Mixing for some non-uniformly hyperbolic systems

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Abstract

In this work we obtain mixing (and in some cases sharp mixing rates) for a reasonable large class of invertible systems preserving an infinite measure. The examples considered here are the invertible analogue of both Markov and non Markov unit interval maps. Moreover, we obtain results on the decay of correlation in the finite case of invertible non Markov maps, which, to our knowledge, were not previously addressed.

The present method consists of a combination of the framework of operator renewal theory, as introduced in the context of dynamical systems by Sarig [39], with the framework of function spaces of distributions developed in the recent years along the lines of Blank, Keller and Liverani [9].

1 Introduction

At present there exist well developed theories that provide subexponential decay of correlation for non-uniformly expanding maps, culminating with the work of Sarig [39]. For systems with subexponential decay of correlations, previous approaches to [39] for estimating decay of correlations provided only upper bounds. These previous approaches include the coupling method of Young [43] (developed upon [42]), Birkhoff cones techniques adapted to general Young towers by Maume-Deschamps [33] and the method of stochastic perturbation developed by Liverani et al [30].

Among other statistical properties, the method of [43] provides polynomial decay of correlation for non uniformly expanding maps that can be modeled by Young towers with polynomially decaying return time tails. The estimates obtained in [43] were shown to be optimal via the method of *operator renewal theory* introduced in [39] to obtain precise asymptotics and thus, sharp mixing rates. The later mentioned

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method is an extension of scalar renewal theory from probability theory to dynamical systems. Later on, the method of operator renewal theory then was substantially extended and refined by Gouëzel [20, 21].

In recent work, Melbourne and Terhesiu [35] developed an operator renewal theory framework that recovers the classical notion of mixing for a very large class of (non-invertible) dynamical systems with infinite measure. We recall that the notion of "mixing" for infinite measure preserving systems is very delicate: given a conservative ergodic infinite measure preserving transformation (X, f, μ) with transfer operator L, we have $\int L^n v \, d\mu \to 0$, as $n \to \infty$, for all $v \in L^1(\mu)$. Thus, to recover the classical notion of mixing, one needs to find a sequence c_n and a reasonably large class of functions v (within L^1) such that $c_n \int L^n v \, d\mu \to C \int v \, d\mu$ for some $C > 0$.

In short, the framework of operator renewal theory has been cast (at least implicitly) in a rather general Banach space setting (see, e.g. [39, 20, 21, 35, 22]) and has been successfully employed to study the statistical properties of both finite and infinite measure preserving, non invertible (eventually expanding) systems. Our aim in this work is to carry out the method of operator renewal theory, in the case of (finite and infinite measure preserving, but focusing on the later) invertible systems. In such a case one would need Banach spaces that allow a direct study of the spectral properties of the transfer operator eliminating altogether, in the uniformly expanding case, the need of coding the system. While until recently it was unclear if such Banach spaces existed at all, the last decade, starting with Blank, Keller and Liverani [9], and reaching maturity with [23, 24, 6, 7, 12, 4, 13, 32, 11, 5, 19, 16], has produced an abundance of such spaces. Yet, all such Banach spaces are necessarily Banach spaces of distributions, hence the need to *explicitly* cast all the renewal theory arguments in a completely abstract form (for example one must avoid implicit assumptions like the Banach space being a subset of some L^p). In this work we provide a set of abstract conditions on dynamical systems (including the non invertible ones) and develop a corresponding renewal theory framework; this set of hypotheses/conditions includes the existence of Banach spaces with certain good properties. Moreover, we provide some examples to show that the above mentioned hypotheses are indeed checkable in non trivial cases. Let us explain the situation in more details.

1.1 Operator renewal theory for invertible systems: the need for new functions spaces

Given a conservative (finite or infinite) measure preserving transformations (X, f, μ), renewal theory is an efficient tool for the study of the long term behavior of the transfer operator $L : L^1(X) \to L^1(X)$. Fix $Y \subset X$ with $\mu(Y) \in (0, \infty)$. Let $\varphi : Y \to \mathbb{Z}^+$ be the first return time $\varphi(y) = \inf\{n \geq 1 : f^n y \in Y\}$ (finite almost everywhere by conservativity). Let $L : L^1(X) \to L^1(X)$ denote the transfer operator for f and define

$$T_n = 1_Y L^n 1_Y, \;\; n \geq 0, \qquad R_n = 1_Y L^n 1_{\{\varphi = n\}}, \;\; n \geq 1. \tag{1.1}$$

Thus T_n corresponds to general returns to Y and R_n corresponds to first returns to Y. The relationship $T_n = \sum_{j=1}^{n} T_{n-j} R_j$ generalises the notion of scalar renewal sequences (see [8, 17] and references therein). The rough idea behind operator renewal theory is that the asymptotic behavior of the sequence T_n can be obtained via a good understanding of the sequence R_n. Apriori assumptions needed to deal with the sequence R_n include the uniform hyperbolicity of the first return map F, along with good spectral properties of the associated transfer operator.

In short, to carry out the method of operator renewal theory to invertible maps f, we need to establish the required spectral results for the transfer operator associated with the uniformly hyperbolic first return map F. In particular, a spectral gap is needed. Since it is well known that the transfer operators for invertible systems do not have a spectral gap on any of the usual spaces (such as $L^p, W^{p,q}$ or BV), unconventional Banach spaces are necessary.

In Section 2 we will specify exactly which conditions are needed and in the sections following Section 2 we obtain several results under such conditions. In section 1.4 we provide examples for which the above conditions are satisfied. This examples are shown to satisfy the above mentioned conditions in sections 6 and 7.

1.2 Mixing for non–invertible infinite measure preserving systems

The techniques in [35] are very different from the ones developed for the framework of operator renewal sequences associated with finite measure [39, 20, 21].

In the infinite mean setting a crucial ingredient for the asymptotics of renewal sequences is that $\mu(y \in Y : \varphi(y) > n) = \ell(n)n^{-\beta}$ where ℓ is slowly varying[1] and $\beta \in (0, 1]$ (see Garsia and Lamperti [18] and Erickson [14] for the setting of scalar renewal sequences). Under suitable assumptions on the first return map T^{φ}, [35] shows that for a (“sufficiently regular”) function v supported on Y and a constant $d_0 = \frac{1}{\pi}\sin\beta\pi$, the following hold: i) when $\beta \in (\frac{1}{2}, 1]$ then $\lim_{n\to\infty} \ell(n)n^{1-\beta}T_n v = d_0 \int_Y v\, d\mu$, uniformly on Y; ii) if $\beta \in (0, \frac{1}{2}]$ and $v \geq 0$ then $\liminf_{n\to\infty} \ell(n)n^{1-\beta}T_n v = d_0 \int_Y v\, d\mu$, pointwise on Y and iii) if $\beta \in (0, \frac{1}{2})$ then $T_n v = O(\ell(n)n^{-\beta})$ uniformly on Y. As shown in [35], the above results on T_n extend to similar results on L^n associated with a large class of non-uniformly expanding systems preserving an infinite measure.

The results for the case $\beta < 1/2$ are *optimal* under the *general assumption* $\mu(\varphi > n) = \ell(n)n^{-\beta}$ (see [18]). Under the *additional assumption* $\mu(\varphi = n) = O(\ell(n)n^{-(\beta+1)})$, Gouëzel [22] obtains first order asymptotic for L^n for *all* $\beta \in (0, 1)$.

A typical example considered for the study of mixing/mixing rates via renewal operator theory associated with, both, finite and infinite measure preserving systems is the family of Pomeau-Manneville intermittency maps [38]. To fix notation, we

[1]We recall that a measurable function $\ell : (0,\infty) \to (0,\infty)$ is slowly varying if $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$. Good examples of slowly varying functions are the asymptotically constant functions and the logarithm.

recall on the version studied by Liverani *et al.* [30]:

$$f_0(x) = \begin{cases} x(1+2^\alpha x^\alpha), & 0 \le x \le \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x \le 1. \end{cases} \tag{1.2}$$

It is well known that the statistical properties for f_0 can be studied by inducing on a 'good' set Y inside $(0,1]$, such as the standard set $Y = [1/2, 1]$. In particular, we recall that the inducing method can be used to show that there exists a unique (up to scaling) σ-finite, absolutely continuous invariant measure μ_0: finite if $\alpha \le 1$ and infinite if $\alpha \ge 1$; equivalently, writing $\beta := 1/\alpha$, μ is finite if $\beta > 1$ and infinite if $\beta \le 1$.

Let φ_0 be the return time function to Y, rescale the f_0 invariant measure μ such that $\mu(Y) = 1$ and set $Y_j = \{\varphi_0 = j\}$. We recall that $\mu(Y_j) \le Cj^{-(\beta+1)}$ and $|f_0'(y_j)|^{-1} \le Cj^{-(\beta+1)}$, for all $y_j \in Y_j$ (see [30]). Hence, $\mu(\varphi_0 = n) = O(n^{-(\beta+1)})$ and the assumption in Gouëzel [22] is satisfied, providing first order asymptotic for L^n for all $\beta \in (0,1)$.

Apart from the above Markov example, the results in [35, 22] apply also to the class of non-Markovian interval maps, with indifferent fixed points studied in Zweimüller [44, 45]. For simplicity, consider the following example that satisfies the above mentioned additional assumption in Gouëzel [22].

Define a map $f_0 : [0,1] \to [0,1]$ that on $[0, \frac{1}{2}]$ agrees with the map defined by (1.2). On $(1/2, 1]$, we assume that there exists a finite partition into open intervals I_p, $p \ge 1$ such that f_0 is C^2 and strictly monotone in each I_p with $|f_0'| > 2$. Moreover, assume that f_0 is topologically mixing. Obviously, the new (not necessarily Markov) map f_0 shares many of the properties of the map defined by (1.2). In particular, there exists a unique (up to scaling) σ-finite, absolutely continuous invariant measure μ: finite if $\alpha \le 1$ and infinite if $\alpha \ge 1$; equivalently, writing $\beta := 1/\alpha$, μ is finite if $\beta > 1$ and infinite if $\beta \le 1$. Moreover, given that $Y = [1/2, 1]$, φ_0 is the return time function of f_0 to Y and $Y_j = \{\varphi_0 = j\}$, one can easily see that $|f_0'(y_j)|^{-1} \le Cj^{-(\beta+1)}$, for all $y_j \in Y_j$ and that $\mu(\varphi_0 = n) = O(n^{-(\beta+1)})$.

For more general classes of mixing (in the sense described above) of non-invertible infinite measure preserving systems (including parabolic maps of the complex plane) we refer to [35]. At present it is not entirely clear how to deal with the infinite measure preserving setting of higher dimensional non uniformly expanding maps considered by Hu and Vaienti [29].

1.3 Mixing rates in the non invertible case

For results on decay of correlation in the finite case of (1.2) we refer to [39, 20] and [28]. For the infinite case, the method developed in [35] yields mixing rates and *higher* order asymptotics of L^n. The results in this work suggest that mixing rates in the infinite case can be regarded as the analogue of the decay of correlation in the finite case.

As shown in [35], mixing rates in the infinite measure setting of f_0 can be obtained by exploiting a good enough expansion of the tail behavior $\mu_0(\varphi_0 > n)$, where φ_0 is the return time function to a 'good' set Y inside $(0,1]$, such as the standard set $Y = [1/2, 1]$.

Exploiting a modest expansion of the tail behavior $\mu_0(\varphi_0 > n)$ and good properties of the induced map F_0, [35] shows that for any Hölder or bounded variation observable $v : [0,1] \to \mathbb{R}$ with v supported on some compact subset of $(0,1)$, we have $L^n v = d_\beta n^{\beta-1} \int v d\mu_0 + O(n^{-(\beta-1/2)})$, uniformly on Y. As noted in [35], this rate is optimal for $\beta \geq 3/4$. Exploiting more properties of the return function φ_0 and of the induced map F_0, improved mixing rates are obtained in [41]. The higher order asymptotic of L^n in [35, 41] is obtained via the study of associated operator renewal sequences $T_n : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is the space of Hölder or bounded variation functions.

1.4 Invertible systems: Markov and non Markov examples.

As explained in the previous paragraphs, to employ the framework of operator renewal theory to invertible systems $f : X \to X$, we need a suitable function space under which the transfer operator associated to the (desirable uniformly hyperbolic) first return map $F : Y \to Y$, for some $Y \subset X$, satisfies specific spectral properties. As already mentioned we use φ to designate the return time to Y and we set $Y_n = \{x \in Y \ : \ \varphi(x) = n\}$.

Our examples below provide a large class of systems, where the task of checking specific spectral properties can be accomplished by using the appropriate anysotropic Banach spaces.

The examples considered below are far from being the most general (see Remark 1.2 for details). Nevertheless, they are fairly representative for both classes of invertible systems: i) preserving and ii) lacking a Markov structure. The needed properties for invertible maps f is established in: a) Section 6 in the Markov case; b) Section 7 in the non Markov case.

The strongest restriction in our examples is given by the requirement that there exists a globally smooth stable foliation. In principle, our methods could be applied to more general cases, but it is not so obvious how to identify the correct functional analytic framework in which to analyze the transfer operator.

The requirement that f preserves a global smooth foliation means that there exists a smooth map $\mathbb{H}(x,y) = (H(x,y), y)$, which can be normalized so that $H(x,0) = x$, such that

$$f \circ \mathbb{H}(x,y) = (H(f_0(x), g(x,y)), g(x,y)) = \mathbb{H}(f_0(x), g(x,y)),$$

for some functions f_0, g. Indeed, the fibre through the point $(x,0)$ can be seen as the graph of the function $H(x,\cdot)$ over the y axis and is mapped by f to the fibre thru the point $(f_0(x), 0)$. In other words the map f is conjugated to the map

$$\mathbb{H}^{-1} \circ f \circ \mathbb{H}(x,y) = (f_0(x), g(x,y)).$$

Accordingly, in the following, we will consider only maps of the latter form.

Example 1: a set of Markov maps. Consider $f : [0,1]^2 \to [0,1]^2$,

$$f(x,y) = (f_0(x), g(x,y)), \tag{1.3}$$

where f_0 is the map defined in (1.2). We require that $g(x,[0,1]) \subset [0,1/2]$ for $x \in [0,1/2)$ and $g(x,[0,1]) \subset [1/2,1]$ for $x \in (1/2,1]$, also there exists $\sigma > 0$ such that $|\partial_y g| > \sigma$. This implies that f is an invertible map. Also we assume that g is C^2 when restricted to $A_1 = (0,1/2) \times [0,1]$ and $A_2 = (1/2,1) \times [0,1]$. Setting $R_i = f(A_i)$, it is possible that the closure of $R_1 \cup R_2$ is strictly smaller than $[0,1]^2$. However, f preserves the Markov structure of the map f_0. In particular, the preimage of a vertical segment $\{x\} \times [0,1]$ consists of two vertical segments of the same type.

Let $Y = (1/2,1] \times [0,1]$ and F be the first return map to such a set. Obviously, it will have the form $F(x,y) = (F_0(x), G(x,y))$ where F_0 is the return map of f_0 to $(1/2,1]$. We assume that, where defined,

$$|\partial_y G| \leq \lambda^{-1} < 1. \tag{1.4}$$

This implies that the stable foliation consists of the vertical segments. Also, we require that there exists $K_0 > 0$ such that

$$\frac{|\partial_x G|}{|F_0'|} \leq K_0. \tag{1.5}$$

This implies that the cone $\mathcal{C} = \{(a,b) \in \mathbb{R}^2 \;:\; |b| \leq K|a|\}$ is invariant for DF, provided $K \geq (1-\lambda^{-1})^{-1}K_0$. This readily implies the existence of an unstable foliation and that it is made by curves that are graphs over the x coordinate.

We note that condition (1.4) does not require that f is uniformly contracting in the vertical direction. If, for example, $|\partial_y g|_\infty \leq 1$ and $\sup_{x\in[1/2,1],y} |\partial_y g(x,y)| < 1$, then one can easily check that (1.4) is satisfied.

As for condition (1.5), we note that setting $f^n = (f_0^n, g_n)$, we have that $g_{n+1}(x,y) = g(f_0^n(x), g_n(x,y))$. Hence, for all $(x,y) \in Y$,

$$\begin{aligned}\partial_x g_{n+1}(x,y) &= (\partial_x g)(x_n,y_n) \cdot (f_0^n)'(x) + \partial_y g(x_n,y_n) \cdot \partial_x g_n(x,y)\\ &= (f_0^n)'(x) \sum_{k=0}^{n} (\partial_x g)(x_k,y_k) \prod_{j=k+1}^{n} \frac{(\partial_y g)(x_j,y_j)}{f_0'(x_j)},\end{aligned} \tag{1.6}$$

where $(x_n,y_n) = f^n(x,y)$. The above displayed equation together with the distortion properties of $F_0 = f_0^{\varphi_0}$ and the fact that $|\partial_y g|_\infty \leq 1$, yields

$$\begin{aligned}\frac{|\partial_x G(x,y)|}{|F_0'(x)|} &\leq C \sum_{k=0}^{\varphi(x)-1} \prod_{j=k+1}^{\varphi(x)-1} \frac{|(\partial_y g)(x_j,y_j)|}{f_0'(x_j)}\\ &\leq C \sum_{k=0}^{\varphi(x)-1} (\varphi(x)-k)^{-1-\beta} \prod_{j=k+1}^{\varphi(x)-1} |(\partial_y g)(x_k,y_k)| \leq C.\end{aligned}$$

The above argument shows that there exists a large class of systems satisfying hypotheses (1.4) and (1.5). For such systems the return map F is uniformly hyperbolic, it has a Markov structure with countably many interval of smoothness, but neither the derivative nor its inverse is, in general, uniformly bounded. Moreover, any SRB invariant measure for f (i.e. any invariant measure absolutely continue with respect to Lebesgue once restricted to the unstable direction) has a marginal in the x direction that is absolutely continuous with respect to Lebesgue and must be an invariant measure of f_0. Hence, there exists a unique (up to scaling) σ-finite, absolutely continuous (on the unstable direction) invariant measure μ: finite if $\alpha \leq 1$ and infinite if $\alpha \geq 1$ (equivalently, writing $\beta := 1/\alpha$, μ is finite if $\beta > 1$ and infinite if $\beta \leq 1$).

Example 2: a set of non Markov maps. In this case we take f_0 to be the one dimensional topologically mixing map described at the end of Section 1.2. We recall that: i) f_0 agrees with the definition in the previous example in $[0, \frac{1}{2}]$; ii) there exists a finite partition of $(1/2, 1]$ into open intervals I_p, $p \geq 1$ such that f_0 is C^2 and strictly monotone in each I_p; iii) $|f_0'| > 2$ in $(1/2, 1]$; iv) f_0 is topologically mixing. Also we require that there exists at least one I_p such that $0 \in \overline{f_0(I_p)}$. If not, then the invariant measure μ_0 of f_0 would be supported away from zero and hence (f_0, μ_0) would be a uniformly expanding systems, so the present theory would be superfluous. Define $f : [0,1]^2 \to [0,1]^2$,

$$f(x, y) = (f_0(x), g(x, y)), \tag{1.7}$$

where $g \in C^2$ in $(0, 1/2) \times [0, 1]$ and in each $I_p \times [0, 1]$. As in Example 1, we ask $0 < \sigma \leq |\partial_y g| \leq 1$ and that g is such that f is invertible.

Set $Y = (1/2, 1] \times [0, 1]$ and let F be the first return map to such a set. So, we can write $F(x, y) = (F_0(x), G(x, y))$ where F_0 is the return map of f_0 to $(1/2, 1]$. It turns out that, to treat this case, conditions of the type (1.4) and (1.5), are not sufficient. Indeed, if the contraction in the stable direction is much slower than the expansion, then it is unclear what is the reasonable result one should expect. To make things simple we ask that the contraction overbeats the expansion. We assume that there exists $C > 0$ such that, for almost all $(x, y) \in [0, 1]^2$,

$$|\partial_y G(x, y)| \leq C\varphi(x, y)^{-1} |F_0'|^{-1}. \tag{1.8}$$

Remark 1.1 We do not claim that condition (1.8) is optimal, yet it is not very strong either. In particular, note that if we assume the rather strong condition $\|\partial_y g\|_\infty \leq \lambda^{-1} < 1$, then the above condition reads $\lambda^{-\varphi} \leq C\varphi^{-2-\beta}$ which is obviously satisfied.[2]

[2] Recall that $F_0' \sim \varphi^{1+\beta}$.

In the following we use (1.8) to obtain certain estimates that will be needed in section 7. In some sense these are the properties that are really needed to apply our results, yet we find condition (1.8) more appealing to state and simpler to check.

Note that, using the notation of Example 1, $\partial_y G(x,y) = \prod_{k=0}^{\varphi(x,y)-1} \partial_y g(x_k, y_k)$. It follows

$$\partial_y^2 G = \sum_{k=0}^{\varphi-1} \frac{\partial_y^2 g(x_k,y_k)}{\partial_y g(x_k,y_k)} \prod_{j=0}^{k-1} \partial_y g(x_j,y_j) \prod_{j=0}^{\varphi-1} \partial_y g(x_k,y_k).$$

We cannot take much advantage of the first product, so we bound it by one. The second product yields $\partial_y G$ and, using (1.8), we have

$$\|F_0' \partial_y G\|_\infty + \|F_0' \partial_y^2 G\|_\infty \leq C \tag{1.9}$$

Moreover, differentiating (1.6), we have

$$\begin{aligned}\partial_y \left(\frac{\partial_x G}{F_0'} \right) =& \sum_{k=0}^{\varphi-1} \partial_y \partial_x g(x_k,y_k) \partial_y G \prod_{j=k+1}^{\varphi-1} f_0'(x_j)^{-1} \\ &+ \sum_{k=0}^{\varphi-1} \partial_x g(x_k,y_k) \sum_{l=k+1}^{\varphi-1} \frac{\partial_y^2 g(x_l,y_l)}{\partial_y g(x_l,y_l)} \prod_{j=k+1}^{\varphi-1} \frac{\partial_y g(x_j,y_j)}{f_0'(x_j)} \cdot \prod_{s=0}^{l-1} \partial_y g(x_s,y_s).\end{aligned}$$

Recall that $\prod_{j=k+1}^{\varphi-1} [f_0'(x_j)]^{-1} \leq C(\varphi-k)^{-1-\beta}$, while the products of the $\partial_y g$ can be used to recover $\partial_y G$. Using such facts in the above expression, we obtain

$$\left\| F_0' \partial_y \frac{\partial_x G}{F_0'}(x,\cdot) \right\|_{C^0} \leq C. \tag{1.10}$$

In this work we focus on mixing rates in the infinite case of (1.3) and mixing in the infinite case of (1.7). However, we mention that the properties established in Section 6 (the Markov case) and Section 7 (the non Markov case), equally allows one to study statistical properties (such as polynomial decay of correlation) in the finite case of (1.3) and (1.7), respectively.

Remark 1.2 As already mentioned, the above classes of examples are not the most general possible. They have been chosen as a reasonable compromise between generality and simplicity of exposition, with the aim of showing how the general theory developed in the next section can be applied to concrete examples. Yet, here is a word on more general possibilities. We note that there is no reason why the contracting direction should be one dimensional, maps with higher dimensional stable manifolds can be treated in exactly the same manner. Also, one can consider the case in which the expanding direction is higher dimensional. The Markov case would be essentially identical. In the finite partition non Markov case, one could model the Banach space on higher dimensional bounded variation functions or spaces of generalized variation (see [26, 40]). Note hoverer that, as already mentioned, this poses non trivial problems already in the expanding case. Provided some appropriate technical condition on the image of the partition is satisfied, the case of (not necessarily Markov) countable partitions can also be treated. But in the latter case, one would have to use the arguments put forward in [40, 31] to prove the relevant spectral properties for the return map.

1.5 Previous results on mixing/mixing rates for invertible systems

Adapting Bowen's technique (see [10]), Melbourne [34] generalizes the results on mixing in [35] to infinite measure preserving systems of the form (1.3) described in subsection 1.4. The method in [34] covers the class of diffeomorphisms that can be modeled by Young towers, where it is explicitly assumed the quotient of the first return map has a Gibbs Markov structure. The results on mixing in [34] are contained in our Corollary 4.3 and Corollary 4.5.

Under the additional assumption of exponential contraction along the stable manifold, [34] generalizes the results on mixing rates in [35, 41]. As mentioned in [34], without this further assumption, the employed method does not provide satisfactory results on mixing rates. Our Theorem 1.4 below provides optimal mixing rates for the infinite case of (1.3), where such uniform contraction along the stable manifold is not required.

Results on (upper bounds for) the decay of correlation in the finite case of (1.3) can be found in [37, Appendix B].

To our knowledge there is no result in the literature that deals with mixing/mixing rates in either the finite or infinite case of (1.7).

1.6 Main results and outline of the paper

In Section 2, we describe an abstract framework for operator renewal sequences associated with non-uniformly hyperbolic systems based on the abstract hypothesis (H1)–(H5), under which results on mixing and mixing rates hold. Our result on mixing and mixing rates are stated and proved in Section 4 (see Lemma 4.1 and Lemma 4.4) and Section 5 (see Corollary 5.3), respectively.

Given a specific map, the task of checking the hypothesis (H1–H5) is a non trivial one and, alone, can constitute the content of a paper. Nonetheless, we claim that these hypotheses are reasonable and can be checked in a manifold of relevant examples. To illustrate how to proceed and to convince the reader that the above claim has some substance, in Section 6 and Section 7 we prove that the abstract hypothesis (H1)–(H5) are satisfied by systems of the form (1.3) and (1.7) described in Section 1.4. The advantage of focusing on these examples is that the technicalities are reduced to a bare minimum, which leads to simpler arguments for the verification of (H1–H5).

We believe that the arguments used in Section 6 and Section 7 can be followed also by a reader unfamiliar with the theory (still in part under construction) of Banach spaces adapted to hyperbolic dynamical systems.

Once verified, hypothesis (H1–H5) allow us to establish the results on mixing and mixing rates below. More precisely, from Corollary 4.5 it follows that

Theorem 1.3 *Assume the setting of maps f of the form (1.3) or (1.7) described in subsection 1.4 with $\beta \in (0,1)$. Let $v, w : [0,1]^2 \to \mathbb{R}$ be C^{1+q}, $q \in (\frac{1+\beta}{2+\beta}, 1]$, observables supported on Y. Then, there exists a positive constant d_0 (depending only on the map f) such that*

$$\lim_{n\to\infty} n^{1-\beta} \int_{[0,1]^2} v\, w \circ f^n \, d\mu = d_0 \int_{[0,1]^2} v \, d\mu \int_{[0,1]^2} w \, d\mu.$$

As already mentioned in Section 1.3, mixing rates for maps of the form (1.2) depend heavily on a good expansion of the tail behavior. A good tail expansion for the invertible map f of the form (1.3) described in subsection 1.4 follows immediately from the tail expansion for the map (1.2) (see subsection 6.8).

On the other hand, we note that at present it is not clear how to obtain the required expansion for non uniformly expanding, non Markov maps such as the one described at the end of Section 1.2. For precisely this reason (although all our hypotheses (H1)–(H5) are shown to hold for the non Markov map of the form (1.7) described in subsection 1.4), the next result provides mixing rates just for the case of (1.3). More precisely, by Corollary 5.3 we obtain

Theorem 1.4 *Assume the setting of maps f of the form (1.3) described in subsection 1.4 with $\beta \in (1/2, 1)$. Let $v, w : [0,1]^2 \to \mathbb{R}$ be C^1 observables supported on Y. Set $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$. Then, there exist real constants $d_0, \dots, d_q$ (depending only on the map f) such that*

$$\int_{[0,1]^2} v\, w \circ f^n \, d\mu = (d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \dots + d_q n^{(q+1)(\beta-1)}) \int_{[0,1]^2} v \, d\mu \int_{[0,1]^2} w \, d\mu + O(n^{-\beta}).$$

Remark 1.5 The above mixing rate is optimal and matches the results on mixing rates in [35, 41] for maps of the form (1.2).

In the finite measure setting we note that [20, Theorem 1.1], which requires our hypothesis (H1) and (H5) and (H4)(iii), together with the standard argument used in the proof of Corollary 4.3 yields

Theorem 1.6 *Assume the setting of maps f of the form (1.3) or (1.7) described in subsection 1.4 with $\beta > 1$. Let $v, w : [0,1]^2 \to \mathbb{R}$ be C^{1+q}, $q \in (\frac{1+\beta}{2+\beta}, 1]$, observables supported on Y. Then*

$$\lim_{n\to\infty} n^{1-\beta} \left| \int_{[0,1]^2} v\, w \circ f^n \, d\mu - \int_{[0,1]^2} v \, d\mu \int_{[0,1]^2} w \, d\mu, d\mu \right| = c_0 \int_{[0,1]^2} v \, d\mu \int_{[0,1]^2} w \, d\mu$$

where $c_0 n^{-\beta+1} = \sum_{k>n} \mu(Y_k)$ *for* n *large enough.*

Apart from the strong property of mixing for invertible infinite measure preserving systems (along with mixing rates), the present framework allows us to deal with the property of weak pointwise dual ergodicity under some weak conditions (under which mixing cannot be proved). For this type of result we refer to subsection 4.3. The property of weak p.d.e. has been recently exploited by Aaronson and Zweimüller in [3]). As shown in this work, weak p.d.e. along with regular variation of the first return time allows one to establish limit theorems (such as Darling Kac) for infinite measure preserving systems that are not pointwise dual ergodic (see subsection 4.3 for details).

Notation We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ as $n \to \infty$ if there is a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \geq 1$.

2 Operator renewal sequences for non-uniformly hyperbolic systems

In this section we present an abstract framework that suffices for concrete results on mixing (for maps such as (1.3) and (1.7)), but general enough to accommodate a large class of dynamical systems. In particular, it extends the framework of [39, 20] and respectively [35] for operator renewal sequences associated with non-uniformly expanding maps to the non-uniformly hyperbolic context (see the explanatory Remark 2.2).

Let M be a manifold and $f : M \to M$ be a non-singular transformation w.r.t. Lesbegue (Riemannian) measure m. We require that there exists $Y \subset M$ such that the first return map $F = f^\varphi$ to Y is uniformly hyperbolic (possibly with singularities) and satisfies the functional analytic assumptions listed below.

Recall that the transfer operator $R : L^1(m) \to L^1(m)$ for the first return map $F : Y \to Y$ is defined by duality on $L^1(m)$ via the formula $\int_Y Rv\, w\, dm = \int_Y v\, w \circ F\, dm$ for all bounded and measurable w. See Remark 2.3 for a more explicit description of the transfer operator R. We assume that there exist two Banach spaces of distributions $\mathcal{B}$, $\mathcal{B}_w$ supported on Y and some $\alpha, \gamma > 0$ such that

(H1) i) $C^\alpha \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^\gamma)'$, where $C^\alpha = C^\alpha(M, \mathbb{C})$ and $(C^\gamma)'$ is the dual of $C^\gamma(M, \mathbb{C})$.[3]

ii) The transfer operator R associated with F admits a continuous extension to $\mathcal{B}$, which we still call R.

iii) With the above convention, we assume that the operator $R : \mathcal{B} \to \mathcal{B}$ has a simple eigenvalue at 1 and the rest of the spectrum is contained in a disk of radius less than 1.

[3] To be precise, by the above inclusion we mean that there exists an injective embedding.

We note that (H1)(i) should be understood in terms of the usual convention (see, for instance, [23, 12]), which we follow thereon: any function $\psi \in C^\alpha$ is identified with a distribution via the duality relation

(*) $h(\phi) = \int h\phi \, dm$.

Note that, via such identification, the Lebesgue measure m can be identified with the constant function one. Hence $m \in \mathcal{B}$. Moreover, since by (H1)(i) it follows $\mathcal{B}' \supset (C^\gamma)'' \supset C^\gamma \ni 1$, we have $m \in \mathcal{B}'$ as well.

Recall that $\varphi : Y \to \mathbb{N}$ is the first return time to Y. Throughout, we assume that

(H2) there exists $C > 0$ such that, for any connected component E of $\varphi^{-1}(n)$ and any $h \in \mathcal{B}$ we have $|\int_E h \, dm| \le C\|h\|_{\mathcal{B}} \, m(E)$.

By (H1), the spectral projection P associated with the eigenvalue 1 is defined by $P = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} R^k$. Note that, for each $\phi \in L^\infty$,

$$m(P\phi) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} m(1 \cdot R^k\phi) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} m(\phi) = m(\phi).$$

Thus, by (H1)(iii), $P\phi = hm(\phi)$, where $Rh = h$ and $R'm = m$ with $h \in \mathcal{B}$. For any $\phi \in C^\gamma$,

$$|h(\phi)| = |P1(\phi)| = \left| \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} R^k m(\phi) \right| \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |m(\phi \circ F^n)| \le m(Y)|\phi|_\infty.$$

Hence, $|h(\phi)| \le C|\phi|_\infty$ and h is a measure.

Summarizing the above, the eigenfunction associated with the eigenvalue 1 is an invariant measure for F, which can be normalized to provide an invariant probability measure μ. Since $m \in \mathcal{B}$ and $1(\phi) = \int \phi \, dm$, m can be viewed as the element 1 of both spaces $\mathcal{B}$ and $(C^\gamma)'$ and we equivalently write $P1 = \mu$ and $P1 = h$.

Given that μ is the physical probability invariant measure for F, a finite or σ finite measure μ for f can be obtained by the standard push forward method[4] (that goes back to [25]). In the *infinite* setting we require that

(H3) $\mu(y \in Y : \varphi(y) > n) = \ell(n)n^{-\beta}$ where ℓ is slowly varying and $\beta \in [0, 1]$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Given $z \in \bar{\mathbb{D}}$, we define the perturbed transfer operator $R(z)$ (acting on $\mathcal{B}, \mathcal{B}_w$) by $R(z)v = R(z^\varphi v)$. Also, for each $n \ge 1$, we define R_n (acting on $\mathcal{B}, \mathcal{B}_w$) by $R_n v = R(1_{\{\varphi=n\}}v)$. We assume (H4)(i) through and one of (H4)(ii), (H4)(iii).

(H4) (i) $R_n : \mathcal{B} \to \mathcal{B}$ are bounded operators satisfying $\sum_{n=1}^\infty \|R_n\|_{\mathcal{B}\to\mathcal{B}} < \infty$.

[4] For any set A in the σ-algebra $\mathcal{A}$, $\mu(A) = \sum_{n=0}^\infty \mu(\{\varphi > n\} \cap f^{-n}A)$.

(ii) $\|R_n\|_{\mathcal{B}\to\mathcal{B}_w} \ll c_n$, where $\sum_{j>n} c_n \ll n^{-(\beta-\epsilon_0)}$ with $\beta \in (1/2,1)$ and $\epsilon_0 < \max\{2\beta-1, 1-\beta\}$.

(iii) $\|R_n\|_{\mathcal{B}} \ll n^{-(\beta+1)}$.

Note that (H4)(i) implies that $R(z) = \sum_{n=1}^{\infty} R_n z^n$. Also, we notice that (H1) and (H4)(i) ensure that $z \mapsto R(z)$, $z \in \bar{\mathbb{D}}$, is a continuous family of bounded operators on $\mathcal{B}$. Throughout we assume:

(H5) i) There exist $C > 0$ and $\lambda > 1$ such that for all $z \in \bar{\mathbb{D}}$ and for all $h \in \mathcal{B}$, $n \geq 0$,

$$\|R(z)^n h\|_{\mathcal{B}_w} \leq C\|h\|_{\mathcal{B}_w}, \quad \|R(z)^n h\|_{\mathcal{B}} \leq C\lambda^{-n}|z|^n\|h\|_{\mathcal{B}} + C\|h\|_{\mathcal{B}_w}$$

ii) For $z \in \bar{\mathbb{D}} \setminus \{1\}$, the spectrum of $R(z) : \mathcal{B} \to \mathcal{B}$ does not contain 1.

In particular, we note that (H1), (H4)(i) and (H5)(ii) imply that for $z \in \mathbb{D}$, $z \mapsto (I - R(z))^{-1}$ is an analytic family of bounded linear operators from on $\mathcal{B}$. Define $T_n : \mathcal{B} \to \mathcal{B}$ for $n \geq 0$ and $T(z) : \mathcal{B} \to \mathcal{B}$ for $z \in \bar{\mathbb{D}}$ by setting $T_0 = I$ and

$$T_n v = \sum_{k=1}^{\infty} \sum_{i_1+\ldots+i_k=n} R_{i_1} \ldots R_{i_k} v, n \geq 1 \; ; \qquad T(z) = \sum_{n=0}^{\infty} T_n z^n.$$

Remark 2.1 We notice that if $L : L^1(m) \to L^1(m)$ is the transfer operator of the original transformation $f : M \to M$ (defined by duality on $L^1(m)$ by $\int_M Lv\, w\, dm = \int_M v\, w \circ f\, dm$, for all bounded and measurable w), then the sequences of operators R_n, T_n defined in this section coincide with the sequences of operators defined in (1.1). For R_n this is simply the bare definition, while for T_n, it follows by decomposing the itinerary of $f : Y \to Y$ into consecutive returns to Y (see, for instance, [20]).

By a standard computation we have that $T(z) = I + R(z)T(z)$ for all $z \in \mathbb{D}$. Then, by (H5)(ii), we have the renewal equation

$$T(z) = (I - R(z))^{-1}.$$

Note that $T(z)$ extends continuously to $\bar{\mathbb{D}} \setminus \{1\}$. Moreover, on $\mathbb{D}$, $z \to T(z) = \sum_{n=0}^{\infty} T_n z^n$ is an analytic family of bounded linear operators from $\mathcal{B}$ to $\mathcal{B}_w$.

Remark 2.2 In the context of non-uniformly expanding maps preserving a *finite* invariant measure μ, the functional analytic assumption on F summarizes as follows. It is assumed that there exists a Banach space $\mathcal{B}$ (for non-uniformly expanding interval maps $\mathcal{B}$ is Hölder or BV) such that H1(ii) and (H5)(ii) hold for $R(1)$ and $R(z)$, respectively, as operators on $\mathcal{B}$. Moreover, one requires that (H4)(ii) holds under the strong norm $\|.\|$ on $\mathcal{B}$ for some $\beta > 1$ (see [39, 20]). We also refer to [37], where (H4)(ii) reduces to $\sum_{n=1}^{\infty} \sum_{j>n} \|R_j\| < \infty$. In the case of non-uniformly expanding maps preserving an *infinite* invariant measure μ, the assumption (H3) is crucial (see [35]).

Remark 2.3 Note that, using convention (*), one has the following. Identifying a measure h that is absolutely continuous w.r.t m with its density (which will be again called h), the space of measures absolutely continuous w.r.t. m can be canonically identified with $L^1(Y, \mathbb{R}, m)$. Restricting to $L^1(Y) \subset (C^\gamma)'$ and writing $DF := |detF|$, we have $Rh = h \circ F^{-1}|DF \circ F^{-1}|^{-1}$. Thus, our operator R on $\mathcal{B}$ is a extension of the usual transfer operator.

Remark 2.4 Recall that a measure ν is *physical* if there exists a measurable set A, $m(A) > 0$, such that, for each continuous function ϕ, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ F^k(x) = \nu(\phi)$, for each $x \in A$. In the present case, by hypothesis (H1)(iii), we have that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ F^k(x) = \mu(\phi)$ for m-almost all x. Thus μ is the unique physical measure of F. Indeed, suppose there exists $B \subset Y$ for which the last limit is larger than $\mu(\phi) + \varepsilon$, for some $\varepsilon > 0$ (the case of the limit being smaller being treated similarly). Then, by Lusin theorem with respect to both m and μ there exists a C^1 function h_ε such that $\|\mathbb{1}_B - h_\varepsilon\|_{L^1(m)} + \|\mathbb{1}_B - h_\varepsilon\|_{L^1(\mu)} < \frac{\varepsilon m(B)}{4|\phi|_\infty}$. But then

$$\begin{aligned}\mu(\phi)m(h_\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m(h_\varepsilon \phi \circ F^k) &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} m(\mathbb{1}_B \phi \circ F^k) - \frac{\varepsilon m(B)}{4} \\ &\geq m(B)\mu(\phi) + \frac{3\varepsilon m(B)}{4}\end{aligned}$$

which implies a contradiction.

3 Asymptotics of $T(z)$

3.1 Asymptotic results under (H1), (H2), (H3), (H5) and (H4)(i)

As in the framework of [35, 36], the asymptotics of $T(z)$, $z \in \bar{\mathbb{D}}$ depends essentially on the asymptotics of the eigenvalue $\lambda(z)$ of $R(z)$ defined in a neighborhood of 1. In the present context, this requires some clarification.

Our aim in this section is to estimate $\|T(z)\|_{\mathcal{B}}$, $z \in \bar{\mathbb{D}}$ under the weak hypothesis (H4)(i).

By (H1), (H4)(i) and (H5), there exist $\delta > 0$ and a continuous family of simple eigenvalues of $R(z)$, namely $\lambda(z)$ for $z \in \bar{\mathbb{D}} \cap B_\delta(1)$ with $\lambda(1) = 1$. Given $c > 0$, let $P(z) = \frac{1}{2\pi i} \int_{|\xi-1|=c} (\xi I - R(z))^{-1} d\xi$ denote the corresponding family of spectral projections with $P(1) = P$ and complementary projections $Q(z) = I - P(z)$.

For $z \in \bar{\mathbb{D}} \cap B_\delta(1)$, $R(z) = \lambda(z)P(z) + R(z)Q(z)$. Hence,

$$T(z) = (1 - \lambda(z))^{-1}P + (1 - \lambda(z))^{-1}(P(z) - P) + (I - R(z))^{-1}Q(z). \tag{3.1}$$

By a standard argument (see, for instance, [35, Proposition 2.9]), we have

Proposition 3.1 *Assume (H1) and (H5)(ii). There exists* $\delta, C > 0$ *such that* $\|(I - R(z))^{-1}Q(z)\|_{\mathcal{B}} \le C$ *for* $z \in \bar{\mathbb{D}} \cap B_\delta(1)$, $z \ne 1$ *and* $\|T(z)\|_{\mathcal{B}} \le C$ *for* $z \in \bar{\mathbb{D}} \setminus B_\delta(1)$.

By (3.1), it remains to obtain the asymptotics of $(1 - \lambda(z))^{-1}$ and $(1 - \lambda(z))^{-1}(P(z) - P)$. First, we note that (H4)(i) implies that as $z \to 1$,

$$\|R(z) - R\|_{\mathcal{B}} \to 0, \quad |\lambda(z) - 1| \to 0.$$

Next, let $v(1)$ be the eigenfunction associated with the eigenvalue 1 with $m(v(1)) = 1$. Choose $\psi \in \mathcal{B}$ such that $m(\psi) = 1$ and define $\tilde{v}(z) := P(z)\psi$. Since, by standard perturbation theory, $\|P(z) - P\|_{\mathcal{B}} \to 0$, we have $\|\tilde{v}(z) - v(1)\|_{\mathcal{B}} \to 0$. Thus, $m(\tilde{v}(z)) \ne 0$ and we can define the normalized eigenfunction $v(z) := \frac{\tilde{v}(z)}{m(\tilde{v}(z))}$.

Since $m(v(z)) = 1$, we can use the formalism in [21] (a simplification of [2]) and write

$$1 - \lambda(z) = 1 - \int_Y \lambda(z)v(z)dm = 1 - \int_Y R(z)v(z)dm = \Psi(z) - V(z),$$

where

$$\begin{aligned} \Psi(z) &= \int_Y (1 - z^\varphi)d\mu, \\ V(z) &= \int_Y (R(1) - R(z))(v(1) - v(z))dm = \int_Y ((1 - z^\varphi)(v(1) - v(z))dm. \end{aligned} \tag{3.2}$$

Now, $|V(z)| = |\sum_{n\ge1}\int_{\{\varphi=n\}}((1-z^\varphi)(v(1)-v(z))\,dm|$. This together with (H2) yields

$$|V(z)| \ll \|v(z) - v(1)\|_{\mathcal{B}} \sum_{n\ge1} |1 - z^n| m(\{\varphi = n\}).$$

Recall that as $z \to 1$, $\|v(z) - v(1)\|_{\mathcal{B}} \to 0$. Also, by (H3) and standard computations (based on [8, Proposition 1.5.8]), $\sum_{n\ge1} |1-z^n|\mu(\{\varphi = n\}) \ll \ell(1/|z-1|)|z-1|^\beta$. Thus,

$$|V(z)| = o(\ell(1/|z - 1|)|z - 1|^\beta) \text{ as } z \to 1. \tag{3.3}$$

The precise asymptotic of $\Psi(z)$ on $\bar{\mathbb{D}}$, a generalization of the more standard result for the precise asymptotic of $\Psi(z)$ on the unit circle (see for instance [18]), reads as

Proposition 3.2 *[36, Proof of Lemma A.4] Assume (H3) with* $\beta \in (0, 1)$. *As* $u, \theta \to 0$, $\Psi(e^{-(u+i\theta)}) = \Gamma(1 - \beta)\ell(1/|u - i\theta|)(u - i\theta)^\beta(1 + o(1))$.

By (3.2), (3.3) and Proposition 3.2, we have the following generalization of well known result of [2] (see [36, Lemma A.4]):

$$1 - \lambda(z) = \Gamma(1 - \beta)\ell(1/|u - i\theta|)(u - i\theta)^\beta(1 + o(1)). \tag{3.4}$$

Recall that $\|P(z) - P\|_{\mathcal{B}} \to 0$. By (3.1) and (3.4),

$$T(e^{-u+i\theta}) = d_\beta \ell(1/|u - i\theta|)^{-1}(u - i\theta)^{-\beta} P + E, \text{ as } u, \theta \to 0, \tag{3.5}$$

where $d_\beta = \Gamma(1 - \beta)^{-1}$ and $\|E\|_{\mathcal{B}} = o(\ell(1/|u - i\theta|)|u - i\theta|^{-\beta})$.

3.2 Asymptotic under (H1), (H2), (H3), (H5) and (H4)(ii)

In this section, we obtain explicit bounds on the continuity of $\|T(z)\|_{\mathcal{B}\to\mathcal{B}_w}$, for z in a neighborhood of 1 (see Proposition 3.6). To do so, we work with (H4)(ii), which again requires (H2). To complete the picture of the asymptotic of $T(z)$, $z\in\mathbb{D}$ we estimate the derivative of $T(z)$ for z outside a neighborhood of 1 (see Corollary 3.9).

The following standard consequence of (H4)(ii) is instrumental in the proof of several results below (see, for instance, step 1 of the proof of [20, Lemma 3.1])).

Proposition 3.3 *Assume (H4)(ii). Then there is a constant $C>0$ such that for all $u\geq 0$, $\theta\in[0,2\pi)$ and $h>0$, $\|R(e^{-u+i\theta})-R(e^{-u+i(\theta-h)})\|_{\mathcal{B}\to\mathcal{B}_w}\leq Ch^{\beta-\epsilon_0}$ and $\|R(e^{-u+i\theta})-R(1)\|_{\mathcal{B}\to\mathcal{B}_w}\leq C|u-i\theta|^{\beta-\epsilon_0}$.*

If assumption (H4)(ii) is replaced by the stronger form in the $\|\cdot\|_{\mathcal{B}}$ norm, then the estimates of Proposition 3.3 hold in the, stronger, norm of $L(\mathcal{B},\mathcal{B})$. Since we want to estimate $\|P(z)-P\|_{\mathcal{B}\to\mathcal{B}_w}$, in what follows we use the arguments in [27]. For $\delta>0$, small enough, we set

$$V_\delta=\{\xi\in\mathbb{C}:|\xi-1|=\delta,\quad dist(\xi,spec(R))\geq\delta\}$$

and state

Lemma 3.4 *Assume (H1), (H4)(ii) and (H5)(i). Then the statements below hold for any $\epsilon_1\in(\epsilon_0,\beta^*)$, where $\beta^*<\max\{2\beta-1,1-\beta\}$ and some positive constant C. More precisely, there exists $\epsilon_*>0$ such that for all $0<\delta<\epsilon_*$, the following hold for any $\xi\in V_\delta$, for all $u\in(0,\delta)$ and for all $\theta\in(-\delta,\delta)$.*

i) $\|(\xi-R(e^{-u+i\theta}))^{-1}-(\xi-R(1))^{-1}\|_{\mathcal{B}\to\mathcal{B}_w}\leq C|u-i\theta|^{\beta-\epsilon_1}$.

ii) *For all* $h\leq\min\{|\theta|,u\}$, $\|(\xi-R(e^{-u+i\theta}))^{-1}-(\xi-R(e^{-u+i(\theta-h)}))^{-1}\|_{\mathcal{B}\to\mathcal{B}_w}\leq Ch^{\beta-\epsilon_1}$.

Proof We provide the argument for item i). Item ii) follows similarly.

In what follows, we adapt the argument of [27, Proof of Theorem 1] to the present context. For $\xi\in V_\delta$ and some $k\geq 1$ (to be specified below) we write

$$(\xi-R(1))^{-1}=\xi^{-k}(\xi-R(1))^{-1}R(1)^n+\sum_{j=0}^{k-1}\xi^{-j-1}R(1)^j.$$

For $h\in\mathcal{B}$ write $g=(\xi-R(1))h$. So, $h=(\xi-R(1))^{-1}g$. This together with (H5)(i) yields

$$\begin{aligned}\|h\|_{\mathcal{B}_w}&\leq|\xi|^{-k}\|(\xi-R(1))^{-1}R(1)^kg\|_{\mathcal{B}_w}+\sum_{j=0}^{k-1}|\xi|^{-j}\|R(1)^j\|_{\mathcal{B}_w}\\&\leq C\lambda^{-k}\|(\xi-R(1))^{-1}\|_{\mathcal{B}}\|g\|_{\mathcal{B}}+C\|(\xi-R(1))^{-1}\|_{\mathcal{B}}\|g\|_{\mathcal{B}_w}+Ck\|g\|_{\mathcal{B}_w}.\end{aligned}$$

Clearly, for any $\xi \in V_\delta$, $\|(\xi - R(1))^{-1}\|_{\mathcal{B}} = O(1)$. Also, by the definition of g and (H1)(ii), $\|g\|_{\mathcal{B}} \le (C + |\xi|)\|h\|_{\mathcal{B}} \le (C + 1 + \delta)\|h\|_{\mathcal{B}}$. Hence, the following holds for some $C > 0$:

$$\|h\|_{\mathcal{B}_w} \le C\lambda^{-k}(C+\delta)\|h\|_{\mathcal{B}} + C\|g\|_{\mathcal{B}_w} + Ck\|g\|_{\mathcal{B}_w}.$$

Next, compute that $g = (R(e^{-u+i\theta}) - R(1))(\xi - R(1))^{-1}g + (\xi - R(e^{-u+i\theta}))(\xi - R(1))^{-1}g$. This together with Proposition 3.3 implies that there exists $C > 0$ such that

$$\|g\|_{\mathcal{B}_w} \le C|u - i\theta|^{\beta-\epsilon_0}\|h\|_{\mathcal{B}} + \|(\xi - R(e^{-u+i\theta}))h\|_{\mathcal{B}_w}.$$

Putting together the last two estimates,

$$\|h\|_{\mathcal{B}_w} \le C\lambda^{-k}(C+\delta)\|h\|_{\mathcal{B}} + Ck|u - i\theta|^{\beta-\epsilon_0}\|h\|_{\mathcal{B}} + k\|(\xi - R(e^{-u+i\theta}))h\|_{\mathcal{B}_w}. \tag{3.6}$$

Choose $k = [(\beta - \epsilon_0)\log(|u - i\theta|)\log(1/\lambda)^{-1}]$. So, for all $u \in (0, \delta)$ and for all $\theta \in (-\delta, \delta)$,

$$\lambda^{-k} \le |u - i\theta|^{\beta-\epsilon_0}, \quad k \ll \log(1/|u - i\theta|)$$

and

$$Ck|u - i\theta|^{\beta-\epsilon_0} \ll \log(1/|u - i\theta|)|u - i\theta|^{\beta-\epsilon_0}.$$

Thus, (3.6) becomes

$$\|h\|_{\mathcal{B}_w} \ll |u - i\theta|^{\beta-\epsilon_0}\|h\|_{\mathcal{B}} + \log(1/|u - i\theta|)\|(\xi - R(e^{-u+i\theta}))h\|_{\mathcal{B}_w}. \tag{3.7}$$

Let $v \in \mathcal{B}$ such that $h = (\xi - R(e^{-u+i\theta}))^{-1}v$. So, (3.7) translates into

$$\|(\xi - R(e^{-u+i\theta}))^{-1}v\|_{\mathcal{B}_w} \ll |u - i\theta|^{\beta-\epsilon_0}\|v\|_{\mathcal{B}} + \log(1/|u - i\theta|)\|v\|_{\mathcal{B}_w}.$$

Applying the above inequality to $v := (R(e^{-u+i\theta}) - R(1))(\xi - R(1))^{-1}w$,

$$\begin{aligned}
\|((\xi - R(e^{-u+i\theta}))^{-1}) - (\xi - R(1))^{-1})w\|_{\mathcal{B}_w} &\ll |u - i\theta|^{\beta-\epsilon_0}\|v\|_{\mathcal{B}} + \log(1/|u - i\theta|)\|v\|_{\mathcal{B}_w} \\
&\ll |u - i\theta|^{\beta-\epsilon_0}\|(R(e^{-u+i\theta}) - R)(\xi - R(1))^{-1}w\|_{\mathcal{B}} \\
&\quad + \log(1/|u - i\theta|)|u - i\theta|^{\beta-\epsilon_0}\|(\xi - R(1))^{-1}w\|_{\mathcal{B}} \\
&\ll |u - i\theta|^{\beta-\epsilon_0}\|w\|_{\mathcal{B}} + |u - i\theta|^{\beta-\epsilon_1}\|w\|_{\mathcal{B}},
\end{aligned}$$

for any $\epsilon_1 \in (\epsilon_0, \beta^*)$, which ends the proof. ∎

An immediate consequence of Lemma 3.4 is

Corollary 3.5 *For all $z \in spec(R(z) \cap B_\delta(1)$, the estimates provided in Lemma 3.4 hold for the families $P(z)$, $Q(z)$ and $v(z)$.*

The next result is based on the estimates provided by Corollary 3.5; it provides explicit bounds on the continuity of $T(z)$, for z a neighborhood of 1.

Proposition 3.6 *Assume (H1), (H2), (H3), (H4)(i), (H4)(ii) and (H5). Let ϵ_* be fixed as in the statement of Lemma 3.4. Then, the following hold for any $\delta < \epsilon_*$, for all $u \in (0, \delta)$, all $\theta \in (-\delta, \delta)$, all $h \le \min\{|\theta|, u\}$ and for any $\epsilon_1 \in (\epsilon_0, \beta^*)$, where $\beta^* < \max\{2\beta - 1, 1 - \beta\}$.*

i) $|\lambda(e^{-u+i\theta}) - \lambda(e^{-u+i(\theta-h)})| \ll h^\beta \ell(1/h) + h^{\beta-\epsilon_1}|u - i\theta|^\beta \ell(1/|u - i\theta)$.

ii) Also,

$$\|T(e^{-u+i\theta}) - T(e^{-u+i(\theta-h)})\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u - i\theta|)^{-2}\ell(1/h)h^\beta|u - i\theta|^{-2\beta} + \ell(1/|u - i\theta|)^{-1}h^{\beta-\epsilon_1}|u - i\theta|^{-\beta}.$$

Proof i)Put $\Delta_\lambda = \lambda(e^{-u+i\theta}) - \lambda(e^{-u+i(\theta-h)})$. By (3.2),

$$\begin{aligned}\Delta_\lambda &= \int_Y (e^{(-u+i(\theta-h))\varphi} - e^{(-u+i\theta)\varphi})\, d\mu + \int_Y (e^{-u+i(\theta-h)\varphi}) - (e^{(-u+i\theta)\varphi}))(v(e^{-u+i(\theta-h)}) - 1)\, dm \\ &+ \int_Y ((e^{(-u+i\theta)\varphi}) - 1)(v(e^{-u+i(\theta-h)}) - v(e^{-u+i\theta}))\, dm \\ &= \int_Y (e^{(-u+i(\theta-h))\varphi} - e^{(-u+i\theta)\varphi})d\mu + V_1(u, \theta) + V_2(u, \theta).\end{aligned}$$

Note that $\int_Y (e^{(-u+i(\theta-h))\varphi} - e^{(-u+i\theta)\varphi})d\mu = \int_0^\infty e^{-(u-i\theta)x}(e^{ihx} - 1)dG(x)$, where $G(x) = \mu(\varphi \le x)$. Under (H3), the estimate $|\int_0^\infty e^{-(u-i\theta)x}(e^{ihx}-1)dG(x)| \ll h^\beta \ell(1/h)$ follows by the argument used in the proof of [18, Lemma 3.3.2].

The argument used in obtaining (3.3) (with Corollary 3.5 instead of $\|v(z) - v(1)\|_\mathcal{B} \to 0$) together with the fact that $h \le \min\{|\theta|, u\}$ yields

$$|V_1(u, \theta)| \ll h^\beta \ell(1/h)|u - i\theta|^\beta, \quad |V_2(u, \theta)| \ll h^{\beta-\epsilon_1}\ell(1/|u - i\theta|)|u - i\theta|^\beta.$$

Item i) follows by putting the above together.

To prove item ii), we proceed as in the proof of [41, Corollary 6.2].

Let $\Delta_T = T(e^u e^{i\theta}) - T(e^{-u}e^{i(\theta-h)})$. Set

$$\Delta_{\lambda,P} = (1 - \lambda(e^{-u}e^{i\theta}))^{-1}P(e^{-u}e^{i\theta}) - (1 - \lambda(e^{-u}e^{i(\theta-h)}))^{-1}P(e^{-u}e^{i(\theta-h)})$$

and

$$\Delta_{(I-R)^{-1}Q} = (I - R(e^{-u}e^{i\theta}))^{-1}Q(e^{-u}e^{i\theta}) - (I - R(e^{-u}e^{i(\theta-h)}))^{-1}Q(e^{-u}e^{i(\theta-h)}).$$

By (3.1), $\Delta_T = \Delta_{\lambda,P} + \Delta_{(I-R)^{-1}Q}$. Next,

$$\begin{aligned}\|\Delta_{\lambda,P}\|_{\mathcal{B}\to\mathcal{B}_w} &\ll \|(1 - \lambda(e^{-u}e^{i\theta}))^{-1}(P(e^{-u}e^{i\theta}) - P(e^{-u}e^{i(\theta-h)}))\|_{\mathcal{B}\to\mathcal{B}_w} \\ &+ \|P(e^{-u}e^{i(\theta-h)})\Big((1 - \lambda(e^{-u}e^{i\theta}))^{-1} - (1 - \lambda(e^{-u}e^{i(\theta-h)}))^{-1}\Big)\|_{\mathcal{B}\to\mathcal{B}_w}.\end{aligned}$$

By (3.4) for all $u \in (0, \delta)$, $\theta \in [-\delta, \delta)$, $|(1-\lambda(e^{-u+i\theta}))^{-1}| \ll \ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}$. This together with Corollary 3.5 yields $\|(1-\lambda(e^{-u}e^{i\theta}))^{-1}(P(e^{-u}e^{i\theta}) - P(e^{-u}e^{i(\theta-h)}))\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u-i\theta|)^{-1}h^{\beta-\epsilon_1}|u-i\theta|^{-\beta}$. By (i) of the present Propostion and (3.4),

$$\left\|\frac{P(e^{-u}e^{i(\theta-h)})(\lambda(e^{-u}e^{i(\theta-h)}) - \lambda(e^{-u}e^{i\theta}))}{((1-\lambda(e^{-u}e^{i\theta}))(1-\lambda(e^{-u}e^{i(\theta-h)})}\right\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u-i\theta|)^{-2}\ell(1/h)h^{\beta}|u-i\theta|^{-2\beta} + \ell(1/|u-i\theta|)^{-1}h^{\beta-\epsilon_1}|u-i\theta|^{-\beta}.$$

Thus,

$$\|\Delta_{\lambda,P}\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u-i\theta|)^{-1}h^{\beta-\epsilon_1}|u-i\theta|^{-\beta} + \ell(1/|u-i\theta|)^{-2}\ell(1/h)h^{\beta}|u-i\theta|^{-2\beta}.$$

To estimate $\|\Delta_{(I-R)^{-1}Q}\|_{\mathcal{B}\to\mathcal{B}_w}$, we compute that

$$\|\Delta_{(I-R)^{-1}Q}\|_{\mathcal{B}\to\mathcal{B}_w} \ll \|(I-R(e^{-u}e^{i\theta}))^{-1}(Q(e^{-u}e^{i\theta}) - Q(e^{-u}e^{i(\theta-h)}))\|_{\mathcal{B}\to\mathcal{B}_w} +$$
$$\|(I-R(e^{-u}e^{i\theta}))^{-1}(R(e^{-u}e^{i\theta}) - R(e^{-u}e^{i\theta-h}))(I-R(e^{-u}e^{i(\theta-h)}))^{-1}Q(e^{-u}e^{i(\theta-h)})\|_{\mathcal{B}\to\mathcal{B}_w}.$$

By (3.5) and Proposition 3.1, $\|(I-R(e^{-u}e^{i\theta}))^{-1}\|_{\mathcal{B}} \ll \ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}$. This together with the previous displayed equation, Corollary 3.5, Proposition 3.3 and Proposition 3.1 implies that

$$\|\Delta_{(I-R)^{-1}Q}\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u-i\theta|)^{-1}h^{\beta-\epsilon_1}|u-i\theta|^{-\beta}.$$

Item ii) follows by putting together the estimates for $\|\Delta_{\lambda,P}\|_{\mathcal{B}\to\mathcal{B}_w}$ and $\|\Delta_{(I-R)^{-1}Q}\|_{\mathcal{B}\to\mathcal{B}_w}$. ∎

Remark 3.7 In the case $\beta = 1$, the scheme above can also be combined with the arguments in [35], providing the desired asymptotics of $T(z)$, $z \in \mathcal{S}^1$ and as such, first and higher order theory for the coefficients T_n of $T(z)$, $z \in \bar{\mathbb{D}}$. To simplify the exposition in what follows we omit the case $\beta = 1$.

The bounds provided by Proposition 3.6 do not allow one to apply directly the argument of [35] for the estimation of the coefficients of $T(z)$, $z \in \bar{\mathbb{D}}$: the arguments in [35] require that $\|T(e^{-u+i\theta}) - T(e^{-u+i(\theta-h)})\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(1/|u-i\theta|)^{-2}\ell(1/h)h^{\beta}|u-i\theta|^{-2\beta}$. However, as explained in Section 4, the modified version of these arguments in [41] applies. In this sense, we note that

Proposition 3.8 *Assume (H4)(ii). Write $z = e^{-(u+i\theta)}$. Then for all $u > 0$,*

$$\left\|\frac{d}{d\theta}(R(z))\right\|_{\mathcal{B}\to\mathcal{B}_w} \ll u^{\beta-\epsilon_0-1}.$$

Proof The result follows by the argument used in the proof of [41, Proposition 4.6]. ∎

As a consequence we have

Corollary 3.9 *Assume (H1), (H3), (H4)(ii) and (H5)(i). Write* $z = e^{-(u+i\theta)}$. *Let* ϵ_* *be fixed as in the statement of Lemma 3.4 and choose* $\delta \in (0, \epsilon_*)$. *Let* θ *such that* $|\theta| > \delta$. *Then for all* $u > 0$,

$$\left\| \frac{d}{d\theta}(T(z)) \right\|_{\mathcal{B} \to \mathcal{B}_w} \ll u^{\beta - \epsilon_0 - 1}.$$

Proof The result follows from Proposition 3.1, Proposition 3.8 and the formula $\frac{d}{d\theta}(I - R(z))^{-1} = (I - R(z))^{-1} \frac{d}{d\theta} R(z)(I - R(z))^{-1}$. ∎

4 First order asymptotic of T_n: mixing.

Given the asymptotic behaviors of $T(z) : \mathcal{B} \to \mathcal{B}_w$ for $z \in \bar{\mathbb{D}} \cap B_\delta(1)$ and of $\frac{d}{d\theta}(T(z))$ for $z \in \bar{\mathbb{D}} \setminus B_\delta(1)$ described in Section 3, the arguments used in [41](a modified version of [35]) for estimating the coefficients T_n of $T(z)$, $z \in \bar{\mathbb{D}}$ apply. We briefly recall the main steps. By exactly the same argument as in [35] (equivalently, by the simplified argument in [36] essentially based on (3.5) and the dominated convergence theorem), the Fourier coefficients of $T(z) : \mathcal{B} \to \mathcal{B}_w$, $z \in \mathcal{S}^1$ coincide with the Taylor coefficients of $T(z) : \mathcal{B} \to \mathcal{B}_w$, $z \in \mathbb{D}$. Hence, first and higher order of T_n can be obtained by estimating either the Fourier or Taylor coefficients of $T(z)$, $z \in \bar{\mathbb{D}}$.

4.1 Mixing under (H4)(ii)

Lemma 4.1 *Assume (H1), (H2), (H3), (H4)(i), (H4)(ii) and (H5). Let* $\beta \in (1/2, 1)$ *and suppose that (H4)(ii) hold. Then, as* $n \to \infty$,

$$\sup_{v \in \mathcal{B}, \|v\|_{\mathcal{B}} = 1} \|\ell(n) n^{1-\beta} T_n v - d_\beta P v\|_{\mathcal{B}_w} \to 0.$$

Remark 4.2 The above result generalizes [35, Theorem 2.1] to the abstract class of transformations described in Section 2.

Proof We argue as in [41, Proof of Theorem 3.3].

Let $\Gamma = \{e^{-u} e^{i\theta} : -\pi \le \theta < \pi\}$ with $e^{-u} = e^{-1/n}$, $n \ge 1$. Choose $\delta \in (0, \epsilon_*)$ such that $\lambda(e^{-1/n} e^{i\theta})$ is well defined for $\theta \in (-\delta, \delta)$. Let $b \in (0, \delta n)$. Let $A = [-\pi, -\delta] \cup [\delta, \pi]$.

With the above specified, we proceed to estimate T_n.

$$
\begin{aligned}
T_n &= \frac{1}{2\pi}\int_\Gamma \frac{T(z)}{z^{n+1}}dz = \frac{e}{2\pi}\int_{-\pi}^{\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{e}{2\pi}\Big(\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta \\
&\quad + \int_{-\delta}^{-b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \int_{b/n}^{\delta} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \int_A T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta\Big) \\
&= \frac{e}{2\pi}\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \frac{e}{2\pi}(I_\epsilon + I_{-\epsilon} + I_A). \qquad (4.1)
\end{aligned}
$$

By [41, Proposition 6.5],

$$
\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{2\pi}{e}\frac{1}{\Gamma(\beta)}P.
$$

Hence, the conclusion will follow once we show that $n^{1-\beta}\ell(n)I_A = o(1)$ and $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)(I_\delta + I_{-\delta}) = 0$.

We first estimate I_A. Compute that

$$
I_A = \frac{i}{n}\int_A T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}(e^{-in\theta})d\theta = \frac{1}{in}\int_A \frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))e^{-in\theta}d\theta + E(n),
$$

where $E(n) \ll n^{-1}(\|T(e^{-1/n}e^{i\epsilon b/n})\|_{\mathcal{B}\to\mathcal{B}_w} + \|T(e^{-1/n}e^{i\pi})\|_{\mathcal{B}\to\mathcal{B}_w})$. By Proposition 3.1, $\|T(e^{-1/n}e^{i\theta})\|_{\mathcal{B}} = O(1)$ for all $\theta \in A$. Hence $E(n) = O(n^{-1})$. Note that for $\theta \in A$, $|\theta| > \epsilon$ and thus Corollary 3.9 applies. It follows that $\|\frac{d}{d\theta}(T(z))\|_{\mathcal{B}\to\mathcal{B}_w} \ll n^{1-\beta+\epsilon_0}$. Putting these together,

$$
|I_A| \ll n^{-(\beta-\epsilon_0)} + n^{-1} \ll n^{-(\beta-\epsilon_0)}. \qquad (4.2)
$$

Since $\epsilon_0 < \beta^*$, where $\beta^* < \max\{2\beta - 1, 1-\beta\}$, we have $n^{1-\beta}\ell(n)|I_A| \ll n^{-(2\beta-1-\epsilon_0)}\ell(n) = o(1)$.

Next, we estimate I_δ. The estimate for $I_{-\delta}$ follows by a similar argument. Recall $b \in (0, n\delta)$. Proceeding as in the proof of [35, Lemma 5.1](see also [18]), we write

$$
I_\delta = \int_{b/n}^{\delta} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta = -\int_{(b+\pi)/n}^{\delta+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta.
$$

Hence

$$
2I_\delta = \int_{b/n}^{\delta} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta - \int_{(b+\pi)/n}^{\delta+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta = I_1 + I_2 + I_3,
$$

where

$$
I_1 = \int_{\delta}^{\delta+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta, \qquad I_2 = \int_{b/n}^{(b+\pi)/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta,
$$

$$
I_3 = \int_{(b+\pi)/n}^{\delta} \{T(e^{-1/n}e^{i\theta}) - T(e^{-1/n}e^{i(\theta-\pi/n)})\}e^{-in\theta}\,d\theta.
$$

By Proposition 3.1, $|I_1| \ll 1/n$. By (3.5) and Proposition 3.1, $\|(I-R(e^{-u}e^{i\theta}))^{-1}\|_{\mathcal{B}} \ll \ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}$. This together with standard calculations (see, for instance, the argument used in the proof of [35, Lemma 5.1] -in estimating I_2 there) implies that $|I_2| \ll \ell(n)^{-1}n^{-(1-\beta)}b^{-(\beta-\gamma)}$, for any $0<\gamma<\beta$. Putting the above together, $n^{1-\beta}\ell(n)I_\delta = n^{1-\beta}\ell(n)I_3 + O(b^{-(\beta-\gamma)})$.

Next, we estimate I_3. By Corollary 3.6, for all $\theta \in ((b+\pi)/n, \delta)$ and for any $\epsilon_1 \in (\epsilon_0, 2\beta-1)$, we have

$$\|T(e^{-1/n}e^{i\theta}) - T(e^{-1/n}e^{i(\theta-\pi/n)})\|_{\mathcal{B}\to\mathcal{B}_w} \ll \ell(n)\ell(n/|1-in\theta|)^{-2}n^{-\beta}|\frac{1}{n}-i\theta|^{-2\beta} + \ell(n/|1-in\theta|)^{-1}n^{-(\beta-\epsilon_1)}|\frac{1}{n}-i\theta|^{-\beta}.$$

Hence,

$$\begin{aligned}|I_3| &\ll n^{-\beta}\ell(n)\int_{(b+\pi)/n}^{\epsilon}\ell(n/|1-in\theta|)^{-2}\theta^{-2\beta}d\theta + n^{-(\beta-\epsilon_1)}\int_{(b+\pi)/n}^{\epsilon}\ell(n/|1-in\theta|)^{-1}\theta^{-\beta}d\theta \\ &\ll \ell(n)^{-1}n^{-\beta}\int_{(b+\pi)/n}^{\epsilon}\theta^{-2\beta}\frac{\ell(n)^2}{\ell(n/|1-in\theta|)^2}d\theta + \ell(n)^{-1}n^{-(\beta-\epsilon_1)}\int_{(b+\pi)/n}^{\epsilon}\theta^{-\beta}\frac{\ell(n)}{\ell(n/|1-in\theta|)}d\theta \\ &= \ell(n)^{-1}n^{-\beta}I_{3,1} + \ell(n)^{-1}n^{-(\beta-\epsilon_1)}I_{3,2}. \qquad (4.3)\end{aligned}$$

Using Potter's bounds (see, for instance, [8]), for any $\gamma>0$,

$$I_{3,1} = n^{2\beta-1}\int_{b+\pi}^{n\delta}\sigma^{-2\beta}\frac{\ell(n)^2}{\ell(n/|1-i\sigma|)^2}\,d\sigma \ll n^{(2\beta-1)}\int_{b+\pi}^{n\delta}\sigma^{-(2\beta-\gamma)}d\sigma.$$

Taking $0<\gamma<2\beta-1$,

$$\ell(n)^{-1}n^{-\beta}|I_{3,1}| \ll \ell(n)^{-1}n^{-\beta}n^{2\beta-1}n^{\delta}b^{2\beta-\gamma-1} = \ell(n)^{-1}n^{\beta-1}b^{-(2\beta-1-\gamma)}.$$

Finally, we estimate $I_{3,2}$. Using Potter's bounds, we obtain that for any $\gamma'>0$,

$$|I_{3,2}| \ll \int_{(b+\pi)/n}^{\delta}\theta^{-\beta}\Big((\theta n)^{\delta'} + (\theta n)^{-\gamma'}\Big)\,d\theta \ll n^{\delta'}\int_{(b+\pi)/n}^{\delta}\theta^{-(\beta+\gamma')}\,d\theta.$$

Hence, $\ell(n)^{-1}n^{-(\beta-\epsilon_1)}I_{3,2} \ll \ell(n)^{-1}n^{-(\beta-\epsilon_1-\gamma')}$ for arbitrary small γ'.

Putting together the estimates for I_1, I_2 and I_3 (using (4.3) and the estimates for $\ell(n)^{-1}n^{-\beta}|I_{3,1}|$ and $\ell(n)^{-1}n^{-(\beta-\epsilon_1)}I_{3,2}$), we obtain that for arbitrary small γ,γ',

$$|I_\delta| \ll n^{\beta-1}\ell(n)^{-1}b^{-(2\beta-1-\gamma)} + n^{-(\beta-\epsilon_1-\gamma')}\ell(n)^{-1}b^{-(\beta-\delta)}.$$

Since $\epsilon_1 < \beta*$,

$$|I_\delta| \ll n^{\beta-1}\ell(n)^{-1}b^{-(2\beta-1-\gamma)}. \qquad (4.4)$$

Hence, $n^{1-\beta}\ell(n)|I_\delta| \ll b^{-(2\beta-1-\gamma)}$and thus, $\lim_{b\to\infty}\lim_{n\to\infty}n^{1-\beta}\ell(n)I_\delta = 0$. By a similar argument, $\lim_{b\to\infty}\lim_{n\to\infty}n^{1-\beta}\ell(n)|I_{-\delta}| = 0$, ending the proof. ■

To note a straightforward consequence of the above result, we recall that $L: L^1(m) \to L^1(m)$ is the transfer operator of the original transformation $f: M \to M$. By Remark 2.1, for any $v \in \mathcal{B}$, $T_n v = 1_Y L^n(1_Y v)$. With this specified, we state

Corollary 4.3 *Assume the setting of Lemma 4.1. Then*

(i) If ν is a probability measure supported on Y and $\nu \in \mathcal{B}$, then $\lim_{n\to\infty} \|\ell(n)n^{1-\beta}L^{*n}\nu - d_\beta P1\|_{\mathcal{B}\to\mathcal{B}_w} = 0$.

(ii) If $v, w : M \to \mathbb{R}$ are C^α (with α as in (H1)(i)) observables supported on Y, then

$$\lim_{n\to\infty} \ell(n)n^{1-\beta} \int_M v\, w \circ f^n\, d\mu = d_\beta \int_M v\, d\mu \int_M w\, d\mu.$$

Proof Item (i), (ii) follow immediately from Lemma 4.1 and Lemma 4.4. For completeness, below we recall the standard argument for (ii).

Recall $P1 = h$, $P1 = \mu$. Note that for any C^α observable $v : M \to \mathbb{R}$, v supported on Y, we have $Pvh = (\int_M v\, d\mu)h$. Also, by Lemma 4.1 and Lemma 4.4, $\|\ell(n)n^{1-\beta}1\!\!1_Y L^n(vh) - d_\beta Pvh\|_{\mathcal{B}\to\mathcal{B}_w} = o(1)$. Puttting these together,

$$\begin{aligned}\ell(n)n^{1-\beta} \int_M v\, w \circ f^n\, d\mu &= \ell(n)n^{1-\beta} \int_M L^n(vh)w\, dm = d_\beta \int_M P(vh)w\, dm + o(1) \\ &= d_\beta \int_M v\, d\mu \int_M hw\, dm + o(1) = d_\beta \int_M v\, d\mu \int_M w\, d\mu + o(1).\end{aligned}$$

∎

4.2 Mixing under (H4)(iii)

Under hypothesis (H4)(iii), and remembering (3.5), the arguments in [22] carry over with no modification, yielding

Lemma 4.4 *[22, Theorem 1.4] Assume that (H1) and (H5) hold. Let $\beta \in (0, 1]$ and suppose that (H3) and (H4)(iii) hold. Then,*

$$\sup_{v\in\mathcal{B},\|v\|_{\mathcal{B}}=1} \|\ell(n)n^{1-\beta}T_n v - d_\beta P v\|_{\mathcal{B}} \to 0.$$

By the argument used in the proof of Corollary 4.3, we obtain the following consequence of f Lemma 4.4:

Corollary 4.5 *Assume the stroger assumtptions of Lemma 4.4. Then i) of Corollary 4.3 holds in the $\mathcal{B}$ norm for all $\beta \in (0, 1)$. Also, item ii) of Corollary 4.3 holds for all $\beta \in (0, 1)$.*

4.3 Weak pointwise dual ergodicity under weak assumptions

As mentioned in the introduction, the present framework allows us to deal with the property of weak pointwise dual ergodicity (weak p.d.e.) under some weak conditions (under which mixing cannot be proved). Below, we provide a result that allows one to check weak p.d.e. in the framework of Section 2 without assuming (H5) and only requiring (H4)(i).

We recall that a conservative ergodic measure preserving transformation $(X, \mathcal{A}, f, \mu)$ is pointwise dual ergodic (p.d.e.) if there exists some positive sequence a_n such that $\lim_{n\to\infty} a_n^{-1} \sum_{j=0}^{n} L^j v = \int_X v\, d\mu$, a.e. on X for all $v \in L^1(\mu)$. The property of weak p.d.e. has been recently exploited and defined in [3]. As noted in [1], if $f$is invertible and $\mu(X) = \infty$ then f cannot be p.d.e., but it can be weak p.d.e.; that is, there exists some positive sequence a_n such that

(i) $a_n^{-1} \sum_{j=0}^{n} L^j v \to^{\nu} \int_X v\, d\mu$ as $n \to \infty$, for all $v \in L^1(\mu)$. Here, $\to^{\nu}$ stands for convergence in measure for any finite measure $\nu \ll \mu$.

(ii) $\limsup_{n\to\infty} a_n^{-1} \sum_{j=0}^{n} L^j v = \int_X v\, d\mu$, a.e. on X for all $v \in L^1(\mu)$.

As shown in [3, Proposition 3.1], weak p.d.e. for infinite c.e.m. p.t. can be established as soon as items (i) and (ii) above are shown to hold for $v = 1_Y$ for some $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$. Moreover, as noted elsewhere (see [3] and reference therein), item (i) follows as soon as the above mentioned convergence in measure is established for $\mu|_Y$ for some $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$.

In the framework of Section 2, the following holds result for original transformations f with first return map $F : Y \to Y$:

Proposition 4.6 *Assume (H1), (H2), (H3) and (H4)(i). Furthermore, set* $a_n = \ell(n) n^{1-\beta} d_\beta$ *and suppose that* $\sup_Y a_n^{-1} \sum_{j=0}^{n} L^j 1_Y \to 1, \mod \mu$, *as* $n \to \infty$. *Then* f *is weak p.d.e.*

Proof Recall that under (H1), (H2), (H3) and (H4)(i), equation (3.5) holds. By same argument combined with the estimates in [35, Proof of Proposition 7.1],

$$\|T(e^{-u}) - d_\beta \ell(1/|u|)^{-1} u^{-\beta} P\|_{\mathcal{B}} \to 0, \text{ as } u \to 0.$$

Let ν be a σ-finite measure on M such that ν is supported on Y and $\nu|_Y$ is a probability measure in $\mathcal{B}$. Recall $P1 = \mu$ and $\mu(Y) = 1$.

By assumption, $\sup_Y \ell(n) n^{1-\beta} d_\beta \sum_{j=0}^{n} L^j 1_Y = O(1)$, as $n \to \infty$. Thus, the argument used in the proof of [36, Lemma 3.5](a Karamata Tauberian theorem for positive operators that generalizes Karamata tauberian theorem for scalar sequences [8, Proposition 4.2]) applies. It follows that

$$\lim_{n\to\infty} \ell(n) n^{1-\beta} \sum_{j=0}^{n-1} L^{j^*} \nu(1_Y) = d_\beta.$$

In particular, the above equation holds for $\nu = \mu$. So, $\ell(n)n^{1-\beta}d_\beta \sum_{j=0}^{n-1} L^{j^*}\mu(1_Y) \to 1$, as $n \to \infty$ and item (i) (in the definition of weak p.d.e.) for the function 1_Y follows. ∎

5 Higher order asymptotic of T_n: mixing rates

As already mentioned in the introduction, mixing rates for non-invertible infinite measure preserving systems have been obtained in [35, 41]. The results in these works depend heavily on a higher order expansion of the tail probability $\mu(\varphi > n)$. The arguments in [35, 41] generalize to set up of Section 2 and (in an obvious notation), we state

Lemma 5.1 *Assume (H1), (H2), (H3), (H4)(i), (H4)(ii) and (H5). Let* $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$. *Then there exist real constants* $d_0, \dots, d_q$ *(depending only on the map f)*[5], *such that the following hold:*

(i) Let $\beta > 1/2$ *and suppose that* $\mu(\varphi > n) = cn^{-\beta} + H(n)$ *for some* $c > 0$ *and* $H(n) = O(n^{-2\beta})$. *Then,*

$$T_n = (d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \dots + d_q n^{(q+1)(\beta-1)})P + D,$$

where $\|D\|_{\mathcal{B}\to\mathcal{B}_w} = O(n^{-(\beta-1/2)})$.

(ii) Let $\beta > 1/2$ *and suppose that* $\mu(\varphi > n) = cn^{-\beta} + b(n) + H(n)$, *for some* $c > 0$, *some function b such that* $nb(n)$ *has bounded variation and* $b(n) = O(n^{-2\beta})$, *and some function H such that* $H(n) = O(n^{-\gamma})$ *with* $\gamma > 2$.

Then (i) holds with the improved rate $\|D\|_{\mathcal{B}\to\mathcal{B}_w} = O(n^{-\beta})$.

Remark 5.2 We note that items i), ii) correspond to the results on mixing rates for non-invertible systems provided by [35, Theorem 9.1] and [41, Theorem 3.1], respectively.

Proof Below we provide the argument for item i). Item ii) follows by the argument used in the proof of [41, Theoreme 1.1].

Choose $\delta > 0$ such that $\lambda(e^{-1/n}e^{i\theta})$ is well defined for $\theta \in (-\delta, \delta)$. Let $b \in (0, \delta n)$, $n \geq 1$. Also, let I_A, I_δ and $I_{-\delta}$ be defined as in the proof of Lemma 4.1. Hence, equation (4.1) holds and we can write

$$T_n = \frac{e}{2\pi}\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \frac{e}{2\pi}(I_\delta + I_{-\delta} + I_A).$$

Let ϵ_0 be as defined in assumption (H4)(ii). By equation (4.2), $|I_A| \ll n^{-(\beta-\epsilon_0)}$.

[5] For the precise form of these constants we refer to in [35, Theoreme 9.1] and [41, Theoreme 1.1].

Let ϵ_1 be as defined in the statement of Proposition 3.6, that is $\epsilon_1 \in (\epsilon_0, \beta^*)$, where $\beta^* < \max\{2\beta - 1, 1 - \beta\}$. By equation (4.4), for any $\gamma > 0$, $|I_\delta + I_{-\delta}| \ll n^{\beta-1}\ell(n)^{-1}b^{-(2\beta-1-\gamma)}$.

Next, we estimate $\frac{e}{2\pi}\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta$. By the argument used in obtaining equation (3.4) together with the assumption on $\mu(\varphi > n)$, $|(1 - \lambda((e^{-1/n}e^{i\theta}))^{-1}| \ll |\frac{1}{n} - i\theta|^{-\beta}$. This together with Corollary 3.5 implies that

$$\|(1 - \lambda((e^{-1/n}e^{i\theta}))^{-1}(P((e^{-1/n}e^{i\theta}) - P)\|_{\mathcal{B}\to\mathcal{B}_w} \ll |\frac{1}{n} - i\theta|^{-\epsilon_1}.$$

Recall $b \in (0, \delta n)$, $n \geq 1$. The above displayed equation together with equation (3.1) and Proposition 3.1 yield

$$\begin{aligned}\Big|\int_{-b/n}^{b/n}(T(e^{-1/n}e^{i\theta}) - (1 - \lambda(e^{-1/n}e^{i\theta}))^{-1}P)e^{-in\theta}\,d\theta\Big| &\ll \int_0^{b/n}\frac{|\frac{1}{n} - i\theta|^{-\epsilon_1}}{\ell(1/|1/n - i\theta|)}\,d\theta\\ &\ll n^{\epsilon_1-1}b.\end{aligned}$$

Recall $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$. By the argument used in the proof of [35, Proposition 9.5] (which exploits exactly the same assumption on $\mu(\varphi > n)$ stated in item i) of the lemma),

$$\int_{-b/n}^{b/n}(1 - \lambda(e^{-1/n}e^{i\theta}))^{-1}e^{-in\theta}\,d\theta = d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \ldots + d_q n^{(q+1)(\beta-1)} + O(bn^{-\beta}),$$

where $d_0, \ldots, d_q$ are real constants, depending only on the map f (again, for the precise form of these constants we refer to [35, Theoreme 9.1]).

Putting together the last two displayed equations,

$$\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta = d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \ldots + d_q n^{(q+1)(\beta-1)} + O(bn^{-\beta}) + O(n^{\epsilon_1-1}b).$$

Take $b = n^{1/2}$ and recall $\epsilon_1 < \max\{2\beta - 1, 1 - \beta\}$. So, $\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta = d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \ldots + d_q n^{(q+1)(\beta-1)} + O(n^{-(\beta-1/2)})$. To conclude, note that $|I_\delta + I_{-\delta} + I_A| \ll n^{-(\beta-1/2)}$. ∎

By the argument used in the proof of Corollary 4.3, we obtain the following consequence of the previous result.

Corollary 5.3 *Assume the setting of Lemma 5.1. Let $v, w : M \to \mathbb{R}$ be C^α observables supported on Y. Then*

$$\int_M v\, w \circ f^n\, d\mu = (d_0 n^{\beta-1} + d_1 n^{2\beta-2} + \ldots + d_q n^{(q+1)(\beta-1)})\int_M v\, d\mu \int_M w\, d\mu + E_n,$$

where $E_n = O(n^{-(\beta-1/2)})$ if the assumption on $\mu(\varphi > n)$ stated in Lemma 5.1, i) holds and $E_n = O(n^{-\beta})$ if the assumption on $\mu(\varphi > n)$ stated in Lemma 5.1, ii) holds.

6 Banach spaces estimates in the Markov case

Let $f : [0,1]^2 \to [0,1]^2$ be the map (1.3) described in subsection 1.4. Let $Y_0 = Y = (1/2, 1] \times [0,1]$ and let $\varphi : Y \to \mathbb{Z}^+$ be the return time to Y. Let $F = f^\varphi$ be the first return map. In this section we show that F satisfies (H1–H5) for some appropriate function spaces $\mathcal{B}, \mathcal{B}_w$ constructed in analogy with [12]. We start by describing the spaces $\mathcal{B}, \mathcal{B}_w$.

6.1 Notation and definitions

It is convenient to introduce the notation $F^n = (F_0^n, G_n)$, for all $n \in \mathbb{N}$. The properties of F can be understood in terms of the map $F_0 = f_0^{\varphi_0}$, where φ_0 is the first return time of f_0 to $X_0 = (1/2, 1]$. For $j \geq 1$, set $(x_{j-1}, x_j] = X_j = \{\varphi_0 = j\}$. For $j \geq 1$ set $Y_j = X_j \times [0,1]$. So, $Y_j = \{\varphi = j\}$. For all $j \geq 1$, let $f^j Y_j = Y_j' = \{(x,y) \in X_0 \times [0,1] \ : \ y \in K_j(x)\}$ for some collection of intervals $K_j(x) \subset [0,1]$. Hence, we can write $F = f^j : Y_j \to Y_j'$.

For $n \geq 0$, let $\mathcal{Y}_n = \{Y_{n,j}\}$ be the corresponding partition of Y associated with (Y, F^n). Since F is invertible, we have $F^{-n}(\{Y'_{n,j}\}) = \{Y_{n,j}\}$. The map F^n is smooth in the interior of each element of the partition $\mathcal{Y}_n$.

Admissible leaves: We start by introducing a set of *admissible leaves* Σ. Such leaves consists of full vertical segments W. A full vertical segment $W(x)$, based at the point $x \in [0,1]$, is given by $\mathbb{G}_x(t) = (x,t)$, $t \in [0,1]$. The definition of the set of admissible leaves differs with the one in [12] and allows for a considerable simplification of the arguments. Yet, it is possible only due to the (very special) fact that the map is a skew product.

Uniform contraction/expansion, distortion properties: Given the simple structure of the stable leaves it is convent to introduce the projection on the second co-ordinate $\pi : [0,1]^2 \to [0,1]$ defined by $\pi(x,y) = y$.

By hypothesis (1.4) we can chose $\lambda > 1$ such that:

- If $x, y \in W$, $W \in \Sigma$ then $|\pi(F^n x) - \pi(F^n y)| \leq C\lambda^{-n}$.

For any $(x,y) \in Y \in \mathcal{Y}_n$, $|\det(DF^n(x,y))| = (F_0^n)'(x) \cdot \partial_y G_n(y)$.
It is well known that there exists $C > 0$ such that, for each $(x,y), (x',y') \in \mathcal{Y}_0$,

$$\left| \frac{|F_0'(x)|}{|F_0'(x')|} - 1 \right| \leq C|x - x'|. \tag{6.1}$$

In fact, more is true,

$$\left| \frac{d}{dx}(F_0'(x))^{-1} \right| \leq C|F_0'(x)^{-1}|. \tag{6.2}$$

Test functions: In what follows, for $W \in \Sigma$ and $q \le 1$ we denote by $C^q(W, \mathbb{C})$ the Banach space of complex valued functions on W with Hölder exponent q and norm

$$|\phi|_{C^q(W,\mathbb{C})} = \sup_{z\in W} |\varphi(z)| + \sup_{z,w\in W} \frac{|\varphi(z) - \varphi(w)|}{|z - w|^q}.$$

Note that $C^q(W(x), \mathbb{C})$ is naturally isomorphic to $C^q([0, 1], \mathbb{C})$ via the identification of the domain given by $t \to (x, t)$. In the following we will use implicitly such an identification, in particular for $\phi \in C^q([0, 1], \mathbb{C})$ we still call ϕ the corresponding function in $C^q(W(x), \mathbb{C})$ and we write

$$\int_{W(x)} h\phi \, dm = \int_0^1 h(x, t)\phi(t)dt.$$

Remark 6.1 Note that we use m both for the one dimensional and two dimensional Lebesgue measure. Also, in the following we will often suppress dm as this does not create any confusion.

Definition of the norms: Given $h \in C^1(Y, \mathbb{C})$, define the *weak norm* by

$$\|h\|_{\mathcal{B}_w} := \sup_{W\in\Sigma} \sup_{|\phi|_{C^1(W,\mathbb{C})}\le 1} \int_W h\phi \, dm. \tag{6.3}$$

Given $q \in [0, 1)$ we define the *strong stable norm* by

$$\|h\|_s := \sup_{W\in\Sigma} \sup_{|\phi|_{C^q(W,\mathbb{C})}\le 1} \int_W h\phi \, dm. \tag{6.4}$$

For some small ϵ_0 (to be specified later), define the *strong unstable norm* by

$$\|h\|_u := C^{-1} \sup_{0<|x-y|\le\epsilon_0} \sup_{|\phi|_{C^1}\le 1} \frac{1}{|x - y|} \left| \int_{W(x)} h\phi \, dm - \int_{W(y)} h\phi \, dm \right|. \tag{6.5}$$

Finally, the *strong norm* is defined by $\|h\|_{\mathcal{B}} = \|h\|_s + \|h\|_u$.

Definition of the Banach spaces: We will see briefly that $\|h\|_{\mathcal{B}_w} + \|h\|_{\mathcal{B}} \le C\|h\|_{C^1}$. We then define $\mathcal{B}$ to be the completion of C^1 in the strong norm and $\mathcal{B}_w$ to be the completion in the weak norm.

The spaces $\mathcal{B}$ and $\mathcal{B}_w$ defined above are simplified versions of functional space defined in [12] (adapted to the setting of (1.3)). The main difference in the present setting is the simpler definition of admissible leaves and the absence of a control on short leaves. The latter is necessary and possible since the discontinuities do no satisfy any transversality condition while, instead, they enjoy some form of Markov structure.

6.2 Embedding properties: verifying (H1)(i)

The next result shows that (H1)(i) holds for $\alpha = \gamma = 1$ and $\mathcal{B}, \mathcal{B}_w$ as described above.

Lemma 6.2 *For all $q \in (0,1)$ in definition (6.4) we have*

$$C^1 \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^1)'.$$

Proof By the definition of the norms it follows that $\|\cdot\|_{\mathcal{B}_w} \le \|\cdot\|_s \le \|\cdot\|_{\mathcal{B}}$, from this the inclusion $\mathcal{B} \subset \mathcal{B}_w$ follows. For each $h \in C^1$ we have

$$\left|\int_{W(x)} h\phi - \int_{W(y)} h\phi\right| = \left|\int_0^1 h(x,t)\phi(t)dy - \int_0^1 h(y,t)\phi(t)\right| \le C\|h\|_{C^1}\|\phi\|_{C^0}|x-y|.$$

The above implies $\|h\|_u \le C\|h\|_{C^1}$. Thus $C^1 \subset \mathcal{B}$.

The other inclusion is an immediate consequence of Proposition 6.3, an analogue of [12, Lemma 3.3]. ∎

Proposition 6.3 *Let I be an interval, $I \subset [0,1]$ and set $E = I \times [0,1]$. Then for all $\phi \in C^1$ and for all $h \in \mathcal{B}_w$, we have*

$$\left|\int_E h\phi\, dV\right| \le \|h\|_{\mathcal{B}_w}\|\phi\|_{C^1},$$

where dV is the normalised measure on E.

Proof By density it suffices to consider $h \in C^1$. By Fubini theorem and the fact that almost all vertical segments are admissible leaves

$$\left|\int_E h\phi\right| \le \int_I dx \left|\int_0^1 dy h(x,y)\phi(x,y)\right| \le \|h\|_{\mathcal{B}_w}\|\phi\|_{C^1}|I| = \|h\|_{\mathcal{B}_w}\|\phi\|_{C^1} m(E).$$

∎

6.3 Verifying (H2)

We note that the connected components of $\varphi^{-1}(n)$ satisfy the assumption on the set E in the statement of Proposition 6.3. Hence,

$$\left|\int_E h\, dm\right| \le \|h\|_{\mathcal{B}_w} m(E),$$

and (H2) follows.

6.4 Transfer operator: definition

If $h \in L^1$, then $R : L^1 \to L^1$ acts on h by

$$\int Rh \cdot v = \int h \cdot v \circ F, \quad v \in L^\infty.$$

By a change of variable we have

$$Rh = \mathbb{1}_{F(Y)} h \circ F^{-1} \det(DF^{-1}). \tag{6.6}$$

Note that, in general, $RC^1 \not\subset C^1$, so it is not obvious that the operator R has any chance of being well defined in $\mathcal{B}$. The next Lemma addresses this problem.

Lemma 6.4 *With the above definition, $R(C^1) \subset \mathcal{B}$.*

Proof Using the notation introduced at the beginning of section 6.1, we have $F(Y) = \cup_j Y_j'$. Moreover, both F^{-1} and $\det(DF^{-1})$ are C^1 on each $\overline{Y_j'}$. For each $j \in \mathbb{N}$ note that Y_j' consists of an horizontal strip bounded by the curves $\gamma_1(x) = G(g_j(x), 1)$, $\gamma_0(x) = G(g_j(x), 0)$ where $g_j : (\frac{1}{2}, 1] \to (\frac{1}{2}, 1]$ is the inverse brach of F_0 corresponding to the return time j. Remark that, by equation (1.5), it follows that $|\gamma_i'|_\infty \leq K_0$, for $i \in \{0, 1\}$. We can then consider a sequence of $\bar{\psi}_n \in C_0^1(\mathbb{R}, [0, 1])$ that converges monotonically to $\mathbb{1}_{[0,1]}$ and define $\tilde{\psi}_n(x, y) = \bar{\psi}_n(y)$. Next, we define the function $\psi_n = \tilde{\psi}_n \circ F^{-1} \cdot \mathbb{1}_{F(Y)}$. Note that ψ_n is smooth and converges monotonically to $\mathbb{1}_{F(Y)}$. We can then define

$$H_n = \psi_n h \circ F^{-1} \det(DF^{-1}) \in C^1.$$

Consider an admissible leaf $W = W(x)$ and a test function ϕ. Note that $F^{-1}W = \cup_j W_j$ where $W_j = F^{-1}W \cap Y_j = W(g_j(x))$ are vertical leaves. Thus, given $h \in C^1$,[6]

$$\begin{aligned}\left| \int_W [Rh - H_n]\phi \right| &\leq \sum_j \int_{W_j} |h| \, |F_0'|^{-1} |\phi \circ F| \, |1 - \tilde{\psi}_n| \\ &\leq \|h\|_\infty \|\phi\|_\infty C \sum_j |F_0'(g_j(x))|^{-1} \int_0^1 |1 - \bar{\psi}_n(t)| dt\end{aligned}$$

since $\det(DF^{-1}) \circ F = \det(DF)^{-1} = (F_0' \cdot \partial_y G)^{-1}$ and where we have used (6.1) in the second line. Also we have used, and will use in the following, a harmless abuse of notation insofar we write $\phi \circ F^n$ to mean $\phi \circ \pi \circ F^n$. Since the sun is convergent and the integral converge to zero, it follows that the right hand side can be made arbitrarily small by taking n large enough. It follows that H_n converges to Rh in $\mathcal{B}_w$.

The above computation also shows that $\lim_{n\to\infty} \|Rh - H_n\|_s = 0$. Thus, it remains to check the unstable norm. Let $x, z \in [0, 1]$, $|x - z| \leq \varepsilon_0$. Let $x_j = g_j(x)$, $(x_j, 0) \in Y_j$,

[6] Since $W_j \subset Y$, $F(W_j) \subset F(Y)$. Thus, $\mathbb{1}_{F(Y)} \circ F$, restricted on W_j, equals one.

and $z_j = g_j(z)$, $(z_j, 0) \in Y_j$. Then, for each ϕ, $\|\phi\|_{C^1} \leq 1$, we have

$$
\begin{aligned}
&\left|\int_{W(x)} [Rh - H_n]\phi - \int_{W(z)} [Rh - H_n]\phi\right| \\
&\leq \sum_j \int_0^1 \left|h(x_j, t) F_0'(x_j)^{-1} \phi \circ F(x_j, t) - h(z_j, t) F_0'(z_j)^{-1} \phi \circ F(z_j, t)\right| \left|1 - \bar{\psi}_n(t)\right|.
\end{aligned}
$$

Since, by hypothesis and equations (1.5), (6.2), $|\frac{d}{dx}[h(\cdot, t)(F_0)'(\cdot)^{-1}\phi \circ F(\cdot, t)]| \leq C$ for some fixed $C > 0$, we have

$$
\left|\int_{W(x)} [Rh - H_n]\phi - \int_{W(x)} [Rh - H_n]\phi\right| \leq C \sum_j |x_j - z_j| \int_0^1 \left|1 - \bar{\psi}_n(t)\right|.
$$

Finally, by equation (6.1), we have $|x_j - z_j| \leq C|F_0'(x_j)|^{-1}|x - z|$ and again we can conclude as above. ∎

6.5 Lasota–Yorke inequality and compactness: verifying (H5)(i) and (H1)(ii).

The next Lemma is the basic result on which all the theory rests.

Proposition 6.5 *(Lasota–Yorke inequality.) For each $z \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$ and $h \in C^1(Y, \mathbb{C})$ we have*

$$
\begin{aligned}
&\|R(z)^n h\|_{\mathcal{B}_w} \leq C\|h\|_{\mathcal{B}_w} \\
&\|R(z)^n h\|_{\mathcal{B}} \leq \lambda^{-nq}\|h\|_{\mathcal{B}} + C\|h\|_{\mathcal{B}_w}.
\end{aligned}
$$

Proof Setting $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ F^k$ we have that $R(z)^n h = R^n(z^{\varphi_n} h)$. Remark that φ_n is constant on the elements of $Y_{n,j}$ of $\mathcal{Y}_n$, moreover $\varphi_n \geq n$, hence $|z^{\varphi_n}| \leq |z|^n$.

Given $W \in \Sigma$, with base point x, we have $F^{-n}(W) = \cup_{j \in \mathbb{N}} W_j$ where $\mathcal{W} = \{W_j\}_{j \in \mathbb{N}} \subset \Sigma$ is the collections of the maximal connected components. Note that each $Y_{n,j} \in \mathcal{Y}_n$ contains precisely one W_j. Then, for $|\phi|_{C^1(W,\mathbb{C})} \leq 1$, we have

$$
\int_W (R(z)^n h)\phi \, dm = \int_W \mathbb{1}_{F^n(Y)} h \circ F^{-n} \det(DF^{-n})\phi.
$$

By the invertibility of F, the connected components of $W \cap F^n(Y)$ are exactly $\{F^n W'\}_{W' \in \mathcal{W}}$. Notice that $F^n(x, y) = (F_0^n(x), G_n(x, y))$, while $F^{-n}(x, y)$ has the more general form $(A(x, y), B(x, y))$. Yet, the function A depends on y only in a limited manner: $A(x, y) = g_j(x)$ for all $(x, y) \in F^n(Y_{n,j})$. Also, it is convenient to call B_j the function B restricted to $F^n(Y_{n,j})$. If $(x_j, 0) \in W_j \in \mathcal{W}$, then $G_n(x_j, t)$ provides a parametrization for the little segment $F^n W_j \subset W$. In addition, for $(x, y) \in F^n W_j$, $F^{-n}(x, y) = (x_j, H_{n,j}(y))$ with $x_j = g_j(x)$ and $H_{n,j}(y) = B_j(x, y)$. We can then write

$$\begin{aligned}\int_{F^n W_j} h \circ F^{-n} \det(DF^{-n})\phi &= \int_{G_n(x_j,0)}^{G_n(x_j,1)} \frac{h(x_j, H_{n,j}(t))}{(F_0^n)'(x_j)\partial G_n(x_j, H_{n,j}(t))}\phi(t)dt \\ &= \int_0^1 \frac{h(x_j, t)}{(F_0^n)'(x_j)}\phi(G_n(x_j, t))dt.\end{aligned}$$

By the above computation we have

$$\left|\int_W (R(z)^n h)\phi\, dm\right| \le \sum_{W_j \in \mathcal{W}} \left|\int_{W_j} z^{\varphi_n} h[(F_0^n)']^{-1}\phi \circ F^n\, dm\right| \le \sum_{Y_{n,j} \in \mathcal{Y}_n} \|h\|_{\mathcal{B}_w} \left|\frac{\phi \circ F^n}{(F_0^n)'}\right|_{C^1(W_j)} |z|^n.$$

Note that (6.2) and (6.1) imply

$$\|[(F_0^n)']^{-1}\phi \circ F^n\|_{C^0(W_j)} \le C \sup_{x \in W_j} [(F_0^n)']^{-1} \le 2Cm(Y_{n,j}).$$

W.r.t. the C^1 norm, we have

$$\|[(F_0^n)']^{-1}\phi \circ F^n\|_{C^1(W_j)} \le \|[(F_0^n)']^{-1}\|_{C^1}\|\phi \circ F^n\|_{C^0} + \|[(F_0^n)']^{-1}\|_{C^0}\|\phi \circ F^n\|_{C^1}.$$

Recall that $\phi \circ F^n$ stands for $\phi \circ \pi \circ F^n$, thus[7]

$$\begin{aligned}\|\phi \circ F^n\|_{C^1} &\le |\phi|_\infty + \sup_t |\frac{d}{dt}\phi \circ \pi \circ F^n(x_j, t)| \le |\phi|_\infty + |\phi' \circ \pi \circ F^n \cdot \partial_y G_n(x_j, \cdot)|_\infty \\ &\le |\phi|_\infty + |\phi'|_\infty \lambda^{-n} \le C\|\phi\|_{C^1}.\end{aligned}$$

Thus

$$\|[(F_0^n)']^{-1}\phi \circ F^n\|_{C^1(W_j)} \le Cm(Y_{n,j}). \tag{6.7}$$

Equation (6.7) allows to estimate the weak norm as follows

$$\left|\int_W (R(z)^n h)\phi\, dm\right| \le C\|h\|_{\mathcal{B}_w}|z|^n \sum_j m(Y_{n,j}) \le C\|h\|_{\mathcal{B}_w}|z|^n. \tag{6.8}$$

The first inequality of the proposition follows. Let us discuss the strong stable norm. Given $|\phi|_{C^q(W,\mathbb{C})} \le 1$, we have

$$\begin{aligned}\left|\int_W (R(z)^n h)\phi\, dm\right| &\le \sum_{W_j \in \mathcal{W}} \left|\int_{W_j} h[(F^n)']^{-1}\phi \circ F^n\, dm\right| |z|^n \\ &\le \sum_{W_j \in \mathcal{W}} \left|\int_{W_j} h\hat{\phi}_j\, dm\right| |z|^n + \left|\int_{W_j} h[(F^n)']^{-1}\bar{\phi}_j\, dm\right| |z|^n,\end{aligned}$$

[7] In the following we will implicitly use standard computations of this type.

where $\bar{\phi}_j = |W_j|^{-1} \int_{W_j} \phi \circ F^n$ and $\hat{\phi}_j = [(F^n)']^{-1}(\phi \circ F^n - \bar{\phi}_j)$. Let $J_{W_j} F^n$ be the stable derivative on the fibre W_j (recall that $|J_{W_j} F^n| \leq \lambda^{-n}$), then

$$
\begin{aligned}
&|\phi \circ F^n - \bar{\phi}_j| \leq |J_{W_j} F^n|_\infty^q |W_j|^q \leq C |F^n(W_j)|^q \\
&\sup_{x,y \in W_j} \frac{|\hat{\phi}_j(x) - \hat{\phi}_j(y)|}{\|x-y\|^q} \leq C |(F^n)'|^{-1}_{L^\infty(W_j)} |J_{W_j} F^n|_\infty^q \leq \frac{C|F^n(W_j)|^q}{|(F^n)'|_{L^\infty(W_j)} |W_j|^q}.
\end{aligned}
$$

Hence,

$$
\|\hat{\phi}_j\|_{C^q(W_j,\mathbb{C})} \leq C \frac{|F^n(W_j)|^q}{|(F^n)'|_{L^\infty(W_j)}} \leq C m(Y_{n,j}) \lambda^{-nq}.
$$

The above bound yields,

$$
\left| \int_W (R(z)^n h) \phi \, dm \right| \leq C \|h\|_s \lambda^{-nq} \, |z|^n + C \|h\|_{\mathcal{B}_w} \, |z|^n.
$$

We are left with the strong unstable norm. Let $\|\phi\|_{C^1} \leq 1$, $x, y \in [0,1]$ with $|x-y| \leq \varepsilon$, $\varepsilon \in (0, \varepsilon_0)$. Let $\mathcal{W}(x)$ be the set of pre images of $W(x)$ under F^n and the same for $\mathcal{W}(y)$. Note that to each element of $W_j(x)$ it corresponds a unique element $W_j(y)$ that belongs to the same set $Y_{n,j} \in \mathcal{Y}_n$. Let $\xi_j, \eta_j \in [0,1]$ be such that $W_j(x) = W(\xi_j)$ and $W_j(y) = W(\eta_j)$. By the usual distortion estimates we have $|\xi_j - \eta_j| \leq C |(F_0^n)'(\xi_j)|^{-1} |x-y|$. We introduce the function $\Phi_n = |(F_0^n)'|^{-1}(\eta_j) \cdot \phi \circ F^n|_{W(\eta_j)}$, and write

$$
\begin{aligned}
&\left| \int_{W(x)} R(z)^n h \phi \, dm - \int_{W(y)} R(z)^n h \phi \, dm \right| \leq \sum_j \left| \int_{W(\xi_j)} h \Phi_n \, dm - \int_{W(\eta_j)} h \Phi_n \, dm \right| \, |z|^n \\
&\quad + \sum_j \left| \int_{W(\xi_j)} h \left[\, |(F_0^n)'(\xi_j)|^{-1} - |(F_0^n)'(\eta_j)|^{-1} \right] \phi \circ F^n \, dm \right| \, |z|^n \\
&\leq \sum_j \lambda^{-n} |x-y| \left[\|h\|_u + C \|h\|_s \right] |(F_0^n)'|^{-1}_{L^\infty(W(x_j))} \leq C \lambda^{-n} |x-y| \|h\|,
\end{aligned} \tag{6.9}
$$

where we have used that $\phi \circ F^n|_{W(\eta_j)} = \phi \circ F^n|_{W(\xi_j)}$. The Lemma follows then by iterating the above formula. ∎

The above Proposition, together with Lemma 6.4, readily implies that $R(z) \in L(\mathcal{B}, \mathcal{B})$, i.e. Hypothesis (H1)(ii) holds true. Note that Proposition 6.5 alone would not suffice, indeed the fact that a function has a bounded norm does not imply that it belongs to $\mathcal{B}$: for this, it is necessary to prove that it can be approximate by C^1 functions in the topology of the Banach space.

The proof of Lemma 6.4 holds essentially unchanged also for the operator $R(z)$, thus $R(z) \in L(\mathcal{B}, \mathcal{B})$. We can then extend, by density, the statement of Proposition 6.5 to all $h \in \mathcal{B}$, whereby proving hypothesis (H5)(i).

6.6 Verifying (H1)(iii) and (H5)(ii)

We start by verifying the compactness property of $R(z)$. This is the starting point to prove (H1)(iii) and (H5)(ii).

Lemma 6.6 *For each $z \in \overline{\mathbb{D}}$ the operator $R(z)$ is quasi-compact with spectral radius bounded by $|z|$ and essential spectral radius bounded by $|z|\lambda^{-\frac{q}{2}}$.*

Proof Let $h \in C^1$ with $\|h\|_{\mathcal{B}} \leq 1$. For each $\varepsilon \geq 0$ let $n \in \mathbb{N}$ such that $\lambda^{-n} = \varepsilon$ and choose $\xi_j \in Y_{n,j}$ for each $Y_{n,j} \in \mathcal{Y}_n$. If $W \in \Sigma$ is contained in $Y_{n,j}$, then

$$\left| \int_W h\phi - \int_{W(\xi_j)} h\phi \right| \leq \varepsilon |\phi|_{C^1}.$$

Choose $N_\varepsilon \in \mathbb{N}$ by such that $\sum_{j \geq N_\varepsilon} m(Y_{n,j}) \leq \varepsilon$. Arguing as in (6.8) we have

$$\begin{aligned}\left| \int_W R(z)^n h \cdot \phi \right| &\leq \sum_{j \leq N_\varepsilon} \left| \int_{W_j} h |(F_0^n)'|^{-1} \phi \circ F^n \, dm \right| |z|^n + C\varepsilon \|h\|_{\mathcal{B}} |z|^n \\ &\leq \sum_{j \leq N_\varepsilon} \left| \int_{W(\xi_j)} h |(F_0^n)'|^{-1} \phi \circ F^n \, dm \right| |z|^n + C\varepsilon \|h\|_{\mathcal{B}} |z|^n\end{aligned}$$

Now for each $W(\xi_j)$ we can choose a a sequence of functions $\{\phi_{j,l}\} \subset C^1$ such that they are C^q $\varepsilon N_\varepsilon^{-1}$-dense. Then

$$\|R(z)^n h\|_{\mathcal{B}_w} \leq \sup_{j \leq N_\varepsilon} \sup_l \left| \int_{W(\xi_j)} h\phi_{j,l} \, dm \right| |z|^n + C\varepsilon \|h\|_{\mathcal{B}} |z|^n.$$

By the Lasota-Yorke inequality then it follows

$$\begin{aligned}\|R(z)^{2n} h\|_{\mathcal{B}} &\leq C\lambda^{-nq} \|h\|_{\mathcal{B}} |z|^{2n} + C \|R(z)^n h\|_{\mathcal{B}_w} |z|^n \\ &\leq C\lambda^{-nq} \|h\|_{\mathcal{B}} |z|^{2n} + C \sup_{j \leq N_\varepsilon} \sup_l \left| \int_{W(\xi_j)} h\phi_{j,l} \, dm \right| |z|^{2n}.\end{aligned}$$

The above inequality implies that the image of the unit ball under $R(z)^{2n}$ can be covered with finitely many balls of radius $\lambda^{-nq}|z|^{2n}$. The conclusion follows by the usual Neussbaum formula. ∎

Note that 1 belongs to the spectrum of R (since the composition with F is the dual operator to R and $1 \circ F = 1$). By the spectral decomposition of R it follows that $\frac{1}{n}\sum_{i=0}^{n-1} R^n$ converges (in uniform topology) to the eigenprojector Π associated to the eigenvalue 1. Let $\mu = \Pi 1$. The next step is the characterization of the peripheral spectrum.

Lemma 6.7 *Let $\nu \in \sigma(R(z))$ with $|\nu| = 1$. Then any associated eigenvector h is a complex measure. Moreover, such measures are all absolutely continuous with respect to μ and have bounded Radon-Nikodym derivative.*

Proof Note that $|z| = 1$ since the spectral radius of $R(z)$ is smaller or equal to $|z|$. Next, let h be an eigenvector with eigenvalue ν, then $h \in \mathcal{B}_w \subset (C^1)'$ and, for each $\phi \in C^1$, we have

$$|h(\phi)| = \left|\nu^{-n} h(z^{\varphi_n} \phi \circ F^n)\right| \leq \sum_j |h(z^j \mathbb{1}_{Y_{n,j}} \phi \circ F^n)| \leq \|h\|_{\mathcal{B}_w} \|\phi \circ F^n\|_{C^1}$$

where we have used Proposition 6.3. Since $\lim_{n\to\infty} \|\phi \circ F^n\|_{C^1} = \|\phi\|_{C^0}$ it follows that $h \in (C^0)'$, i.e. it is a measure. Since, by Lemma 6.6, the projector $\Pi_\nu(z)$ on the eigenspace associated to ν can be obtained as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \nu^{-i} R(z)^i$$

and since the range of $\Pi_\nu(z)$ is finite dimensional, there must exists $\psi \in C^1$ such that $h = \Pi_\nu(z)\psi$. Then, for each $\phi \in C^1(Y, \mathbb{R}_+)$,

$$\left| \int \phi \frac{1}{n} \sum_{i=0}^{n-1} \nu^{-i} R(z)^i \psi \right| = |\psi|_\infty \int \frac{1}{n} \sum_{i=0}^{n-1} R^i 1 \phi$$

which, taking the limit for $n \to \infty$, implies $|h(\phi)| \leq |\psi|_\infty \mu(\phi)$ and the Lemma. ∎

Now suppose that $Rh = e^{i\theta}h$. Then, by the above Lemma, there exists $v \in L^\infty(\mu)$ such that $h = v\mu$. Hence

$$\mu(v\phi) = h(\phi) = e^{-i\theta} h(\phi \circ F) = e^{-i\theta} \mu(v\phi \circ F) = e^{-i\theta} \mu(\phi v \circ F^{-1})$$

implies $v = e^{i\theta} v \circ F$ μ-almost surely. By similar arguments, if $z = e^{i\theta}$ and $R(z)h = R(e^{i\theta\varphi} h) = h$, then there exists $v \in L^\infty(\mu)$ such that $ve^{i\theta\varphi} = v \circ F$ μ-almost surely.

Proposition 6.8 *Hypotheses (H1)(iii) and (H5)(ii) hold true.*

Proof As the proof of the two hypotheses is essentially the same, we limit ourselves to the proof of (H5)(ii). Let $v : Y \to \mathbb{C}$ be a (non identically zero) measurable solution to the equation $v \circ F = e^{i\theta\varphi} v$ a.e. on Y, with $\theta \in (0, 2\pi)$. By Lusin's theorem, v can be approximated in $L^1(\mu)$ by a C^0 function, which in turn can be approximated by a C^∞ function. Hence, there exists a sequence ξ_n of C^1 functions such that $|\xi_n - v|_{L^1(\mu)} \to 0$, as $n \to \infty$. So, we can write

$$v = \xi_n + \rho_n,$$

where $|\rho_n|_{L^1(\mu)} \to 0$, as $n \to \infty$.

Starting from $v = e^{-i\theta\varphi} v \circ F$ and iterating forward m times (for some m large enough to be specified later),

$$v = e^{-i\theta \sum_{j=0}^{m-1} \varphi_0 \circ F_0^j} (\xi_n \circ F^m + \rho_n \circ F^m).$$

Clearly,

$$|e^{-i\theta \sum_{j=0}^{m-1} \varphi_0 \circ F_0^j} \rho_n \circ F^m|_{L^1(\mu)} = |\rho_n \circ F^m|_{L^1(\mu)} = |\rho_n|_{L^1(\mu)} \to 0, \tag{6.10}$$

as $n \to \infty$.

Next, put $A_{n,m} := e^{-i\theta \sum_{j=0}^{m-1} \varphi_0 \circ F_0^j} \xi_n \circ F^m$ and note that for all n and m

$$|\partial_y A_{n,m}| \le |\partial_y \xi_n|_\infty |\partial_y F^m|.$$

By condition (1.4), there exists $0 < \tau < 1$ such that $|\partial_y F| = \tau$. Hence, for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that

$$|\partial_y A_{n,m}|_\infty < \varepsilon.$$

It is then convenient to use $\mathbb{E}_\mu$ for the expectation with respect to μ and $\mathbb{E}_\mu(\cdot \mid x)$ for the conditional expectation with respect to the σ-algebra generated by the set of admissible leaves. As a consequence, $|A_{n,m}(x,y) - \mathbb{E}_\mu(A_{n,m} \mid x)| \le \varepsilon$. For arbitrary $\psi \in L^\infty(\mu)$, we can then write

$$\begin{aligned} \mathbb{E}(\psi v) &= \mathbb{E}_\mu(\psi A_{n,m}) + O(\varepsilon) = \mathbb{E}_\mu(\psi \mathbb{E}_\mu(A_{n,m} \mid x)) + O(\varepsilon) \\ &= \mathbb{E}_\mu(\mathbb{E}_\mu(\psi \mid x) A_{n,m}) + O(\varepsilon) = \mathbb{E}_\mu(\psi \mathbb{E}_\mu(v \mid x)) + O(\varepsilon). \end{aligned} \tag{6.11}$$

By the arbitrariness of ε and ψ it follows $v = \mathbb{E}_\mu(v \mid x)$. But this implies that $v \circ F_0 = v \circ F = e^{i\theta\varphi_0} v$, but this has only the trivial solution $v = 0$ (see [2, Theorem 3.1]). ∎

6.7 Verifying (H4): bounds for $\|R_n\|_{\mathcal{B}}$.

The next result shows that the strongest form of (H4), that is (H4)(iii), holds.

Lemma 6.9 *For each $n \in \mathbb{N}$ we have the bound*

$$\|R_n\|_{\mathcal{B}} \le C n^{-\beta-1}.$$

Proof Note that φ is constant on $\mathcal{Y}_1$, hence there exists j_n such that $\varphi|_{Y_{j_n}} = n$. Thus for each $\|\phi\|_{C^q} \le 1$ and $W \in \Sigma$,

$$\left| \int_W (R_n h) \phi \, dm \right| \le \left| \int_{W_{j_n}} h |(F_0^n)'|^{-1} \phi \circ F^n \, dm \right| \le \|h\|_s \, |(F_0^n)'|^{-1}_{C^q(W_{j_n})} \le C n^{-\beta-1}.$$

Next, let $x, y \in [0,1]$, $|x-y| \le \varepsilon_0$, and $\phi \in C^1$. Setting $\Phi_n = |(F_0^n)'|^{-1}(\eta_j) \cdot \phi \circ F^n|_{W(\eta_j)}$,

$$\begin{aligned}\left|\int_{W(x)} R_n h\phi \, dm - \int_{W(y)} R_n h\phi \, dm\right| &\le \left|\int_{W(\xi_{jn})} h\Phi_n \, dm - \int_{W(\eta_{jn})} h\Phi_n \, dm\right| \\ &+ \left|\int_{W(\xi_j)} h\left[|(F_0^n)'(\xi_j)|^{-1} - |(F_0^n)'(\eta_j)|^{-1}\right]\phi \circ F^n \, dm\right| \\ &\le \lambda^{-1}|x-y|\left[\|h\|_u + C\|h\|_s\right]|(F_0^n)'(\xi_j)|^{-1} \le Cn^{-\beta-1}|x-y|\|h\|_{\mathcal{B}}.\end{aligned}$$

∎

6.8 Verifying (H3)

To conclude we must verify (H3). Again the strategy is to reduce to the one dimensional map F_0. Indeed, consider ψ such that $\psi(x,y) = \mathbb{E}_\mu(\psi \mid x)$, then

$$\mu(\psi \circ F_0) = \mu(\psi \circ F) = \mu(\psi)$$

this implies that the marginal of μ is the invariant measure μ_0 of the map F_0. Since φ does not depend on y, (H3) holds for F since it holds for F_0 (see, for instance, [30]).

The argument above together with the the tail expansion of $\mu_0(\varphi_0 > n)$ (associated with f_0) obtained in [35, 41] shows that the conditions on the tail behavior $\mu(\varphi > n)$ (associated with the f) stated in Lemma 5.1, (i)-(ii) are satisfied.

7 Banach spaces estimates in the non Markov setting

Let $f : [0,1]^2 \to [0,1]^2$ be the non Markov map (1.7) introduced in subsection 1.4. Let $Y_0 = Y = (1/2, 1] \times [0,1]$ and let $\varphi : Y \to \mathbb{Z}^+$ be the return time to Y. Let $F = f^\varphi$ be the first return map and write $F^n = (F_0^n, G_n)$, for all $n \in \mathbb{N}$ and $F_0 = f_0^{\varphi_0}$, where φ_0 is the first return time of f_0 to $X_0 = (1/2, 1]$. In this section we show that F satisfies (H1–H5) for some appropriate function spaces $\mathcal{B}, \mathcal{B}_w$ described below.

In what follows we use the notation introduced in Section 6 for the study of the Markov example (1.3) keeping in mind the new definition of f_0, f, F_0, F. Note that, for functions that depend only on x, the Banach space in the previous section was essentially reducing to the space of Lipsichtz functions. Here instead it will reduce to BV. This is natural, since BV is the standard Banach space on which to analyse the spectrum of the transfer operator of a piecewise expanding map.

7.1 Banach spaces

Consider the set of test functions $\mathcal{D}_q = \{\phi \in C^0([0,1]^2, \mathbb{C}) \mid \|\phi(x,\cdot)\|_{\mathcal{C}^q} \le 1, \text{ for almost all } x\}$ and $\mathcal{D}_q^0 = \mathcal{D}_q \cap \text{Lip}$.[8] With this, given $q \in (\frac{1+\beta}{2+\beta}, 1]$, for all $h \in BV$ we define the norms

$$\|h\|_{\mathcal{B}_w} = \sup_{\phi \in \mathcal{D}_{1+q}} \int_Y h \cdot \phi$$
$$\|h\|_0 = \sup_{\phi \in \mathcal{D}_q} \int_Y h \cdot \phi$$
$$\|h\|_1 = \sup_{\phi \in \mathcal{D}_{1+q}^0} \int_Y h \cdot \partial_x \phi$$

and set $\|h\|_{\mathcal{B}} = \|h\|_1 + \|h\|_0$.

Lemma 7.1 *For each $h \in BV$ we have*

$$\|h\|_{\mathcal{B}_w} \le \|h\|_{\mathcal{B}} \le \|h\|_{BV}.$$

Proof The first follows from $\|h\|_{\mathcal{B}_w} \le \|h\|_0$, which is obvious since he sup is taken on a larger set of functions. To see the second for each $\phi \in \mathcal{D}_{1+q}^0$, let $\Phi = (\phi, 0) \in C^0([0,1]^2, \mathbb{C}^2)$. Then, for each $h \in BV$,

$$\|h\|_1 = \sup_{\phi \in \mathcal{D}_{1+q}^0} \int h \partial_x \phi = \sup_{\phi \in \mathcal{D}_{1+q}^0} \int h \operatorname{div} \Phi \le \sup_{\|\Psi\|_{C^0} \le 1} \int h \operatorname{div} \Phi = \|h\|_{BV},$$

where, in the last equation, we have used the definition of the BV norm in any dimension [15]. ∎

We can then define the Banach spaces $\mathcal{B}_w$, $\mathcal{B}$ obtained, respectively, by closing BV with respect to $\|\cdot\|_{\mathcal{B}_w}$ and $\|\cdot\|_{\mathcal{B}}$. Note that such a definition (together with Lemma 7.1) implies $BV \subset \mathcal{B} \subset \mathcal{B}_w$. In fact, the next Lemma gives a more stringent embedding property.

Lemma 7.2 *The unit ball of $\mathcal{B}$ is relatively compact in $\mathcal{B}_w$.*

Proof For each $\phi \in \mathcal{D}_{1+q}^0$ define $\Psi(x,y) = \int_{1/2}^x \phi(z,y)dz$. Next, for each $\varepsilon > 0$ define $a_k = \frac{1}{2} + k\varepsilon$ and consider the piecewise linear function

$$\theta(x,y) = \int_{1/2}^{a_k} \phi(z,y)dy + \frac{x - a_k}{\varepsilon} \int_{a_k}^{a_{k+1}} \phi(z,y)dz.$$

[8] By Lip we mean the set of Lipsichtz functions.

One can easily check that $\varepsilon^{-1}(\Psi - \theta) \in \mathcal{D}_{1+q}$. Thus, for each $h \in BV$ belonging to the unit ball of $\mathcal{B}$, we have

$$\int_Y h\phi = \int_Y h\partial_x \Psi = \int_Y h\partial_x(\Psi - \theta) + \int_Y h\partial_x\theta = \int_Y h\partial_x\theta + O(\varepsilon).$$

Setting $\alpha_k(y) = \varepsilon^{-1}\int_{a_k}^{a_{k+1}} \phi(z, y)dz$ we have $\|\alpha_k\|_{\mathcal{C}^{1+q}} \le 1$ and

$$\partial_x\theta(x, y) = \sum_k \mathbb{1}_{[a_k,a_{k+1}]}(x)\alpha_k(y).$$

We can set $b_j = \varepsilon j$ and define

$$\begin{aligned}
\ell_k(y) &= \alpha_k(b_j) + \frac{\alpha_k(b_{j+1}) - \alpha_k(b_j)}{\varepsilon}(y - b_j) \qquad \text{for all } y \in [b_j, b_{j+1}] \\
\ell(y) &= \sum_k \mathbb{1}_{[a_k,a_{k+1}]}(x)\ell_k(y).
\end{aligned}$$

Note that $\varepsilon^{-1}[\partial_x\theta - \ell] \in \mathcal{D}_q$, thus

$$\int_Y h\phi = \int_Y h\ell + O(\varepsilon).$$

To conclude note that, by construction, for each ε the functions ℓ belong to a uniformly bounded set in a finite dimensional space (hence are contained in a compact set). This implies that, for each ε there exists a set of $\{\ell_i\}_{i=1}^{N_\varepsilon}$ such that

$$\|h\|_{\mathcal{B}_w} \le \sup_{i \le N_\varepsilon} \left| \int_Y h\ell_i \right| + O(\varepsilon).$$

By the standard diagonalization trick the above suffices to prove sequential compactness which implies compactness since $\mathcal{B}_w$ is a metric space. ∎

7.2 Lasota-Yorke type inequality

In the remaining of the paper, R stands for the transfer operator associated with F defined by 6.6 (with the current definition of F). With this specified we state

Lemma 7.3 *For each $h \in BV$ and $z \in \mathbb{D}$, we have*

$$\begin{aligned}
&\|R(z)h\|_{\mathcal{B}_w} \le |z|\|h\|_{\mathcal{B}_w} \\
&\|R(z)h\|_{\mathcal{B}} \le \max\{2\lambda^{-1}, \lambda^{-q}\}|z|\|h\|_{\mathcal{B}} + C|z|\|h\|_{\mathcal{B}_w}.
\end{aligned}$$

Proof For each $\phi \in \mathcal{D}_{1+q}$ and $h \in BV$ we have

$$\int_Y R(z)h \cdot \phi = \int_Y hz^{\varphi}\phi \circ F.$$

Note that, for almost all $x \in (1/2, 1]$, $\psi_x(\cdot) = z^{\varphi_0(x)}\phi \circ F(x, \cdot)$ is a C^{1+q} function by condition (1.10). Moreover, $\|\psi_x\|_{\mathcal{C}^{1+q}} \leq |z|$ by the stable contraction of F. Hence, the first inequality follows.

If we have $\phi \in \mathcal{D}_q$

$$\int_Y R(z)h \cdot \phi = \int_Y h(z^{\varphi}\phi \circ F - \theta_z) + \int_Y h\theta_z,$$

where $\theta_z(x, y) = z^{\varphi_0(x)}\phi \circ F(x, 0)$. Note that, for almost all $x \in (1/2, 1]$, $\|z^{\varphi(x,\cdot)}\phi \circ F(x, \cdot) - \theta_z(x, \cdot)\|_{\mathcal{C}^q} \leq \lambda^{-q}|z|$, while $\|\theta_z\|_{\mathcal{C}^{1+q}} \leq |z|$. Hence, $\lambda^q|z|^{-1}[z^{\varphi}\phi \circ F - \theta_z] \in \mathcal{D}_q$ and $|z|^{-1}\theta_z \in \mathcal{D}_{1+q}$. Accordingly,

$$\left|\int_Y R(z)h \cdot \phi\right| \leq \lambda^{-q}|z|\|h\|_0 + C|z|\|h\|_{\mathcal{B}_w}.$$

To conclude, let $\phi \in \mathcal{D}^0_{1+q}$. Then

$$\int_Y R(z)h \cdot \partial_x\phi = \sum_j \int_{Y_j} hz^{\varphi}(\partial_x\phi) \circ F.$$

Since $F \in C^2$ in each Y_j, we compute that

$$\partial_x \frac{\phi \circ F}{F_0'} = [\partial_x\phi)] \circ F + \frac{\partial_y\phi \circ F \cdot \partial_x G}{F_0'} + \phi \circ F \cdot \partial_x(F_0')^{-1}. \tag{7.1}$$

First of all, notice that there exists $C > 0$ such that $C^{-1}z^{\varphi}\phi \circ F\partial_x(F_0')^{-1} \in \mathcal{D}_{1+q}$. Next, let $\theta_z(x, y) = \frac{z^{\varphi_0(x)}\partial_y\phi\circ F(x,0)\cdot\partial_x G(x,0)}{F_0'(x)}$. Notice that $\|\theta_z\|_{\mathcal{C}^{1+q}} \leq |z|$ and

$$|z|^{-1}\lambda^q\left[\frac{z^{\varphi}\partial_y\phi \circ F \cdot \partial_x G}{F_0'} - \theta_z\right] \in \mathcal{D}_q.$$

Putting the above together,[9]

$$\left|\int_Y R(z)h \cdot \partial_x\phi\right| \leq \left|\sum_j \int_{\overline{Y}_j} h\partial_x\left[z^{\varphi}\frac{\phi \circ F}{F_0'}\right]\right| + \lambda^{-q}|z|\|h\|_0 + C|z|\|h\|_{\mathcal{B}_w}.$$

Since $\Psi := z^{\varphi}\frac{\phi\circ F}{F_0'}$ is discontinuous it does not belong to $\mathcal{D}^0_{1+q}$. To take care of such a problem we introduce appropriate counter terms.

[9] Note that φ is constant on each Y_j and hence can be moved inside the derivative.

Remark that each Y_j is the union of, at most finitely many, connect sets of the form $Y_{j,m} = [a_{j,m}, b_{j,m}] \times [0,1]$ for some $a_{j,m}, b_{j,m} \in [0,1]$. On each $Y_{j,m}$ we define the function

$$\ell_{j,m}(x,y) = \Psi(a_{j,m}, y) + \frac{\Psi(b_{j,m}, y) - \Psi(a_{j,m}, y)}{b_{j,m} - a_{j,m}}(x - a_{j,m}),$$

and $\ell = \sum_{j,m} \mathbb{1}_{Y_{j,m}} \ell_{j,m}$. Note that $I_p^1 = \{x \in I_p \;:\; \varphi_0(x) > 1\}$ consist of one single interval. Thus $f_0(I_p^1) \subset [0, \frac{1}{2}]$ is the union of intervals whose image will eventually cover all $(\frac{1}{2}, 1]$ apart, at most, for two intervals at the boundary of $f_0(I_p^1)$. By the usual distortion estimates this implies that there exists a constant $C > 0$ such that, for all but finitely many of the above mentioned intervals $[a_{j,m}, b_{j,m}]$, have

$$|b_{j,m} - a_{j,m}| \geq C |(F_0^j)'(a_{j,m})|^{-1}.$$

But then the same estimates, possibly with a smaller C, for all intervals.

The above considerations together with condition (1.10) imply that $\frac{\lambda}{2|z|}(\Psi - \ell) \in \mathcal{D}_{1+q}^0$ while $C^{-1}\partial_x \ell \in \mathcal{D}_{1+q}$. We can then conclude

$$\left| \int_Y R(z)h \cdot \partial_x \phi \right| \leq \max\{2\lambda^{-1}, \lambda^{-q}\}|z| \|h\|_{\mathcal{B}} + C|z| \|h\|_{\mathcal{B}_w}.$$

∎

7.3 Checking Hypoteses (H1)-(H5)

In this section we check the hypotheses needed to apply the abstract theory. As many arguments are similar to the ones in Section 6 we will go over them very quickly.

By Lemma 7.1 $C^1 \subset BV \subset \mathcal{B} \subset \mathcal{B}_w$. Moreover, if $h \in C^1$

$$\left| \int h\phi \right| \leq \|h\|_{\mathcal{B}_w} \|\phi\|_{C^{1+q}}.$$

Hence, $\mathcal{B}_w \subset (C^{1+q})'$ and (H1)(i) is satisfied with $\alpha = 1$ and $\gamma = 1 + q$.

Next, let us discuss (H2). In fact, for later convenience, we will prove a slightly stronger result. Note that the connect components of $\varphi^{-1}(Y)$ have the form $E = (a,b) \times [0,1]$ for some $(a,b) \subset (1/2, 1]$. Hence, for all $\phi \in \mathcal{D}_{1+q}^0$,

$$\begin{aligned} \int_E h\phi = \int_Y h\partial_x \int_0^x \phi \mathbb{1}_E \leq \|h\|_{\mathcal{B}} \left\| \int_0^1 \phi(t,\cdot)\mathbb{1}_E(t,\cdot)dt \right\|_{C^{1+q}} \\ \leq \|h\|_{\mathcal{B}} |b - a| = \|h\|_{\mathcal{B}} m(E). \end{aligned} \tag{7.2}$$

Hypothesis (H3) does not depend on the Banach space; it is rather an assumption on the map, and is proven as in Section 6.

Next, we look at the hypotheses involving the transfer operator. Note that, for $h \in BV$, $R(z)h$ might fail to be in BV due to possible unbounded oscillations in the vertical direction. Thus, even though the $\mathcal{B}$ norm of $R(z)h$ is bounded by Lemma 7.3, the function $R(z)h$ might fail to belong to $\mathcal{B}$, since the latter is defined as the objects that are approximated by BV function. Note however that $R_n h \in BV$ for each $n \in \mathbb{N}$.[10] Since

$$R(z)h = \sum_n z^n R_n h$$

it follows that (H4)(i) implies $R(z)(BV) \subset \mathcal{B}$, and hence (H1)(ii).

We proceed thus to prove the, stronger, (H4)(iii). Let $\phi \in \mathcal{D}_q$, then

$$\int R_n h\phi = \int h(\phi \circ F \mathbb{1}_{Y_n} - \theta) + \int h\theta,$$

where $\theta(x, y) = \phi \circ F(x, 0)\mathbb{1}_{Y_n}(x, 0)$. Note that, for almost all x, by (1.9),

$$\|\phi \circ F(x, \cdot)\mathbb{1}_{Y_n}(x, \cdot) - \theta(x, \cdot)\|_{C^q} \leq \|\mathbb{1}_{Y_n}\partial_y G\|_\infty^q \leq Cm(Y_n),$$

where we have used the limitation on the possible values of q.[11] Thus, since $Y_n = [a_n, b_n] \times [0, 1]$, arguing as in (7.2),

$$\left|\int R_n h\phi\right| \leq \|h\|_0 Cm(Y_n) + \int h\partial_x \int_0^x \theta \leq C\|h\|_{\mathcal{B}} m(Y_n).$$

This takes care of $\|R_n h\|_0$. Next, let $\phi \in \mathcal{D}^0_{1+q}$. Arguing like in the proof of Lemma 7.3 we obtain

$$\begin{aligned}\left|\int R_n h\phi\right| &= \left|\sum_m \int_{Y_{n,m}} h(\partial_x\phi) \circ F\right| \\ &= \left|\sum_m \int_{Y_{n,m}} h\left[\partial_x\left(\frac{\phi \circ F}{F_0'}\right) - \frac{\partial_y\phi \circ F \cdot \partial_x G}{F_0'} - \phi \circ F\partial_x(F_0')^{-1}\right]\right|.\end{aligned}$$

Then, again as in Lemma 7.3, we introduce θ_m linear in x, so that the functions $\mathbb{1}_{Y_{n,m}}\left[\frac{\phi \circ F}{F_0'} - \theta_m\right]$ are continuous. Remembering (1.9) and (1.10) we readily obtain

$$\left|\int R_n h \cdot \partial_x\phi\right| \leq Cm(Y_n)\|h\|_0 + Cm(Y_n)\|h\|_1$$

from which the hypothesis follows.

The proof of (H1)(iii) and (H5)(ii) goes more or less as in the Markov case (with trivial changes due to the different norm) once one remembers the topological mixing assumption of f. Lemma 7.3 proves (H5)(i).

[10] This follows since $\mathbb{1}_{Y_n}$ is a multiplier in BV and, for each smooth function $T : Y_n \to Y$, $h \in \mathrm{BV}$ implies $h \circ T \in BV$.

[11] In fact, here is the only place were such a condition is used.

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