gamma(x, y, s) = (F_\gamma(x, y), \psi_\gamma(x, s)))$$

$$(5.13)$$

$$\psi_\gamma(x, s) = \frac{1 - \gamma\varphi'(x) + s}{2 - \gamma\varphi'(x) + s},$$

and of the transfer operator $\mathcal{L}_{\widehat{\mathbb{F}}_\gamma} = \Xi\,\theta^*\mathcal{L}_{\mathbb{F}_\gamma}(\theta^*)^{-1}\,\Xi^{-1}$ which reads

$$(5.14) \qquad \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}\boldsymbol{g}(x, y, s) = (1 - s)\,\boldsymbol{g} \circ \widehat{\mathbb{F}}_\gamma^{-1}(x, y, s).$$

For small γ the above operator is a perturbation of $\mathcal{L}_{\widehat{\mathbb{F}}_0}$, the spectrum of which we have computed. So by the perturbation theory in [40] it will have eigenvalues close to the ones of $\mathcal{L}_{\widehat{\mathbb{F}}_0}$. Hence, the second eigenvalue of $\mathcal{L}_{\mathbb{F}_\gamma}$ will be close to $\bar{\nu} < 1$, thus it will not have any influence on the growth of ergodic integrals. Accordingly, for small γ it persists the conclusion that either a function has non zero average, and hence the ergodic integral grows like t, or it has zero average and then it is a continuous coboundary. This is a bit disappointing, yet it does provide non trivial information on the flow.

Remark 5.8. *Note that, for $\gamma \neq 0$ there is no obvious reason to expect that the projection of the eigendistributions are automatically trivial (as in the linear case). Hence, having ergodic integrals with a growth t^β, $\beta \notin \{0, 1\}$ seems to be related to having a transfer operator, associated to the measure of maximal entropy, with the second largest eigenvalue outside the disk of radius one. Note that, when $\gamma = 1$ the map is no longer Anosov. In fact, we have a map of the class studied in [46] where it is shown that the decay of correlations with respect to Lebesgue, is only polynomial. In particular, this shows that the Ruelle transfer operator (associated to Lebesgue) cannot have a spectral gap. This is suggestive, although the relevance of such a fact for the present context is doubtful since we are studying different operators.[44] In conclusion, it might be possible to have non trivial obstructions to the boundedness of the ergodic integral, although no such example is currently known.[45]*

Next, let us investigate the coboundary regularity. This amounts to understanding the growth of $\langle \boldsymbol{v}, \nabla\overline{H}_{\gamma,T}(g) \rangle$ where, as in (1.9),

$$\overline{H}_{\gamma,T}(g)(x) = -\int_{\mathbb{R}_\geq} dt\, \chi \circ \tau_{n_T}(x, t) g \circ \phi_{\gamma,t}(x).$$

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[44] In fact, in [10] it is proven that the operator $\mathcal{L}_{\mathbb{F}_1}$ has a spectral gap. More in general, in the case of area preserving Anosov maps Giovanni Forni gave us an argument showing that there should not be obstructions to the boundedness of the ergodic averages, the general case is however unclear.

[45] Though some hope is given by the construction of generic examples, for the operator associated to the SRB measure, with spectrum different from $\{0, 1\}$ by Alexander Adam [1]. See also the more recent results in [8] based on special examples described in [54] for which the spectrum can be explicitly computed.

Almost all the rest of Section 5.2 is devoted to proving the following claim.

Lemma 5.9. *For r large enough, there exist $N_2 \in \{2, 3, 4, 5\}$, and distributions $\mathcal{O}_{2,\gamma}, \ldots, \mathcal{O}_{N_2,\gamma} \in (\mathcal{C}^r)'$ such that, if $g \in \mathcal{C}^r$ belongs to the kernel of $\mathcal{O}_{1,\gamma}$ and $\mathcal{O}_{2,\gamma}$, then*

$$\|\nabla \overline{H}_{\gamma,T}(g)(x)\| \leq C_\# T^{1+c_\# \gamma}, \tag{5.15}$$

while if g belongs to the kernels of all the $\mathcal{O}_{1,\gamma}, \ldots, \mathcal{O}_{N_2,\gamma}$, then

$$\sup_{T \in \mathbb{R}_>} \sup_{x \in \mathbb{T}^2} \|\nabla \overline{H}_{\gamma,T}(g)(x)\| < \infty, \tag{5.16}$$

and g is a Lipschitz coboundary. In addition, there exists $C_* > 0$, and a non zero function $\Omega_*^\dagger \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$ such that, for all $\gamma \in (0, \gamma_0)$, $T \geq C_* \ln \gamma^{-1}$ and φ $\mathcal{C}^2$-generic, we have that $\mathcal{O}_{2,\gamma}$ is a measure and

$$|\langle V^\perp, \nabla \overline{H}_{\gamma,T}(g)(x) \rangle| \geq \gamma^2 \left| \Omega_*^\dagger(x) \mathcal{O}_{2,\gamma}(g) \right| \bar{\nu}^{-3n_T} e^{-c_\# \gamma n_T} - C_* \gamma \|g\|_{\mathcal{C}^r} \bar{\nu}^{-n_T} e^{c_\# \gamma n_T}.$$

Proof. The first claims are just a particular instance of Theorem 2.12, apart from the fact that $N_2 \in \{2, 3, 4, 5\}$. In other words the distributions are the one constructed in the proof of Theorem 2.12 while the fact that $N_2 \leq 5$ follows by perturbation theory, Lemma 4.4 and formula (4.30). Also the inequalities (5.15), (5.16) follow by equations (4.30), (4.31), the spectral analysis of $\mathcal{L}_{\mathbb{F}_0}, \widehat{\mathcal{L}}_{\mathbb{F}_0}$ carried out in sections 5.1.1, 5.1.2 (which shows that at most one eigenvalue of $\mathcal{L}_{\mathbb{F}_0}$ and three eigenvalues of $\widehat{\mathcal{L}}_{\mathbb{F}_0}$ can contribute to the growth in (5.16)) and standard perturbation theory for transfer operators [40]. The assertion that $\mathcal{O}_{2,\gamma}$ is a measure and the last inequality require much more work and will be proven in Section 5.2.6 (more precisely, the first assertion follows from Remark 5.15, while the inequality follows from equation (5.75)). $\qquad\square$

Remark 5.10. *Note that the $\mathcal{O}_{i,\gamma}$, which are defined in the proof of Theorem 2.12, could be trivial (this is the case for $\mathcal{O}_{2,\gamma}$, see Lemma 5.11). If they were all trivial, then the case $\gamma \neq 0$ would behave like the linear model: either the ergodic integral of g grows linearly or g is a Lipschitz coboundary. If this is the case or not it remains to be seen.*

Lemma 5.11. *There exists $\gamma_0 > 0$ such that for all $|\gamma| < \gamma_0$ the obstruction $\mathcal{O}_{2,\gamma}$ is invariant for the flow, hence it is identically zero or proportional to $\mathcal{O}_{1,\gamma}$. Moreover, for all $g \in \mathcal{C}^r$, $\lim_{\gamma \to 0} \mathcal{O}_{2,\gamma}(g) = 0$.*

Proof. Assume that $\mathcal{O}_{2,\gamma}(g) \neq 0$ for some $g \in \mathcal{C}^{1+\alpha}$. Then, for each $\varepsilon > 0$ there exists $g_\varepsilon \in \mathcal{C}^r$ such that $\|g - g_\varepsilon\|_{\mathcal{C}^0} \leq C_\# \varepsilon$, $\|g - g_\varepsilon\|_{\mathcal{C}^1} \leq C_\# \varepsilon^\alpha$ and $\|g\|_{\mathcal{C}^r} \leq C_\# \varepsilon^{-r+1}$. By (4.24) and Lemma 4.4

$$|\langle V^\perp, \nabla \overline{H}_{\gamma,T}(g - g_\varepsilon)(x) \rangle| \leq C_\# \bar{\nu}^{-3n_T} e^{c_\# \gamma n_T} \varepsilon.$$

Thus, by Lemma 5.9,

$$|\langle V^\perp, \nabla \overline{H}_{\gamma,T}(g)(x) \rangle| \geq \gamma^2 \left| \Omega_*^\dagger(x) \mathcal{O}_{2,\gamma}(g) \right| \bar{\nu}^{-3n_T} e^{-c_\# \gamma n_T}$$
$$- C_* \gamma \varepsilon^{-r+1} \bar{\nu}^{-n_T} e^{c_\# \gamma n_T} - C_\# \bar{\nu}^{-3n_T} e^{c_\# \gamma n_T} \varepsilon.$$

Choosing $\varepsilon = e^{-c_\# \gamma n_T}$ shows that there exists $c_1 > 0$ such that

$$|\langle V^\perp, \nabla \overline{H}_{\gamma,T}(g)(x) \rangle| \geq \gamma^2 \left| \Omega_*^\dagger(x) \mathcal{O}_{2,\gamma}(g) \right| \bar{\nu}^{-3n_T} e^{-c_1 \gamma n_T} \tag{5.17}$$
$$- C_* \gamma \bar{\nu}^{-n_T} e^{c_\# \gamma n_T} - C_\# \bar{\nu}^{-3n_T} e^{-2c_1 \gamma n_T}.$$

Next, we show that equation (5.17) implies that g cannot be a $\mathcal{C}^{1+\alpha}$ coboundary if $\mathcal{O}_{2,\gamma}(g) \neq 0$.

Note that, differentiating (4.13) and doing a rough bound on the derivative yields

$$\left\| D^2 \overline{H}_{\gamma,T}(g) \right\| \leq C_\# \nu_{\gamma,n_T}^{-4} \leq C_\# T^{4+c_\#\gamma}.$$

Since $\bar{\nu}^{-3n_T} e^{-c_1 \gamma n_T} \geq T^{3-c_\#\gamma}$, by equation (5.17) it follows that there exists a ball of radius at least $C_g T^{-1-c_\#\gamma}$ in which the derivative of $\overline{H}_{\gamma,T}(g)$ in the $V^\perp$ direction, is larger than $\gamma^2 C_g T^{3-c_\#\gamma}$, provided $T \geq C_g \ln \gamma^{-1}$. This implies the existence of a positive measure set of points x_0, x_1, $\|x_0 - x_1\| = C_g T^{-3-c_\#\gamma}$ such that

$$(5.18) \qquad\qquad |\overline{H}_{\gamma,T}(g)(x_0) - \overline{H}_{\gamma,T}(g)(x_1)| \geq C_{g,\gamma} T^{-c_\#\gamma}.$$

To conclude it suffices to prove that (5.18) is impossible if g is a ϖ-Hölder coboundary with $\varpi > 1/3$. We argue by contradiction, assume that g is a ϖ-Hölder coboundary, then there exists a function $h_\gamma \in \mathcal{C}^\varpi$ such that $h_\gamma \circ \phi_{\gamma,t} - h_\gamma = \int_0^t g \circ \phi_{\gamma,s} ds$. Accordingly, by (1.9) and (1.5)

$$\overline{H}_{\gamma,T}(g)(x_0) - \overline{H}_{\gamma,T}(g)(x_1) = h_\gamma(x_0) - h_\gamma(x_1)$$
$$+ \int_{\mathbb{R}_\geq} h_\gamma \circ \phi_{\gamma,s}(x_0) \chi' \circ \tau_{n_T}(x_0, s) \nu_{\gamma,n_T}(\phi_{\gamma,s}(x_0)) ds$$
$$- \int_{\mathbb{R}_\geq} h_\gamma \circ \phi_{\gamma,s}(x_1) \chi' \circ \tau_{n_T}(x_1, s) \nu_{\gamma,n_T}(\phi_{\gamma,s}(x_1)) ds.$$

By (4.25) and Lemma 5.16 it then follows

$$|\overline{H}_{\gamma,T}(g)(x_0) - \overline{H}_{\gamma,T}(g)(x_1)| \leq C_{g,\gamma} T^{1+c_\#\gamma} T^{-3\varpi - c_\#\gamma} \leq C_{g,\gamma} T^{1-3\varpi + c_\#\gamma}$$

which, as announced, for γ small enough, is incompatible with (5.18), provided T is large enough.

It follows that for all $f \in \mathcal{C}^2$, $\mathcal{O}_{2,\gamma}(\langle V_\gamma, \nabla f \rangle) = 0$, since $\langle V_\gamma, \nabla f \rangle$ is a $\mathcal{C}^{1+\alpha}$ coboundary. But this implies that $\mathcal{O}_{2,\gamma}$ is an invariant measure for the flow $\phi_{\gamma,t}$. Since the invariant measure is unique, either $\mathcal{O}_{2,\gamma}$ is identically zero or is proportional to $\mathcal{O}_{1,\gamma}$.

The last assertion in the Lemma follows from (5.78). $\square$

By the above Lemma it follows an interesting fact.

Corollary 5.12. *If $g \in \mathcal{C}^r$ and $\mathcal{O}_{1,\gamma}(g) = 0$, then g is a $\mathcal{C}^{\frac{1}{2} - c_\#\gamma}$ coboundary.*

Proof. By equation (5.15) and Lemma 5.11 we have

$$\|\nabla \overline{H}_{\gamma,T}(g)(x)\| \leq C_g T^{1+c_\#\gamma}.$$

By equation (4.9), calling h_γ the coboundary, it follows, for each $T > 0$,

$$\|\overline{H}_{\gamma,T}(g) - h_\gamma\|_{\mathcal{C}^0} \leq C_g \bar{\nu}^{n_T} e^{c_\#\gamma n_T},$$

since, in the present case, the spectral radius θ of $\mathcal{L}_{\mathbb{F}_\gamma}$, restricted to the kernel of $\mathcal{O}_{1,\gamma}$, is given by $\bar{\nu} e^{c_\#\gamma}$. Accordingly,

$$|h_\gamma(x) - h_\gamma(y)| \leq |\overline{H}_{\gamma,T}(g)(x) - \overline{H}_{\gamma,T}(g)(y)| + C_g T^{-1+c_\#\gamma n_T}$$
$$\leq C_g |x - y| T^{1+c_\#\gamma} + C_g T^{-1+c_\#\gamma},$$

and the claim follows by choosing $T = |x - y|^{-\frac{1}{2}}$. $\square$

To complete the proofs of Lemmata 5.9 and 5.11 we must first identify explicitly $\mathcal{O}_{2,\gamma}$. To this end we start by studying the operators (4.27) that, in the present context, read (recall (2.10))

$$(5.19) \quad \begin{aligned} \mathcal{L}_{\gamma,A}\boldsymbol{g} &= \Xi\,\theta^*\mathcal{L}_{\mathbb{F}_\gamma,\widehat{A}}(\theta^*)^{-1}\,\Xi^{-1}\boldsymbol{g} = (1-s)^{-1}\,\boldsymbol{g}\circ\widehat{\mathbb{F}}_\gamma^{-1}(x,y,s) \\ \left[\widehat{\mathcal{L}}_{\widehat{\mathbb{F}}_\gamma}\mathbf{g}\right]_{(x,y,s)} &= \left[\Xi\,\theta^*\widehat{\mathcal{L}}_{\mathbb{F}_\gamma}(\theta^*)^{-1}\,\Xi^{-1}\mathbf{g}\right]_{(x,y,s)} = (1-s)\left[(\widehat{\mathbb{F}}_\gamma^{-1})^*\mathbf{g}\right]_{(x,y,s)}. \end{aligned}$$

As in Subsection 5.1.2 we can write $\widehat{\mathcal{L}}_{\widehat{\mathbb{F}}_\gamma}\langle\boldsymbol{\eta},(dx,dy)\rangle = \langle\mathcal{L}_\gamma\boldsymbol{\eta},(dx,dy)\rangle$ where

$$(5.20) \quad [\,\mathcal{L}_\gamma\boldsymbol{\eta}\,](x,y,s) = (1-s)\left[(DF_\gamma^*)^{-1}\boldsymbol{\eta}\right]\circ\widehat{\mathbb{F}}_\gamma^{-1}(x,y,s).$$

Remark 5.13. *Note that we are interested in applying the above to the case in which $\mathbf{g} = \boldsymbol{\pi}^*dg$ (recall that $\boldsymbol{\pi}(x,v) = x$) for some function $g \in \mathcal{C}^r(\mathbb{T}^2,\mathbb{R})$. This means that, in (5.20) we are interested in $\boldsymbol{\eta} = (\nabla g)\circ\boldsymbol{\pi}$.*

5.2.2. *Some preliminary facts.*

We are thus left with the task of studying the spectrum of $\mathcal{L}_{\gamma,A}, \mathcal{L}_\gamma$, for small γ. We will use the perturbation theory developed in [33, Section 8] which shows, in particular, that all the spectral data are differentiable in γ, but first we need to establish some facts and notations.

By direct computation we have, setting $\pi_x(x,y,s) = x$,

$$(5.21) \quad \begin{aligned} \partial_\gamma\widehat{\mathbb{F}}_\gamma(x,y,s) &= -(\varphi(x),\varphi(x),\varphi'(x)(2-\gamma\varphi'(x)+s)^{-2}) \\ &= -(\varphi(x),\varphi(x),\varphi'(x)\partial_s\psi_\gamma(x,s)) \\ (D\widehat{\mathbb{F}}_\gamma)^{-1} &= \begin{pmatrix} 1 & -1 & 0 \\ -1+\gamma\varphi' & 2-\gamma\varphi' & 0 \\ -\partial_x\psi_\gamma/\partial_s\psi_\gamma & \partial_x\psi_\gamma/\partial_s\psi_\gamma & 1/\partial_s\psi_\gamma \end{pmatrix} \\ (D\widehat{\mathbb{F}}_\gamma)^{-1}\partial_\gamma\widehat{\mathbb{F}}_\gamma(x,y,s) &= -(0,\varphi(x),\varphi'(x)) \\ \partial_\gamma\widehat{\mathbb{F}}_\gamma^{-1}(x,y,s) &= -\left[(D\widehat{\mathbb{F}}_\gamma)^{-1}\partial_\gamma\widehat{\mathbb{F}}_\gamma\right]\circ\widehat{\mathbb{F}}_\gamma^{-1}(x,y,s) \\ &= (0,\varphi,\varphi')\circ\pi_x\circ\widehat{\mathbb{F}}_\gamma^{-1}(x,y,s)\,. \end{aligned}$$

In alternative, one can compute the above noticing that

$$(5.22) \quad \begin{aligned} \mathbb{F}_\gamma^{-1}(x,y,s) &= \left(x-y,2y-x+\gamma\varphi(x-y),\psi_\gamma^-(x,y,s)\right) \\ \psi_\gamma^-(x,y,s) &= \frac{2s-1+\gamma\varphi'(x-y)(1-s)}{1-s}\,;\quad \partial_s\psi_\gamma^-(x,y,s) = (1-s)^{-2}. \end{aligned}$$

Hence, for each $\boldsymbol{\eta} = (\eta_1,\eta_2) \in \mathcal{C}^\infty(\mathbb{T}^2\times(-\beta_1,-\beta_2),\mathbb{R}^2)$,

$$(5.23) \quad \begin{aligned} \partial_\gamma\mathcal{L}_\gamma\boldsymbol{\eta} &= (1-s)\left[(DF_\gamma^*)^{-1}\Upsilon\boldsymbol{\eta}\right]\circ\widehat{\mathbb{F}}_\gamma^{-1} = \mathcal{L}_\gamma\Upsilon\boldsymbol{\eta} \\ \Upsilon\boldsymbol{\eta} &= \left[\varphi\circ\pi_x\partial_y\boldsymbol{\eta}+\varphi'\circ\pi_x\partial_s\boldsymbol{\eta}+\varphi'\circ\pi_x\langle e_2,\boldsymbol{\eta}\rangle e_1)\right]. \end{aligned}$$

We are interested in the maximal eigenvalue μ_γ of $\mathcal{L}_\gamma$, which we know to be simple, and the associated left and right eigenvectors $\boldsymbol{\ell}_\gamma, \boldsymbol{h}_\gamma$. Thus, for all $\boldsymbol{\eta} \in \mathcal{C}^\infty(\mathbb{T}^2\times(-\beta_1,-\beta_2),\mathbb{R}^2)$,

$$(5.24) \quad \mathcal{L}_\gamma\boldsymbol{h}_\gamma = \mu_\gamma\boldsymbol{h}_\gamma \quad \boldsymbol{\ell}_\gamma(\mathcal{L}_\gamma\boldsymbol{\eta}) = \mu_\gamma\boldsymbol{\ell}_\gamma(\boldsymbol{\eta})$$

and, recalling (5.6),

$$\mu_0 = \bar{\nu}^{-2}$$

(5.25)
$$\boldsymbol{h}_0 = Vf(s)$$

$$\boldsymbol{\ell}_0(\boldsymbol{\eta}) = \int_{\mathbb{T}^2} \langle V, \boldsymbol{\eta}(x,y,\bar{s})\rangle dxdy.$$

Moreover, calling $V_{+,\gamma}$, $\|V_{+,\gamma}\| = 1$, the unstable distribution of F_γ, we have

(5.26)
$$DF_\gamma^n V_{+,\gamma} = \nu_{\gamma,n}^u V_{+,\gamma} \circ F_\gamma^n.$$

Since

$$\langle DF_\gamma^* V_{+,\gamma}^\perp \circ F_\gamma, V_{+,\gamma}\rangle = \langle V_{+,\gamma}^\perp \circ F_\gamma, DF_\gamma V_{+,\gamma}\rangle = 0,$$

we have $DF_\gamma^* V_{+,\gamma}^\perp \circ F_\gamma = \tau V_{+,\gamma}^\perp$ and, multiplying the latter relation by V_γ yields

$$DF_\gamma^* V_{+,\gamma}^\perp \circ F_\gamma = \frac{\nu_{\gamma,1}\langle V_\gamma, V_{+,\gamma}^\perp\rangle \circ F_\gamma}{\langle V_\gamma, V_{+,\gamma}^\perp\rangle} V_{+,\gamma}^\perp.$$

Thus

(5.27)
$$(DF_\gamma^*)^{-1} V_{+,\gamma}^\perp = \frac{\langle V_\gamma^\perp, V_{+,\gamma}\rangle}{\nu_{\gamma,1}\langle V_\gamma^\perp, V_{+,\gamma}\rangle \circ F_\gamma} V_{+,\gamma}^\perp \circ F_\gamma.$$

Analogously,

$$(DF_\gamma^*)^{-1} V_\gamma^\perp = \frac{\langle V_\gamma^\perp, V_{+,\gamma}\rangle}{\nu_{\gamma,1}^u \langle V_\gamma^\perp, V_{+,\gamma}\rangle \circ F_\gamma} V_\gamma^\perp \circ F_\gamma.$$

Note that we can always write $\boldsymbol{\eta} = \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma}\rangle}\left[aV_\gamma^\perp + bV_{\gamma,+}^\perp\right]$, hence we have

$$\mathcal{L}_\gamma \boldsymbol{\eta} = \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma}\rangle}\left[V_{\gamma,+}^\perp \frac{1-s}{\nu_{\gamma,1}} b \circ \mathbb{F}_\gamma^{-1} + V_\gamma^\perp \frac{1-s}{\nu_{\gamma,1}^u} a \circ \mathbb{F}_\gamma^{-1}\right].$$

Thus, setting[46]

$$\mathcal{L}_{0,\gamma} f = \frac{1-s}{\nu_{\gamma,1}^u} f \circ \mathbb{F}_\gamma^{-1}$$

$$\mathcal{L}_{2,\gamma} f = \frac{1-s}{\nu_{\gamma,1}} f \circ \mathbb{F}_\gamma^{-1},$$

we have, for each $n \in \mathbb{N}$,

$$\mathcal{L}_\gamma^n \boldsymbol{\eta} = \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma}\rangle}\left[V_{\gamma,+}^\perp \mathcal{L}_{2,\gamma}^n b + V_\gamma^\perp \mathcal{L}_{0,\gamma}^n a\right].$$

Note that the maximal eigenvalue of $\mathcal{L}_{0,0}$ is one, while the maximal eigenvalue of $\mathcal{L}_{2,0}$ is $\mu_0 = \bar{\nu}^{-2}$. Also note that the operators $\mathcal{L}_{2,\gamma}$ are well defined on $\mathcal{B}^{1,\alpha}$ and, on such spaces, it has essential spectral radius bounded by $\mu_\gamma \bar{\nu}^\alpha e^{c\#\gamma}$. Thus, by perturbation theory, $\sigma_{\mathcal{B}^{1,\alpha}}(\mathcal{L}_{2,\gamma}) \subset \{\mu_\gamma\} \cup \{z \in \mathbb{C} \ : \ |z| \leq \mu_\gamma \bar{\nu}^\alpha e^{c\#\gamma}\}$.

Let us set $\bar{s}_\gamma \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R}_<)$ so that $V_\gamma = (1, \bar{s}_\gamma)[1 + \bar{s}_\gamma]^{-\frac{1}{2}}$ and, for each $\psi \in \mathcal{C}^0(\Omega, \mathbb{R})$, we define

(5.28)
$$\bar{\pi}_\gamma \psi(x,y) = \psi(x, y, \bar{s}_\gamma(x,y)).$$

Then,

$$\|\mathcal{L}_{2,\gamma}^n \psi - \mathcal{L}_{2,\gamma}^n \bar{\pi}_\gamma \psi\|_{\mathcal{C}^0} \leq C_\#(\|\psi\|_{\mathcal{C}^1} e^{c\#\gamma n}).$$

---

[46] The operators are well defined on $\mathcal{B}^{1,\alpha}$, which suffices for the present purpose to characterise the maximal eigenvectors.

By the above considerations it follows

$$
\begin{aligned}
\boldsymbol{h}_\gamma \boldsymbol{\ell}_\gamma(\boldsymbol{\eta}) &= \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma} \rangle} \mu_\gamma^{-n} V_{\gamma,+}^\perp \mathcal{L}_{2,\gamma}^n(b) + \mu_\gamma^{-n} \mathcal{O}(\|\boldsymbol{\eta}\|_{\mathcal{C}^0} e^{c_\# \gamma n}) \\
&= \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma} \rangle} \mu_\gamma^{-n} V_{\gamma,+}^\perp \mathcal{L}_{2,\gamma}^n(\bar{\pi}_\gamma b) + \mu_\gamma^{-n} \mathcal{O}(\|\boldsymbol{\eta}\|_{\mathcal{C}^0} e^{c_\# \gamma n}) \\
&= \frac{1}{\langle V_\gamma^\perp, V_{+,\gamma} \rangle} h_{2,\gamma} V_{\gamma,+}^\perp \ell_{2,\gamma}(\bar{\pi}_\gamma b) + \bar{\nu}^{\alpha n} \mathcal{O}(\|\boldsymbol{\eta}\|_{\mathcal{C}^{1+\alpha}} e^{c_\# \gamma n}),
\end{aligned}
$$

where $h_{2,\gamma}, \ell_{2,\gamma}$ are the right and left maximal eigenvectors of $\mathcal{L}_{2,\gamma}$, respectively normalised so that $\ell_{2,\gamma}(1) = \ell_{2,\gamma}(h_{2,\gamma}) = 1$. Taking the limit $n \to \infty$ it follows

$$
\boldsymbol{\ell}_\gamma(\boldsymbol{\eta}) = \ell_{2,\gamma}\left(\bar{\pi}_\gamma \langle V_\gamma, \boldsymbol{\eta} \rangle\right)
$$

(5.29)
$$
\boldsymbol{h}_\gamma = \frac{1}{\langle V_\gamma, V_{+,\gamma}^\perp \rangle} V_{\gamma,+}^\perp h_{2,\gamma}.
$$

Remark 5.14. *Note that $\ell_{2,\gamma}$ is the maximal left eigenvalue of a transfer operator with a $\mathcal{C}^{1+\alpha}$ potential. It follows from [34, Lemma 4.9, 4.10] (by applying them to the case $\iota = 0$, $r = 2 + \alpha$, $p = 1$ and $q = \alpha$) that $h_{2,\gamma}, \ell_{2,\gamma}$ are measures, thus the same is true for $\boldsymbol{h}_\gamma, \boldsymbol{\ell}_\gamma$.*

The above provides some global information, next we need to compute the derivatives with respect to γ of the various object of interest. Recall that we have seen in Subsection 5.1.2 that, outside of the essential spectrum, $\sigma(\widehat{\mathcal{L}}_0) = \{\bar{\nu}^{2k-2}\}_{k \in \mathbb{N}}$. On the other hand the spectrum of $\mathcal{L}_{0,A}$ can be computed as in section 5.1.1 yielding $\sigma(\mathcal{L}_{0,A}) = \{\bar{\nu}^{2k+1}\}_{k \in \mathbb{N}}$.

5.2.3. *Perturbation theory.*
Differentiating equations (5.24) and remembering (5.23) yields

$$
\begin{aligned}
\partial_\gamma \mu_\gamma &= \boldsymbol{\ell}_\gamma\left(\partial_\gamma \mathcal{L}_\gamma \boldsymbol{h}_\gamma\right) \\
\partial_\gamma \boldsymbol{h}_\gamma &= \sum_{k=0}^\infty \mu_\gamma^{-k-1} \mathcal{L}_\gamma^k [\partial_\gamma \mathcal{L}_\gamma \boldsymbol{h}_\gamma - \boldsymbol{\ell}_\gamma(\partial_\gamma \mathcal{L}_\gamma \boldsymbol{h}_\gamma)\boldsymbol{h}_\gamma] \\
(5.30) \quad \partial_\gamma \boldsymbol{\ell}_\gamma(\boldsymbol{\eta}) &= \boldsymbol{\ell}_\gamma\left(\mu_\gamma^{-1} \partial_\gamma \mathcal{L}_\gamma(\mathbb{1} - \mu_\gamma^{-1}\mathcal{L}_\gamma)^{-1}[\boldsymbol{\eta} - \boldsymbol{h}_\gamma \boldsymbol{\ell}_\gamma(\boldsymbol{\eta})]\right) \\
&= \sum_{k=0}^\infty \mu_\gamma^{-k} \boldsymbol{\ell}_\gamma\left(\Upsilon \mathcal{L}_\gamma^k[\boldsymbol{\eta} - \boldsymbol{h}_\gamma \boldsymbol{\ell}_\gamma(\boldsymbol{\eta})]\right).
\end{aligned}
$$

Remembering equation (5.23), (5.25) and since $f(\bar{s}) = 1$ (see the line before (5.6)), the first of (5.30) yields

$$
\partial_\gamma \mu_\gamma|_{\gamma=0} = \boldsymbol{\ell}_0\left(\partial_\gamma \mathcal{L}_\gamma|_{\gamma=0} \boldsymbol{h}_0\right) = \int_{\mathbb{T}^2} \langle V, [\mathcal{L}_0 \Upsilon V f](x, y, \bar{s}) \rangle dx dy.
$$

Next, we can use the definition (5.20) and (5.23) again to obtain

$$
(5.31) \qquad \partial_\gamma \mu_\gamma|_{\gamma=0} = \bar{\nu}^{-2} \int_{\mathbb{T}^2} [\langle V, e_1 \rangle \langle e_2, V \rangle + f'(\bar{s})]\varphi'(x) dx dy = 0.
$$

Next, (5.23) implies

$$
\partial_\gamma \mathcal{L}_\gamma \boldsymbol{h}_\gamma|_{\gamma=0} = \mathcal{L}_\gamma \left[f'(\bar{s})\varphi'V + \varphi'f(\bar{s})\langle e_2, V \rangle A(1, -1)\right] \circ \widehat{\mathbb{F}}^{-1}(x, y, s).
$$

Hence

$$(5.32) \qquad \partial_\gamma \boldsymbol{h}_\gamma|_{\gamma=0} = \sum_{k=0}^\infty \bar{\nu}^{-2k-2} \mathcal{L}_0^{k+1} \left[f'(\bar{s})\varphi' V + \varphi' \langle e_2, V \rangle e_1 \right].$$

On the other hand, for each $\boldsymbol{\eta} \in \mathcal{C}^\infty(\Omega^*, \mathbb{R}^2)$, $\Omega^* = \mathbb{T}^2 \times (-\beta_1, -\beta_2)$, such that $\ell_0(\boldsymbol{\eta}) = 0$,

$$\partial_\gamma \ell_\gamma(\boldsymbol{\eta})|_{\gamma=0} = \sum_{n=0}^\infty \int_{\mathbb{T}^2} \varphi(x) \langle V, \partial_y(\bar{\nu}^{2n} \mathcal{L}_0^n(\boldsymbol{\eta}))(x,y,\bar{s}) \rangle$$

$$+ \sum_{n=0}^\infty \int_{\mathbb{T}^2} \varphi'(x) \langle V, \bar{\nu}^{2n} \partial_s(\mathcal{L}_0^n(\boldsymbol{\eta}))(x,y,\bar{s}) \rangle$$

$$+ \sum_{n=0}^\infty \int_{\mathbb{T}^2} \varphi'(x) \langle e_2, \bar{\nu}^{2n} \mathcal{L}_0^n(\boldsymbol{\eta})(x,y,\bar{s}) \rangle \langle V, e_1 \rangle.$$

Note that the terms in the first sum are all identically zero by integration by parts with respect to y. We must compute the s-derivative in the second term of the above equation

$$\partial_s \widehat{\mathbb{F}}_\gamma^{-1}(x,y,s) = (0, 0, (\partial_s \psi_\gamma)^{-1} \circ \widehat{\mathbb{F}}_\gamma^{-1}) = (0, 0, (1-s)^{-2})$$

$$(5.33) \qquad \partial_s(\mathcal{L}_\gamma^n \boldsymbol{\eta}) = -(1-s)^{-1} \mathcal{L}_\gamma^n \boldsymbol{\eta} + (1-s)^{-2} \mathcal{L}_\gamma \partial_s(\mathcal{L}_\gamma^{n-1} \boldsymbol{\eta})$$

$$= -\sum_{k=0}^{n-1} (\nu_*^2 \mathcal{L}_\gamma)^k \nu_* \mathcal{L}_\gamma^{n-k} \boldsymbol{\eta} + (\nu_*^2 \mathcal{L}_\gamma)^n \partial_s \boldsymbol{\eta},$$

where we have set $\nu_*(s) = (1-s)^{-1}$ and the last formula can be checked by induction. Accordingly,

$$\partial_\gamma \ell_\gamma(\boldsymbol{\eta})|_{\gamma=0} = -\sum_{n=0}^\infty \frac{1 - \bar{\nu}^{2n}}{1 - \bar{\nu}^2} \bar{\nu} \int_{\mathbb{T}^2} \varphi'(x) \langle V, \boldsymbol{\eta} \circ \mathbb{F}_0^{-n}(x,y,\bar{s}) \rangle$$

$$+ \sum_{n=0}^\infty \bar{\nu}^{2n} \int_{\mathbb{T}^2} \varphi'(x) \langle V, (\partial_s \boldsymbol{\eta}) \circ \mathbb{F}_0^{-n}(x,y,\bar{s}) \rangle$$

$$+ \sum_{n=0}^\infty \int_{\mathbb{T}^2} \bar{\nu}^{1+n} \varphi'(x) \langle A^{-n} e_2, \boldsymbol{\eta} \circ \mathbb{F}_0^{-n}(x,y,\bar{s}) \rangle \langle V, (1,-1) \rangle.$$

In the following we are interested only in the case when the one form is given by dg, $g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$. This yields $\boldsymbol{\eta}(x,y,s) = \varsigma(s) \nabla g(x,y)$, for $\varsigma(s) = \sqrt{1+s^2}$. Thus the last term of the above formula becomes

$$\sum_{n=0}^\infty \int_{\mathbb{T}^2} \bar{\nu}^{1+n} \varphi'(x) \varsigma(\bar{s}) \langle A^{-n} e_2, (\nabla g) \circ F_0^{-n}(x,y) \rangle \langle V, (1,-1) \rangle$$

$$= \sum_{n=0}^\infty \int_{\mathbb{T}^2} \bar{\nu}^{1+n} \varphi'(x) \varsigma(\bar{s}) \langle e_2, \nabla(g \circ F_0^{-n})(x,y) \rangle \langle V, (1,-1) \rangle$$

$$= \sum_{n=0}^\infty \int_{\mathbb{T}^2} \bar{\nu}^{1+n} \varphi'(x) \varsigma(\bar{s}) \partial_y(g \circ F_0^{-n})(x,y) \rangle \langle V, (1,-1) \rangle$$

which is again zero by integration by part (in the y variable). We are left with

$$
\begin{aligned}
\partial_\gamma \ell_\gamma(\varsigma \nabla g)|_{\gamma=0} = &-\sum_{n=0}^{\infty} \frac{1-\bar{\nu}^{2n}}{1-\bar{\nu}^2} \bar{\nu}_\varsigma(\bar{s}) \int_{\mathbb{T}^2} \varphi' \circ F_0^n(x,y) \langle V, \nabla g(x,y) \rangle \\
&+ \sum_{n=0}^{\infty} \bar{\nu}^{2n} \varsigma'(\bar{s}) \int_{\mathbb{T}^2} \varphi'(x) \langle V, (\nabla g) \circ F_0^{-n}(x,y) \rangle \\
= &-\sum_{n=0}^{\infty} \bar{\nu}^n \langle V, e_2 \rangle \langle V, e_1 \rangle \int_{\mathbb{T}^2} \varphi'' \circ F_0^n \cdot g,
\end{aligned}
$$
(5.34)

where, in the last line, we have used the algebraic identities $\bar{\nu}_\varsigma(\bar{s}) = -(1-\bar{\nu}^2)\varsigma'(\bar{s})$ and $\varsigma'(\bar{s}) = \langle V, e_2 \rangle$.

5.2.4. *Leading obstruction: an explicit formula.*
We assume that g is of zero average with respect to the invariant measure of $\phi_{\gamma,t}$. Accordingly, by Lemma 2.11, $\varsigma g \circ \boldsymbol{\pi}$ belongs to the kernel of the largest left eigenvector ℓ_γ of $\mathcal{L}_{\hat{\mathbb{F}}_\gamma}$. Hence, by perturbation theory we have

$$
\|\mathcal{L}_{\hat{\mathbb{F}}_\gamma}^n g \circ \boldsymbol{\pi}\|_{p,q} \le C_\# \bar{\nu}^n e^{c_\# n\gamma} \|g\|_{\mathcal{C}^r}.
$$
(5.35)

Also, again by perturbation theory, for each $\boldsymbol{g} \in \mathcal{B}^{p,q}$,

$$
\|\mathcal{L}_{\gamma,A}^n \boldsymbol{g}\|_{p,q} \le C_\# \bar{\nu}^n e^{c_\# n\gamma} \|\boldsymbol{g}\|_{\mathcal{C}^r}.
$$
(5.36)

Using the above facts, recalling (4.30), (4.31) and Lemma 4.4, and using perturbation theory on the second largest eigenvalues of $\widehat{\mathcal{L}}_{\mathbb{F}_\gamma}$, we can write

$$
\begin{aligned}
\langle \boldsymbol{v}(x), \nabla \overline{H}_{\gamma,T}(g)(x) \rangle = &\sum_{l=1}^{K_T} \mathbb{H}^1_{F_\gamma^{ln_*}(x), \chi_* \mathbb{F}_{\gamma,*}^{ln_*}(\boldsymbol{v},0)} \Bigg[-\widehat{\mathcal{L}}_{\mathbb{F}_\gamma}^{ln_*} \boldsymbol{\pi}^* dg \\
&+ \sum_{j=0}^{ln_*-1} \widehat{\mathcal{L}}_{\mathbb{F}_\gamma}^{ln_*-j} ((\mathcal{L}_{\mathbb{F}_\gamma}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_B) \\
&- \sum_{j=0}^{ln_*-1} \sum_{m=0}^{ln_*-j-1} \widehat{\mathcal{L}}_{\mathbb{F}_\gamma}^{ln_*-j-m} (\mathcal{L}_{\mathbb{F}_\gamma,\widehat{A}}^m \widehat{E} \mathcal{L}_{\mathbb{F}_\gamma}^j \boldsymbol{g}) \cdot \hat{\boldsymbol{\omega}}_\Gamma \Bigg] \\
&+ \mathcal{O}(\|g\|_{\mathcal{C}^r} \|\boldsymbol{v}\| e^{c_\# \gamma n_T}).
\end{aligned}
$$
(5.37)

Next, note that, by (5.10), (5.20), we have

$$
\widehat{\mathcal{L}}_{\mathbb{F}_\gamma}^n \boldsymbol{\pi}^* dg = \langle \left[\frac{1}{\varsigma} \mathcal{L}_\gamma^n(\varsigma \nabla g) \right] \circ \theta^{-1}, (dx, dy) \rangle.
$$

Accordingly, recalling (5.24), (5.35), (5.36) and setting $\boldsymbol{h}_\gamma^* = \langle \left[\frac{1}{\varsigma}\boldsymbol{h}_\gamma\right] \circ \theta^{-1}, (dx, dy)\rangle$, we have

$$\langle \boldsymbol{v}(x), \nabla \overline{H}_{\gamma,T}(g)(x)\rangle = \sum_{l=1}^{K_T} \mathbb{H}^1_{F_\gamma^{ln_*}(x), \chi_* \mathbb{F}_{\gamma,*}^{ln_*}(\boldsymbol{v},0)}\left(\boldsymbol{h}_\gamma^*\right) \mu_\gamma^{ln_*} \Theta_{l,\gamma}$$
$$+ \mathcal{O}(\|g\|_{\mathcal{C}^r}\|\boldsymbol{v}\|\bar{\nu}^{-n_T} e^{c_\# \gamma n_T})$$

(5.38)
$$\Theta_{l,\gamma} = -\boldsymbol{\ell}_\gamma(\varsigma(\nabla g) \circ \boldsymbol{\pi}) + \sum_{j=0}^{ln_*-1} \mu_\gamma^{-j} \boldsymbol{\ell}_\gamma(\widehat{B} \circ \theta(\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^j \varsigma g \circ \boldsymbol{\pi}))$$

$$- \sum_{j=0}^{ln_*-1} \sum_{m=0}^{ln_*-j-1} \mu_\gamma^{-j-m} \boldsymbol{\ell}_\gamma(\Gamma \circ \theta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma, \widehat{A}\circ\theta}^m \widehat{E} \circ \theta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^j \varsigma g \circ \boldsymbol{\pi}))).$$

It is educative to write $\Theta_{l,\gamma}$ differently. This is done by using the identity, for $n \in \mathbb{N}$,

(5.39)
$$\widehat{\nabla}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^n(\varsigma g) = \mathcal{L}_\gamma^n(\widehat{\nabla}(\varsigma g)) + \gamma \sum_{k=0}^{n-1} \mathcal{L}_\gamma^k \eta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} \varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}})^{n-1-k}\partial_s(\varsigma g)$$
$$- \gamma \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-1} \mathcal{L}_\gamma^k \eta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} \varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^{n-1-j}\nu_* \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^{j-k}\varsigma g,$$

where $\widehat{\nabla}g = (\partial_x g, \partial_y g)$ and $\boldsymbol{w} = (1, -1)$. The identity (5.39) can be checked by induction using (5.33) and recalling (5.20), $\nu_*(s) = (1-s)^{-1}$. Applying $\boldsymbol{\ell}_\gamma$ to (5.39) yields, for all $n \in \mathbb{N}$,

$$\boldsymbol{\ell}_\gamma(\varsigma(\nabla g) \circ \boldsymbol{\pi}) = \mu_\gamma^{-n} \boldsymbol{\ell}_\gamma(\widehat{\nabla}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^n(\varsigma g)) - \gamma \sum_{k=0}^{n-1} \mu_\gamma^{-k-1}\boldsymbol{\ell}_\gamma\left(\boldsymbol{w}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}\varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^k \varsigma' g\right)$$
$$+ \gamma \sum_{k=0}^{n-1}\sum_{j=0}^{k-1} \mu_\gamma^{-k-1}\boldsymbol{\ell}_\gamma(\boldsymbol{w}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}\varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^j \nu_* \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^{k-j}\varsigma g)$$

We can then write

$$\Theta_{l,\gamma} = -\mu_\gamma^{-ln_*}\boldsymbol{\ell}_\gamma(\widehat{\nabla}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^{ln_*}(\varsigma g)) + \Theta_{l,\gamma}^*$$

$$\Theta_{l,\gamma}^* = \gamma \sum_{k=0}^{ln_*-1} \mu_\gamma^{-k-1}\boldsymbol{\ell}_\gamma\left(\boldsymbol{w}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}\varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^k \varsigma' g\right)$$

(5.40)
$$- \gamma \sum_{k=0}^{ln_*-1}\sum_{j=0}^{k-1} \mu_\gamma^{-k-1}\boldsymbol{\ell}_\gamma(\boldsymbol{w}\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}\varphi''(\nu_*^2 \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^j \nu_* \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^{k-j}\varsigma g)$$
$$+ \sum_{j=0}^{ln_*-1} \mu_\gamma^{-j} \boldsymbol{\ell}_\gamma(\widehat{B} \circ \theta(\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^j \varsigma g \circ \boldsymbol{\pi}))$$
$$- \sum_{j=0}^{ln_*-1} \sum_{m=0}^{ln_*-j-1} \mu_\gamma^{-j-m} \boldsymbol{\ell}_\gamma(\Gamma \circ \theta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma, \widehat{A}\circ\theta}^m \widehat{E} \circ \theta \mathcal{L}_{\widehat{\mathbb{F}}_\gamma}^j \varsigma g \circ \boldsymbol{\pi}))).$$

Note that

$$|\Theta_{\infty,\gamma}^* - \Theta_{l,\gamma}^*| \leq C_\# \|g\|_{\mathcal{C}^r} \mu_\gamma^{ln_*}\bar{\nu}^{ln_*} e^{c_\# \gamma ln_*}$$

Thus we can finally identify $\mathcal{O}_{2,\gamma}$ and rewrite (5.38) as

(5.41)
$$\langle \boldsymbol{v}(x), \nabla \overline{H}_{\gamma,T}(g)(x) \rangle = \gamma \sum_{l=1}^{K_T} \mathbb{H}^1_{F_\gamma^{ln*}(x), \chi_* \mathbb{F}_{\gamma,*}^{ln*}(\boldsymbol{v},0)} \left(\boldsymbol{h}_\gamma^* \right) \mu_\gamma^{ln*} \mathcal{O}_{2,\gamma}$$
$$+ \mathcal{O}(\|g\|_{\mathcal{C}^r} \|\boldsymbol{v}\| \bar{\nu}^{-n_T} e^{c_\# \gamma n_T})$$

$$\mathcal{O}_{2,\gamma}(g) = \gamma^{-1} \Theta_{\infty,\gamma}^*.$$

Remark 5.15. *Note that, by Remark 5.14, $\mathcal{O}_{2,\gamma}$ is a measure for each γ. We will see shortly that it does not blow up for $\gamma \to 0$.*

Equation (5.38) shows the form of the obstruction. However one must rule out a nasty possibility: $\mathbb{H}^1_{F^{ln*}(x), \chi_* \mathbb{F}_*^{ln*}(\boldsymbol{v},0)}(\boldsymbol{h}_\gamma^*)$ could be identically zero. To dismiss such a conspiracy we need to better understand the flow derivative. We take the opportunity for an interesting and useful digression.

5.2.5. *Interlude: Truly parabolic.*
The next result, a refinement of Lemma 4.2 adapted to the present context, shows that our flows are typically not elliptic.

Lemma 5.16. *For each $t \in \mathbb{R}_>$ we have*[47]

(5.42) $D_\xi \phi_{\gamma,t} = |V_\gamma \circ \phi_{\gamma,t}\rangle\langle V_\gamma| + a_\gamma(\xi,t)|V_\gamma \circ \phi_{\gamma,t}\rangle\langle V_\gamma^\perp| + b_\gamma(\xi,t)|V_\gamma^\perp \circ \phi_{\gamma,t}\rangle\langle V_\gamma^\perp|$

where, for some $C_0, b_*, t_* > 0$ and for all $t \geq t_*$,
$$C_0 t^{-b_* \gamma} \leq b_\gamma(\xi,t) \leq C_0^{-1} t^{b_* \gamma}$$
$$|a_\gamma(\xi,t)| \leq \gamma C_0^{-1} t^{1+b_* \gamma}.$$

Moreover, provided γ is small enough, generically[48] *there exists $c_* > 0$ such that*
$$\sup_x \limsup_{t \to \infty} b_\gamma(x,t) t^{-c_* \gamma^2} = \infty.$$

Proof. We follow the logic of Lemma 4.2 but using the special properties of the flows under consideration. To this end note that since $\det DF_\gamma = 1$, for all $m \in \mathbb{N}$,

$$DF_\gamma^m = \nu_{\gamma,m} |V_\gamma \circ F_\gamma^m\rangle\langle V_\gamma| + \langle V_\gamma \circ F_\gamma^m, DF_\gamma^m V_\gamma^\perp\rangle |V_\gamma \circ F_\gamma^m\rangle\langle V_\gamma^\perp|$$

(5.43)
$$+ \nu_{\gamma,m}^{-1} |V_\gamma^\perp \circ F_\gamma^m\rangle\langle V_\gamma^\perp|$$
$$DF_\gamma^{-m} \circ F_\gamma^m = \nu_{\gamma,m}^{-1} |V_\gamma\rangle\langle V_\gamma \circ F_\gamma^m| - \langle V_\gamma \circ F_\gamma^m, DF_\gamma^m V_\gamma^\perp\rangle |V_\gamma\rangle\langle V_\gamma^\perp \circ F_\gamma^m|$$
$$+ \nu_{\gamma,m} |V_\gamma^\perp\rangle\langle V_\gamma^\perp \circ F_\gamma^m|.$$

By (4.5) we can write[49]

(5.44)
$$b_\gamma(\xi,t) = \nu_{\gamma,m} \circ \phi_{\gamma,t} \langle V_\gamma^\perp(F_\gamma^m \circ \phi_{\gamma,t}(\xi)), D_{F_\gamma^m(x)} \phi_{\gamma,\tau_m(\xi,t)} \cdot D_\xi F_\gamma^m V_\gamma^\perp(\xi) \rangle$$
$$= \frac{\nu_{\gamma,m} \circ \phi_{\gamma,t}(\xi)}{\nu_{\gamma,m}(\xi)} \langle V_\gamma^\perp(F_\gamma^m \circ \phi_{\gamma,t}(\xi)), D_{F_\gamma^m(\xi)} \phi_{\gamma,\tau_m(\xi,t)} V_\gamma^\perp(F_\gamma^m(\xi)) \rangle.$$

By the arbitrariness of m it follows

(5.45)
$$b_\gamma(\xi,t) = \lim_{m \to \infty} \frac{\nu_{\gamma,m} \circ \phi_{\gamma,t}(\xi)}{\nu_{\gamma,m}(\xi)}.$$

---

[47] Here, and in the following, we use the quantum mechanical notation $|v\rangle\langle w|$ for the tensor product $v \otimes w$ as we find it more convenient.

[48] In the $\mathcal{C}^2$ topology in the set $\|\varphi\|_{\mathcal{C}^2} < 1$.

[49] Where we have used (5.43) and that $D_{F_\gamma^m(x)} \phi_{\gamma,\tau_m(\xi,t)} V_\gamma \circ F_\gamma^m(\xi) = V_\gamma(F_\gamma^m(\phi_{\gamma,t}(\xi)))$.

Also, for future use, note that (5.42) implies

$$b_\gamma(\xi, t) = \det(D\phi_t).$$

For $m_t = C_\# \frac{\ln t}{h_{\text{top}}}$ and $s \in [0, t]$ we have, by [31, Lemma C.3],[50]

$$\|F_\gamma^{m_t}(\xi) - F_\gamma^{m_t} \circ \phi_{\gamma, s}(\xi)\| \le C_\# st^{-3}.$$

Accordingly, provided $s \in [0, t]$, $t \ge 1$,

$$(5.46) \qquad b_\gamma(\xi, s) = \prod_{j=0}^\infty \frac{\nu_{\gamma, 1} \circ F_\gamma^j \circ \phi_{\gamma, s}(\xi)}{\nu_{\gamma, 1}(F_\gamma^j \xi)} = \frac{\nu_{\gamma, m_t} \circ \phi_{\gamma, s}(\xi)}{\nu_{\gamma, m_t}(\xi)} \left[1 + \mathcal{O}(t^{-2})\right].$$

The above formula implies, for γ small enough,

$$(5.47) \qquad C_\# t^{-c_\# \gamma / h_{\text{top}}} \le |b_\gamma(\xi, t)| \le e^{c_\# \gamma m_t} \le C_\# t^{c_\# \gamma / h_{\text{top}}} \le C_\# \sqrt{t}.$$

One might expect that b_γ oscillates in time, however it cannot be always small. To see this consider

$$\mathcal{E}_t^* = \int_0^t b_\gamma(\xi, s) = \int_0^t \frac{\nu_{\gamma, m_t} \circ \phi_{\gamma, s}(\xi)}{\nu_{\gamma, m_t}(\xi)} ds + \mathcal{O}(1/\sqrt{t}).$$

For each $\xi \in \mathbb{T}^2$ let $m_t^*(\xi)$ be an integer so that $\tau_{m_t^*(\xi)}(\xi, t) \in (c_*, c_*^{-1})$. Since [31, Lemma C.3] implies that $e^{h_{\text{top}} m_t^*(\xi)} \ge C_\# t$,[51] it follows that we can choose $m_t^*(\xi) = m_t^*$, independent on ξ, provided c_* has been chosen small enough. Then, recalling (1.5),

$$\mathcal{E}_t^*(\xi) \ge e^{-c_\#} \int_0^t \frac{\nu_{\gamma, m_t^*} \circ \phi_{\gamma, s}(\xi)}{\nu_{\gamma, m_t^*}(\xi)} ds + \mathcal{O}(1) \ge \frac{C_\#}{\nu_{\gamma, m_t^*}(\xi)} + \mathcal{O}(1)$$

$$\ge C_\# t \frac{e^{-h_{\text{top}} m_t^*}}{\nu_{\gamma, m_t^*}(\xi)} + \mathcal{O}(1).$$

On the other hand, if $\overline{\mu}_{e,\gamma}$ is the (invariant) measure of maximal entropy of F_γ generically, for $\gamma > 0$, it will not be the SRB measure and hence, by Ruelle inequality, [42, Theorem 1.5] and structural stability we have[52]

$$\overline{\mu}_{e,\gamma}(\ln \nu_{\gamma, 1}^{-1}) = \overline{\mu}_{e,\gamma}(\ln \nu_\gamma^u) > h_{\text{top}} = \overline{\mu}_{h,0}(\ln \nu_{0,1}^{-1}).$$

In fact, using perturbation theory (as in the proof of [34, Proposition 8.1], but doing a painful second order computation), one can get the following more precise result which (rather lengthy) proof we momentarily postpone.

Sub-Lemma 5.17. *There exists $\gamma_0, \bar{c} > 0$ such that, for $|\gamma| \le \gamma_0$,*

$$\overline{\mu}_{e,\gamma}(\ln \nu_{\gamma, 1}^{-1}) \ge h_{top} + \bar{c}\gamma^2.$$

---

[50] Recall that, by structural stability, the topological entropy is constant for the family F_γ.

[51] Here h_{top} is the topological entropy of F_γ^{-1}, which coincides with the topological entropy of F_γ (since two trajectories are ε-separated for F_γ iff they are ε-separated for F_γ^{-1}).

[52] The first equality follows from the fact that F_γ is area preserving and the invariance of $\overline{\mu}_{e,\gamma}$. Indeed, let θ_γ be the angle between $V_\gamma, V_{+,\gamma}$, that is $\|V_\gamma \wedge V_{+,\gamma}\| = |\sin \theta_\gamma|$, then

$$|\sin \theta_\gamma| = \|F_\gamma^*(V_\gamma \circ F_\gamma^{-1} \wedge V_{+,\gamma} \circ F_\gamma^{-1})\| = \|(DF_\gamma V_\gamma) \wedge (DF_\gamma V_{+,\gamma})\| = \nu_{\gamma, 1} \nu_\gamma^u |\sin \theta_\gamma \circ F_\gamma|.$$

Then, by Birkhoff theorem, for $\overline{\mu}_{e,\gamma}$-almost all $\xi \in \mathbb{T}^2$, provided t is large enough,

$$\mathcal{E}_t^*(\xi) \geq C_\# t^{1+c_\# \gamma^2}, \tag{5.48}$$

from which the last statement of the Lemma follows. To study a_γ, let us set $\zeta(t) = D\phi_{\gamma,t}$. By the smooth dependence with respect to the initial conditions and recalling (4.14), we have

$$\dot{\zeta}(t) = \left[V_\gamma^\perp \otimes p_\gamma \right] \circ \phi_{\gamma,t}\, \zeta(t), \tag{5.49}$$

hence $\langle V_\gamma, \dot{\zeta} V_\gamma^\perp \rangle = 0$ and, differentiating our representation of ζ,

$$\dot{a}_\gamma(t) = \langle V_\gamma, p_\gamma \rangle \circ \phi_{\gamma,t}\, b_\gamma(t), \tag{5.50}$$

from which the wanted bound follows by (5.47) and integrating. $\qquad\square$

Proof of Sub-Lemma 5.17. By [34, Theorem 6.4] it follows that the transfer operator associated to the measure of maximal entropy is exactly $\mathcal{L}_{\mathbb{F}_\gamma}$.[53] Hence, recalling (5.14), and calling ℓ_γ, h_γ the left and right eigenvector of $\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}$ associated to the maximal eigenvalue $\tilde{\mu}_\gamma$ we have, for all $g \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$,

$$\overline{\mu}_{e,\gamma}(g) = \ell_\gamma(g h_\gamma).$$

In addition [54]

$$\tilde{\mu}_\gamma = e^{h_{\text{top}}(F_\gamma)} = e^{h_{\text{top}}(F_0)} = \bar{\nu}^{-1} =: \tilde{\mu}. \tag{5.51}$$

Also, by [34, equation (8.10)] and using the relation between the eigenvectors of $\mathcal{L}_{\mathbb{F}_\gamma}$ and $\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}$, we have

$$\begin{aligned} \overline{\mu}_{e,\gamma}(\ln \nu_{\gamma,1}^{-1}) &= \ell_\gamma(h_\gamma \ln(1-s)) + \ell_\gamma(h_\gamma \ln \varsigma \circ \widehat{\mathbb{F}}_\gamma^{-1}) - \ell_\gamma(h_\gamma \ln \varsigma) \\ &= \ell_\gamma(h_\gamma \ln(1-s)), \end{aligned} \tag{5.52}$$

where we have used the invariance of the measure $\ell_\gamma(h_\gamma \cdot)$ with respect to $\widehat{\mathbb{F}}_\gamma$. Recall, from section 5.1.1, that

$$\begin{aligned} h_0(x,y,s) &= f(s) = \prod_{k=0}^{\infty} \bar{\nu}(1 - \bar{s} + \psi^{-k}(\bar{s}) - \psi^{-k}(s)) \\ \ell_0(\varphi) &= \int \varphi(x,y,\bar{s}) dx dy \\ \psi^{-1}(s) &= \frac{2s-1}{1-s}; \quad 1 - \bar{s} = \bar{\nu}^{-1}. \end{aligned} \tag{5.53}$$

---

[53] Recall that the definition of the space here is a bit different with respect to [34]. However the two spaces, and hence the two transfer operators, are related by the continuous isomorphism (B.4).

[54] The second equality follows from structural stability.

Clearly $f(\bar{s}) = 1$, also, for further reference, note that a direct computation yields[55]

$$f'(\bar{s}) = -\frac{\bar{\nu}}{1 - \bar{\nu}^2}$$

(5.54)

$$f''(\bar{s}) = f'(\bar{s})^2 - \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k-1} 2\bar{\nu}^{2k+2j+2} + \bar{\nu}^{4k+2} \right\} = 0.$$

Then, $\ell_0(h_0 \ln(1 - s)) = -\ln \bar{\nu} = h_{\text{top}}$. As the expression on the right hand side of (5.52) is smooth in γ (see [33, Theorem 2.7]), it suffices to compute the derivatives in zero. Note that we can normalize h_γ, ℓ_γ so that $\ell_\gamma(h_\gamma) = 1$ and $\ell_\gamma(\partial_\gamma h_\gamma) = 0$. It follows $[\partial_\gamma \ell_\gamma](h_\gamma) = 0$.

We start with the analogous of (5.30):[56]

$$\partial_\gamma \tilde{\mu}_\gamma = \ell_\gamma(\partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} h_\gamma) = 0$$

(5.55)

$$\partial_\gamma h_\gamma = (\tilde{\mu}_\gamma - \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^{-1} \partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} h_\gamma$$

$$\partial_\gamma \ell_\gamma(g) = \ell_\gamma \left(\partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} (\tilde{\mu}_\gamma - \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^{-1} [g - h_\gamma \ell_\gamma(g)] \right).$$

Recalling (5.14), (5.21) we have

$$\partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} g = (1 - s) \langle \nabla g, (0, \varphi, \varphi') \circ \pi_x \rangle \circ \widehat{\mathbb{F}}_\gamma^{-1}(x, y, s) = \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} \widehat{\Upsilon} g$$

(5.56)

$$\widehat{\Upsilon} g = \langle \nabla g, (0, \varphi, \varphi') \circ \pi_x \rangle.$$

It follows that

$$\partial_\gamma h_\gamma|_{\gamma=0} = (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_0} h_0$$

(5.57)

$$\partial_\gamma \ell_\gamma|_{\gamma=0}(g) = \ell_0 \left(\partial_\gamma \mathcal{L}_{\widehat{\mathbb{F}}_0} (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} [g - h_0 \ell_0(g)] \right).$$

Thus, setting $\Gamma(\gamma) = \ell_\gamma(\ln(1 - s) h_\gamma)$, we have $\Gamma(0) = \ln \bar{\nu}^{-1}$ and

$$\Gamma'(\gamma) = \tilde{\mu} \ell_\gamma \left(\langle \nabla(\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^{-1} [\ln(1 - s) - \Gamma(\gamma)] h_\gamma, (0, \varphi, \varphi') \rangle \right)$$

(5.58)

$$+ \ell_\gamma \left(\ln(1 - s)(\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^{-1} \mathcal{L}_{\widehat{\mathbb{F}}_\gamma} \langle \nabla h_\gamma, (0, \varphi, \varphi') \rangle \right).$$

Accordingly,

$$\Gamma'(0) = \sum_{k=0}^{\infty} \ell_0 \left(\langle \nabla \bar{\nu}^k \mathcal{L}_{\widehat{\mathbb{F}}_0}^k [\ln(1 - s) + \ln \bar{\nu}] f(s), (0, \varphi, \varphi') \rangle \right)$$

$$+ \sum_{k=0}^{\infty} \ell_0 \left(\ln(1 - s) \bar{\nu}^{k+1} \mathcal{L}_{\widehat{\mathbb{F}}_0}^{k+1} \langle \nabla h_0, (0, \varphi, \varphi') \rangle \right)$$

$$= \sum_{k=0}^{\infty} \bar{\nu}^k \ell_0 \left(\partial_s \left\{ \mathcal{L}_{\widehat{\mathbb{F}}_0}^k [\ln(1 - s) + \ln \bar{\nu}] f(s) \right\} \varphi' \right).$$

---

[55] Note that

$$f'(s) = -f(s) \sum_{k=0}^{\infty} \frac{\bar{\nu} \prod_{j=0}^{k-1} [\nu_* \circ \psi^{-j}(s)]^2}{1 + \bar{\nu} \psi^{-k}(\bar{s}) - \bar{\nu} \psi^{-k}(s)}.$$

[56] The last equality in the first line follows from (5.51).

Note that, using (5.13) and computing as in (5.33) we have (recall that $\nu_*(s) = (1-s)^{-1}$), for each $g \in \mathcal{C}^1$ we have

$$(5.59) \qquad \partial_s(\mathcal{L}^k_{\widehat{\mathbb{F}}_\gamma} g) = -\sum_{j=0}^{k-1}(\nu_*^2\mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^j\nu_*\mathcal{L}^{k-j}_{\widehat{\mathbb{F}}_\gamma} g + (\nu_*^2\mathcal{L}_{\widehat{\mathbb{F}}_\gamma})^k\partial_s g.$$

Thus, setting $\widehat{g}(x,y,s) = [\ln(1-s) + \ln\bar\nu]\,f(s)$,

$$(5.60) \qquad \ell_0\left(\varphi'\partial_s\mathcal{L}^k_{\widehat{\mathbb{F}}_0}\widehat{g}\right) = -\sum_{j=0}^{k-1}\bar\nu^{2j-k+1}\widehat{g}(\bar s)\int_{\mathbb{T}^2}\varphi'(x) + \bar\nu^k\partial_s\widehat{g}(\bar s)\int_{\mathbb{T}^2}\varphi'(x) = 0.$$

It follows

$$(5.61) \qquad\qquad\qquad\qquad \Gamma'(0) = 0.$$

We must then compute the second derivative.

Differentiating (5.58) we have[57]

$$
\begin{aligned}
(5.62)\quad \Gamma''(0) = {} & \tilde\mu\ell_0'\left(\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& + \sum_{k=0}^{\infty}\sum_{j=0}^{k-1}\tilde\mu^{-k}\ell_0\left(\langle\nabla\mathcal{L}^{k-j-1}_{\widehat{\mathbb{F}}_0}\mathcal{L}'_{\widehat{\mathbb{F}}_0}\mathcal{L}^j_{\widehat{\mathbb{F}}_0}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& + \tilde\mu\ell_0\left(\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0', (0,\varphi,\varphi')\rangle\right) \\
& + \ell_0'\left(\ln(1-s)(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\mathcal{L}_{\widehat{\mathbb{F}}_0}\langle\nabla h_0, (0,\varphi,\varphi')\rangle\right) \\
& + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\tilde\mu^{-k}\ell_0\left(\ln(1-s)\mathcal{L}^{k-j-1}_{\widehat{\mathbb{F}}_0}\mathcal{L}'_{\widehat{\mathbb{F}}_0}\mathcal{L}^j_{\widehat{\mathbb{F}}_0}\langle\nabla h_0, (0,\varphi,\varphi')\rangle\right) \\
& + \ell_0\left(\ln(1-s)(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\mathcal{L}_{\widehat{\mathbb{F}}_0}\langle\nabla h_0', (0,\varphi,\varphi')\rangle\right).
\end{aligned}
$$

Next we must compute, one by one, the above six terms. We will use (5.57), (5.56), (5.59), (5.60) and (5.53). Let us proceed in the order in which the terms appear

$$
\begin{aligned}
& \tilde\mu\ell_0\left(\mathcal{L}'_{\widehat{\mathbb{F}}_0}(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& = \tilde\mu^2\ell_0\left(\varphi'\partial_s\left\{(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right\}\right) \\
& = -\sum_{k=0}^{\infty}\sum_{j=0}^{k-1}\bar\nu^{k+1}\ell_0\left(\varphi'(\nu_*^2\mathcal{L}_{\widehat{\mathbb{F}}_0})^j\nu_*\mathcal{L}^{k-j}_{\widehat{\mathbb{F}}_0}\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& \quad + \sum_{k=0}^{\infty}\bar\nu^{k+1}\ell_0\left(\varphi'(\nu_*^2\mathcal{L}_{\widehat{\mathbb{F}}_0})^k\partial_s\langle\nabla(\tilde\mu - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& = -\sum_{k=0}^{\infty}\sum_{j=0}^{k-1}\sum_{i=0}^{\infty}\bar\nu^{2j+i+1}\ell_0\left(\varphi'\circ F_0^k\langle\nabla\mathcal{L}^i_{\widehat{\mathbb{F}}_0}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right) \\
& \quad + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar\nu^{2k+i}\ell_0\left(\varphi'\circ F_0^k\partial_s\langle\nabla\mathcal{L}^i_{\widehat{\mathbb{F}}_0}\left[\ln(1-s) + \ln\bar\nu\right]h_0, (0,\varphi,\varphi')\rangle\right).
\end{aligned}
$$

---

[57] To simplify notation we abuse it and use ℓ_0' to signify $\partial_\gamma\ell_\gamma|_{\gamma=0}$ and similarly for h_γ and $\mathcal{L}_{\widehat{\mathbb{F}}_\gamma}$.

To continue note that

$$\ell_0 \left(\varphi' \circ F_0^k \langle \nabla \mathcal{L}_{\widehat{\mathbb{F}}_0}^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right)$$

(5.63)

$$= -\tilde{\mu}^i \ell_0 \left(\left[\partial_y (\varphi' \circ F_0^k \varphi) \right] \circ F_0^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \rangle \right)$$

$$+ \ell_0 \left(\varphi' \circ F_0^k \varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right)$$

$$= -\ell_0 \left(\varphi' \circ F_0^k \varphi' \bar{\nu}^{2i+1} \mathcal{L}_{\widehat{\mathbb{F}}_0}^i h_0 \right) = -\bar{\nu}^{i+1} \mu_{e,0} (\varphi' \circ F_0^k \varphi').$$

Also,

$$\ell_0 \left(\varphi' \circ F_0^k \partial_s \langle \nabla \mathcal{L}_{\widehat{\mathbb{F}}_0}^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right)$$

$$= \ell_0 \left(\left\{ \partial_y \left[\varphi \cdot \varphi' \circ F_0^k \right] \right\} (\bar{\nu}^2 \mathcal{L}_{\widehat{\mathbb{F}}_0})^i \partial_s \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right)$$

$$+ \ell_0 \left(\varphi' \varphi' \circ F_0^k \partial_s^2 \mathcal{L}_{\widehat{\mathbb{F}}_0}^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right).$$

Next, remark that if $g(x, y, \bar{s}) = 0$, then

(5.64)
$$\partial_s^2 \mathcal{L}_{\widehat{\mathbb{F}}_0}^i g(x, y, \bar{s}) = \bar{\nu}^{4i} \left[\mathcal{L}_{\widehat{\mathbb{F}}_0}^i \partial_s^2 g \right] (x, y, \bar{s}),$$

hence

$$\ell_0 \left(\varphi' \circ F_0^k \partial_s \langle \nabla \mathcal{L}_{\widehat{\mathbb{F}}_0}^i \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right)$$

$$= \bar{\nu}^{2+3i} \frac{1 + \bar{\nu}^2}{1 - \bar{\nu}^2} \bar{\mu}_{e,0} \left(\varphi' \varphi' \circ F_0^k \right).$$

To further simplify the above expression, note the $\bar{\mu}_{e,0}$, when applied to functions that do not depend on s, is just Lebesgue measure. In addition, for each zero average function $g \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R})$ and $g_1 \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R})$ such that $\partial_y g = \partial_y g_1 = 0$ we have that, for all $k > 0$,

(5.65)
$$\int_{\mathbb{T}^2} g \circ F_0^k \cdot g_1 = 0.$$

Indeed, calling G the primitive of g, we have

$$0 = \int_{\mathbb{T}^2} g_1 \partial_y (G \circ F_0^k) = \langle DF_0^k e_2, e_1 \rangle \int_{\mathbb{T}^2} g_1 \cdot g \circ F_0^k,$$

from which the claim follows since $\langle DF_0^k e_2, e_1 \rangle \neq 0$.

Collecting all the above it follows

$$\tilde{\mu} \ell_0 \left(\mathcal{L}_{\widehat{\mathbb{F}}_0}' (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \langle \nabla (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right)$$

(5.66)

$$= \frac{\bar{\nu}^2}{(1 - \bar{\nu}^2)^2} \operatorname{Leb}((\varphi')^2).$$

We can now compute the second term of (5.62). Recalling (5.56), (5.59) we have

$$\ell_0 \left(\langle \nabla \mathcal{L}_{\widehat{\mathbb{F}}_0}^{k-j-1} \mathcal{L}'_{\widehat{\mathbb{F}}_0} \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right)$$

$$= \ell_0 \left(\varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^{k-j} \left[\varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right] \right)$$

$$= - \sum_{i=0}^{k-j-1} \bar{\nu}^{2i+1} \ell_0 \left(\varphi' \circ F_0^{k-j} \varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right)$$

$$+ \bar{\nu}^{k-j} \ell_0 \left(\varphi' \circ F_0^{k-j} \varphi' \partial_s^2 \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \left[\ln(1-s) + \ln \bar{\nu} \right] h_0 \right)$$

$$= \bar{\nu}^{j+2} \frac{1+\bar{\nu}^2}{1-\bar{\nu}^2} \operatorname{Leb}((\varphi')^2) \delta_{kj},$$

where in the last line we have used (5.64). Thus

$$(5.67) \qquad \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} \tilde{\mu}^{-k} \ell_0 \left(\langle \nabla \mathcal{L}_{\widehat{\mathbb{F}}_0}^{k-j-1} \mathcal{L}'_{\widehat{\mathbb{F}}_0} \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \left[\ln(1-s) + \ln \bar{\nu} \right] h_0, (0, \varphi, \varphi') \rangle \right) = 0.$$

The third term reads, setting $\eta = (0, \varphi, \varphi')$,

$$\tilde{\mu} \ell_0 \left(\langle \nabla (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \left[\ln(1-s) + \ln \bar{\nu} \right] (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \mathcal{L}_{\widehat{\mathbb{F}}_0} \langle \nabla h_0, \eta \rangle, \eta \rangle \right)$$

$$(5.68)$$

$$= \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \bar{\nu}^{j+k} \ell_0 \left(\varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^k \left[\ln(1-s) + \ln \bar{\nu} \right] \mathcal{L}_{\widehat{\mathbb{F}}_0}^j (f' \varphi') \right) = 0.$$

The fourth term can be computed analogously yielding as well

$$(5.69) \qquad \ell'_0 \left(\ln(1-s)(\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \mathcal{L}_{\widehat{\mathbb{F}}_0} \langle \nabla h_0, (0, \varphi, \varphi') \rangle \right) = 0.$$

For the fifth term we must compute

$$\tilde{\mu}^{-k} \ell_0 \left(\ln(1-s) \mathcal{L}_{\widehat{\mathbb{F}}_0}^{k-j-1} \mathcal{L}'_{\widehat{\mathbb{F}}_0} \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \langle \nabla h_0, (0, \varphi, \varphi') \rangle \right)$$

$$(5.70)$$

$$= \bar{\nu}^j \ln \bar{\nu}^{-1} \ell_0 (\varphi' \partial_s \mathcal{L}_{\widehat{\mathbb{F}}_0}^j \varphi' f') = \bar{\nu}^{j+2} \ln \bar{\nu}^{-1} \frac{1 - \bar{\nu}^{2j}}{(1 - \bar{\nu}^2)^2} \operatorname{Leb}(\varphi' \varphi' \circ F_0^j) = 0,$$

where we used (5.54) and (5.65).

At last we consider the sixth term

$$\ell_0 \left(\ln(1-s)(\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \mathcal{L}_{\widehat{\mathbb{F}}_0} \langle \nabla h'_0, (0, \varphi, \varphi') \rangle \right)$$

$$(5.71)$$

$$= \ln \bar{\nu}^{-1} \sum_{k=1}^{\infty} \ell_0 (\varphi' \partial_s (\tilde{\mu} - \mathcal{L}_{\widehat{\mathbb{F}}_0})^{-1} \mathcal{L}_{\widehat{\mathbb{F}}_0} f' \varphi') = 0.$$

Collecting equations (5.66), (5.67), (5.68), (5.69), (5.70) and (5.71) yields

$$\bar{\mu}_{e,\gamma} (\ln \nu_{\gamma,1}^{-1}) = h_{\text{top}} + \frac{\bar{\nu}^2 \int (\varphi')^2}{(1 - \bar{\nu}^2)^2} \gamma^2 + \mathcal{O}(\gamma^3)$$

from which the claim follows. □

To conclude the section let us note a property of a_γ which is useful to check the correctness of the subsequent computations.

Lemma 5.18. *For al $\gamma \in \mathbb{R}$ small enough and all $t \geq 0$, we have*

$$\int_{\mathbb{T}^2} a_\gamma(t, \xi) d\xi = 0.$$

Proof. By (4.20), there exists ω_γ, with $\|\omega_\gamma\|_\infty \le C_\#$, such that

$$(5.72) \qquad\qquad\qquad\qquad p_\gamma = \gamma\omega_\gamma.$$

Note that

$$\int_{\mathbb{T}^2} \langle V_\gamma, p_\gamma \rangle \circ \phi_{\gamma,t}\, b_\gamma(t) d\xi = \int_{\mathbb{T}^2} \langle V_\gamma, p_\gamma \rangle \circ \phi_{\gamma,t} \det(D_\xi \phi_{\gamma,s}) d\xi = \int_{\mathbb{T}^2} \langle V_\gamma, p_\gamma \rangle d\xi.$$

On the other hand, recalling (4.14),

$$0 = \int_{\mathbb{T}^2} \partial_{\xi_k} V_{\gamma,j}(\xi) d\xi = \int_{\mathbb{T}^2} V^\perp_{\gamma,j}(\xi) p_{\gamma,k}(\xi) d\xi$$

implies

$$\int_{\mathbb{T}^2} V_{\gamma,j}(\xi) \omega_{\gamma,k}(\xi) d\xi = 0$$

for all k, j. Hence,

$$\int_{\mathbb{T}^2} \langle V_\gamma, p_\gamma \rangle \circ \phi_{\gamma,t}\, b_\gamma(t) d\xi = 0.$$

Hence, using (5.50), we can write

$$a_\gamma(\xi,t) = \gamma \int_0^t \langle V_\gamma, \omega_\gamma \rangle \circ \phi_{\gamma,s}(\xi)\, b_\gamma(\xi,s) ds,$$

which, by Fubini, concludes the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.2.6. *Characterization of the obstruction: conclusion.*

We can now continue our estimate left at (5.41). As already mentioned the first problem is to investigate the prefactor. Using (4.29) and (5.29), we can write

$$\mathbb{H}^1_{F^{ln*}(x), \chi_* \mathbb{F}^{ln*}_*(\boldsymbol{v},0)}(\boldsymbol{h}^*_\gamma) = \int_{\mathbb{R}} \langle \boldsymbol{h}^*_\gamma(\phi_{\gamma,s}(z_l), \bar{s}_\gamma \circ \phi_{\gamma,s}(z_l)), D_{z_l}\phi_{\gamma,s} D_x F^{ln*}_\gamma \boldsymbol{v} \rangle \chi_*(z_l,s) ds$$

$$= \int_{\mathbb{R}} \frac{\bar{h}_{2,\gamma}}{\varsigma(\bar{s}_\gamma)\langle V_\gamma, V^\perp_{+,\gamma}\rangle} \circ \phi_{\gamma,s}(z_l)) \langle V^\perp_{+,\gamma}(\phi_{\gamma,s}(z_l)), D_{z_l}\phi_{\gamma,s} D_x F^{ln*}_\gamma \boldsymbol{v} \rangle \chi_*(z_l,s) ds,$$

where we used the notation $z_l = F^{n_* l}(x)$. To continue, we need the following.

Lemma 5.19. *For each $t \in (0,1)$ it holds true that*

$$\langle V^\perp_{+,\gamma} \circ \phi_{\gamma,t}, D\phi_{\gamma,t} V_{+,\gamma} \rangle = \gamma \langle \boldsymbol{e}, V^\perp \rangle \langle e_1, V \rangle \sum_{k=1}^\infty \bar{\nu}^{2k+1} \left[\varphi' \circ F^{-k}_\gamma \circ \phi_{\gamma,t} - \varphi' \circ F^{-k}_\gamma \right]$$

$$+ \mathcal{O}(\gamma^2).$$

Proof. Since $\|V_{+,\gamma}\| = 1$, we have $\partial_{x_k} V_{+,\gamma} = \gamma V^\perp_{+,\gamma} \omega^+_{\gamma,k}$ for some vector function ω^+_γ. It turns out to be convenient to take the derivative in the flow direction. Doing so we have, recalling (5.49), (5.72),

$$\frac{d}{ds} \langle V^\perp_{+,\gamma} \circ \phi_{\gamma,s}, D\phi_{\gamma,s} V_{+,\gamma} \rangle = -\gamma \langle V_{+,\gamma} \circ \phi_{\gamma,s}, D\phi_{\gamma,s} V_{+,\gamma} \rangle \langle \omega^+_\gamma, V_\gamma \rangle \circ \phi_{\gamma,s}$$

$$+ \gamma \langle V^\perp_{+,\gamma}, V^\perp_\gamma \rangle \circ \phi_{\gamma,s} \langle \omega_\gamma \circ \phi_{\gamma,s}, V_{+,\gamma} \rangle$$

$$= -\gamma \langle \omega^+_\gamma, V_\gamma \rangle \circ \phi_{\gamma,s} + \mathcal{O}(\gamma^2).$$

We are left with the task of computing ω_γ^+. This is done as in (4.15):

$$(\partial_{x_k} DF_\gamma)V_{+,\gamma} + \gamma\omega_{\gamma,k}^+ DF_\gamma \hat{V}_{+,\gamma}^\perp = \partial_{x_k}\nu_{\gamma,1}^u V_{+,\gamma}\circ F_\gamma$$
$$+ \gamma\nu_{\gamma,1}^u \sum_j \partial_{x_k} F_{\gamma,j}\omega_{\gamma,j}^+ \circ F_\gamma \hat{V}_{+,\gamma}^\perp \circ F_\gamma.$$

Which, remembering footnote 52 and multiplying the above by $V_{+,\gamma}^\perp \circ F_\gamma$, yields

$$-\varphi'' e_1\langle V_{+,\gamma}^\perp, e_1\rangle\langle e, V_{+,\gamma}\rangle + \omega_\gamma^+\langle V_{+,\gamma}^\perp \circ F_\gamma, DF_\gamma V_{+,\gamma}^\perp\rangle = \nu_{\gamma,1}^u(DF_\gamma)^*\omega_\gamma^+ \circ F_\gamma.$$

Note that,

$$\langle V_{+,\gamma}^\perp \circ F_\gamma, DF_\gamma V_{+,\gamma}^\perp\rangle = \frac{\nu_\gamma\langle V_{+,\gamma}^\perp, V_\gamma\rangle \circ F_\gamma}{\langle V_{+,\gamma}^\perp, V_\gamma\rangle}.$$

Thus,

$$\omega_\gamma^+ = \sum_{k=1}^\infty \frac{\nu_{\gamma,k}\circ F_\gamma^{-k}\langle V_{+,\gamma}^\perp, V_\gamma\rangle}{\nu_{\gamma,k}^u \circ F_\gamma^{-k}\langle V_{+,\gamma}^\perp, V_\gamma\rangle \circ F_\gamma^{-k}}(DF_\gamma^{-k})^*\Xi \circ F_\gamma^{-k}$$

$$\Xi = -(\nu_{\gamma,1}^u)^{-1}\langle V_{+,\gamma}^\perp, \nabla\varphi'\rangle\langle e, V_{+,\gamma}\rangle e_1 = -\bar{\nu}\langle V_\gamma, \nabla\varphi'\rangle\langle e, V^\perp\rangle e_1 + \mathcal{O}(\gamma).$$

Accordingly, remembering (5.42) and (5.45),

$$\frac{d}{ds}\langle V_{+,\gamma}^\perp \circ \phi_{\gamma,s}, D\phi_{\gamma,s}V_{+,\gamma}\rangle = \gamma\sum_{k=1}^\infty \bar{\nu}^{2k+1}\langle V_\gamma, \nabla(\varphi'\circ F_\gamma^{-k})\rangle\circ\phi_{\gamma,s}\langle e, V^\perp\rangle\langle e_1, V\rangle + \mathcal{O}(\gamma^2)$$

$$= \gamma\langle e, V^\perp\rangle\langle e_1, V\rangle\sum_{k=1}^\infty \bar{\nu}^{2k+1}\frac{d}{ds}\varphi'\circ F_\gamma^{-k}\circ\phi_{\gamma,s} + \mathcal{O}(\gamma^2)$$

from which the Lemma follows by integration. $\qquad\square$

Using the Lemma 5.19 and recalling Remark 5.14 we have

$$\mathbb{H}^1_{F^{ln_*}(x),\chi_*\mathbb{F}_*^{ln_*}(\boldsymbol{v},0)}(\boldsymbol{h}_\gamma^*) = \mathcal{O}(\nu_{\gamma,ln_*}\|\boldsymbol{v}\| + \gamma^2\nu_{\gamma,ln_*}^{-1}\|\boldsymbol{v}\|) + \gamma\frac{\langle e, V^\perp\rangle\langle e_1, V\rangle\langle V^\perp, \boldsymbol{v}\rangle}{\nu_{\gamma,ln_*}\varsigma(\bar{s})}$$

$$\times \sum_{k=1}^\infty \bar{\nu}^{2k+1}\int_\mathbb{R} h_{2,\gamma}(\phi_{\gamma,s}(z_l))\left[\varphi'\circ F_\gamma^{-k}\circ\phi_{\gamma,s}(z_l) - \varphi'\circ F_\gamma^{-k}(z_l)\right]\chi_*(z_l, s)ds$$

It is convenient to define

$$\Omega^\dagger(z, \boldsymbol{v}) = \frac{\langle e, V^\perp\rangle\langle e_1, V\rangle}{\varsigma(\bar{s})}\sum_{k=1}^\infty \bar{\nu}^{2k+1}\int_\mathbb{R} \Xi_{\gamma,k}^*(z, s)\chi_*(z, s)ds$$

(5.73)
$$+ \mathcal{O}(\nu_{\gamma,ln_*}^2\|\boldsymbol{v}\| + \gamma\|\boldsymbol{v}\|)\langle V^\perp, \boldsymbol{v}\rangle^{-1}$$

$$\Xi_{\gamma,k}^*(z, s) = h_{2,\gamma}(\phi_{\gamma,s}(z))\left[\varphi'\circ F_\gamma^{-k}\circ\phi_{\gamma,s}(z) - \varphi'\circ F_\gamma^{-k}(z)\right].$$

The function $\Omega^\dagger$ is generically not identically zero for large l. To see it just consider the case in which x is a periodic point, say $x = 0$ (hence $z_l = 0$), then any perturbation that leaves $\varphi'(0)$ invariant but changes the value in a neighborhood will change the value of the integral. On the other hand, for $l \geq C_\# \ln\gamma^{-1}$ the integral is the dominating term in the above expression.

We can finally write, for $\langle V^\perp, \boldsymbol{v}\rangle \neq 0$,

(5.74)
$$\mathbb{H}^1_{F^{ln_*}(x),\chi_*\mathbb{F}_*^{ln_*}(\boldsymbol{v},0)}(\boldsymbol{h}_\gamma^*) = \gamma\nu_{\gamma,ln_*}^{-1}\langle V^\perp, \boldsymbol{v}\rangle\Omega^\dagger(z_l, \boldsymbol{v}).$$

Hence (5.41) becomes

$$\langle V^{\perp}, \nabla \overline{H}_{\gamma,T}(g)(x)\rangle = \gamma^2 \nu_{\gamma,n_T}^{-1} \mu_{\gamma}^{n_T} \Omega_*^{\dagger}(x)\mathcal{O}_{2,\gamma}(g)$$
$$+ \mathcal{O}(\|g\|_{\mathcal{C}^r}\|\boldsymbol{v}\|\bar{\nu}^{-n_T} e^{c_{\#}\gamma n_T}),$$

(5.75)

$$\Omega_*^{\dagger}(x) = \nu_{\gamma,n_T}\mu_{\gamma}^{-n_T}\sum_{l=1}^{K_T}\Omega^{\dagger}(F_{\gamma}^{n_*l}(x), V^{\perp})(\nu^{-1}\mu_{\gamma})^{ln_*}.$$

Our last task it to show that the term $\mathcal{O}_{2,\gamma}(g) = \gamma^{-1}\Theta_{\infty,\gamma}^*$ in equation (5.75) does not blow up when $\gamma \to 0$.

Recalling (4.26), (4.17) and setting $Q(s) = (1+s)^2 + (2+s)^2$ we have

$$\widehat{A} \circ \theta \circ \widehat{\mathbb{F}}_{\gamma}^{-1}(x,y,s) = \frac{1+s^2}{\|D_x F_{\gamma}^{-1}(1,s)\|^2}$$

$$= \frac{1+s^2}{2-6s+5s^2} - \gamma\frac{2(1+s^2)(2s-1)(1-s)}{[2-6s+5s^2]^2}\varphi'(x) + \mathcal{O}(\gamma^2)$$

$$= A_0(s) + \gamma A_1(s)\varphi'(x) + \mathcal{O}(\gamma)$$

$$\widehat{B} \circ \theta(x,y,s) = -\gamma\varphi''(x)\frac{3-2\gamma\varphi'(x)+2s}{(1-\gamma\varphi'(x)+s)^2 + (2-\gamma\varphi'(x)+s)^2}e_1$$

$$= -\gamma\frac{\varphi''(x)[3+2s]}{Q(s)}e_1 - 2\gamma^2\frac{(\varphi'(x)^2)'[2+3s+s^2]}{Q(s)^2}e_1 + \mathcal{O}(\gamma^3)$$

$$= \gamma B_0(s)\varphi''(s)e_1 + \gamma^2 B_1(s)(\varphi'(x)^2)'e_1 + \mathcal{O}(\gamma^3)$$

$$\widehat{E} \circ \theta(x,y,s) = \frac{3(s^2+s-1)}{Q(s)} + \gamma\varphi'(x)\frac{4(s^4+3s^3+3s^2+3s+2)}{Q(s)^2} + \mathcal{O}(\gamma^2)$$

$$= E_0(s,\gamma) + \gamma\varphi'(x)E_1(s) + \mathcal{O}(\gamma^2)$$

$$\Gamma \circ \theta(x,s) = -\gamma\varphi''(x)\varsigma(s)^{-2}e_1.$$

Note, for future reference, that $A_0(\bar{s}) = \bar{\nu}^2$, $E_0(\bar{s},\gamma) = 0$.

Thus, remembering (5.40), our assumption that $\varsigma g \circ \boldsymbol{\pi}$ belongs to the kernel of ℓ_{γ} and equations (5.31), (5.56),(5.28) we have

(5.76) $$\Theta_{\infty,\gamma}^* = \gamma\Delta_1 + \mathcal{O}(\gamma^2\|g\|_{\mathcal{C}^r})$$

where

$$\Delta_1 = \sum_{k=0}^{\infty}\bar{\nu}^{3k+1}\frac{(1-\bar{s})\varsigma'(\bar{s})}{\varsigma(\bar{s})}\,\mathrm{Leb}\left(\varphi'' \circ F_0^k g\right)$$

(5.77)

$$-\sum_{k=0}^{\infty}\bar{\nu}^{k+2}(1-\bar{s})\frac{1-\bar{\nu}^{2k}}{1-\bar{\nu}^2}\,\mathrm{Leb}(\varphi'' \circ F_0^k g)$$

$$-\sum_{j=0}^{\infty}\bar{\nu}^k\frac{3+2\bar{s}}{(1+\bar{s})^2+(2+\bar{s})^2}\,\mathrm{Leb}(\varphi'' \circ F_0^k g).$$

Note that, since $\varsigma'(\bar{s})/\varsigma(\bar{s}) = -\bar{\nu}/(1-\bar{\nu}^2)$ and $\bar{s}^2 + \bar{s} - 1 = 0$, $\Delta_1 = 0$, hence

(5.78) $$\mathcal{O}_{2,\gamma}(g) = \mathcal{O}(\gamma\|g\|_{\mathcal{C}^r}).$$

Remark 5.20. *We could have reached the same conclusion by using equations* (5.34), (5.38). *Also, using the above formulae and* (5.34) *one can check that also*

$\partial_\gamma \mathcal{O}_{2,\gamma}|_{\gamma=0}$ *is a measure (and, with a lot more work, one could compute it). This lends credibility to the possibility* $\mathcal{O}_{2,\gamma} \equiv 0$.

APPENDIX A. A LITTLE CLASSIFICATION

Here we provide the proof of a partial classification of the flows that satisfy our conditions.

Proof of Lemma 1.1. The map F is topologically conjugated to a linear automorphism [39, Theorem 18.6.1]. Such conjugation shows that the flow is topologically orbit equivalent to a rigid rotation. Hence one can chose a global Poincarè section and the associated Poincarè map. Such a map will have a rotation number determined by the foliation of the total automorphism, which a straightforward computation shows to have the claimed property.

Conversely, if ϕ_t has no fix points nor periodic orbits, then there exists a global section uniformly transversal to the flow (see [51] for the original work, or [30] for a brief history of the problem and references) and the associated Poincarè map is a $\mathcal{C}^{1+\alpha}$ map of the circle with irrational rotation number ω. To claim that the Poincarè map is conjugated to a rigid rotation requires however some regularity. In particular, if $\alpha \geq 1$, then Denjoy Theorem [39, Theorem 12.1.1] implies that the Poincarè map is topologically conjugated to a rigid rotation. If ω is Diophantine, then for $\alpha \geq 2$ it is possible to show that the conjugation is $\mathcal{C}^\beta$ for all $\beta < \alpha$, [37, Théorèm fundamental, page 8]. Then, if ω satisfies property (1.2), we can view a linear foliation as the stable foliation of a toral automorphism. We then obtain a $\mathcal{C}^\beta$ Anosov map with the wanted properties by conjugation. $\square$

APPENDIX B. ANISOTROPIC BANACH SPACES: DISTRIBUTIONS

In this section we first construct the Banach spaces used in Section 3, then we discuss the relation with the Banach spaces constructed in [34], finally, we prove Proposition 2.7 and show that $\mathbb{H}$ is a bounded functional.

The construction of the Banach spaces are based on the definition of appropriate norms. The Banach spaces are then obtained by closing $\mathcal{C}^r(\Omega, \mathbb{C})$ with respect to such norms.[58] The basic idea is to control not the functions themselves but rather their integrals along curves close to the stable manifolds. Hence the first step is to define the set of relevant curves.[59] To do so we need to fix $\delta \in (0, 1/2)$ and $K \in \mathbb{R}_>$.

Definition B.1 (Admissible leaves). *Given* $r \in \mathbb{R}_>$, *an admissible leave* $W \subset \mathbb{T}^2$ *is a* $\mathcal{C}^r$ *curve with length in the interval* $[\delta/2, \delta]$. *We require that there exists a parametrization* $\omega : [0,1] \to W$ *of such a curve such that* $\omega'(\tau) \in C^s(\omega(\tau))$, *for all* $\tau \in [0,1]$, *and* $\|\omega\|_{\mathcal{C}^r([0,1],\mathbb{T}^2)} \leq K$. *Moreover we ask* $(\omega(\tau), \omega'(\tau)\|\omega'(\tau)\|^{-1}) \in \Omega$, *that is the curves have all the chosen orientation. We call* Σ *the set of admissible curves where to any* $W \in \Sigma$ *is associated a parameterization* ω_W *satisfying the properties mentioned above.*

The above set is not empty as it contains pieces of stable manifolds, provided K has been chosen large enough, since the stable manifolds are uniformly $\mathcal{C}^r$, [39]. The basic fact about admissible curves is that if $W \in \Sigma$, then, for each $n \in \mathbb{N}$,

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[58] We consider complex valued functions because we are interested in having nice spectral theory.
[59] In fact, in the simple case at hand, we could consider directly pieces of stable manifolds. We do not do it to make easier to use already existing results.

$F^{-n}W \subset \cup_{i=1}^{N_n} W_i$ for some finite set $\{W_i\}_{i=1}^{N_n} \subset \Sigma$. This is quite intuitive but see [33] for a detailed proof in a more general setting.

Next, we define the integral of an element $\boldsymbol{g} \in \mathcal{C}^r(\Omega, \mathbb{C})$ along an element $W \in \Sigma$ against any $\varphi \in \mathcal{C}^0(W, \mathbb{C})$:

$$(B.1) \qquad \int_W \varphi \boldsymbol{g} := \int_0^1 ds\, \varphi \circ \omega_W(s) \cdot \boldsymbol{g}(\omega_W(s), \omega'_W(s)\|\omega'_W(s)\|^{-1})\|\omega'_W(s)\|.$$

Also, given $W \in \Sigma$ and $\varphi : W \to \mathbb{R}$ we set, for all $s \leq r$,

$$(B.2) \qquad \|\varphi\|_{\mathcal{C}^s(W,\mathbb{R})} \doteq \|\varphi \circ \omega_W\|_{\mathcal{C}^s([0,1],\mathbb{R})}.$$

We are now ready to define the relevant semi-norms:[60]

$$(B.3) \qquad \|\boldsymbol{g}\|_{p,q} := \sup_{W \in \Sigma} \sup_{|\alpha| \leq p} \sup_{\|\varphi\|_{\mathcal{C}_0^{q+|\alpha|}(W,\mathbb{C})} \leq 1} \int_W \varphi \cdot \partial^\alpha(\boldsymbol{g}),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the usual multi–index and $1, 2$ refer to the x co-ordinate while 3 refers to v.[61] It is easy to check that the $\| \cdot \|_{p,q}$ are indeed semi-norms on $\mathcal{C}^r(\Omega, \mathbb{C})$.

Definition B.2 ($\mathcal{B}^{p,q}$ spaces). *Let $p \in \mathbb{N}^*, q \in \mathbb{R}$, $p + q \leq r$ and $q > 0$. We define $\mathcal{B}^{p,q}$ to be the closure of $\mathcal{C}^r(\Omega, \mathbb{C})$ with respect to the semi-norm $\| \cdot \|_{p,q}$.*[62]

Remark B.3. *Note that $\|\boldsymbol{g}\|_{p,q} \leq \|\boldsymbol{g}\|_{\mathcal{C}^{p+q}}$.*

The Banach spaces defined above are well suited for the tasks at hand but, unfortunately, they are not exactly the one introduced in [34] where a more general theory is put forward. To avoid having to develop the theory from scratch, it is convenient to show how to relate the present setting to the one in [34]. To this end let us briefly recall the construction in [34], then we will explain the relation with the present one. This will allow us to apply the general results in [34] to the present context.

We start by recalling, particularizing them to our simple situation, the basic objects used in [34]: the r times differentiable sections $\mathcal{S}^r$ of a line bundle over the Grassmannian of one dimensional subspaces. More precisely, let $\mathcal{G} = \{(x, E)\}$ where $x \in \mathbb{T}^2$ and $E \subset \mathbb{R}^2$ is a linear one dimensional subspace, then $h \in \mathcal{S}^r$ is a $\mathcal{C}^r$ map $(x, E) \to E^*$.[63] Note that there is a strict relation between $\mathcal{S}^r$ and $\mathcal{C}^r(\Omega, \mathbb{C})$: for each $(x, v) \in \Omega$ let $E_v = \{\mu v\}_{\mu \in \mathbb{R}}$, then for each $h \in \mathcal{S}^r$ define $\boldsymbol{i} : \mathcal{S}^r \to \mathcal{C}^r(\Omega, \mathbb{C})$ by

$$(B.4) \qquad [\boldsymbol{i}h](x, v) = h(x, E_v)(v).$$

The important fact is that the elements of $\mathcal{S}^r$, when restricted to the tangent bundle of W, are volume forms on W, hence can be integrated. Let us be explicit: given

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[60] By $\mathcal{C}_0^s(W, \mathbb{C})$ we mean the $\mathcal{C}^s$ functions with support contained in $\mathrm{Int}(W)$. The fact that the test functions must be zero at the boundary of W is essential for the following arguments.

[61] To be more explicit, if we choose a chart $v = (\cos\theta, \sin\theta)$, then α_3 refers to the derivative with respect to θ.

[62] To be precise the elements of $\mathcal{B}^{p,q}$ are the equivalence classes determined by the equivalence relation $h \sim \bar{h}$ if and only if $\|h - \bar{h}\|_{p,q} = 0$.

[63] To be precise, since we are going to do spectral theory, we should consider the complex dual. We do not insist on this since the complexification is totally standard.

$W \in \Sigma$, $h \in \mathcal{S}^r$ and $\varphi \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{C})$, by (B.1) and [34, Section 2.2.1] we have

$$(B.5) \qquad \int_W \varphi h := \int_0^1 ds\, \varphi \circ \omega_W(s)\, h(\omega_W(s), E_{\omega'_W(s)})(\omega'_W(s)) = \int_W \varphi\, \boldsymbol{i} h.$$

Finally, note that the norm in [34] is also given by integrals along curves in Σ. Accordingly, if $h, \tilde{h} \in \mathcal{S}^r$ differ only for (x, E) such that E does not belong to $C^s(x)$, then any norm of the difference based on integrations along curves in Σ will be zero. The readers can then check that the norms defined in [34] are equivalent to $\|\boldsymbol{i} h\|_{p,q}$. Thus $\boldsymbol{i}$ extends, by density, to a Banach space isomorphism between the spaces defined in [34] and the $\mathcal{B}^{p,q}$ presently defined.[64] Finally, we have to understand how the operator $\mathcal{L}_{\mathbb{F}}$ reads in the corresponding language of [34]. To this end it is useful to introduce the operator $\Xi : \mathcal{C}^r(\Omega, \mathbb{C}) \to \mathcal{C}^r(\Omega, \mathbb{C})$ defined by

$$(\Xi \boldsymbol{g})(x, v) := \boldsymbol{g}(x, v) \|V(x)\|.$$

Note that, by the assumptions of Definition 2.3, Ξ is invertible and both the operator and its inverse can be extended to a continuous operator on $\mathcal{B}^{p,q}$. It then follows by equations (B.1), (B.5), (2.10), and [34, Section 3.2] that, for all $W \in \Sigma$ and $\varphi \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{C})$, we have

$$\int_W \varphi\, \boldsymbol{i}^{-1} \Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \boldsymbol{i} h = \int_0^1 ds\, \varphi(\omega_W(s))(\Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \boldsymbol{i} h)(\omega_W(s), \widehat{\omega}'_W(s)) \|\omega'_W(s)\|$$

$$= \int_0^1 ds\, \varphi(\omega_W(s))(\boldsymbol{i} h) \left(F^{-1} \omega_W(s), \frac{D_{\omega_W(s)} F^{-1} \omega'_W(s)}{\|D_{\omega_W(s)} F^{-1} \omega'_W(s)\|} \right) \|D_{\omega_W(s)} F^{-1} \omega'_W(s))\|$$

$$= \int_{F^{-1}W} \varphi \circ F\, \boldsymbol{i} h = \int_{F^{-1}W} \varphi \circ F\, h = \int_W \varphi F_* h,$$

where we used the notation $\widehat{\omega}'_W(s) = \omega'_W(s) \|\omega'_W(s))\|^{-1}$. Hence we conclude that

$$(B.6) \qquad\qquad \boldsymbol{i}^{-1} \Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \boldsymbol{i} h = F_* h := (F^{-1})^* h,$$

that is $\mathcal{L}_{\mathbb{F}}$ is conjugated to the push-forward of F on $\mathcal{S}^r$.

Proof of Proposition 2.7. Since (B.6) states that our operator is conjugated to the push-forward F_*, all the spectral properties of F_*, acting on $\mathcal{B}^{p,q,1}$, and $\mathcal{L}_{\mathbb{F}}$, acting on $\mathcal{B}^{p,q}$, coincide. It thus suffices to note that [34, Proposition 4.4, Theorem 5.1, Theorem 6.4] state that, for $q \in \mathbb{R}_>$, $p \in \mathbb{N}_>$ and $p + q \leq r$, F_* can be extended continuously to $\mathcal{B}^{p,q,1}$, that the logarithm of the spectral radius of F_* is given by the topological entropy (which is the maxim of the metric entropy), that the maximal eigenvalue is simple and F_* has a spectral gap and the essential spectral radius is bounded by $e^{h_{\text{top}}} \lambda^{-\min\{p,q\}}$. $\qquad\square$

We have thus seen that the operator $\mathcal{L}_{\mathbb{F}}$ acts very nicely on the spaces $\mathcal{B}^{p,q}$. The next important fact is that the functionals we are interested in are well behaved on such spaces.

Lemma B.4. *There exists $C > 0$ such that, for each $x \in \mathbb{T}^2$, $q \in \mathbb{R}_>$, $p \in \mathbb{N}_>$, $p + q \leq r$, and $\varphi \in \mathcal{C}_0^r(\mathbb{R}_\geq, \mathbb{R})$, $\boldsymbol{g} \in \mathcal{C}^r(\Omega, \mathbb{R})$ we have*

$$|\mathbb{H}_{x,\varphi}(\boldsymbol{g})| \leq C |\operatorname{supp} \varphi| \|\boldsymbol{g}\|_{p,q} \|\varphi\|_{\mathcal{C}^{p+q}}.$$

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[64] Note that, not by chance, the Banach spaces in [34] are named similarly: $\mathcal{B}^{p,q,1}$. The superscript 1 refers there to the fact that, as we will see briefly, in the present language we do not need to have a weight in the transfer operator.

Proof. Let us start by considering the case supp $\varphi \subset [a, a + \delta]$, for some $a > 0$. Then, $\{\phi_t(x)\}_{t \in [a,a+\delta]}$ is the re-parametrization of a curve W in Σ, provided the constant K in Definition B.1 has been chosen large enough. To see it just consider the parametrization $\omega_W(s) = \phi_{a+\delta s}(x)$. Moreover, setting $\tilde{\varphi}(\phi_s(x)) = \varphi(s),$[65] by (B.1) and (3.1)

$$\int_W \tilde{\varphi}\boldsymbol{g} = \int_0^1 ds \; \varphi \circ \phi_{a+\delta s}(x)\boldsymbol{g}(\phi_{a+\delta s}(x), \widehat{V}(\phi_{a+\delta s}(x)))\|\widehat{V}(\phi_{a+\delta s}(x))\|\delta$$
$$= \int_{\mathbb{R}} ds \; \varphi(s)(\Xi\boldsymbol{g}) \circ \boldsymbol{\phi}_s(x, \widehat{V}(x)) = \mathbb{H}_{x,\varphi}(\Xi\boldsymbol{g}).$$

Since the first quality on the left is exactly one of the functionals used in (B.3) to define the norm ($p = 0$) and Ξ^{-1} is a bounded operator on each space $\mathcal{B}^{p,q}$, we have

$$\|\mathbb{H}_{x,\varphi}(\boldsymbol{g})\| \leq C_{\#}\|\Xi^{-1}\|_{0,p}\|\varphi\|_{\mathcal{C}^q}\|\boldsymbol{g}\|_{0,q} \leq C_{\#}\|\varphi\|_{\mathcal{C}^{p+q}}\|\boldsymbol{g}\|_{p,q}.$$

The Lemma follows then by using a partition of unity. $\qquad\square$

Appendix C. Anisotropic Banach spaces: currents

In this appendix we briefly describe the Banach spaces of currents used in our second results and sketch the needed facts. We will be much faster than in Appendix B, we will omit several details as the construction is very similar to the previous one and no essentially new ideas are present.

We consider the same set of admissible leaves detailed in Definition B.1. For each $W \in \Sigma$, let $\mathcal{V}^q$ be the set of $\mathcal{C}^q$ vector fields compactly supported on W and with $\mathcal{C}^q$ norm bounded by one. Then, for each smooth one form $\mathfrak{g}$ on Ω we define

$$(C.1) \qquad \|\mathfrak{g}\|_{p,q} := \sup_{W \in \Sigma} \sup_{|\alpha| \leq p} \sup_{\varphi \in \mathcal{V}^{q+|\alpha|}} \int_W [\partial^\alpha(\mathfrak{g})](\varphi),$$

where the integral is defined as in the previous section.

Note that there exists a standard isomorphism $\boldsymbol{i}$ from vector fields to one forms, so that $\mathfrak{g}(\varphi) = \langle \mathfrak{g}, \boldsymbol{i}(\varphi) \rangle$.[66] Thus the above norm is equivalent to the norm $\|\cdot\|_{p,q,1}$ used in [31]. Let $\mathcal{A}$ be the set of $\mathcal{C}^\infty$ one forms on Ω such that, for all $v \in \mathbb{R}^2$, $\mathfrak{g}((0, v)) = 0$. If we define $\widehat{\mathcal{B}}^{p,q}$ as the closure of $\mathcal{A}$ with respect to the above norm, we obtain a space isomorphic to a subspace of the space $\mathcal{B}^{p,q,1}$ defined in [31].

Unfortunately, the transfer operator used here differs from the one studied in [31] insofar it has a potential, which was absent in [31]. In principle, we should therefore prove the Lasota-Yorke inequality for our operator and compute the spectral radius for the present operator via a variational principle (as in [34]). Since such a computation is completely standard but a bit lengthy, we just state a partial result that suffices for our goals (in particular we do not bother computing exactly the spectral radius). Such a result follows by copying the computations made in [31] to obtain the Lasota-Yorke inequality. Such computations are exactly the same, apart from the need to keep track of the potential, which can be done easily:

- The operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ extends continuously on $\widehat{\mathcal{B}}^{p,q}$, has spectral radius ρ and essential spectral radius strictly bounded by $\lambda^{-\min\{p,q\}}\rho$.

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[65] Note that, since the stable manifolds are uniformly $\mathcal{C}^r$, [39], $\|\tilde{\varphi}\|_{\mathcal{C}^r(W,\mathbb{R})} \leq C_{\#}\|\varphi\|_{\mathcal{C}^r(\mathbb{R}_>,\mathbb{R})}$.

[66] See [31] for the relevant definition of scalar product between forms in the present context.

- For all $w \in \mathcal{C}^r$ and $x \in \mathbb{T}^2$ and $p + q \leq r$ we have[67]

$$\left| \mathbb{H}^1_{x,w}(\mathbf{g}) \right| \leq C_\# |\operatorname{supp} w| \|\mathbf{g}\|_{p,q} \|w\|_{\mathcal{C}^{p+q}}.$$

The above two facts are all we presently need.

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