EXPONENTIAL DECAY OF CORRELATIONS FOR PIECEWISE CONE HYPERBOLIC CONTACT FLOWS

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Abstract. We prove exponential decay of correlations for a realistic model of piecewise hyperbolic flows preserving a contact form, in dimension three. This is the first time exponential decay of correlations is proved for continuous-time dynamics with singularities on a manifold. Our proof combines the second author's version [30] of Dolgopyat's estimates for contact flows and the first author's work with Gouëzel [6] on piecewise hyperbolic discrete-time dynamics.

1. Introduction, definitions, and statement of the main theorem

1.1. **Introduction.** Many chaotic continuous-time dynamical systems posess an ergodic physical (or SRB) measure [49], the simplest case being when volume is preserved (and ergodic). When such systems are mixing, it is natural to ask at which speed decay of correlations takes place, for Hölder observables, say. Controlling the rate of decay of correlations is notouriously more difficult for continuous-time than for discrete-time dynamics: For mixing smooth Anosov (uniformly hyperbolic) flows, exponential decay of correlations was obtained only in the late nineties, in dimension three (or under a bunching assumption) in a groundbreaking work of Dolgopyat [20], while the analogous result for Anosov diffeomorphisms had been known for almost twenty years, see e.g. [49]. Dolgopyat's result implies that geodesic flows on surfaces of variable strictly negative curvature are exponentially mixing, a generalisation of the result in constant negative curvature ([34], [36]) obtained more than a dozen years before. Liverani [30] was then able to discard the bunching assumption when the flow preserved a contact form, generalising Dolgopyat's result to geodesic flows on manifolds of variable strictly negative curvature in any dimension.

Some natural chaotic systems are only piecewise smooth. The most prominent example is given by dispersive (Sinai) billiards (see [16] and references therein). Exponential decay of correlations was obtained for a discrete-time version of the billiard (and other piecewise hyperbolic maps) by Young [47] (see also the work of Chernov [13]–[14] and earlier work of Liverani [29] on piecewise hyperbolic maps). For the actual billiard flow, only a stretched exponential upper bound is known, recently proved by Chernov [15]. (Since then, Melbourne [33] has proved superpolynomial decay of correlations — a weaker result — in a more general setting.)

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Some results of exponential decay of correlations for piecewise hyperbolic flows or semi-flows do exist, but under assumptions which avoid the main difficulties, making them unfit for generalisation to realistic flows with singularities (such as the Sinai billiard): Baladi–Vallée [9] extended Dolgopyat's results to some systems with infinite Markov partitions, but although this idea could later be applied to Teichmüller flows ([2]) and Lorenz-type flows ([1]), it does not seem applicable to billiards (in the infinite Markov partition given by Young's tower [47], the relation between the metric and the initial euclidean structure is lost, making it impossible to exploit uniform non joint integrability). Stoyanov [39] obtained exponential decay for *open* billiard flows, where the discontinuities in fact do not play a role, and for Axiom A flows with C^1 laminations [40].

We believe that a new approach is needed to attack exponential decay of correlations for chaotic flows with singularities. Most proofs [1] of exponential decay of correlations for a map F, or a flow T_t, boil down to a spectral gap for a transfer operator (or a one parameter semigroup of operators). Classical approaches [47] first reduce the hyperbolic dynamics F or T_t to an expanding Markov system, and a great amount of information is lost in this procedure. We think that studying the original transfer operators

$$\mathcal{L}\psi = \frac{\psi \circ F^{-1}}{|\det DF| \circ F^{-1}}\,, \qquad \mathcal{L}_t\psi = \frac{\psi \circ T_{-t}}{|\det DT_t| \circ T_{-t}}$$

on a suitable space of (anisotropic) distributions on the manifold will be the key to obtaining exponential decay of correlations for many systems which have resisted the traditional techniques.

Appropriate anisotropic Banach spaces were first introduced by Blank, Keller, and Liverani [11] to give a new proof of exponential decay of correlations for smooth Anosov diffeomorphisms (as well as other results). This approach was developed in the next few years for smooth discrete-time hyperbolic dynamics by Baladi [3], Gouëzel-Liverani [23]–[24], and Baladi–Tsujii [7]–[8], and more recently for some smooth hyperbolic flows (Butterley-Liverani [12], Tsujii [45] [46]). Except for the Sobolev–Triebel spaces used in [3] (where a strong assumption of regularity of the dynamical foliation was required), it turns out that the Banach spaces appropriate for smooth hyperbolic dynamics are not suitable for systems with discontinuities, because multiplication by the characteristic function of a domain, however nice, is not a bounded operator for the corresponding norms. For this reason, we unfortunately cannot exploit directly Tsujii's [46] remarkable work on smooth contact hyperbolic flows, which would give much more than exponential decay. Very recently, new anisotropic spaces which satisfy this bounded multiplier property were introduced by Demers–Liverani [18] and Baladi–Gouëzel [5]–[6] to obtain in particular exponential decay of correlations for various piecewise hyperbolic maps. The approach of [5]–[6] consists in adapting the Sobolev–Triebel space results of [3] to the piecewise hyperbolic case, exploiting a key work of Strichartz [41], and using families of (noninvariant) foliations to replace the actual stable foliation, which is only measurable in general for piecewise smooth systems.

In the present paper, building on the analysis from [6], we introduce anisotropic spaces adapted to piecewise hyperbolic *continuous-time* systems. Using these spaces, we then adapt Liverani's [30] version of the Dolgopyat estimate for Anosov contact flows to obtain the first result of exponential decay of correlations for hyperbolic

[1]An important exception is given by the "coupling" methods introduced in this context by L-S.Young in [48] and greatly generalised by D.Dolgopyat in [21] giving rise to the "coupling of standard pairs" method. See Chernov and others [16] for a review on how to obtain exponential mixing for the discrete-time Sinai billiard via coupling of standard pairs. Implementing this strategy for flows, let alone the billiard flow, does not currently seem an approachable task.

systems with (true) singularities. Our result applies to various natural examples (Subsection 1.3), and we explain in Remark 1.6 below how close we are to solving the actual Sinai billiard flow problem.

After this paper was completed, we learned that Demers and Zhang [19] obtained a spectral proof of exponential decay of correlations for the discrete time Sinai billiard.

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1.2. **Definitions and the main theorem.** In this subsection, we state our main result and outline the structure of our argument (and of the paper), ending by the main formal definitions.

Let $T_t : X_0 \to X_0$ be a piecewise C^2 cone hyperbolic flow defined on a closed subset X_0 of a compact d-dimensional ($d = 2k + 1 \geq 3$) C^∞ manifold M (see Definition 1.3, and note that X_0 is the union of finitely many closed flow boxes). Let α be a C^2 contact form on M. Recall that a flow generated by a vector field V is the *Reeb flow* of α if $V \in \mathrm{Ker}(d\alpha)$ and $\alpha(V) = 1$. This implies $\frac{d}{dt}\alpha(DT_t v) = 0$ for each $v \in TM$, i.e., the flow is contact. Assume also that T_t is the Reeb flow of α. In particular, T_t preserves the volume $dx = \wedge^k d\alpha \wedge \alpha$, and $|\det DT_t| \equiv 1$. If $M = X_0$ and T_t is a hyperbolic geodesic flow, or more generally an Anosov flow preserving a contact form α, then T_t is the Reeb flow of α, up to replacing α by $\alpha(V)^{-1}\alpha$, where V is the vector field generating the flow, see [45, p. 1496 and §2]. More generally, in Appendix A we show that all ergodic piecewise smooth hyperbolic contact flows are Reeb. (For example, a billiard flow with speed one is the Reeb flow of the contact form $p\,dq$.)

Our main result is the following theorem (its proof can be found at the end of Section 3):

Theorem 1.1 (Exponential mixing for piecewise hyperbolic contact flows)**.** *Let M be a compact 3-dimensional manifold. Let T_t be a piecewise C^2 cone hyperbolic flow on a closed subset X_0 of M, which is the Reeb flow of a C^2 contact form α. Assume in addition that T_t is ergodic for dx, that complexity grows subexponentially (Definition 1.5), and that T_t satisfies the transversality condition of Definition 1.4. Then for each $\xi > 0$ there exist $K_\xi > 0$ and $\sigma_\xi > 0$, so that for any C^ξ functions $\psi_1, \psi_2 : M \to \mathbb{C}$*

$$|\int \psi_1(\psi_2 \circ T_t)\, dx - \int \psi_1\, dx \int \psi_2\, dx| \le K_\xi \|\psi_1\|_{C^\xi} \|\psi_2\|_{C^\xi} e^{-\sigma_\xi t}\,, \quad \forall t \ge 0\,.$$

Our proof is based on a spectral analysis of the linear operator $\mathcal{L}_t\psi = \psi \circ T_{-t}$, defined initially on bounded functions, e.g. (By definition, $\mathcal{L}_t^*$ preserves dx.) The strategy, following [30], is to study $\mathcal{L}_t$ as an operator on a suitable Banach space $\widetilde{\mathbf{H}}$ of anisotropic distributions, and to prove Dolgopyat-like estimates. Just like in [30], we do not claim that the transfer operator $\mathcal{L}_1$ associated to the time-one map T_1 has a spectral gap. However, the resolvent method we adapt from [30] gives us a precise description of the spectrum of the generator X of the semigroup of operators $\mathcal{L}_t$ in a half-plane large enough to deduce exponential decay of correlations (see Corollaries 3.6 and 3.10). The spaces $\widetilde{\mathbf{H}}_p^{r,s,q}$ that we shall use are a modification (see Subsection 2.2) of the spaces $\mathbf{H}_p^{r,s}$ ($s < 0 < r$ and $1 < p < \infty$) of [6] for piecewise hyperbolic maps (the spaces in [6] generalise earlier constructions in [5] and [3], more directly related to standard Triebel spaces). In particular, the norm is defined by taking a supremum over a class of admissible foliations (which are compatible with the stable cones and satisfy some regularity property). The main difference between $\mathbf{H}_p^{r,s}$ and $\widetilde{\mathbf{H}}_p^{r,s,q}$, is due to the direction of the time that must be added to the foliation class (leading to the additional regularity parameter $q \ge 0$). As a consequence the proof of the key Lemma 3.3 from [6] (invariance of the class of admissible foliations under the action of the dynamics) had to be rewritten in full detail, because a new phenomenon appears in continuous time (see Lemma C.2): We get invariance only modulo precomposition by a perturbation Δ limited to the flow direction. This can be dealt with, up to a worsening of the regularity exponents in the time direction (Lemma B.8). It follows that the "bounded" term in our Lasota-Yorke bound (Lemma 3.1) is not really bounded. The "compact" term in the Lasota–Yorke bound is not compact either, due to a loss of regularity in the flow direction of a more elementary origin (see Lemma 4.1), which also played a part in Liverani's [30] proof. Like in [30], we may overcome these problems because we work with the resolvent $\mathcal{R}(z) = \int_0^\infty e^{-zt}\mathcal{L}_t\, dt$ which involves integration along the time direction. A price needs to be paid, in the form of a power of the imaginary part of z in the estimates, see Lemma 3.4, and note that our Lasota-Yorke estimate for the resolvent is Lemma 3.8. Another difference with respect to [6] is that we need to decompose the time t, taking into account the Poincaré maps and the return times, so that the proof of the Lasota-Yorke estimate Lemma 3.1 needs to be rewritten in full (the use of the Strichartz bound Lemma B.2 in the argument is also a bit different). With respect to Liverani's argument [30] for contact Anosov flows, the key Dolgopyat estimate Lemma 6.1 (leading to Proposition 3.9) uses the same idea of "averaging in the (un)stable direction" (see the definition of $\mathbb{A}_\delta$ in Section 5). The main nontrivial difference is that, to prove Lemma 6.1, instead of the actual strong stable foliation W^s used in [30], but which is only measurable in the present setting, we work with "fake stable foliations" which lie in the stable cones and *belong to the kernel of the contact form.* This is possible because the arguments in [30, §6, App B] (in particular Lemma B.7 there) do not require the fact that W^s is the actual invariant foliation of the flow. What matters is that the contact form α vanishes along the leaves of the fake stable foliation.

This is why, although the contact assumption is not needed for smooth Anosov flows in dimension three [20] since the foliation is C^1 (by this we mean that the tangent space to the leaves vary in a C^1 manner), the contact assumption is essential in the present setting, where the foliation is only measurable. In fact, we show in Appendix D that, locally, one can effectively approximate the unstable foliation by a Lipschitz foliation, yet this alone does not suffice to apply Dolgopyat's argument. The contact form is our leverage towards the lower bounds which yield the "oscillatory integral"-type cancellations we need.

For smooth contact flows it is well-known [27, Thm 3.6] that ergodicity implies mixing, and a similar result holds for two-dimensional dispersing billiards ([16, §6.9] and references therein). Yet, we are not aware of such a general theorem for contact systems with discontinuities, even though it is probably true. In any case, we do not deduce mixing directly from ergodicity and the contact property: In our uniformly hyperbolic setting it follows from the Dolgopyat estimate, Lemma 6.1, which gives our stronger spectral/exponential mixing result. (Our proof is thus organised a bit differently from [30], where mixing was given by [27, Thm 3.6].)

We emphasize also that Lemma 6.1 does not involve the anisotropic norms: It is formulated as an upper bound on the supremum norm, with respect to the supremum and H^1_∞ norms. We are able to exploit this upper bound by using the fact that our spaces $\widetilde{\mathbf{H}}_p^{r,0,0}$ (when $s=0$ and $q=0$) are isomorphic to the ordinary Sobolev spaces H^r_p, and by using mollification operators (see Section 5), and Sobolev embeddings. Finally, note that we restrict to the three-dimensional setting in this work to simplify as much as possible the intricate estimate in Section 6. The other arguments hold in general odd dimension $d \ge 3$, and do not become shorter or simpler for $d=3$. We hope that the three-dimensional assumption limitation can be removed (bunching, however, is necessary with the present technology, as in [6], to prove invariance of admissible charts, see Appendix C).

The paper is organised as follows: Subsection 1.3 discusses a simple class of examples to which our result applies. After introducing the anisotropic Banach spaces in Section 2, we show in Section 3 how to reduce our theorem to Lasota-Yorke estimates (Lemma 3.1) and Dolgopyat estimates (Proposition 3.9, which hinges on Lemma 6.1). Lemma 3.1 is proved in Section 4. In Section 5 we study mollification operators $\mathbb{M}_\epsilon$, and stable-averaging operators $\mathbb{A}_\delta$. These operators are used to reduce to Lemma 6.1, the bound in Section 6, which is the heart of the paper. In Section 6, we follow the lines of [30, §5, §6], but we must take into account the fact that our Banach spaces are different. Section 7 contains the proof of Proposition 3.9. Finally Appendix A contains some useful facts about contact flows and changes of coordinates, Appendices B and C detail several basic results needed to construct and study our Banach spaces, and Appendices E and D contain constructions fundamental for the arguments in Section 6.

We end this subsection by defining piecewise C^2 cone hyperbolic flows and the assumptions needed for our theorem.

Definition 1.2 (Cones in $\mathbb{R}^d$)**.** A k-dimensional cone in $\mathbb{R}^d$, for an integer $1 \le k \le d-1$, is a closed subset $\mathcal{C}$ of $\mathbb{R}^d$ so that there exists a linear coordinate system $\mathbb{R}^{d-k} \times \mathbb{R}^k$ and a maximal rank linear map $\mathcal{A} : \mathbb{R}^k \to \mathbb{R}^d$ for which

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^{d-k} \times \mathbb{R}^k \mid |x| \le |\mathcal{A}y|\} . \tag{1.1}$$

In particular [2], a cone $\mathcal{C}$ has nonempty interior, it is invariant under scalar multiplication, and its dimension is the maximal dimension of a vector subspace included in $\mathcal{C}$. If $\mathcal{C}$ is a k-dimensional cone in $\mathbb{R}^d$ and $\mathcal{C}'$ is a k'-dimensional cone in $\mathbb{R}^d$ (not

[2]See [6, Def. 2.1] and the remark thereafter for a more general notion of cone and the corresponding notion of transversality.

necessarily for the same coordinate systems), we say that $\mathcal{C}$ and $\mathcal{C}'$ are transversal if $\mathcal{C} \cap \mathcal{C}' = \{0\}$. We say that $\mathcal{C}(w)$ depends continuously on w if both the coordinate system and the map $\mathcal{A}(w)$ depend continuously on w.

Note that in our main application to three dimensional flows, only one-dimensional cones are needed.

Let M be a smooth d-dimensional compact manifold, with $d = d_u + d_s + 1$, for integers $d_u \geq 1$, $d_s \geq 1$. Again, the reader only interested in the application to three-dimensional flows can focus on visualising the case $d_u = d_s = 1$. (It would not shorten the exposition to restrict to that case.)

Definition 1.3 (Piecewise C^2 cone hyperbolic flows)**.** A measurable flow $T_t : X_0 \to X_0$ is a piecewise C^2 hyperbolic flow on a closed subset X_0 of M if there exist $\epsilon_0 > 0$ and finitely many codimension-one C^2 open hypersurfaces $\{O_i, i \in I\}$ of M, uniformly transversal to the flow direction, and for each $i \in I$, there exists $J_i \subset I$ so that:

(0) For each $j \in J_i$ there exists an open subset (in the sense of hypersurfaces) $O_{i,j} \subset O_i$ so that $O_i = \cup_{j \in J_i} O_{i,j}$ (modulo a zero Lebesgue measure set), this union is disjoint, and each boundary $\partial O_{i,j}$ is a finite union of codimension-two C^1 hypersurfaces. For each $j \in J_i$ there exists a C^2 real-valued and strictly positive function $\tau_{i,j}$ defined on a neighbourhood $\widetilde{O}_{i,j}$ of $\overline{O_{i,j}}$ (as hypersurfaces), so that

$$T_t(w) \in X_0 \, , \, \forall w \in O_{i,j} \, , \forall t \in [0, \tau_{i,j}(z)) \, , \quad T_{\tau_{i,j}(w)}(w) \in O_j \, , \, \forall w \in O_{i,j} \, ,$$

and, setting,

$$B_{i,j} = \cup_{z \in O_{i,j}} \cup_{t \in [0, \tau_{i,j}(z))} T_t(z) \, ,$$

the sets $B_{i,j}$, $i \in I$, $j \in J_i$ (called "flow boxes") are two by two disjoint, and

$$X_0 = \cup_{i \in I} \cup_{j \in J_i} B_{i,j} \text{ (modulo a zero Lebesgue measure set).}$$

For $w \in B_{i,j}$, we let $z(w) \in O_{i,j}$ and $t(w) \in (0, \tau_{i,j}(z(w))$ be such that

$$w = T_{t(w)}(z(w)) \, . \tag{1.2}$$

Note that $\tau_{i,j}$ is the restriction to $O_{i,j}$ of the first return time of T_t to the section

$$M_0 := \cup_k O_k \, .$$

(1) For each $j \in J_i$ there exists a neighbourhood $\widetilde{B}_{i,j}$ of the closure of the flow box $B_{i,j}$ and a C^2 flow $T_{i,j,t}$ defined in $\widetilde{B}_{i,j}$ such that for each $w \in B_{i,j}$ and every t such that $T_t(w) \in B_{i,j}$ we have $T_t(w) = T_{i,j,t}(w)$. In addition, there exists a neighbourhood $\widehat{O}_{i,j}$ in M of the closure of $\widetilde{O}_{i,j}$ so that $T_{\tau_{i,j}(w)}(w) : O_{i,j} \to O_j$ extends to a C^2 map $\mathbb{P}_{i,j} : \widehat{O}_{i,j} \to M$, which is a diffeomorphism onto its image. The C^2 map

$$P_{i,j} := \mathbb{P}_{i,j}|_{\widetilde{O}_{i,j}}$$

restricted to $O_{i,j}$ is the first return (Poincaré) map to the section M_0.

(2) For each $j \in J_i$, there exist two continuous families $\mathcal{C}^{(u)}_{i,j}(w)$ and $\mathcal{C}^{(s)}_{i,j}(w)$ of cones on $\overline{B}_{i,j}$, where $\mathcal{C}^{(u)}_{i,j}(w) \subset T_w M$ is d_u-dimensional, $\mathcal{C}^{(s)}_{i,j}(w) \subset T_w M$ is d_s-dimensional, and, denoting the flow direction by $\text{flowdir}(w) \subset T_w M$,

$$\mathcal{C}^{(u)}_{i,j}(w) \cap \mathcal{C}^{(s)}_{i,j}(w) = \{0\} \, , \quad \mathcal{C}^{(u)}_{i,j}(w) \cap \text{flowdir}(w) = \text{flowdir}(w) \cap \mathcal{C}^{(s)}_{i,j}(w) = \{0\} \, ;$$

in addition, for any $t_{00} > 0$, there exist a smooth norm on TM and continuous functions $\lambda_{i,j,u} : \overline{B}_{i,j} \to (1, \infty)$ and $\lambda_{i,j,s} : \overline{B}_{i,j} \to (0, 1)$ such that, for each $w \in \overline{B}_{ij}$

and each $t \in (t_{00}, \tau_{i,j}(z(w)) - t(w)]$, letting (k, ℓ) be [3] such that $T_{i,j,t}(w) \in B_{k,\ell}$, we have

$$DT_{i,j,t}(w)\mathcal{C}^{(u)}_{i,j}(w) \subset \mathcal{C}^{(u)}_{k,\ell}(T_{i,j,t}(w)) \text{ and } |DT_{i,j,t}(w)v| \ge \lambda^t_{i,j,u}(w)|v|\,,$$

for all $v \in \mathcal{C}^{(u)}_{i,j}(w)$, and

$$DT^{-1}_{i,j,t}(T_{i,j,t}(w))\mathcal{C}^{(s)}_{k,\ell}(T_{i,j,t}(w)) \subset \mathcal{C}^{(s)}_{i,j}(w) \text{ and } |DT^{-1}_{i,j,t}(T_{i,j,t}(w))v| \ge \lambda^{-t}_{i,j,s}(w)|v|\,,$$

for all $v \in \mathcal{C}^{(s)}_{k,\ell}(w)$.

We must still formulate the transversality and complexity conditions. We shall do this at the level of the Poincaré maps $P_{i,j}$.

For $n \ge 1$, and $\mathbf{i} \in I^{n+1}$, we let $P^n_{\mathbf{i}} = P_{i_{n-1}i_n} \circ \cdots \circ P_{i_0 i_1}$, which is defined on a neighbourhood of $O_{\mathbf{i}} \subset M_0$, where $O_{(i_0 i_1)} = O_{i_0,i_1}$, and

$$O_{(i_0,\ldots,i_n)} = \{z \in O_{i_0,i_1} \mid P_{i_0 i_1}(z) \in O_{(i_1,\ldots,i_n)}\}\,. \tag{1.3}$$

Conditions (0)-(1)-(2) imply that for each iterate of the Poincaré map $P^n_{\mathbf{i}}$, and every $z \in O_{\mathbf{i}}$, there exist weakest contraction and expansion constants $\lambda^{(n)}_{\mathbf{i},s}(z) < 1$ and $\lambda^{(n)}_{\mathbf{i},u}(z) > 1$, and a strongest expansion constant $\Lambda^{(n)}_{\mathbf{i},u}(z) \ge \lambda^{(n)}_{\mathbf{i},u}(z)$. We put

$$\lambda_{s,n}(z) = \sup_{\mathbf{i}} \lambda^{(n)}_{\mathbf{i},s}(z) < 1\,, \quad \lambda_{u,n}(z) = \inf_{\mathbf{i}} \lambda^{(n)}_{\mathbf{i},u}(z) > 1\,.$$

We can now formulate the bunching condition on a piecewise C^2 hyperbolic flow: For some $n \ge 1$

$$\sup_{\mathbf{i}\in I^n, z\in M_0} \left(\lambda^{(n)}_{\mathbf{i},s}(z)\lambda^{(n)}_{\mathbf{i},u}(z)^{-1}\Lambda^{(n)}_{\mathbf{i},u}(z)\right) < 1\,. \tag{1.4}$$

(The bunching condition (1.4) is automatically satisfied if $d_u = 1$, which implies that $\Lambda^{(n)}_{\mathbf{i},u}(z)/\lambda^{(n)}_{\mathbf{i},u}(z)$ tends to 1 as $n \to \infty$, uniformly.) If (1.4) holds for n, there exists $\beta > 0$ so that

$$\sup_{\mathbf{i}\in I^n, z\in M_0} \left((\lambda^{(n)}_{\mathbf{i},s}(z))^{1-\beta}\lambda^{(n)}_{\mathbf{i},u}(z)^{-1}(\Lambda^{(n)}_{\mathbf{i},u}(z))^{1+\beta}\right) < 1\,. \tag{1.5}$$

(It is in fact the above condition (1.5) which appears in our argument.)

As is usual in piecewise hyperbolic settings (see e.g. [47]), we assume transversality and subexponential complexity. In view of the transversality definition, it is convenient to assume that the cone fields $\mathcal{C}^{(s)}_{i,j}$ do not depend on i and j, i.e., they are continuous throughout (see [6] for an alternative definition of transversality in the general case, and Remark 2.4 there) and we shall do so. (This allows us to use a simplified definition of the norm (2.18), and is useful also in the proof of Lemma 3.4 below: Otherwise a further argument is needed since we cannot apply Strichartz' result [41], [5] for $q = 1$, except if we have continuity at least in the time direction.)

Definition 1.4 (Transversality)**.** Let T_t be a piecewise C^2 hyperbolic flow. We say that the flow T_t satisfies the transversality condition if

- the cone fields $\mathcal{C}^{(s)}_{i,j}$ do not depend on i, j;
- each $\partial O_{i,j}$ is a finite union of C^1 hypersurfaces $K_{i,j,k}$, the image of each of which by the Poincaré map is transversal to the stable cone (i.e., for all $z \in K_{i,j,k}$, the tangent space $T_z(P_{i,j}(K_{i,j,k}))$ contains a d_u-dimensional subspace which intersects $\mathcal{C}^{(s)}_i$ only at 0).

[3] Note that either $(k, \ell) = (i, j)$, or $t = \tau_{i,j}(w)$ with $T_{i,j,t}(w) \in \overline{O_{k,\ell}}$ for $k = j$ and $\ell \in J_j$.

Definition 1.5 (Subexponential complexity)**.** Let T_t be a piecewise C^2 hyperbolic flow. For $n \geq 1$ and $\mathbf{i} = (i_0, \ldots, i_n) \in I^{n+1}$, set

$$D^b_n = \max_{z \in M_0} \operatorname{Card}\{\mathbf{i} = (i_0, \ldots, i_n) \mid z \in \overline{O_{\mathbf{i}}}\},$$

and

$$D^e_n = \max_{z \in M_0} \operatorname{Card}\{\mathbf{i} = (i_0, \ldots, i_n) \mid z \in \overline{P^n_{\mathbf{i}}(O_{\mathbf{i}})}\}.$$

We say that complexity is subexponential if

$$\limsup_{n\to\infty} \frac{1}{n} \ln(D^e_n) = 0 \text{ and } \limsup_{n\to\infty} \frac{1}{n} \ln(D^b_n) = 0. \tag{1.6}$$

1.3. **Examples.** We present a simple example to which our main theorem applies.

Given $M = \mathbb{T}^2 \times \mathbb{R}$ and $\tau \in L^\infty(M, \mathbb{R})$, we define the set $X_0 = \{(x, y, z) \in M : (x, y) \in \mathbb{T}^2, z \in [0, \tau(x, y)]\}$.

To define the dynamics we consider a piecewise C^2 hyperbolic symplectic (with respect to the symplectic form $dx \wedge dy$) map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Let $\tau : [0, 1]^2 \to \mathbb{R}_+$. Let $\{\hat{O}_i\}$ be the domains on $\mathbb{T}^2$ in which f is smooth and define $O_i = \{(x, y, z) \in M : (x, y) \in \hat{O}_i, z = 0\}$, we assume that the O_i are simply connected and that f has a C^2 extension in a neighbourhood of each $\hat{O}_i$.

We now define the flow on X_0 by

$$T_t(x, y, z) = (x, y, z + t)$$

for all $t \in [0, \tau(x, y) - z)$, while

$$T_{\tau(x,y)-z}(x, y, z) = (f(x, y), 0) = (f_1(x, y), f_2(x, y), 0).$$

The contact form is the standard one: $\alpha = dz - y dx$. In order to check that α is preserved by the flow, we must ensure that the form does not change while going through the roof. A direct computation shows that this is equivalent to requiring

$$\begin{aligned} \partial_x \tau &= y - f_2(x, y)\partial_x f_1(x, y) =: a, \\ \partial_y \tau &= -f_2(x, y)\partial_y f_1(x, y) =: b. \end{aligned}$$

By the symplecticity of f it follows that $a\, dx + b\, dy$ is a closed form, and hence τ is uniquely defined on each $\hat{O}_i$ apart from a constant. In particular, we can chose such constants as to ensure that there exists $\tau_- > 0$ such that $\inf \tau \geq \tau_-$. We have thus a piecewise smooth contact flow. The sets $O_{i,j}$ are defined in the obvious way, and $P_{i,j} = T_\tau|_{O_{i,j}}$. If the map f is uniformly hyperbolic, then one can define continuous cones $\hat{\mathcal{C}}^u_{i,j} \subset \mathbb{R}^2$ that are mapped strictly inside themselves by df and such that each vector in them is expanded at least by some $\lambda > 1$. We can then define the cones $\mathcal{C}^u_{i,j}(x, y, z) = \{(\eta, \xi, \zeta) \in \mathbb{R}^3 : (\eta, \xi) \in \mathcal{C}^u_{i,j}(x, y), \delta|\zeta| \leq \|(\eta, \xi)\|\}$. One can verify that this cone family is strictly invariant under the Poincaré maps and that the Poincaré map is hyperbolic, provided δ is chosen small enough. The transversality hypothesis is then satisfied by the flow if it is satisfied by the map f.

Next we provide an open set of examples in which this construction yields a flow that satisfies all our hypotheses, many other similar examples can be constructed. Consider the map $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$f_0(x, y) = \begin{cases} (x + y, \frac{x}{2} + \frac{3y}{2}) \mod 1 & \text{for } x \in [0, 1), y \in [0, 1 - x] \\ (x + y, \frac{x}{2} + \frac{3y}{2} - \frac{1}{2}) \mod 1 & \text{for } x \in [0, 1), y \in (1 - x, 1). \end{cases}$$

Note that any cone of the type $\mathcal{C}_a = \{(x, y) : |x - y| \leq a|x + 2y|\}$ is strictly invariant. In particular $Df_0\mathcal{C}_a \subset \mathcal{C}_{\frac{a}{4}}$.[4] To prove hyperbolicity one can first define the norm $\|(x, y, z)\| = \|(x, y)\| + \delta|\zeta|$ under which $T_t((x, y, z)$ is hyperbolic for

[4]The eigenvalues of the matrix are $2, \frac{1}{2}$, for eigenvectors $(1, 1)$ and $(1, -\frac{1}{2})$. One must write the cone is such coordinates to have standard form used in Definition 1.3.

$t \geq \tau(x, y) - z$, provided a, δ are small enough. Then one modifies the norm as to distribute the expansion and contraction in the return map evenly between $(x, y, 0)$ and $(x, y, \tau(x, y))$. The discontinuity manifold is given by $\{y = 1 - x\}$, while the discontinuity line of the inverse map is $\{x = 0\}$. Note that the discontinuity is not contained in the cone $\mathcal{C}_1$, while its image is contained in $\mathcal{C}_1$, thus, by defining the Poincaré map to be some higher power of f_0, the transversality holds. Since such properties are open, they hold also for all maps $f = f_0 \circ \phi$, where $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ is a smooth symplectic map sufficiently close to the identity. For such maps, we have thus both transversality and also subexponential complexity growth since we can apply Theorem 2 of [35].

The only property left to check is ergodicity. This follows from the ergodicity of f that can be proved by applying the Main Theorem in [32, section7].

Remark 1.6. Our original motivation was to understand billiard flows (in dimension two, i.e., the flow acts on a three-dimensional manifold), let us explain now how close we are to this goal: Sinai dispersive billiard flows are of course contact flows. It follows from well-known results (see e.g. [16]) that the ergodicity, transversality, and subexponential complexity assumptions are satisfied for the flows of two-dimensional Sinai dispersive billiards with finite horizon. Sinai billliards are piecewise cone hyperbolic (see [16], and also [32, Section 3], noting that our Poincaré map P_{ij} includes the contribution of what is called the "collision map" Γ there), except for the requirement that the flow is smooth all the way to the boundary of the domains B_{ij} (billiards flows are smooth on the open domains, but their derivatives blow up along some of the boundaries). This is a nontrivial difficulty, and we hope that the tools being developed in [4] will allow to solve it eventually. It should be remarked that some other natural examples suffer from the same "blowup of derivatives along boundaries" problem that affects discrete and continuous-time billiards, and hence do not fit in our framework: For example, consider a compact connected manifold partitioned in regions B_i with nice boundaries. Put on each region a different metric, all with strictly negative curvature, and consider the resulting geodesic flow. Generically, there will be geodesics tangent to the boundaries of the regions with non degenerate tangency. If we now consider a geodesic in B_i tangent to the boundary between B_i and B_j, then there exists an ε-close geodesic that will spend a time $\sqrt{\varepsilon}$ in B_j, and this means that the derivative of the flow at the boundary will be infinite, exactly as in the case of tangent collisions for billiards. Thus our result does not applies to examples obtained as patchwork of different geodesic flows, unless the cutting and pasting is done in the unitary tangent bundle (rather than on the manifold), where one can easily construct regions with boundaries uniformly transversal to the flow.

2. Definition of the Banach spaces $\mathbf{H}_p^{r,s,q}(R)$

Throughout the paper $C_\#$ denotes a generic constant that may vary from line to line.

Let $\beta \in (0, 1)$ satisfy (1.5). For $p \in (1, \infty)$ and real numbers r, s, and q, we shall introduce in Subsection 2.2 scales of Banach spaces $\mathbf{H}_p^{r,s,q}(R)$ of (anisotropic) distributions on M, supported in X_0, and parametrised by β, p, r, s, q, and a large zoom parameter $R > 1$, and auxiliary real parameters $C_0 > 1$, $C_1 > 2C_0$. When the meaning is clear, we write $\mathbf{H}_p^{r,s,q}(R)$, or just $\mathbf{H}_p^{r,s,q}$. Just like in [6], the spaces will depend on the stable cones and[5] on β. In Lemma 3.2, we prove

[5]See (C.1) in Appendix C for the use of β, which will also play a role later in the compact embedding Lemma 3.2.

that $\mathbf{H}_p^{r,0,0}(R)$ is isomorphic to the usual Sobolev space $H_p^r(X_0)$ on X_0 whenever $\max(-\beta, -1+1/p) < r < 1/p$.

2.1. **Anisotropic spaces $H_p^{r,s,q}$ in $\mathbb{R}^d$ and the class $\mathcal{F}(z_0, \mathcal{C}^s)$ of local foliations.** In this subsection, we recall the anisotropic spaces $H_p^{r,s,q}$ in $\mathbb{R}^d$ (generalising those used in [5] and [6]), and we define a class of cone admissible local foliations in $\mathbb{R}^d$ with uniformly bounded C^1 norms (in Lemma C.2 we show that this class is invariant under the action of the transfer operator). These are the two building blocks we shall use in Subsection 2.2 to define our spaces of anisotropic distributions.

Let $d = d_u + d_s + 1$ with $d_s \geq 1$ and $d_u \geq 1$. (Once more, the reader is welcome to concentrate on the case $d_s = d_u = 1$.) We write $x \in \mathbb{R}^d$ as $x = (x^u, x^s, x^0)$ with

$$x^u = (x_1, \ldots, x_{d_u}), \; x^s = (x_{d_u+1}, \ldots, x_{d-1}), \; x^0 = x_d .$$

The subspaces $\{x^u\} \times \mathbb{R}^{d_s} \times \{x^0\}$ of $\mathbb{R}^d$ will be referred to as the *stable leaves* in $\mathbb{R}^d$, and the lines $\{(x^u, x^s)\} \times \mathbb{R}^1$ are the *flow directions* in $\mathbb{R}^d$. We say that a diffeomorphism of $\mathbb{R}^d$ *preserves stable leaves* or *flow directions* if its derivative has this property. For $C > 0$ and $x \in \mathbb{R}^d$, let us write

$$B(x, C) = \{y \in \mathbb{R}^d \mid |y^u - x^u| \leq C, |y^s - x^s| \leq C, |y^0 - x^0| \leq C\},$$

$$B(x^u, x^s, C) = \{y \in \mathbb{R}^{d_u+d_s} \mid |y^u - x^u| \leq C, |y^s - x^s| \leq C\},$$

$$B(x^u, C) = \{y^u \in \mathbb{R}^{d_u} \mid |y^u - x^u| \leq C\}, \; B(x^s, C) = \{y^s \in \mathbb{R}^{d_s} \mid |y^s - x^s| \leq C\}.$$

We denote the Fourier transform in $\mathbb{R}^d$ by $\mathbf{F}$. An element ξ of the dual space of $\mathbb{R}^d$ will be written as $\xi = (\xi^u, \xi^s, \xi^0)$ with $\xi^u \in \mathbb{R}^{d_u}$, $\xi^s \in \mathbb{R}^{d_s}$ and $\xi^0 \in \mathbb{R}$.

The anisotropic Sobolev spaces $H_p^{r,s,q} = H_p^{r,s,q}(\mathbb{R}^d)$ belong to a class of spaces first studied by Triebel [42]:

Definition 2.1 (Sobolev spaces $H_p^{r,s,q}$ and H_p^r in $\mathbb{R}^d$)**.** For $1 < p < \infty$, r, s, and $q \in \mathbb{R}$, let $H_p^{r,s,q}$ be the set of (tempered) distributions v in $\mathbb{R}^d$ such that

$$\|v\|_{H_p^{r,s,q}} := \left\|\mathbf{F}^{-1}(a_{r,s,q}\mathbf{F}v)\right\|_{L^p} < \infty, \tag{2.1}$$

where

$$a_{r,s,q}(\xi) = (1 + |\xi^u|^2 + |\xi^s|^2 + |\xi^0|^2)^{r/2}(1 + |\xi^s|^2)^{s/2}(1 + |\xi^0|^2)^{q/2}. \tag{2.2}$$

We set $H_p^r = H_p^{r,0,0}$.

Triebel proved that rapidly decaying C^∞ functions are dense in each $H_p^{r,s,q}$ (see e.g. [5, Lemma 18]). So we could equivalently define $H_p^{r,s,q}$ to be the closure of rapidly decaying C^∞ functions for the norm (2.1). Triebel also obtained complex interpolation results which apply to the spaces $H_p^{r,s,q}$, see [5, Lemma 18]. Section 3.1 of [5] contains reminders about complex interpolation and references (such as [10] and [43]). In particular, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two Banach spaces forming a compatible couple [10, §2.3] and $0 < \theta < 1$ is real then $[\mathcal{B}_1, \mathcal{B}_2]_\theta$ denotes their complex interpolation [10, §4.1]. In Appendix B, we adapt the results in [6, Section 4] on the anisotropic spaces used there to our current setting.

We next move to the definition of the cone-admissible foliations (also called admissible charts). We shall work with local foliations indexed by points m in appropriate finite subsets of $\mathbb{R}^d$ (the sets will be introduced in Section 2.2). We view $\beta \in (0, 1]$ as fixed, satisfying (1.5), while the constants $C_0 > 1$ and $C_1 > 2C_0$ will be chosen later in Lemma C.2 (see also the quantifiers for the estimate (4.3) in the proof of the Lasota-Yorke-type bound Lemma 3.1). These constants play the following role: If C_0 is large, then the admissible foliation covers a large domain; if C_1 is large, then the leaves of the foliation are almost parallel. (We use the notation $\mathcal{F}(m, \mathcal{C}^s, \beta, C_0, C_1)$ introduced in [6], despite the fact that the spaces are slightly different in view of the additional time direction.)

Definition 2.2 (Sets $\mathcal{F}(m, \mathcal{C}^s, \beta, C_0, C_1)$ of cone-admissible foliations)**.** Let $\mathcal{C}^s$ be a d_s-dimensional cone in $\mathbb{R}^d$, transversal to $\mathbb{R}^{d_u} \times \{0\} \times \mathbb{R}$, let $\beta \in (0,1)$, let $1 < C_0 < C_1/2$, and let $m = (m^u, m^s, m^0) \in \mathbb{R}^d$. Then $\mathcal{F}(m, \mathcal{C}^s, \beta, C_0, C_1)$ is the set of maps

$$\phi = \phi_F : B(m, C_0) \to \mathbb{R}^d \,, \quad \phi_F(x^u, x^s, x^0) = (F(x^u, x^s), x^s, \widetilde{F}(x^u, x^s, x^0)) \,,$$

where $F : B(m^u, m^s, C_0) \to \mathbb{R}^{d_u}$ and $\widetilde{F} : B(m, C_0) \to \mathbb{R}$ are C^1, with $F(x^u, m^s) = x^u$, $\forall |x^u| \le C_0$, $\widetilde{F}(x^u, m^s, x^0) = x^0$, $\forall |x^0| \le C_0$, $\forall |x^u| \le C_0$,

$$(\partial_{x^s} F(x)v, v, \partial_{x^s} \widetilde{F}(x)v) \in \mathcal{C}^s, \ \forall v \in \mathbb{R}^{d_s} \,, \forall x \in B(m^u, m^s, m^0, C_0) \,,$$

and for all $(x^u, x^s, x^0), (y^u, y^s, y^0) \in B(m^u, m^s, C_0)$, on the one hand

$$|DF(x^u, x^s) - DF(x^u, y^s)| \le \frac{|x^s - y^s|}{C_1} \,, \tag{2.3}$$

$$|DF(x^u, x^s) - DF(y^u, x^s)| \le \frac{|x^u - y^u|^\beta}{C_1} \,, \tag{2.4}$$

and

$$|DF(x^u, x^s) - DF(x^u, y^s) - DF(y^u, x^s) + DF(y^u, y^s)| \le \frac{|x^s - y^s|^{1-\beta} |x^u - y^u|^\beta}{C_1} \,, \tag{2.5}$$

and on the other hand,

$$\partial_{x_0} \widetilde{F}(x^u, x^s, x^0) \equiv 1 \,, \tag{2.6}$$

where, writing $\widetilde{F}(x^u, x^s, x^0) = x^0 + \widetilde{f}(x^u, x^s)$

$$|D\widetilde{f}(x^u, x^s) - D\widetilde{f}(x^u, y^s)| \le \frac{|x^s - y^s|}{C_1} \,. \tag{2.7}$$

One easily proves that the set $\mathcal{F}(m, \mathcal{C}^s, \beta, C_0, C_1)$ is large, adapting the argument in [6], below Definition 2.8 there. We refer to [6, Remarks 2.10 and 2.11] for comments on the conditions (2.3), (2.4) and (2.5), in particular, why they are natural in view of the graph transform argument used in the proof of Lemma C.2.

The fact that F does not depend on x^0 is useful e.g. in Lemma 3.4. The smoothness condition on $\widetilde{f}$ is new with respect to [6]. Beware that we shall need to use Lemma B.8 below because of Step 2 in the proof of Lemma C.2, but that Steps 4–5 of the same proof imply that we cannot force $\widetilde{F}(x^u, x^s, x^0) = f(x^0)$ in general (which is intuitively clear: otherwise, all stable leaves would lie in planes x^0 =constant, which means that the ceiling times are cohomologous to a constant, a situation we do not allow).

The following lemma will justify our "foliation" terminology: The graphs

$$\{(F(x^u, x^s), x^s, \widetilde{F}(x^u, x^s, x^0)), x^s \in B(m^s, C_0)\} \,, \ x^u \in B(m^u, C_0) \,, \, x^0 \in B(m^0, C_0) \,,$$

form a partition of a neighbourhood of m of size of the order C_0 [6], into sets whose (d_s-dimensional) tangent space is everywhere contained in $\mathcal{C}^s$. The maps F, $\widetilde{F}$ thus define a local foliation, and the map ϕ_F is a diffeomorphism straightening this foliation, i.e., the leaves of the foliation are the images of the stable leaves of $\mathbb{R}^d$ under the map ϕ_F. (The conditions in the definition imply that the local foliation defined by F, $\widetilde{F}$ is C^{1+Lip} along the stable leaves.) Note that the image of a flow direction by ϕ_F is again a flow direction, parametrised at constant unit speed.

[6] Through the R-zoomed charts to be introduced in Section 2.2, this will correspond to a neighbourhood of size of the order C_0/R in the manifold.

Lemma 2.3 (Admissible foliations are $C^{1+\beta}$ foliations)**.** *Let $\mathcal{C}^s$ be a d_s-dimensional cone which is transversal to $\mathbb{R}^{d_u}\times\{0\}\times\mathbb{R}$. Then there exists a constant $C_\#$ depending only on $\mathcal{C}^s$ such that, for any $1<C_0<C_1/2$, and any $\phi_F\in\mathcal{F}(m,\mathcal{C}^s,\beta,C_0,C_1)$, the map ϕ_F is a diffeomorphism onto its image with*

$$\|D\phi_F\|_{C^\beta}\le C_\# \text{ and } \left\|D\phi_F^{-1}\right\|_{C^\beta}\le C_\#\,.$$

Moreover, $\phi_F(B(m,C_0))$ contains $B(m,C_\#^{-1}C_0)$.

The proof of Lemma 2.3 does not require (2.5).

Proof. Lemma 2.3 can be proved like [6, Lemma 2.9], using [6, Appendix A.1]. We just explain how to show that the C^1 norm of F is uniformly bounded (the argument for $\widetilde F$ is similar, using that $\mathcal{C}^s$ is transversal to $\{0\}\times\{0\}\times\mathbb{R}$ and (2.7)): First, $\partial_{x^s}F$ is bounded since the cone $\mathcal{C}^s$ is transversal to $\mathbb{R}^{d_u+1}\times\{0\}$. Next, $F(x^u,m^s)=x^u$, so that $\partial_{x^u}F(x^u,m^s)=\mathrm{id}$, hence

$$|\partial_{x^u}F(x^u,x^s)-\mathrm{id}|=|\partial_{x^u}F(x^u,x^s)-\partial_{x^u}F(x^u,m^s)|\le\frac{|x^s-m^s|}{C_1}\le\frac{C_0}{C_1}<1\,. \tag{2.8}$$

Finally, estimate (2.8) implies that DF is everywhere invertible, and its inverse has uniformly bounded norm. □

2.2. **Spaces of distributions.** In this subsection, we introduce appropriate C^2 coordinate patches $\kappa_{i,j,\ell}=\kappa_\zeta$ on the manifold and cones $\mathcal{C}^s_{i,j}$ in $\mathbb{R}^d$ (recall that the flow is piecewise C^2). Combining them with admissible charts in suitable families $\mathcal{F}(m,\mathcal{C}^s,\beta,C_0,C_1)$ we glue together the local spaces $H^{r,s,q}_p$ via a partition of unity, and, zooming by a large factor R, we define the space $\mathbf{H}^{r,s,q}_p(R)$ of distributions.[7]

Definition 2.4. An *extended cone* $\mathcal{C}$ is a set of four closed cones $(\mathcal{C}^s,\mathcal{C}^s_0,\mathcal{C}^u,\mathcal{C}^u_0)$ in $\mathbb{R}^d$ such that:
$\mathcal{C}^s\cap\mathcal{C}^u=\{0\}$; $\mathcal{C}^s\cap(\{0\}\times\{0\}\times\mathbb{R})=\mathcal{C}^u\cap(\{0\}\times\{0\}\times\mathbb{R})=\{0\}$.
$\mathcal{C}^s_0$ is d_s-dimensional and contains $\{0\}\times\mathbb{R}^{d_s}\times\{0\}$,
$\mathcal{C}^u_0$ is d_u-dimensional and contains $\mathbb{R}^{d_u}\times\{0\}\times\{0\}$;
$\mathcal{C}^s_0\setminus\{0\}$ is contained in the interior of $\mathcal{C}^s$, $\mathcal{C}^u_0\setminus\{0\}$ is contained in the interior of $\mathcal{C}^u$.

Given two extended cones $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, we say that an invertible matrix $\mathcal{A}:\mathbb{R}^d\to\mathbb{R}^d$ *sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly* if $\mathcal{A}\mathcal{C}^u$ is contained in $\widetilde{\mathcal{C}}^u_0$, and $\mathcal{A}^{-1}\widetilde{\mathcal{C}}^s$ is contained in $\mathcal{C}^s_0$.

For all $i\in I$ and $j\in J_i$, we fix once and for all a finite number of open sets $U_{i,j,\ell,0}$ of M, for $\ell\in N_{i,j}$, covering $\overline{B_{i,j}}$, and included in the open neighbourhood of $\overline{B_{i,j}}$ where the stable and unstable cones extend continuously (recall Definition 1.3). Let also $\kappa_{i,j,\ell}:U_{i,j,\ell,0}\to\mathbb{R}^d$, for $i\in I$, $j\in J_i$, and $\ell\in N_{i,j}$, be a finite family of C^2 charts mapping flow orbits to flow directions, i.e., for each τ, each (z,t) with $z\in\widetilde{O}_{i,j}$ and each t in a neighbourhood of $[0,\tau_{i,j}(z)-\tau]$ (given by Definition 1.3) so that $T_{i,j,\tau}(z,t)\in U_{i,j,\ell,0}$

(2.9)
$$\exists(x^u,x^s)(z)\in\mathbb{R}^{d-1}\,,\text{ s.t.}\quad \kappa_{i,j,\ell}(T_{i,j,\tau}(z,t))=(x^u(z),x^s(z),0)+(0,0,\tau+t)\,,$$

(where the function $(x^u,x^s)(\cdot)$ is C^2, recalling (2) in Definition 1.3), and so that the images of the cones $\mathcal{C}^{(s)}_{i,j}$ and $\mathcal{C}^{(u)}_{i,j}$ satisfy

(2.10)
$$\{0\}\times\mathbb{R}^{d_s}\times\{0\}\subset D\kappa_{i,j,\ell}(w)\mathcal{C}^{(s)}_{i,j}(w)\,,\quad \mathbb{R}^{d_u}\times\{0\}\times\{0\}\subset D\kappa_{i,j,\ell}(w)\mathcal{C}^{(u)}_{i,j}(w)\,,\ \forall w\,.$$

We shall take as "standard" contact form in $\mathbb{R}^d$ the following one form

$$\alpha_0=dx^0-x^s dx^u\,. \tag{2.11}$$

[7] This is a modification of the space $\mathbf{H}^{r,s}_p$ in [6].

We require in addition that each chart $\kappa_{i,j,\ell}$ is Darboux (see [45, §2]), i.e., it satisfies

$$\kappa^*_{i,j,\ell}(\alpha) = \alpha_0 \,. \tag{2.12}$$

(Condition (2.12) is used to study the averaging operator $\mathbb{A}_\delta$ in Section 5.)

Finally, for any fixed small $t_{00} > 0$, and each $i \in I$, $j \in J_i$, let $\mathcal{C}_{i,j,\ell}$ be extended cones in $\mathbb{R}^d$ such that, for every $t \geq t_{00}$ and each $x \in \mathbb{R}^d$ so that $\kappa_{i',j',\ell'} \circ T_t \circ \kappa^{-1}_{i,j,\ell}(x)$ is defined,

$$D(\kappa_{i',j',\ell'} \circ T_t \circ \kappa^{-1}_{i,j,\ell})_x \text{ sends } \mathcal{C}_{i,j,\ell} \text{ to } \mathcal{C}_{i',j',\ell'} \text{ compactly.} \tag{2.13}$$

Such charts and cones exist, as we explain now. Since the flow is hyperbolic (recall (2) in Definition 1.3) and the image of the unstable cone is included in the unstable cone, small enlargements of the unstable cones are sent strictly into themselves by the map T_t for any $t \geq t_{00}$ (and similarly for the stable cones). Since T_t is the Reeb flow of α, we can assume that the same charts satisfy (2.12) (see [45, p. 1496 and §2]), and Appendix A). Therefore, if one considers charts with small enough supports satisfying (2.9), (2.10), and (2.12), locally constant cones $\mathcal{C}^s_{i,j,\ell}$, $\mathcal{C}^u_{i,j,\ell}$ slightly larger than the cones $D\kappa_{i,j,\ell}(w)\mathcal{C}^{(s)}_{i,j}(w)$, $D\kappa_{i,j,\ell}(w)\mathcal{C}^{(u)}_{i,j}(w)$, and finally slightly smaller cones $\mathcal{C}^s_{i,j,\ell,0}, \mathcal{C}^u_{i,j,\ell,0}$, they satisfy the previous requirements. We also fix open sets $U_{i,j,\ell,1}$ covering X_0 such that $\overline{U_{i,j,\ell,1}} \subset U_{i,j,\ell,0}$, and we let $V_{i,j,\ell,k} = \kappa_{i,j,\ell}(U_{i,j,\ell,k})$, $k = 0, 1$.

The spaces of distributions will depend on a large "zoom" parameter $R \geq 1$: If $R \geq 1$ and W is a subset of $\mathbb{R}^d$, denote by W^R the set $\{R \cdot z \mid z \in W\}$. Let also $\kappa^R_{i,j,\ell}(w) = R\kappa_{i,j,\ell}(w)$, so that $\kappa^R_{i,j,\ell}(U_{i,j,\ell,k}) = V^R_{i,j,\ell,k}$. Let

$$\mathcal{Z}_{i,j,\ell}(R) = \{m \in V^R_{i,j,\ell,0} \cap \mathbb{Z}^d \mid B(m, C_0) \cap V^R_{i,j,\ell,1} \neq \emptyset\} \,, \tag{2.14}$$

and

$$\mathcal{Z}(R) = \{(i,j,\ell,m) \mid i \in I, j \in J_i, \ell \in N_{i,j}, m \in \mathcal{Z}_{i,j,\ell}(R)\} \,. \tag{2.15}$$

To $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$ is associated the point $w_\zeta := (\kappa^R_{i,j,\ell})^{-1}(m)$ of M. These are the points around which we shall construct local foliations, as follows. Let us first introduce useful notations: We write, for $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$,

$$O_\zeta = O_{i,j} \,, \quad B_\zeta = B_{i,j} \,, \quad U_{\zeta,k} = U_{i,j,\ell,k} \,, \, k = 0,1 \,, \quad \kappa^R_\zeta = \kappa^R_{i,j,\ell} \text{ and } \mathcal{C}_\zeta = \mathcal{C}_{i,j,\ell} \,. \tag{2.16}$$

These are respectively the partition set, the chart and the extended cone that we use around w_ζ. Let us fix some constants $C_0 > 1$ and $C_1 > 2C_0$. If R is large enough, say $R \geq R_0(C_0, C_1)$, then, for any $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$ and any chart $\phi_\zeta \in \mathcal{F}(m, \mathcal{C}^s_\zeta, \beta, C_0, C_1)$, we have $\phi_\zeta(B(m, C_0)) \subset V^R_{i,j,\ell,0}$. For $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$, we can therefore consider the set of charts (R, C_0 and C_1 do not appear in the notation for the sake of brevity)

$$\mathcal{F}(\zeta) := \{\Phi_\zeta = (\kappa^R_\zeta)^{-1} \circ \phi_\zeta : B(m, C_0) \to M \,, \, \phi_\zeta \in \mathcal{F}(m, \mathcal{C}^s_\zeta, \beta, C_0, C_1)\} \,. \tag{2.17}$$

The image under a chart $\Phi_\zeta \in \mathcal{F}(\zeta)$ of the stable foliation in $\mathbb{R}^d$ is a local foliation around the point w_ζ, whose tangent space is everywhere contained in $(D\kappa^R_\zeta)^{-1}(\mathcal{C}^s_\zeta)$. This set is almost contained in the stable cone $\mathcal{C}^{(s)}_i(w_\zeta)$, by our choice of charts $\kappa_{i,j,\ell}$ and extended cones $\mathcal{C}_{i,j,\ell}$.

Let us fix once and for all a C^∞ function[8] $\rho : \mathbb{R}^d \to [0,1]$ such that

$$\rho(z) = 0 \text{ if } |z| \geq d \qquad \text{and} \qquad \sum_{m \in \mathbb{Z}^d} \rho(z - m) = 1 \,.$$

For $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$, let $\rho_m(z) = \rho(z - m)$, and

$$\rho_\zeta := \rho_\zeta(R) = \rho_m \circ \kappa^R_\zeta : M \to [0,1] \,.$$

[8]Such a function exists since the balls of radius d centered at points in $\mathbb{Z}^d$ cover $\mathbb{R}^d$.

Since ρ_m is compactly supported in $\kappa^R_{i,j,\ell}(U_{i,j,\ell,0})$ if $m \in \mathcal{Z}_{i,j,\ell}(R)$ (and R is large enough, depending on d), the above expression is well-defined. This gives a partition of unity in the following sense:

$$\sum_{m\in\mathcal{Z}_{i,j,\ell}(R)} \rho_{i,j,\ell,m}(w) = 1\,, \forall w \in U_{i,j,\ell,1}\,, \quad \rho_{i,j,\ell,m}(w) = 0\,, \forall w \notin U_{i,j,\ell,0}\,.$$

Our choices ensure that the intersection multiplicity of this partition of unity is bounded, uniformly in R, i.e., for any point w, the number of functions such that $\rho_\zeta(w) \neq 0$ is bounded independently of R.

The space we shall consider depends in an essential way on the parameters p, r, s, and q. It will also depend, in an inessential way, on the choices we have made (i.e., the reference charts $\kappa_{i,j,\ell}$, the extended cones $\mathcal{C}_{i,j,\ell}$, the constants C_0 and C_1, the function ρ, and $R \geq R_0(C_0, C_1)$): Different choices would lead to different spaces, but all such spaces share the same features.

Definition 2.5 (Spaces $\mathbf{H}^{r,s,q}_p(R, C_0, C_1)$ of distributions on M)**.** Let $1 < p < \infty$, $r, s, q \in \mathbb{R}$, let $1 < C_0 < C_1/2$, and let $R \geq R_0(C_0, C_1)$. For any system of charts $\Phi = \{\Phi_\zeta \in \mathcal{F}(\zeta) \mid \zeta \in \mathcal{Z}(R)\}$, let for $\psi \in L^\infty(X_0)$

$$\|\psi\|_\Phi = \left(\sum_{\zeta\in\mathcal{Z}(R)} \|(\rho_\zeta(R)\psi)\circ\Phi_\zeta\|^p_{H^{r,s,q}_p}\right)^{1/p}\,, \tag{2.18}$$

and put $\|\psi\|_{\mathbf{H}^{r,s,q}_p(R,C_0,C_1)} = \sup_\Phi \|\psi\|_\Phi$, the supremum ranging over all such systems of charts Φ.

The space $\mathbf{H}^{r,s,q}_p(R, C_0, C_1)$ is the closure, for the norm $\|\psi\|_{\mathbf{H}^{r,s,q}_p(R,C_0,C_1)}$, of $\{\psi \in L^\infty(X_0)\ :\ \|\psi\|_{\mathbf{H}^{r,s,q}_p(R,C_0,C_1)} < \infty\}$.

Recall that our assumption from Definition 1.4 implies that the cones are continuous, this is why we replace $(\rho_\zeta(R)\cdot\mathbb{1}_{B_\zeta}\psi)\circ\Phi_\zeta$ in the definition of [6] by $(\rho_\zeta(R)\cdot\psi)\circ\Phi_\zeta$.

Remark 2.6. In general, $\mathbf{H}^{r,s,q}_p(R, C_0, C_1)$ is not isomorphic to a Triebel space $H^{r,s,q}_p(X_0)$. However, Lemma 3.2 implies that the Sobolev space $H^\sigma_p(X_0)$ is isomorphic with the Banach space $\mathbf{H}^{\sigma,0,0}_p(R, C_0, C_1)$ if $\max(-\beta, -1+1/p) < \sigma < 1/p$, and that $\mathbf{H}^{\sigma,0,0}_p(R, C_0, C_1) \subset \mathbf{H}^{r,s,q}_p(R, C_0, C_1)$ if $s \leq 0$, $q \leq 0$ and $r \leq \sigma$.

3. Reduction of the theorem to Dolgopyat-like estimates

For each $t \in \mathbb{R}$ we define an operator on $L^\infty(X_0)$ by setting

$$\mathcal{L}_t(\psi) = \psi \circ T_{-t}\,.$$

For $\beta \in (0, 1)$ satisfying (1.5), $p \in (1, \infty)$, and real numbers r, s, and q, we introduced in Subsection 2.2 a space $\mathbf{H}^{r,s,q}_p(R)$ of (anisotropic) distributions on M, supported in X_0. Since $L^\infty(X_0)$ is dense in $\mathbf{H}^{r,s,q}_p(R)$, it makes sense to talk about the extension of $\mathcal{L}_t$ to the Banach space $\mathbf{H}^{r,s,q}_p$. In Section 4, adapting the bounds in [6], we prove:

Lemma 3.1 (Lasota-Yorke type estimate)**.** *Let T_t be a piecewise C^2 hyperbolic flow satisfying transversality (Definition 1.4) and Let $\beta \in (0, 1)$ satisfy* (1.5)*. Fix $\epsilon > 0$. Then, for all large enough C_0 and C_1, for all $1 < p < \infty$, all real numbers s, s', and r, r' satisfying*

$$-1 + 1/p < s' < s < 0 \leq r' < r < 1/p\,, \quad -\beta < r + s < 0\,, \tag{3.1}$$

all $q \geq 0$ satisfying

$$(1 + q/r)(r - s) < 1 \tag{3.2}$$

and

$$1/p - 1 < s(1 + \frac{q}{r}) \leq 0 \leq r(1 + \frac{q}{r}) < 1/p\,. \tag{3.3}$$

there exist $C_\#>1$ (independent of C_1 and C_0), t_0, $\widetilde{\tau}>0$, and $A_0\geq 0$, so that for any $t\geq t_0$ there exists $R(t)$ so that for all $R\geq R(t)$

$$\|\mathcal{L}_t(\psi)\|_{\mathbf{H}_p^{r,s,q}(R)} \leq C_\# e^{A_0 t}\|\psi\|_{\mathbf{H}_p^{r,s,q}(R)}\,, \tag{3.4}$$

and

$$\begin{aligned}\|\mathcal{L}_t(\psi)\|_{\mathbf{H}_p^{r,s,q}(R)} \leq C_\#\lambda^t\|\psi\|_{\mathbf{H}_p^{r,s,q+r-s}(R)}\\ + C_\# R^{2(r-r'+s-s')}e^{A_0 t}\|\psi\|_{\mathbf{H}_p^{r',s',q+2r-r'-s}(R)}\,,\end{aligned} \tag{3.5}$$

where

$$\lambda := (1+\epsilon)^{1/\widetilde{\tau}}[(D^e_{\lceil t/\widetilde{\tau}\rceil})^{(p-1)/p}(D^b_{\lceil t/\widetilde{\tau}\rceil})^{1/p}\|\max(\lambda^{-r}_{u,\lceil t/\widetilde{\tau}\rceil},\lambda^{-(r+s)}_{s,\lceil t/\widetilde{\tau}\rceil})\|_{L^\infty}]^{1/\lceil t\rceil}\,.$$

Throughout, $C_\#$ denotes a constant (which can vary from line to line) depending on r, s, q, p, r', q', and the dynamics, possibly on C_0, but not on C_1 or R, not on the iterate t, and not on the parameter $z=a+ib$ of the resolvent $\mathcal{R}(z)$ to be introduced soon.

Let t_0 be as in Lemma 3.1, and set

$$\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)} = \sup_{t\in[0,t_0]}\|\mathcal{L}_t(\psi)\|_{\mathbf{H}_p^{r,s,q}(R)}\,. \tag{3.6}$$

We get as an easy corollary of (3.4) that there exists $A\geq A_0$ so that for all large enough C_1 and R and all $t\geq 0$

$$\|\mathcal{L}_t(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)} \leq C_\# e^{At}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)}\,. \tag{3.7}$$

(It is not clear in general that $\mathcal{L}_t$ is bounded for small $t<t_0$, for the norm $\mathbf{H}_p^{r,s,q}(R)$. It seems that it is necessary in particular to find charts so that the changes of charts preserve stable leaves.) The bounds (3.7) and (3.5) imply that if $q=2r-r'-s$ satisfies (3.2) then for every $t>0$

$$\begin{aligned}&\|\mathcal{L}_t(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)} \leq\\ &\quad C_\#^{t_0}\lambda^t\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,q+r-s}(R)} + C_\# R^{2(r-r'+s-s')}e^{At}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r',s',q+2r-r'-s}(R)}\,.\end{aligned} \tag{3.8}$$

For r,s,q,p and R as in Lemma 3.1, we define $\widetilde{\mathbf{H}}_p^{r,s,q}(R)$ to be the closure of [9]

$$\{\mathcal{L}_t(\psi)\mid \psi\in C^1(X_0)\,,\ t\geq 0\,,\ \|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)}<\infty\} \tag{3.9}$$

for the norm $\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)}$, setting also $\widetilde{\mathbf{H}}=\widetilde{\mathbf{H}}_p^{r,s,0}(R)$ and $\|\cdot\|=\|\cdot\|_{\widetilde{\mathbf{H}}}$. Note that $\widetilde{\mathbf{H}}$ is not included in $\widetilde{\mathbf{H}}_p^{r',s',2r-r'-s}(R)$, and $\widetilde{\mathbf{H}}$ is not included in $\widetilde{\mathbf{H}}_p^{r,s,r-s}(R)$ (a fortiori there is no compact embedding). Therefore, the inequality (3.5) does not give a "true" Lasota-Yorke inequality. Like in [30], we shall overcome this problem by working with the resolvent $\mathcal{R}(z)$ (there is a difference with [30] here: even the "bounded term" of our Lasota-Yorke inequality is unbounded!).

For $1<p<\infty$ and $\sigma\in\mathbb{R}$, denote by $H_p^\sigma(M)$ the standard (generalised) Sobolev space on M and set [10]

$$H_p^\sigma(X_0)=\{\psi\in H_p^\sigma(M)\mid \text{support}\,(\psi)\subset X_0\}\,,$$

---

[9] In [6], we took the closure of $L^\infty(X_0)$ in the analogous definition. The present definition is adapted in particular for (3.12).

[10] For $\max(-\beta,-1+1/p)<\sigma<1/p$, Corollary 4.2 implies that $H_p^\sigma(X_0)$ coincides with the "Whitney" definition, i.e., the restriction to X_0 of elements in $H_p^\sigma(M)$; beware however that if $\sigma\geq 1$ then, for example, $1_{X_0}\notin H_p^\sigma(X_0)$.

endowed with the $H_p^\sigma(M)$-norm. We put $L^p(X_0) = H_p^0(X_0)$, also for $p = \infty$. We have the following embedding and compact embedding properties [11]:

Lemma 3.2 (Bounded and compact embeddings)**.** *Let $1 < p < \infty$. Then for all* $\max(-\beta, -1 + 1/p) < \sigma < 1/p$*, the Banach space $H_p^\sigma(X_0)$ is isomorphic with $\mathbf{H}_p^{\sigma,0,0}(R)$ and with $\widetilde{\mathbf{H}}_p^{\sigma,0,0}(R)$.*

If $r' \le r$, $s' \le s$, and $q' \le q$ we have the following continuous inclusions

$$\widetilde{\mathbf{H}}_p^{r,s,q}(R) \subset \widetilde{\mathbf{H}}_p^{r',s',q'}(R)\,, \quad \widetilde{\mathbf{H}}_p^{r,s,q}(R) \subset \widetilde{\mathbf{H}}_p^{r-|s|-|q|,0,0}(R)\,, \tag{3.10}$$

(and similarly for $\mathbf{H}_p^{r,s,q}(R)$). If

$$\max(-\beta, r') < r - |s|\,,$$

the following inclusion is compact

$$\widetilde{\mathbf{H}}_p^{r,s,0}(R) \subset \widetilde{\mathbf{H}}_p^{r',0,0}(R)\,. \tag{3.11}$$

Proof of Lemma 3.2. We fix R, C_0, C_1. (For fixed R, the sum in (2.18) involves a uniformly bounded number of terms.)

The continuous embedding claims (3.10) follow from the definitions and properties of Triebel spaces, taking the supremum over all admissible charts (for example, since $H_p^{r,s,q}(\mathbb{R}^d)$ is included in $H_p^{r-|s|-|q|}(\mathbb{R}^d)$, it follows by taking the supremum over the admissible charts that $\mathbf{H}_p^{r,s,q}$ is included in $\mathbf{H}_p^{r-|s|-|q|,0,0}$). We thus only need to prove the compact embedding statement and the relation with Sobolev spaces.

Fix $s \le 0 \le r$, q, and $r_0 < r$, with $r_0 - |s| - |q| > -\beta$. For any admissible charts $\phi_1, \phi_2 \in \mathcal{F}(\zeta)$ for some ζ (recall (2.17)), the change of coordinates $\phi_2 \circ \phi_1^{-1}$ is C^1 and has a (uniformly) C^β Jacobian. Since $r - |s| - |q| > -\beta$, it follows from the functional analytic preliminary in [6, Lemma 4.4] (which requires Lemma B.1 for $\widetilde{\beta} = \beta$) that changing the system Φ of charts in the definition of the $\mathbf{H}_p^{\sigma,0,0}$-norm gives equivalent norms. Hence, $\mathbf{H}_p^{\sigma,0,0}$ is isomorphic to the Sobolev space $H_p^\sigma(X_0)$.

To prove that $\|\psi\|_{\widetilde{\mathbf{H}}_p^{\sigma,0,0}(R)} \le C_\#\|\psi\|_{\mathbf{H}_p^{\sigma,0,0}(R)}$, it suffices to apply (4.22) and the definition. For the converse bound, $\|\psi\|_{\mathbf{H}_p^{\sigma,0,0}(R)} \le \|\psi\|_{\widetilde{\mathbf{H}}_p^{\sigma,0,0}(R)}$ just use the definition and $\mathcal{L}_0(\psi) = \psi$.

To prove the compact embedding statement (3.11), we use $\widetilde{\mathbf{H}}_p^{r,s,0} \subset \mathbf{H}_p^{r-|s|,0,0}$ and that the inclusion $H_p^{r-|s|}(X_0) \subset H_p^{r'}(X_0)$ is compact since $r' < r - |s|$ and X_0 is compact (see e.g. [3, Lemma 2.2]). □

Clearly, $\mathcal{L}_0$ is the identity and $\mathcal{L}_{t'} \circ \mathcal{L}_t = \mathcal{L}_{t'+t}$ for all $t, t' \in \mathbb{R}_+$. We claim that for any fixed $\psi \in \widetilde{\mathbf{H}}_p^{r,s,0}$

$$\lim_{t \downarrow 0} \mathcal{L}_t(\psi) = \psi\,. \tag{3.12}$$

Indeed, by the definition, in particular (3.9), we can approach any ψ in $\widetilde{\mathbf{H}}_p^{r,s,0}$ by a sequence, $\mathcal{L}_{t_n}(\psi_n)$ with $\psi_n \in C^1$ and $t_n \ge 0$. Then we write

$$\mathcal{L}_t(\psi) - \psi = \mathcal{L}_t(\psi - \mathcal{L}_{t_n}\psi_n) - \mathcal{L}_{t_n}(\mathcal{L}_t(\psi_n) - \psi_n) + (\mathcal{L}_{t_n}\psi_n - \psi)\,.$$

Recall (3.7). The first and last term in the right-hand-side above tend to zero in $\widetilde{\mathbf{H}}$ as $n \to \infty$, uniformly in $0 \le t \le 1$. Then, using the bounded inclusion $H_p^1(X_0) \subset \widetilde{\mathbf{H}}$ from Lemma 3.2,

$$\|\mathcal{L}_{t_n}(\mathcal{L}_t(\psi_n) - \psi_n)\| \le Ce^{At_n}\|\mathcal{L}_t(\psi_n) - \psi_n\|_{H_p^1(X_0)}\,,$$

[11] The proof requires the Jacobian of the charts in Definition 2.2 to be β-Hölder.

so that it suffices to see that $\lim_{t\downarrow 0} \|\mathcal{L}_t\psi - \psi\|_{H^1_p(X_0)} = 0$ for any C^1 function ψ. Now this is easy to check, because for any fixed small t, the flow T_t is C^2 except on a set of Lebesgue measure zero. By [17, Prop 1.18] it follows that the map $(t, \psi) \mapsto \mathcal{L}_t(\psi)$ is jointly continuous on $\mathbb{R}_+ \times \widetilde{\mathbf{H}}^{r,s,0}_p$.

In particular, $\mathcal{L}_t$ acting on $\widetilde{\mathbf{H}}$ is a one parameter semigroup, and we can define its infinitesimal generator X by

$$X(\psi) = \lim_{t\downarrow 0} \frac{\mathcal{L}_t(\psi) - \psi}{t} ,$$

the domain of X being the set of ψ for which the limit exists.

Using Lemma 3.2 to get $H^\sigma_p(X_0) \subset \widetilde{\mathbf{H}}$ boundedly for any $\sigma \ge r$ (we take $\sigma < 1$), and since a piecewise C^2 flow is C^2 except on a set of Lebesgue measure 0, it is not difficult to see that $C^2(X_0)$ is included in the domain of X. A priori, X is not a bounded operator on $\widetilde{\mathbf{H}}$, but it is closed (see e.g. [17]). For z not in the spectrum of X (i.e., so that $z - X$ is invertible from the domain $D(X)$ of X to $\widetilde{\mathbf{H}}$, or, equivalently, bijective from $D(X)$ to $\widetilde{\mathbf{H}}$), we shall consider the resolvent

$$\mathcal{R}(z) = (z - X)^{-1} ,$$

which is a bounded operator. For $z \in \mathbb{C}$ with $\Re z > A$, classical results [17] imply

$$\mathcal{R}(z)(\psi) = \int_0^\infty e^{-zt}\mathcal{L}_t(\psi)\, dt . \tag{3.13}$$

Remark 3.3. We will deduce below from the ergodicity and contact assumptions that $\mathcal{L}_t$ does not have any eigenvalue of modulus strictly larger than 1 on $\widetilde{\mathbf{H}}$, and that its only eigenvalue of modulus 1 is the simple eigenvalue corresponding to the fixed point $\psi \equiv 1$. However, since we do not know if the essential spectral radius of $\mathcal{L}_t$ is strictly smaller than 1, we cannot deduce from this eigenvalue control that $A = 0$. We shall overcome this problem by working with the resolvent $\mathcal{R}(z)$.

The following lemma, together with the compact embeddings from Lemma 3.2, will allow us to deduce from Lemma 3.1 a bound on the essential spectral radius of $\mathcal{R}(z)$ (Lemma 3.5). In Section 7, Lemma 3.4 will also be used with the Dolgopyat bound to obtain the key estimate, Proposition 3.9, on the resolvent.

Lemma 3.4 ($\mathcal{R}(z)$ improves regularity in the flow direction)**.** *Fix $1 < p < \infty$, $s < 0 < r$, and $q \ge 0$ as in Lemma 3.1. Then for any $q' \ge q$ there exists $C_\# > 0$ so that for each $z = a + ib \in \mathbb{C}$ with $a > A$, $|b| \ge 1$*

$$\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,q'}_p(R)} \le C_\# \big(1 + \frac{|z|}{R}\big)^{q'-q} \big(\frac{1}{a - A} + 1\big) \|\psi\|_{\widetilde{\mathbf{H}}^{r,s,q}_p(R)} . \tag{3.14}$$

Proof. We first check that for any $q = q' \ge 0$ satisfying (3.2),

$$\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,q}_p} \le \frac{C_\#}{a - A} \|\psi\|_{\widetilde{\mathbf{H}}^{r,s,q}_p} . \tag{3.15}$$

This is a simple computation, using (3.7):

$$\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,q}_p} \le \int_0^\infty e^{-at} \|\mathcal{L}_t\psi\|_{\widetilde{\mathbf{H}}^{r,s,q}_p}\, dt \le C \int_0^\infty e^{-(a-A)t} \|\psi\|_{\widetilde{\mathbf{H}}^{r,s,q}_p}\, dt \le C \frac{\|\psi\|}{a - A} ,$$

just note that $\int_0^\infty e^{-u}\, du = 1$.

To show the claim for $q' > q$, we shall use the easily proved fact that

$$\partial_t(\mathcal{R}(z)(\psi)) \circ \mathcal{L}_t|_{t=t_1} = z\big(\mathcal{R}(z)(\mathcal{L}_{t_1}(\psi))\big) - \mathcal{L}_{t_1}(\psi) \quad \forall t_1 \in \mathbb{R}_+ . \tag{3.16}$$

The rest of the proof uses the Triebel norms and the admissible charts defined in Section 2, as well as interpolation tricks presented in Section 5, and is postponed to Subsection 5.3. □

Using Lemma 3.1, following the arguments of [30, §2], and simplifying the argument in [30, §4], we next estimate the essential spectral radius of $\mathcal{R}(z)$:

Lemma 3.5 (Essential spectral radius of $\mathcal{R}(z)$)**.** *In the setting of Lemma 3.1, assume that $-\beta < r+s$ and in addition that complexity is subexponential so that, up to choosing a larger t_0, we have $\lambda < 1$.*

Then, for each space $\widetilde{\mathbf{H}}_p^{r,s,0}(R)$ so that (3.4) and (3.8) hold for some $A_0 \le A$, and for each $z \in \mathbb{C}$ with $a = \Re z > A$, the operator $\mathcal{R}(z)$ on $\widetilde{\mathbf{H}}_p^{r,s,0}(R)$ has essential spectral radius bounded by $(a + \ln(1/\lambda))^{-1}$.

Proof. By induction, one gets from (3.13) that if $\Re z > A$ then (see [17, §2])

$$\mathcal{R}(z)^n(\psi) = \frac{1}{(n-1)!}\int_0^\infty t^{n-1} e^{-zt}\mathcal{L}_t(\psi)\,dt\,. \tag{3.17}$$

From (3.17) and the consequence (3.8) of Lemma 3.1, we obtain for all large enough C_0, C_1, and R a constant $C_\#$ so that, for all $a > A$, $n \ge 0$ and $\psi \in \widetilde{\mathbf{H}}$, writing $z = a + ib$

$$\begin{aligned} \|\mathcal{R}(z)^{n+1}(\psi)\| &\le C_\# \int_0^\infty \frac{t^{n-1}}{(n-1)!}\big(e^{-t(a+\ln(\lambda^{-1}))}(\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,r-s}} \\ &\qquad + e^{-t(a-A)}\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r',s',2r-r'-s}})\big)\,dt \\ &\le C_\#\Big(\frac{\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,r-s}}}{(a+\ln(1/\lambda))^n} + \frac{\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r',s',2r-r'-s}}}{(a-A)^n}\Big)\,. \end{aligned} \tag{3.18}$$

Lemma 3.4 applied to $s' < s$, $-\beta < r' < s + s'$ and $q = 2r - r' - s$ implies that

$$\|\mathcal{R}(a+ib)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r',s',2r-r'-s}} \le C_\#\Big(\frac{|z|}{R}+1\Big)^{2r-r'-s-s'}\Big(\frac{1}{a-A}+1\Big)\|\psi\|_{\widetilde{\mathbf{H}}_p^{r',s',0}}\,,$$

and applied to r, s and $q = r - s$ gives

$$\|\mathcal{R}(a+ib)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,r-s}} \le C_\#\Big(\frac{|z|}{R}+1\Big)^{r-s}\Big(\frac{1}{a-A}+1\Big)\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,0}}\,.$$

Since $s' \le 0$, Lemma 3.2 gives

$$\|\psi\|_{\widetilde{\mathbf{H}}_p^{r',s',0}(R)} \le C_\#\|\psi\|_{\widetilde{\mathbf{H}}_p^{r',0,0}(R)} \le C_\#\|\psi\|_{H_p^{r'}(X_0)}\,.$$

To prove the bound on the essential spectral radius, take a high enough iterate n, depending on $|z|/R$ and on r, r', s, and use that $\widetilde{\mathbf{H}}_p^{r,s,0} \subset \widetilde{\mathbf{H}}_p^{r-|s|,0,0}$ and that the inclusion $\widetilde{\mathbf{H}}_p^{r-|s|,0,0} \subset H_p^{r'}(X_0)$ is compact by (3.11) in Lemma 3.2 since $r' < s < r - |s|$ and $-\beta < r - |s| = r + s$. (We use Hennion's theorem [25].) □

It will next be easy to bound the spectral radius of $\mathcal{R}(z)$:

Lemma 3.6 (Spectral radius of $\mathcal{R}(z)$)**.** *Under the assumptions of Lemma 3.5, for each $z \in \mathbb{C}$ with $a = \Re z > A$, the operator $\mathcal{R}(z)$ on $\widetilde{\mathbf{H}}_p^{r,s,0}$ has spectral radius bounded by a^{-1}. In addition, if there exists $\psi \in \widetilde{\mathbf{H}}_p^{r,s,0}$ and $\rho \in \mathbb{C}$, $|\rho| = a^{-1}$, such that $\mathcal{R}(z)(\psi) = \rho\psi$, then $\psi \in L^\infty$. Conversely, if there exists $\psi \in L^1$ and $|\rho| = a^{-1}$, such that $\mathcal{R}(z)(\psi) = \rho\psi$,*[12] *then $\psi \in \widetilde{\mathbf{H}}_p^{r,s,0} \cap L^\infty$.*

Proof. Lemma 3.5 implies that the spectrum of $\mathcal{R}(z)$ outside of the disk $\{|\rho| \le (a + \ln(1/\lambda))^{-1}\}$, for $\Re z = a > A$, consists only of isolated eigenvalues of finite multiplicity. Let us assume that there is a unique maximal eigenvalue $\rho(z)$ (the case of finitely many maximal eigenvalues is treated in exactly the same way, apart for

[12] Remember that $\mathcal{R}(z)$ is a well-defined operator both on L^∞ and L^1, abusing notation we use the same name for the operator defined on different spaces.

the necessity of a heavier notation). If $|\rho(z)| \geq a^{-1}$ then by spectral decomposition, [26], we can write

$$\mathcal{R}(z) = \rho(z)\Pi(z) + N(z) + Q(z)\,,$$

where[13] Π, N, Q commute and $\Pi Q = NQ = 0$; Π, N are finite rank, $\Pi^2 = \Pi$, $\Pi N = N$, and, if $N \neq 0$, there exists $d(z) \in \mathbb{N}$ such that $N(z)^{d(z)+1} = 0$ while $N(z)^{d(z)} \neq 0$. In addition there exist $C(z) > 0$, $\rho_0(z) < \rho(z)$ such that $\|Q^n(z)\| \leq C(z)\rho_0(z)^n$. Note that[14]

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^{d+1}} \sum_{k=0}^{n-1} \rho^{-k}\mathcal{R}(z)^k &= \lim_{n\to\infty} \frac{1}{n^{d+1}} \sum_{k=0}^{n-1} \left[\sum_{j=0}^{d} \binom{k}{j} \rho^{-j}\Pi N^j + \mathcal{O}(\rho^{-k}\rho_0^k) \right] \\ &= C_d N^d\,,\end{aligned}$$

for some appropriate constant $C_d > 0$.

In the following, we will use two properties of $\widetilde{\mathbf{H}}_p^{r,s,0}$: There exists a set $\mathcal{D} \subset L^\infty(X_0)$ that is dense in $\widetilde{\mathbf{H}}_p^{r,s,0}$ (see (3.9) and remember that $\mathcal{L}_t$ is a contraction in L^∞); if $\psi \in \widetilde{\mathbf{H}}_p^{r,s,0}$ and $\int_M \psi\varphi\, dx = 0$ for all $\varphi \in C^2$, then $\psi = 0$ (this follows from the embedding properties stated in Lemma 3.2 and the fact that C^∞ is dense in the usual Sobolev spaces).

Let $\psi \in \mathcal{D}$, then for $k \geq 1$, using that $\mathcal{L}_t$ preserves volume,

$$|\rho^{-k}\mathcal{R}(z)^k(\psi)|_\infty \leq |\rho^{-k}| \int_0^\infty \frac{t^{k-1}e^{-at}}{(k-1)!} |\psi \circ T_{-t}|_\infty\, dt \leq |\rho^{-k}||a^{-k}||\psi|_\infty\,. \tag{3.19}$$

But then, if $|\rho(z)| > a^{-1}$, for each $\varphi \in C^2$, we have

$$\begin{aligned} C_{d(z)} \left| \int N^{d(z)}(z)\psi \cdot \varphi\, dx \right| &\leq \lim_{n\to\infty} \frac{1}{n^{d(z)+1}} \sum_{k=0}^{n-1} \left| \rho(z)^{-k} \int \mathcal{R}(z)^k(\psi) \cdot \varphi\, dx \right| \\ &\leq \lim_{n\to\infty} \frac{1}{n^{d(z)+1}} \sum_{k=0}^{n-1} |\rho(z)|^{-k} a^{-k} |\psi|_\infty |\varphi|_{L^1} = 0\,. \end{aligned} \tag{3.20}$$

Hence $N^d\psi = 0$, but then the density of $\mathcal{D}$ implies $N^d = 0$ contrary to the hypotheses. The only possibility left is that $N = 0$, but then, arguing as before, one would obtain $\Pi = 0$, also a contradiction.

Next, note that if $|\rho(z)| = a^{-1}$, then (3.20) implies again $N = 0$ (no Jordan blocks). In other words we have the spectral representation

$$\mathcal{R}(z) = \sum_k a^{-1} e^{i\theta_k(z)}\Pi_k(z) + Q(z)\,, \tag{3.21}$$

where $\theta_k \in \mathbb{R}$, $\Pi_k\Pi_j = \delta_{jk}\Pi_k$, $\Pi_k Q = Q\Pi_k = 0$ and $\|Q^n\| \leq C\rho_0^n$ for some constants $C > 0$, $\rho_0 \in (0, a^{-1})$.

Moreover, for $\theta \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} a^m e^{-im\theta}\mathcal{R}(z)^m = \begin{cases} \Pi_k(z) & \text{if } \theta = \theta_k(z) \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

Hence, for each $\psi \in \mathcal{D}$ and $\varphi \in C^\infty$, arguing as in (3.19),

$$\left| \int \Pi_k(z)\psi \cdot \varphi\, dx \right| \leq |\psi|_{L^\infty} |\varphi|_{L^1}\,. \tag{3.23}$$

This implies that $\Pi_k(\mathcal{D}) \subset L^\infty$, but since the range of Π_k is finite-dimensional, it follows that the $\Pi_k(z)$ are bounded operators from $\widetilde{\mathbf{H}}_p^{r,s,0}$ to L^∞.

---

[13]To ease notation, we will suppress the z dependence when irrelevant or no confusion can arise.

[14]The convergence is in $\widetilde{\mathbf{H}}_p^{r,s,0}$.

On the other hand, suppose that there exist $\psi(z) \in L^1 \setminus \{0\}$ and $\rho(z)$, $|\rho(z)| = a^{-1}$, such that $\mathcal{R}(z)(\psi(z)) = \rho(z)\psi(z)$, then we can consider a sequence $\{\psi_\varepsilon(z)\} \subset C^\infty$ that converges to $\psi(z)$ in L^1 and consider as before

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \rho^{-k}\mathcal{R}(z)^k(\psi_\varepsilon)\,.$$

Note that by (3.22) such a limit always exists, and it is zero if $\rho(z) \notin \sigma(\mathcal{R}(z))$, while it equals $\Pi(z)(\psi_\varepsilon(z))$ if $\rho(z) \in \sigma(\mathcal{R}(z))$, where $\Pi(z)$ is the eigenprojector associated to $\rho(z)$. In the first case, for each $\varphi \in C^2$,

$$\begin{aligned}\left|\int \psi(z)\varphi\,dx\right| &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} a^k\int_0^\infty e^{-at}\frac{t^{k-1}}{(k-1)!}|\psi_\varepsilon(z)-\psi(z)|_{L^1}\cdot|\varphi\circ T(t)|_{L^\infty}\,dt\\ &\le \lim_{\varepsilon\to 0}|\psi_\varepsilon(z)-\psi(z)|_{L^1}\cdot|\varphi|_{L^\infty} = 0\,.\end{aligned}$$

We would then have $\psi(z) = 0$, which is a contradiction. Hence $\rho(z)$ must be an eigenvalue, and by the same computation as above

$$\left|\int(\psi(z)-\Pi(z)(\psi_\varepsilon(z)))\varphi\right| \le |\psi_\varepsilon(z)-\psi(z)|_{L^1}\cdot|\varphi|_{L^\infty}\,. \tag{3.24}$$

Since $\Pi(z)$ is a projector we can write it as $\Pi(z)(\widetilde\psi) = \sum_k \psi_k(z)[\ell_k(z)](\widetilde\psi)$, where $\psi_k(z) \in \widetilde{\mathbf{H}}_p^{r,s,0} \cap L^\infty$, the $\ell_k(z)$ belong to the dual of $\widetilde{\mathbf{H}}_p^{r,s,0}$ and $\ell_k(\psi_j) = \delta_{kj}$. Then (3.24) shows that the sequences $\ell_k(\psi_\varepsilon)$ are bounded. We can then extract a subsequence $\{\psi_{\varepsilon_j}(z)\}$ such that $\Pi(z)(\psi_{\varepsilon_j}(z))$ is convergent. In turn, this shows that $\psi(z)$ is a linear combination of the $\psi_k(z)$, which concludes the proof. □

As a consequence of Lemmata 3.5 and 3.6, we get:

Corollary 3.7 (Spectrum of X)**.** *Under the assumptions of Lemma 3.5, the spectrum of X on $\widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_p^{r,s,0}(R)$ is contained in the left half-plane $\Re z \le 0$. Also, the spectrum of X on $\widetilde{\mathbf{H}}$ in the half-plane $\{z \in \mathbb{C} \mid \Re z > \ln\lambda\}$ consists of at most countably many isolated points, which are all eigenvalues of finite multiplicity. The spectrum on the imaginary axis is a finite union of discrete additive subgroup of $\mathbb{R}$. If the flow is ergodic, then the eigenvalue zero has algebraic multiplicity one.*

Proof. A nonzero $\rho \in \mathbb{C}$ lies in the spectrum of $\mathcal{R}(z)$ on $\widetilde{\mathbf{H}}$ if and only if $\rho = (z-\rho_0)^{-1}$, where ρ_0 lies in the spectrum of X as a closed operator on $\widetilde{\mathbf{H}}$ (see e.g. [17, Lemma 2.11]). The first claim then follows from Lemma 3.6. Indeed, if ρ_0 is in the spectrum of X, then for all $a \ge A$ and all b, we have $|\rho_0 - a - ib| \ge a$. The corresponding complement of union of discs is contained in the left-half plane $\Re(\rho_0) \le 0$. Similarly, the second claim follows from Lemma 3.5.

To prove the third claim note that if $X(\psi) = ib\psi$, $b \ne 0$, then, for $z = a + ib$, $\mathcal{R}(z)(\psi) = a^{-1}\psi$. Another simple computation, using [17, Theorem 1.7] (noting that $\widetilde\psi_t = e^{ibt}\psi$ satisfies $\partial_t\widetilde\psi_t|_{t=s} = X(\widetilde\psi_s)$ so that $\mathcal{L}_t(\widetilde\psi_0) = \widetilde\psi_t$) gives

$$\psi\circ T_t = e^{-ibt}\psi\,. \tag{3.25}$$

Moreover Lemma 3.6 implies that $\psi \in L^\infty$. Then if $X(\psi_k) = ib_k\psi_k$, $k \in \{1,2\}$, we have $\psi_1, \psi_2 \in L^\infty$ and

$$\begin{aligned}\mathcal{R}(z)(\psi_1\psi_2) &= \int_0^\infty e^{-zt}(\psi_1\circ T_{-t})(\psi_2\circ T_{-t})\,dt = \psi_1\psi_2\int_0^\infty e^{-zt+i(b_1+b_2)t}dt\\ &= (z-ib_1-ib_2)^{-1}\psi_1\psi_2\,.\end{aligned}$$

By Lemma 3.6 again, it follows that either $\psi_1\psi_2 = 0$ or $ib_1 + ib_2 \in \sigma(X)$. On the other hand, a similar argument applied to $\bar\psi_k$ shows that $-ib_k \in \sigma(X)$. Thus $|\bar\psi_k|^2$

belongs to the finite dimensional eigenspace of the eigenvalue zero and $\{imb_k\}_{m\in\mathbb{Z}} \subset \sigma(X)$. Finally, if A is a positive measure invariant set, then $1\!\!1_A$ is a eigenvector associate to zero and if ψ is an eigenvector associated to zero then $\{\psi \geq \lambda\}$ are invariant sets,[15] that is ψ must be piecewise constant (otherwise zero would have infinite multiplicity). In other words the eigenspace of zero is spanned by the characteristic functions of the ergodic decomposition of Lebesgue. □

In the setting of [30], it was straightforward to bound the norm of $\mathcal{R}(z)^n$ by Ca^{-n}. Here, by Lemma 3.6 we have for each $\eta > 0$ a constant $C_\eta(z)$ so that $\|\mathcal{R}(z)^n\| \leq C_\eta(z)(a-\eta)^{-n}$ for all n. This abstract nonsense bound (with no control on the z-dependence of C_η) will not suffice. In addition, we shall need in Section 7 a Lasota-Yorke type estimate for $\mathcal{R}(z)$ improving the one obtainable from (3.18) (which contains an unfortunate $(a-A)^n$ factor). This is the purpose of the following lemma:

Lemma 3.8 (Lasota-Yorke estimate for $\mathcal{R}(z)$)**.** *For $1 < p < \infty$, $s < -r < 0 < r$, and $R > 1$ as in setting of Lemma 3.1, assume in addition that complexity is subexponential so that $\lambda < 1$ (up to choosing a larger t_0), and assume that $|s| \in (0, 2r)$. Then there exists $\Lambda > 0$, depending on p, but not on r, s, and there exists $C_\#$, so that for any $N \geq 1$ such that*

$$(1+3N)r < \min\left\{\frac{1}{3}, \frac{1}{p}, 1-\frac{1}{p}\right\}, \tag{3.26}$$

then, for all $z = a + ib$, for $a > A$ (with A given by (3.7)), and all $n \geq 0$, we have

$$\|\mathcal{R}(z)^{n+1}\|_{\widetilde{\mathbf{H}}_p^{r,s,0}(R)} \leq C_\#^N \frac{(1+\frac{|z|}{R})^{N(r-s)}}{(a-\Lambda|s|-\frac{A}{N})^n}, \tag{3.27}$$

and, for every $-1+1/p < s' \leq s$,

(3.28)
$$\|\mathcal{R}(z)^{n+1}(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,0}(R)} \leq C_\#(\frac{1}{a-A}+1)\frac{(1+\frac{|z|}{R})^{N(r-s)}}{(a+\ln(1/\lambda))^n}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,0}(R)} + C_\#^N \frac{(1+\frac{|z|}{R})^{(N-1)(r-s)+r-s'}}{(a-\Lambda|s|-\frac{A}{N})^n}\|\psi\|_{\widetilde{\mathbf{H}}_p^{s',0,0}(R)}.$$

This lemma is proved in Section 4, after the proof of Lemma 3.1.

The main bound in the paper is the following Dolgopyat-like estimate (in the style of [30, Prop 2.12]), which will be proved in Section 7:

Proposition 3.9. *Under the assumptions of Lemma 3.5, and if $d = 3$, then, up to taking larger $p > 1$ and smaller $|r|$ and $|s|$, there exist $C_A \geq 10$, $b_0 \geq 1$, $0 < \widetilde{c}_1 < \widetilde{c}_2$, and $\nu \in (0,1)$, so that*

$$\|\mathcal{R}(a+ib)^n\|_{\widetilde{\mathbf{H}}_p^{r,s,0}(R)} \leq \left(\frac{1}{a+\nu}\right)^n,$$

for all $|b| > b_0$, $a \in [C_A A, b]$ and $n \in \lceil \widetilde{c}_1 a \ln|b|, \widetilde{c}_2 a \ln|b| \rceil$.

(The assumption that $d = 3$ is only used to prove Lemma 6.1 which is a key ingredient of the proof of Proposition 3.9.)

Proposition 3.9 immediately implies the following strengthening of Corollary 3.7 (just like the proof of [30, Cor. 2.13]):

[15]We can assume ψ real since, if not, then its real and imaginary part must also be invariant.

Corollary 3.10. *Under the assumptions of Lemma 3.5, if $d = 3$ and $s < 0 < r$, $1 < p < \infty$ are given by Proposition 3.9, then there exists $\delta_0 > 0$ so that the spectrum of X on $\widetilde{\mathbf{H}}_p^{r,s,0}(R)$ in the half-plane*

$$\{z \in \mathbb{C} \mid \Re(z) > -\delta_0\}$$

consists only of the eigenvalue 0. If the flow is ergodic zero is a simple eigenvalue.

Proof. By Proposition 3.9 and Corollary 3.7 the set $\{z \in \mathbb{C} : \Re(z) > -\nu, |\Im(z)| > b_0\}$ and $\{z \in \mathbb{C} : \Re(z) > 0\}$ is included in the resolvent set of X. This can be deduced using, for $a, A', b \in \mathbb{R}$,

$$\mathcal{R}(a + ib) = (1 + (a - A')\mathcal{R}(A' + ib))^{-1}\mathcal{R}(A' + ib)\,. \tag{3.29}$$

On the other hand, Corollary 3.7 implies that in the region $\{z \in \mathbb{C} : \Re(z) > -\nu, |\Im(z)| \le b_0\}$ there can be only finitely many eigenvalues. The first statement of the lemma would then follow if we could prove that zero is the only eigenvalue on the imaginary axis. Suppose this is not the case and there exists ψ such that $X(\psi) = ib\psi$, $0 \ne |b| \le b_0$. Then Corollary 3.7 implies that, for each $m \in \mathbb{Z}$, $ibm \in \sigma(X)$, but this leads to a contradiction with Proposition 3.9 by choosing $m > b_0 b^{-1}$. The last statement follows from Corollary 3.7 again. □

Our main theorem will then follow:

Proof of Theorem 1.1. Exponential decay for ξ-Hölder observables can be deduced from exponential decay for C^1 observables by a standard approximation argument (which may modify the decay rate). So it suffices to show that there exists $\sigma > 0$ and $C > 0$ so that for each ψ, φ in C^1 with $\int \psi\, dx = 0$

$$|\int \varphi \mathcal{L}_t(\psi)\, dx| \le Ce^{-\sigma t}\|\psi\|_{C^1}\|\varphi\|_{C^1}\,. \tag{3.30}$$

Indeed, for any ψ, φ in C^1, since $\mathcal{L}_t$ fixes constant functions,

$$|\int \psi(\varphi \circ T_t)\, dx - \int \psi\, dx \int \varphi\, dx| = \int \mathcal{L}_t\Big(\psi - 1 \cdot \int \psi\, dx\Big)\varphi\, dx\,.$$

To show (3.30), like in [30, Proof of Theorem 2.4], we shall apply the following easily checked fact: Let $\mathcal{B}$ be a Banach space on which $\mathcal{L}_t$ is bounded. Then for any z in the resolvent set of X (for $\mathcal{B}$) and any ψ in the domain of X^2 (for $\mathcal{B}$) we have (in $\mathcal{B}$)

$$\mathcal{R}(z)(\psi) = z^{-1}\psi + z^{-2}X(\psi) + z^{-2}\mathcal{R}(z)(X^2(\psi))\,. \tag{3.31}$$

In view of applying (3.31) to $\mathcal{B} = \widetilde{\mathbf{H}} = \widetilde{\mathbf{H}}_p^{r,s,0}$ (which will be necessary to exploit Proposition 3.9 below), we fix a C^∞ function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, supported in $(0, 1)$, with $\int \phi(u)\, du = 1$, and, for $\psi \in C^1$, we define (as in [30, Proof of Theorem 2.4])

$$\psi_\epsilon = \int_0^\infty \frac{\phi(t/\epsilon)}{\epsilon}\mathcal{L}_t(\psi)\, dt\,.$$

(Note that $\int \psi_\epsilon\, dx = 0$.) For each $m \ge 1$ the function ψ_ϵ belongs to the domain of X^m for $\widetilde{\mathbf{H}}$, and, letting $r_0 = r + |s|$ (we assume that $r_0 < 1/p$), an easy computation shows

$$\|X^m(\psi_\epsilon)\|_{H_p^{r_0}(X_0)} \le C\epsilon^{-m}\|\phi^{(m)}\|_{L^1}\|\psi\|_{C^1}\,,\ m = 0, 1, 2\,. \tag{3.32}$$

In addition

$$\|\psi_\epsilon - \psi\|_{L^\infty} \le \int_0^\infty \frac{\phi(u/\epsilon)}{\epsilon}\|\psi \circ T_{-u} - \psi\|_{L^\infty}\, du \le \epsilon\|\psi\|_{C^1}\,. \tag{3.33}$$

Next, if $\psi, \varphi \in C^1$ then

$$
\begin{aligned}
|\int \varphi \mathcal{L}_t(\psi)\, dx| &\le |\int \varphi \mathcal{L}_t(\psi_\epsilon)\, dx| + |\int \varphi \mathcal{L}_t(\psi_\epsilon - \psi)\, dx| \\
&\le |\int \varphi \mathcal{L}_t(\psi_\epsilon)\, dx| + \|\varphi\|_{L^1} \cdot \|\psi - \psi_\epsilon\|_{L^\infty}
\end{aligned} \tag{3.34}
$$

To prove (3.30) it suffices then to prove that, for each zero average $\psi \in C^1$,

$$
|\int \varphi \mathcal{L}_t(\psi)| \le C e^{-\sigma_0 t} (\|X^2(\psi)\|_{H^{r_0}_p(X_0)} + \|X(\psi)\|_{H^{r_0}_p(X_0)}) \|\varphi\|_{C^1} . \tag{3.35}
$$

Indeed, using (3.33), (3.35) and (3.32) to estimate (3.34) the result follows with $\sigma = \frac{\sigma_0}{3}$ after choosing $\varepsilon^3 = e^{-\sigma_0 t}$.

Let us prove (3.35). We claim that (3.31) implies that for each $a_0 > 0$

$$
\mathcal{L}_t(\psi) = \frac{1}{2i\pi} \lim_{w \to \infty} \int_{-w}^{w} e^{a_0 t + ibt} \mathcal{R}(a_0 + ib)\psi \, db , \, \forall t > 0 , \tag{3.36}
$$

in the $L^\infty(X_0)$ norm. Indeed, noting that the resolvent set of X for $L^\infty(X_0)$ contains the half-plane $\Re(z) > 0$, the identity (3.31) for $\mathcal{B} = L^\infty$ implies that for any z with $|z| > \zeta$

$$
\|\mathcal{R}(z)(\psi)\|_{L^\infty} \le \frac{(\|\psi\|_{L^\infty} + \zeta^{-1} \|X(\psi)\|_{L^\infty} + \zeta^{-1} \|\mathcal{R}(z)(X^2(\psi))\|_{L^\infty})}{|z|} . \tag{3.37}
$$

Hence (adapting [30, proof of (2.8), footnote 9]), for almost all $x \in X_0$ and each fixed $a > \zeta$ the function $b \mapsto (\mathcal{R}(a + ib)\psi)(x)$ is in $L^2(\mathbb{R})$. One can thus apply the inverse Laplace transform for such x and get (3.36) pointwise (that is, the limit (3.36) takes place in the $L^2([0, \infty], e^{-at}\, dt)$ sense, as a function of t). On the other hand, $t \mapsto \mathcal{L}_t(\psi) \in L^\infty(X_0)$ is continuous, and, using again (3.31), $b \mapsto \mathcal{R}(a + ib)\psi - (a + ib)^{-1}\psi$ is in $L^1(\mathbb{R}, L^\infty)$, while, clearly, $b \mapsto \frac{e^{(a+ib)t}}{a+ib}$ is in $L^1(\mathbb{R})$. Hence, the limit in (3.36) converges in the $L^\infty(X_0)$ norm for each $t \in \mathbb{R}^+$.

The inverse Laplace transform expression (3.36) will be our starting point. We shall use a change of contour to obtain an integral over a vertical in the left half-plane $\Re(z) < 0$. For this, we first study the map $z \mapsto \mathcal{R}(z)(\psi)$.

Since the simple eigenvalue zero for X corresponds to the eigenvector dx for X^*, and since $\int \psi \, dx = 0$, Corollary 3.10 implies that for suitable $s < 0 < r$, $1 < p < \infty$, $R > 1$, and $\delta_0 > 0$, the function $z \mapsto \mathcal{R}(z)(\psi)$ is analytic from the strip $\{\Re(z) > -\delta_0\}$ to $\widetilde{\mathbf{H}}^{r,s,0}_p(R)$. Pick $-\sigma_0 \in (-\delta_0, 0)$ and fix $b_0 > 1$.

We may assume that $a_0 \le 1$, $\sigma_0 \le 1$ and $A \ge 1$ (recall (3.7)). We claim that Proposition 3.9 implies that, up to taking smaller r and $|s|$, and larger b_0 (possibly depending on r and s), there exists $K_1 > 0$ so that[16] for $b_0 \le |b|$,

$$
\sup_{a \in [-\sigma_0, a_0]} \|\mathcal{R}(a + ib)\| \le K_1 \sqrt{|b|} . \tag{3.38}
$$

Let us check (3.38). Following [30, Proof of Thm 2.4] we shall use (3.29). Fix $A' \in [C_A A, b_0]$ and $a \in [-\sigma_0, a_0]$ (in particular $|a| < A'$). Setting $n = \lceil \widetilde{c}_2 A' \ln |b| \rceil$, Proposition 3.9 implies that for some $\nu > 0$

$$
\|(a - A')^n \mathcal{R}(A' + ib)^n\| \le (A' - a)^n (A' + \nu)^{-n} \le (1 + \nu/A')^{-n} .
$$

Clearly, if $|b| > b_0 > 1$ and A' is large enough, then $(1 + \nu/A')^{-\widetilde{c}_2 A' \ln |b|} < |b_0|^{-\widetilde{c}_2 \nu}$.

[16]We pick 1/2 because any exponent < 1 suffices, we could get arbitrarily small exponent > 0.

Next, if $N \geq 1$ is such that $(1+3N)r < \min(1/3, 1/p, 1-1/p)$, then (3.27) applied to $z = A' + ib$ and N gives, for all $0 \leq j < n$

$$\begin{aligned}\|(a-A')^j \mathcal{R}(A'+ib)^j\| &\leq C_\#^N \frac{(1+|z|^{N(r-s)})(A'-a)^j}{(A'-\Lambda|s| - A/N)^j} \\ &\leq C_\#^N \frac{(1+|z|^{N(r-s)})}{(1-(\Lambda|s|+A/N)/A')^j}\,.\end{aligned}$$

Up to taking smaller r, larger N, smaller $|s|$, we may assume that A' is large enough so that

$$\widetilde{c}_2 \ln\Big[\big(\frac{1}{1-((\Lambda|s|+A/N))/A'}\big)^{A'}\Big] < 2\widetilde{c}_2 \ln(e^{\Lambda|s|+A/N}) < 1/8\,.$$

Finally, we can assume that $b_0 > 2A'$ and $N(r-s) < 1/4$ so that

$$(1+|z|)^{N(r-s)} \leq C_\# |b|^{N(r-s)} \leq C_\# |b|^{1/4}\,.$$

Therefore, we can find a constant K_1 so that for any $a \in [-\sigma_0, a_0]$, $b_0 \leq |b|$, and large enough A'

$$\begin{aligned}&\|(1+(a-A')\mathcal{R}(A'+ib))^{-1}\| \\ &\quad\leq \sum_{k=0}^{\infty} \|(a-A')^{kn}\mathcal{R}(A'+ib)^{kn}\| \sum_{j=0}^{n-1} \|(a-A')^j \mathcal{R}(A'+ib)^j\| \\ &\quad\leq \frac{C_\#}{1-(1+\frac{\nu}{A'})^{-\widetilde{c}_2 A' \ln|b|}}(1+|b|^{N(r-s)})c_2 A' \ln|b| \left(\frac{1}{1-\frac{\Lambda|s|+A/N}{A'}}\right)^{\widetilde{c}_2 A' \ln|b|} \\ &\quad\leq K_1 |b|^{1/2}\,,\ \text{proving (3.38).}\end{aligned}$$

Now, since $\|X^m(\psi)\|_{\widetilde{\mathbf{H}}} \leq C_\# \|X^m(\psi)\|_{H_p^{r_0}(X_0)}$ for $m = 1, 2$, the identity (3.31) gives the following upper bound for $\|\mathcal{R}(z)\psi\|_{\widetilde{\mathbf{H}}}$:

$$\frac{\|\psi\|_{\widetilde{\mathbf{H}}}}{|z|} + C(R)\left(\frac{\|X(\psi)\|_{H_p^{r_0}(X_0)}}{|z|^2} + \frac{\|\mathcal{R}(z)\|_{\widetilde{\mathbf{H}}} \|X^2(\psi)\|_{H_p^{r_0}(X_0)}}{|z|^2}\right). \tag{3.39}$$

(The constant $C(R)$ may depend on R, but R is fixed, so we shall replace it by $C_\#$, slightly abusing notation.)

Therefore, the bound (3.38) implies that for each fixed b with $|b| \geq b_0$ the integral over the horizontal segments satisfies

$$\begin{aligned}\|\int_{-\sigma_0}^{a_0} e^{at+ibt}\mathcal{R}(a_0+ib)\psi\, da\|_{\widetilde{\mathbf{H}}} &\leq \|\psi\|_{\widetilde{\mathbf{H}}} \left|\int_{-\sigma_0}^{a_0} \frac{e^{at+ibt}}{a+ib}\, da\right| \\ &+ \frac{2C_\# |a+\sigma_0| e^{\max(\sigma_0,a)t}}{2\pi}\left(\frac{\|X(\psi)\|_{H_p^{r_0}}}{|b|^2} + K_1 \frac{\|X^2(\psi)\|_{H_p^{r_0}}}{|b|^{3/2}}\right).\end{aligned}$$

Thus, since

$$\left|\int_{-\sigma_0}^{a_0} \frac{e^{at+ibt}}{a+ib}\, da\right| \leq \frac{e^{\max(\sigma_0,a)t}}{|b|}\,,$$

and since the isomorphism $H_p^s(X_0) \sim \mathbf{H}_p^{s,0,0}(R)$ given by Lemma 3.2 imply that for any $\Psi \in \widetilde{\mathbf{H}}_p^{r,s,0}(R)$ and each $\varphi \in C^{|s|}$

$$|\int \Psi\varphi\, dx| \leq C_\# \|\Psi\|_{H_p^s(X_0)} \|\varphi\|_{H_{p'}^{|s|}(X_0)} \leq C_\# \|\Psi\|_{\widetilde{\mathbf{H}}} \|\varphi\|_{C^{|s|}}\,, \tag{3.40}$$

we get

$$\| \int_{-\sigma_0}^{a_0} e^{at+ibt}\mathcal{R}(a_0+ib)\psi \, da\|_{(C^{|s|})^*} \le C_\# \| \int_{-\sigma_0}^{a_0} e^{at+ibt}\mathcal{R}(a_0+ib)\psi \, da\|_{\widetilde{\mathbf{H}}}$$
$$\le K_2 e^{\max(\sigma_0,a)t} \frac{\sum_{m=0}^2 \|X^m(\psi)\|_{H_p^{r_0}}}{|b|} ,$$

which tends to zero for each fixed t as $|b|\to\infty$. Therefore, changing contours (the residue of the pole at $z=0$ vanishes since $\int \psi \, dx = 0$) transforms (3.36) into

$$\mathcal{L}_t(\psi) = \frac{1}{2i\pi} \lim_{w\to\infty} \int_{-w}^{w} e^{-\sigma_0 t+ibt}\mathcal{R}(-\sigma_0+ib)(\psi)\, db ,$$

where both sides above are viewed in $(C^1)^*$. Since $|\int_{-w}^{w} \frac{e^{-\sigma_0 t+ibt}}{-\sigma_0+ib}\, db| = O(|w|^{-1})$, uniformly in t, we have

$$\mathcal{L}_t(\psi) = \frac{1}{2i\pi} \lim_{w\to\infty} \int_{-w}^{w} e^{-\sigma_0 t+ibt}(\mathcal{R}(-\sigma_0+ib)(\psi) - \frac{\psi}{-\sigma_0+ib})\, db ,$$

Thus, using again (3.31), (3.38), and (3.40), we find a constant $C_\# > 0$ so that, for $\varphi\in C^1$, and arbitrary $t>0$,

$$|\int \varphi \mathcal{L}_t(\psi)\, dx| \le C_\# \frac{e^{-\sigma_0 t}\|\varphi\|_{C^1}}{2\pi} \int_{\mathbb{R}} \|\mathcal{R}(-\sigma_0+ib)(\psi) - \frac{\psi}{-\sigma_0+ib}\|_{\widetilde{\mathbf{H}}}\, db$$
$$= C_\# \frac{e^{-\sigma_0 t}\|\varphi\|_{C^1}}{2\pi} \int_{\mathbb{R}} \frac{\|\mathcal{R}(-\sigma_0+ib)(X^2(\psi)) + X(\psi)\|_{\widetilde{\mathbf{H}}}}{|-\sigma_0+ib|^2}\, db \tag{3.41}$$
$$\le C_\# \frac{\|\varphi\|_{C^1}}{2\pi e^{\sigma_0 t}} \big(\|X^2(\psi)\|_{H_p^{r_0}(X_0)} \int \frac{K_1 |b|^{1/2}}{\sigma_0+b^2}\, db + \int \frac{\|X(\psi)\|_{H_p^{r_0}(X_0)}}{\sigma_0+b^2}\, db \big) .$$

This proves equation (3.35) and ends the proof of our main theorem. □

4. The Lasota-Yorke estimates

In this section, we prove the basic Lasota-Yorke estimate Lemma 3.1 on $\mathcal{L}_t$ and the Lasota-Yorke estimate Lemma 3.8 on $\mathcal{R}(z)$. The section also includes Lemma 4.2, about multiplication by 1_{X_0}.

The following easy lemma is the heart the proof of Lemma 3.1. It is the analogue of [6, Lemma 4.6]. Note however that a new phenomenon appears in the present setting: the loss of smoothness (of $r-r'$) in the time direction

Lemma 4.1. *For all $s<-r\le 0\le r$, for all $p\in(1,\infty)$, $q\ge 0$, and every $r'<r$, $s'\le s$, there exists a constant $C_\#$, depending only on r, s, p, s', r', such that the following holds:*

Let $D = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be a block diagonal matrix, with A of dimension d_u, B of dimension d_s, and 1 a scalar. Assume that there exist $\lambda_u > 1$, $\lambda_s < 1$ such that $|Av| \ge \lambda_u |v|$ and $|Bv|\le \lambda_s|v|$. Then

$$\|w\circ D^{-1}\|_{H_p^{r,s,q}} \le C_\# |\det D|^{-1/p} \big(\max(\lambda_u^{-r}, \lambda_s^{-(r+s)})\, \|w\|_{H_p^{r,s,q}} + \|w\|_{H_p^{r',s',q+r-r'}} \big)$$
$$\|w\circ D^{-1}\|_{H_p^{r,s,q}} \le C_\# |\det D|^{-1/p}\, \|w\|_{H_p^{r,s,q}} .$$

Proof of Lemma 4.1. Write ${}^tD^{-1} = \begin{pmatrix} U & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $|U\xi^u| \le \lambda_u^{-1}|\xi^u|$ and $|S\xi^s| \ge \lambda_s^{-1}|\xi^s|$. Let

$$b(\xi^u,\xi^s,\xi^0) = a_{r,s,q}\circ {}^tD^{-1}(\xi^u,\xi^s,\xi^0)$$
$$= (1+|U\xi^u|^2 + |S\xi^s|^2 + |\xi^0|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2} .$$

Let us prove that there exist K_1 and K_1' depending only on r and s (but not on D), and K_2 depending only on r, s, $s' \le s$, $r' < r$ (but not on D) so that we have

$$b \le K_1 \max(\lambda_u^{-r}, \lambda_s^{-(r+s)}) a_{r,s,q} + K_2 a_{r',s',q+r-r'}\,, \qquad b \le K_1' a_{r,s,q}\,. \tag{4.1}$$

Assume that we can prove this bound, as well as the corresponding estimates for the successive derivatives of b. Then the Marcinkiewicz multiplier theorem applied to $b/(K_1 \max(\lambda_u^{-r}, \lambda_s^{-(r+s)}) a_{r,s,q} + K_2 a_{r',s',q+r-r'})$ as in [5, Lemma 25] gives

$$\left\|\mathbf{F}^{-1}(b\mathbf{F}v)\right\|_{L^p} \le C \left\|\mathbf{F}^{-1}((K_1 \max(\lambda_u^{-r}, \lambda_s^{-(r+s)}) a_{r,s,q} + K_2 a_{r',s',q+r-r'})\mathbf{F}v)\right\|_{L^p}\,,$$
$$\left\|\mathbf{F}^{-1}(b\mathbf{F}v)\right\|_{L^p} \le C \left\|\mathbf{F}^{-1}((K_1' a_{r,s,q})\mathbf{F}v)\right\|_{L^p}\,,$$

(recall that $\mathbf{F}$ denotes the Fourier transform) which yields the two claims of the lemma, using the arguments in the first part of the proof of [5, Lemma 25].

Let us now prove (4.1) (the proof for the derivatives of b is similar). We shall freely use the following trivial inequalities: for $x \ge 1$ and $\lambda \ge 1$,

$$\frac{1}{\lambda}(1+\lambda x) \le 1 + x \le \frac{2}{\lambda}(1+\lambda x)\,. \tag{4.2}$$

Assume first $|S\xi^s|^2 > |U\xi^u|^2$ and $|S\xi^s|^2 \ge 1$. Then, if $|S\xi^s|^2 \ge |\xi^0|^2$, we get since $r \ge 0$ and $r+s < 0$,

$$\begin{aligned}
b(\xi^u,\xi^s,\xi^0) &\le (1+3|S\xi^s|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+|S\xi^s|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+\lambda_s^{-2}|\xi^s|^2)^{(r+s)/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(\lambda_s^{-2}/2)^{(r+s)/2}(1+|\xi^s|^2)^{(r+s)/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}2^{-(r+s)/2}\lambda_s^{-(r+s)} a_{r,s,q}(\xi^u,\xi^s,\xi^0)\,,
\end{aligned}$$

and if $|S\xi^s|^2 < |\xi^0|^2$, we get since $r \ge 0$ and $s < 0$

$$\begin{aligned}
b(\xi^u,\xi^s,\xi^0) &\le (1+3|\xi^0|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+|\xi^0|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+|\xi^u|^2+|\xi^s|^2+|\xi^0|^2)^{r/2}(1+\lambda_s^{-2}|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(\lambda_s^{-2}/2)^{s/2} a_{r,s,q}(\xi^u,\xi^s,\xi^0)\\
&\le 3^{r/2}2^{-s/2}\lambda_s^{-(r+s)} a_{r,s,q}(\xi^u,\xi^s,\xi^0)\,.
\end{aligned}$$

If $|U\xi^u|^2 > \max(|S\xi^s|^2, |\xi^0|^2)$ and $|U\xi^u|^2 \ge 1$, then since $r \ge 0$ and $s < 0$,

$$\begin{aligned}
b(\xi^u,\xi^s,\xi^0) &\le (1+3|U\xi^u|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+|U\xi^u|^2)^{r/2}(1+\lambda_s^{-2}|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+\lambda_u^{-2}|\xi^u|^2)^{r/2}(1+|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(2\lambda_u^{-2})^{r/2}(1+|\xi^u|^2)^{r/2}(1+|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r}\lambda_u^{-r} a_{r,s,q}(\xi^u,\xi^s,\xi^0)\,.
\end{aligned}$$

If $|\xi^0|^2 \ge |U\xi^u|^2 \ge |S\xi^s|^2$, we get if $|S\xi^s|^2 \ge 1$, since $r \ge 0$ and $s < 0$,

$$\begin{aligned}
b(\xi^u,\xi^s,\xi^0) &\le (1+3|\xi^0|^2)^{r/2}(1+|S\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(1+|\xi^0|^2)^{r/2}(1+\lambda_s^{-2}|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(\lambda_s^{-2}/2)^{s/2}(1+|\xi^0|^2)^{r/2}(1+|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(\lambda_s^{-2}/2)^{s/2}(1+|\xi^u|^2+|\xi^s|^2+|\xi^0|^2)^{r/2}(1+|\xi^s|^2)^{s/2}(1+|\xi^0|^2)^{q/2}\\
&\le 3^{r/2}(\lambda_s^{-2}/2)^{(r+s)/2} a_{r,s,q}(\xi^u,\xi^s,\xi^0)\,,
\end{aligned}$$

and, finally, if $|S\xi^s|^2 \leq 1$, on the one hand,

$$\begin{aligned} b(\xi^u, \xi^s, \xi^0) &\leq (1 + 3|\xi^0|^2)^{r/2}(1 + |S\xi^s|^2)^{s/2}(1 + |\xi^0|^2)^{q/2} \\ &\leq C_s 3^{r/2}(1 + |\xi^0|^2)^{r/2}(1 + |\xi^s|^2)^{s/2}(1 + |\xi^0|^2)^{q/2} \\ &\leq C_s 3^{r/2}(1 + |\xi^u|^2 + |\xi^s|^2 + |\xi^0|^2)^{r/2}(1 + |\xi^s|^2)^{s/2}(1 + |\xi^0|^2)^{q/2}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} b(\xi^u, \xi^s, \xi^0) &\leq (1 + 3|\xi^0|^2)^{r/2}(1 + |S\xi^s|^2)^{s/2}(1 + |\xi^0|^2)^{q/2} \\ &\leq C_{s,s'} 3^{r/2}(1 + |\xi^0|^2)^{r/2}(1 + |\xi^s|^2)^{s'/2}(1 + |\xi^0|^2)^{q/2} \\ &\leq C_{s,s'} 3^{r/2}(1 + |\xi^0|^2)^{r'/2}(1 + |\xi^0|^2)^{(q+r-r')/2}(1 + |\xi^s|^2)^{s'/2} \\ &\leq C_{s,s'} 3^{r/2}(1 + |\xi^u|^2 + |\xi^s|^2 + |\xi^0|^2)^{r'/2}(1 + |\xi^0|^2)^{(q+r-r')/2}(1 + |\xi^s|^2)^{s'/2} \\ &\leq C_{s,s'} 3^{r/2} a_{r',s',q+r-r'}(\xi^u, \xi^s, \xi^0). \end{aligned}$$

Thus, (4.1) follows by choosing K_2 large enough depending on s and s'. □

Combining the lemma we just proved with the results in Appendix B and Appendix C, we now prove the Lasota-Yorke type estimate:

Proof of Lemma 3.1. We start with (3.5). We claim that there exist $\Lambda > 1$, $\widetilde{\tau}_0$, $\widetilde{\tau}_1$, and $C_\# > 0$ so that, for any $N \geq 1$ there exists $C_1(N)$ so that, for any $C_1 \geq C_1(N)$ there exists $t_0(C_1) \geq N$ so that, for any $t \geq t_0$ there exists $R(t)$, so that for any $R \geq R(t)$, setting $Y = 2(r - r' + s - s')$

(4.3)

$$\begin{aligned} &\|\mathcal{L}_t(\psi)\|_{\mathbf{H}_p^{r,s,q}(R,C_0,C_1)} \leq \\ &C_\# \Bigg(\sum_{n=[t/\widetilde{\tau}_0]}^{[t/\widetilde{\tau}_1]} (C_\# N^p)^{n/N} (D_n^e)^{\frac{(p-1)}{p}} (D_n^b)^{\frac{1}{p}} \max(\lambda_{u,n}^{-r}, \lambda_{s,n}^{-(r+s)}) \Bigg) \|\psi\|_{\mathbf{H}_p^{r,s,q+r-s}(R,C_0,C_1)} \\ &\quad + C_\# R^Y \Lambda^{3t} \Bigg(\sum_{n=[t/\widetilde{\tau}_0]}^{[t/\widetilde{\tau}_1]} (C_\# N^p)^{n/N} (D_n^e)^{\frac{(p-1)}{p}} (D_n^b)^{\frac{1}{p}} \Bigg) \|\psi\|_{\mathbf{H}_p^{r',s',q+2r-r'-s}(R,C_0,C_1)}. \end{aligned}$$

The bound (3.5) immediately follows from the above estimate.

All the ingredients of the proof of Lemma 5.1 in [6] are at our disposal to prove (4.3): The analogue of the iteration lemma for charts [6, Lemma 3.3] is Lemma C.2 in Appendix C. Our Lemma 4.1 plays the part of [6, Lemma 4.6] on composition with hyperbolic matrices. The analogues of Lemmata 4.1 (Leibniz formula), 4.2 (multiplication with a characteristic function), 4.3 (localisation), 4.5 (partition of unity), and 4.7 (composition with a C^1 diffeomorphism which is C^{1+Lip} along stable leaves) from [6] are Lemmata B.1, B.2, B.3, B.4, and B.6 in Appendix B below. Note however that Lemma C.2 does not have such a nice form as [6, Lemma 3.3], and this will force us in Step 2 of the proof below to invoke a result specific to our continuous-time setting, Lemma B.8. This point, together with the comparison of the "continuous-time" and "discrete-time" complexities (via the set $\mathcal{N}(t)$) are the main differences between the present proof and that of [6, Lemma 5.1].

Before giving a detailed account of the proof of (4.3), let us describe the order in which we choose the constants. First, N is fixed in the statement (it will be used in the second step of the proof in order to apply Lemma B.2). Then, we choose C_1 very large, depending on N, also in the second step below, so that the admissible charts ϕ_ζ are close enough to linear maps. Then, in Step 3 we fix t very large depending on C_1 (large enough so that every branch of T_t is hyperbolic enough so that Lemma C.2 applies). Finally, we choose R very large so that, at scale $1/R$, all the iterates T_s up to time t look like linear maps, and all the boundaries of the sets we are interested in look like hyperplanes. For the presentation of the argument, we will start the proof with some values of C_1, t, R, and increase them whenever necessary, checking each time that C_1 does not depend on t, R, and that t does not depend on R, to avoid bootstrapping issues. We will denote by $C_\#$ a constant which could depend on r, s, p, C_0 but does not depend on N, C_1, t, R, and may vary from line to line. We shall use that for any $s' < s$

$$\|\cdot\|_{H^{r'+s'-s,s,q}} \le C_\# \|\cdot\|_{H^{r',s',q}} \,. \tag{4.4}$$

For every $\mathbf{i} \in I^{n+1}$, we fix a small neighbourhood $\widetilde{O}_{\mathbf{i}}$ of $\overline{O_{\mathbf{i}}}$ (as a hypersurface) such that $P^n_{\mathbf{i}}$ admits an extension to $\widetilde{O}_{\mathbf{i}}$ with the same hyperbolicity properties as the original $P^n_{\mathbf{i}}$. Reducing these sets if necessary, we can ensure that their intersection multiplicity is bounded by D^b_n, and that the intersection multiplicity of the sets $P^n_{\mathbf{i}}(\widetilde{O_{\mathbf{i}}})$ is bounded by D^e_n.

Our assumptions (in particular the fact that the return times are bounded from above and from below) imply that there exist $\widetilde{\tau}_0 > 0$ and $\widetilde{\tau}_1 > 0$, with the following properties: For each flow box B_{ij}, for every $w \in B_{ij}$, we let $z(w) \in O_{i,j}$ and $t(w) \in (0, \tau_{i,j}(z(w)))$ be as in (1.2). Then for every large enough $t > 0$, there exist uniquely defined

$$\begin{aligned}
&n = n(t,w) \ge 1 \,,\ \mathbf{i} = \mathbf{i}(t,w) = \mathbf{i}(n,w) \in I^{n+1} \,,\ i_0 = i \,,\ i_1 = j \,,\\
&0 \le t_{n+1}(w) \le \tau_{i_{n-1}i_n}(P^n_{i_1\dots i_n}(z(w))) \,,
\end{aligned}$$

so that, setting $t_0(w) = \tau_{ij}(z(w)) - t(w)$

$$t = t_0(w) + t_{n+1}(w) + \sum_{\ell=1}^{n-1} \tau_{i_\ell i_{\ell+1}}(P^\ell_{i_1\dots i_\ell}(z(w))) \,, \tag{4.5}$$

with $n \in [t/\widetilde{\tau}_0 - \Lambda^{-1}, t/\widetilde{\tau}_1 - \Lambda^{-1}]$ for some constant Λ depending only on the dynamics.

Fix $t > 0$ large. Let $\mathcal{N}(t)$ be the finite set of possible values of $n(t,w)$, when i and j range in I, as w ranges over B_{ij}. We define for $n \in \mathcal{N}(t)$ the set

$$\mathcal{I}(n,t) = \{\mathbf{i} \in I^{n+1} \mid \mathbf{i} = (i_0, \dots, i_n) \text{ appears in a decomposition (4.5) for } t\} \,,$$

and we put $\mathcal{I}(t) = \cup_{m \in \mathcal{N}(t)} \mathcal{I}(m,t)$. Introduce for $n \in \mathcal{N}(t)$ and $\mathbf{i} = (i_0, \dots, i_{n+1}) \in \mathcal{I}(n,t)$ the refined flow boxes

$$B_{\mathbf{i},t} = \{w \in B_{i_0 i_1} \mid n = n(t,w) \,, \mathbf{i}(t,w) = \mathbf{i}\} \,.$$

Note that if $w \in B_{\mathbf{i},t}$ then $z(w) \in O_{\mathbf{i}}$, while $t(w)$ lies between the graphs of two piecewise C^2 functions of $z(w)$, which coincide either with 0 or the ceiling time, or are the images by the flow of the transversals $O_{i_0 i_1}$ or $O_{i_1 i_2}$. We let $T_{\mathbf{i},t}$ be the restriction of T_t to $B_{\mathbf{i},t}$. By definition, $T_{\mathbf{i},t}$ admits a C^2 extension to a neighbourhood $\widetilde{B}_{\mathbf{i},t}$ of $B_{\mathbf{i},t}$.

For $\zeta = (i,j,\ell,m) \in \mathcal{Z}(R)$, let us write

$$A(\zeta) = A(\zeta, R) = (\kappa^R_\zeta)^{-1}(B(m,d)) \subset M \,.$$

The set $A(\zeta)$ is a neighbourhood of w_ζ, of diameter bounded by $C_\# R^{-1}$, and containing the support of ρ_ζ.

Let us fix some system of charts Φ as in the Definition 2.5 of the $\mathbf{H}^{r,s,q}_p(R)$-norm. We want to estimate $\|\mathcal{L}_t \psi\|_\Phi$.

First step: Complexity at the end. For any $n \in \mathcal{N}(t)$ the closures of the sets $\{T_{\mathbf{i},t}(B_{\mathbf{i},t}) \mid \mathbf{i} \in \mathcal{I}(n,t)\}$, or (up to taking a smaller neighbourhood) $\{T_{\mathbf{i},t}(\widetilde{B}_{\mathbf{i},t}) \mid \mathbf{i} \in$

$\mathcal{I}(n,t)\}$ have intersection multiplicity at most D^e_n (the partition cannot be refined along the time direction). Therefore, writing[17]

$$\mathcal{L}_t(\psi) = \sum_{n\in\mathcal{N}(t)} \sum_{\mathbf{i}\in\mathcal{I}(n,t)} \mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}(\psi)\circ T_{\mathbf{i},-t}\,,$$

we get by Lemma B.4 that for each $\zeta\in\mathcal{Z}(R)$

$$\begin{aligned}\|(\rho_\zeta\mathcal{L}_t(\psi))\circ\Phi_\zeta\|^p_{H^{r,s,q}_p} &\le C_\# \sum_{n\in\mathcal{N}(t)} (D^e_n)^{p-1} \sum_{\mathbf{i}\in\mathcal{I}(n,t)} \left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}\psi\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r,s,q}_p}\\ &\quad + C_\# R^{Y/2} \sum_{\substack{n\in\mathcal{N}(t)\\ \mathbf{i}\in\mathcal{I}(n,t)}} \lambda_{s,n}^{-Y/2}(D^e_n)^{p-1}\left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}\psi\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r',s',q}_p}\,.\end{aligned}$$

(The factor $C_\#(\lambda_{s,n}^{-1}R)^{Y/2} = C_\#(\lambda_{s,n}^{-1}R)^{r-r'+s-s'}$ comes from the $C_{r-r'+s-s'}$ norm in (B.6) and (4.4).) Summing over $\zeta\in\mathcal{Z}(R)$, we obtain

$$\begin{aligned}&\|\mathcal{L}_t(\psi)\|^p_\Phi \le\\ &C_\# \sum_{n\in\mathcal{N}(t)}(D^e_n)^{p-1} \sum_{\zeta\in\mathcal{Z}(R),\mathbf{i}\in\mathcal{I}(n,t)} \left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}\psi\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r,s,q}_p}\\ &+ C_\# R^{Y/2} \sum_{\substack{n\in\mathcal{N}(t)\\ \zeta\in\mathcal{Z}(R),\mathbf{i}\in\mathcal{I}(n,t)}} \lambda_{s,n}^{-(r-r'+s-s')}(D^e_n)^{p-1}\left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}\psi\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r',s',q}_p}\,.\end{aligned}$$

For $i\in I$, $j\in J_i$, let $U_{i,j,\ell,2}$, $j\in N_{i,\ell}$, be arbitrary open sets covering a fixed neighbourhood $\widetilde{B}^0_{i,j}$ of $\overline{B_{i,j}}$, such that $\overline{U_{i,j,\ell,2}}\subset U_{i,j,\ell,1}$ (they do not depend on t and R). For each $\zeta\in\mathcal{Z}(R)$, $n\in\mathcal{N}(t)$, and $\mathbf{i}=(i_0,\dots,i_n)\in\mathcal{I}(n,t)$ such that $T_{\mathbf{i},t}(B_{\mathbf{i},t})$ intersects $A(\zeta)$, the point $T_{\mathbf{i},-t}(w_\zeta)$ belongs to $\widetilde{B}^0_{i_0,i_1}$ if R is large enough. We can therefore consider k such that $T_{\mathbf{i},-t}(w_\zeta)$ belongs to $U_{i_0,i_1,k,2}$. Then $\sum_{\ell\in\mathcal{Z}_{i_0,k}(R)}\rho_{i_0,i_1,k,\ell}$ is equal to 1 on a neighbourhood of fixed size of $T_{\mathbf{i},-t}(w_\zeta)$, so that $\sum_{\ell\in\mathcal{Z}_{i_0,i_1,k}(R)}\rho_{i_0,i_1,k,\ell}\circ T_{\mathbf{i},-t}$ is equal to 1 on $A(\zeta)$ if R is large enough (depending on t but not on Φ or ζ). Therefore, claim (B.4) in Lemma B.3 (note that (B.3) is (3.2)) gives, if R is large enough (uniformly in Φ, ζ, k, $\mathbf{i}$)

$$\begin{aligned}&\left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}\psi\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r,s,q}_p}\\ &\le C_\# \sum_{\ell\in\mathcal{Z}_{i_0,i_1,k}(R)} \left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}(\rho_{i_0,i_1,k,\ell}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r,s,q}_p}\,.\end{aligned}\tag{4.6}$$

Taking R large enough and summing over $\zeta\in\mathcal{Z}(R)$, $n\in\mathcal{N}(t)$, $\mathbf{i}\in\mathcal{I}(n,t)$ and k in $N_{i_0i_1}$ such that $T_{\mathbf{i},-t}(w_\zeta)\in U_{i_0,i_1k,2}$, we get (writing $\zeta'=(i_0,i_1,k,\ell)\in\mathcal{Z}(R)$)

(4.7)
$$\begin{aligned}\|\mathcal{L}_t(\psi)\|^p_\Phi &\le C_\# \sum_{n,\zeta,\mathbf{i},\zeta'}(D^e_n)^{p-1}\left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r,s,q}_p}\\ &\quad + C_\#\cdot R^{Y/2}\sum_{n,\zeta,\mathbf{i},\zeta'}(D^e_n)^{p-1}\lambda_{s,n}^{-Y/2}\left\|(\rho_\zeta\mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\right\|^p_{H^{r',s',q}_p}\,,\end{aligned}$$

where the sum is restricted to those $(\zeta,\mathbf{i},\zeta')$ such that the support of $\rho_{\zeta'}$ is included in $\widetilde{B}_{\mathbf{i},t}$, the support of ρ_ζ is included in $T_{\mathbf{i},t}(\widetilde{B}_{\mathbf{i},t})$, and $B_{\zeta'}=B_{i_0i_1}$ (this restriction will be implicit in the rest of the proof).

[17] Elements of L^∞ are defined almost everywhere, and the transfer operator is defined initially on $L^\infty(X_0)$, so the fact that $\bigcup_{ij}B_{ij}=X_0$ only modulo a zero Lebesgue measure set is irrelevant.

Second step: Getting rid of the characteristic function. We claim that, if R is large enough, then for any ζ, n, $\mathbf{i}$, ζ' as in the right-hand-side of (4.7), we have

$$\|(\rho_\zeta \mathbb{1}_{T_{\mathbf{i},t}B_{\mathbf{i},t}}(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\|^p_{H^{r,s,q}_p} \le C_\#(C_\# N^p)^{n/N}\,\|(\rho_\zeta(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\|^p_{H^{r,s,q}_p}\ . \tag{4.8}$$

To prove the above inequality, it is sufficient to show that multiplication by $\mathbb{1}_{T_{\mathbf{i},t}(B_{\mathbf{i},t})}\circ \Phi_\zeta$ acts boundedly on $H^{r,s,q}_p$, with norm bounded by $(C_\# N^p)^{n/N}$. First note that there exist C^2 functions $\widetilde\tau_{\mathbf{i},k}(z,t)$, $k=0,1$, so that the images of the dynamically refined flow boxes are of the form

$$T_{\mathbf{i},t}(B_{\mathbf{i},t}) = \{T_\tau(z)\mid z\in P^n_{\mathbf{i}}(O_{\mathbf{i}})\,,\ \tau\in[\widetilde\tau_{\mathbf{i},0}(z,t),\widetilde\tau_{\mathbf{i},1}(z,t))\}\,.$$

For $n\in\mathcal{N}(t)$, let $\kappa=n/N$ and decompose $\mathbf{i}=(i_0,\dots,i_n)$ into subsequences of length N, as $(\mathbf{i}_0,\dots,\mathbf{i}_{\kappa-1})$. Then $\mathbb{1}_{P^n_{\mathbf{i}}O_{\mathbf{i}}}\prod_{j=0}^{\kappa-1}\mathbb{1}_{O_{\mathbf{i}_j}}\circ P^{-(\kappa-j)N}_{\mathbf{i}_j\mathbf{i}_{j+1}\dots\mathbf{i}_{\kappa-1}}$. Define a set

$$\Omega_j = P^{(\kappa-j)N}_{\mathbf{i}_j\mathbf{i}_{j+1}\dots\mathbf{i}_{\kappa-1}}(O_{\mathbf{i}_j})\,,$$

it is therefore sufficient to show that multiplication by $\mathbb{1}_{\{w\in B_{i_{n-jN},i_{n-jN+1}}|z(w)\in\Omega_j\}}\circ \Phi_\zeta$ (this takes care of the lateral boundaries) acts boundedly on $H^{r,s,q}_p$, with norm at most $C_\# N^p$ and multiplication by (this takes care of the top and bottom boundaries)

$$\mathbb{1}_{\{w|z(w)\in O_{i_{n-1},i_n}\ ,\ \tau(w)\in[\widetilde\tau_{\mathbf{i},0}(w,t),\widetilde\tau_{\mathbf{i},1}(w,t))\}}\circ\Phi_\zeta \tag{4.9}$$

acts boundedly on $H^{r,s,q}_p$, with norm at most $C_\#$.

Recall that (3.3) holds. Working with our flow box charts (see (2.9)), the assertion on (4.9) is an immediate application of Lemma B.2 (the number M_{cc} of connected components being then uniform in N), since the functions $\widetilde\tau_{\mathbf{i},k}$ are C^2, uniformly in t, k, $\mathbf{i}$ (they are obtained by composing the original roof functions by a C^2 and hyperbolic restriction of the composition of the Poincaré maps). Next, we assume for a moment that each $\partial O_{i,j}$ is a finite union of C^1 hypersurfaces $K_{i,j,k}$, each of which is transversal to the stable cone. Then the second step of the proof of [6, Lemma 5.1] allows us to apply Lemma B.2 (using Fubini in flow box coordinates) which implies that, if R and C_1 are large enough (depending only on N), then the multiplication by the characteristic function of each set $\{w\in B_{i_{n-jN},i_{n-jN+1}}\mid z(w)\in\Omega_j\}$ has operator-norm on $H^{r,s,q}_p$ bounded by $C_\# NL$. Definition 1.4 required only transversality in the image, but, using the fact that the cone fields $C^{(s)}_{i,j}$ do not depend on i (see Definition 1.4), we can apply the arguments in [6, Appendix C], in particular [6, Theorem C.1] there in the case where the set called Π_1 there coincides with the entire manifold M. This proves (4.8).

Combining (4.7) with (4.8) and with the analogous bound in $H^{r',s',q}_p$ for $-1+r<-1+1/p<r'<r$, we get

$$\begin{aligned}\|\mathcal{L}_t\psi\|^p_\Phi \le{}& C_\#\sum_{\zeta,n,\mathbf{i},\zeta'}(C_\# N^p)^{n/N}(D^e_n)^{p-1}\,\|(\rho_\zeta(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\|^p_{H^{r,s,q}_p}\\ &+C_\# R^{Y/2}\sum_{\zeta,n,\mathbf{i},\zeta'}\lambda_{s,n}^{-Y/2}(D^e_n)^{p-1}(C_\# N^p)^{n/N}\,\|(\rho_\zeta(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\|^p_{H^{r',s',q}_p}\ .\end{aligned} \tag{4.10}$$

Third step: Using the composition lemma. In this step, we shall use Lemma C.2, to pull the charts Φ_ζ in the right hand side of (4.10) back at time $-t$ (glueing some pulled-back charts together to get rid of the summation over ζ), and exploit Lemma 4.1 to obtain decay from hyperbolicity.

Let us partition $\mathcal{Z}(R)$ into finitely many subsets $\mathcal{Z}^1,\dots,\mathcal{Z}^E$, such that $\mathcal{Z}^e$ is included in one of the sets $\mathcal{Z}_{i,j,\ell}(R)$, and $|m-m'|\ge C(C_0)$ whenever $(i,j,\ell,m)\neq$

$(i, j, \ell, m') \in \mathcal{Z}^e$, where $C(C_0)$ is the constant C constructed in Lemma C.2 (it only depends on C_0). The number E may be chosen independently of t and $n \in \mathcal{N}(t)$.

We shall prove the following: For any $\zeta' \in \mathcal{Z}(R)$, any $t > 0$, any $n \in \mathcal{N}(t)$ and $\mathbf{i} \in \mathcal{I}(n, t)$ (such that the support of $\rho_{\zeta'}$ is included in $\widetilde{B}_{\mathbf{i},t}$ and $B_{\zeta'} = B_{i_0 i_1}$), and any $1 \le e \le E$, there exists an admissible chart $\Phi' = \Phi'_{\zeta',\mathbf{i},e} \in \mathcal{F}(\zeta')$ such that

$$\sum_{\zeta \in \mathcal{Z}^e} \|(\rho_\zeta(\rho_{\zeta'} \cdot \psi) \circ T_{\mathbf{i},-t}) \circ \Phi_\zeta\|^p_{H_p^{r,s,q}} \le C_\# \chi_n \left\|(\rho_{\zeta'} \cdot \psi) \circ \Phi'_{\zeta',\mathbf{i},e}\right\|^p_{H_p^{r,s,q+r-s}} + C_\# \lambda_{u,n} \cdot \lambda_{s,n}^{-1} \left\|(\rho_{\zeta'} \cdot \psi) \circ \Phi'_{\zeta',\mathbf{i},e}\right\|^p_{H_p^{r',s',q+2r-r'-s}} , \tag{4.11}$$

where

$$\chi_n = \left\| \max(\lambda_{u,n}^{-r}, \lambda_{s,n}^{-(s+r)})^p \right\|_{L^\infty} . \tag{4.12}$$

As always, the sum on the left hand side of (4.11) is restricted to those values of ζ such that the support of ρ_ζ is included in $T_{\mathbf{i},t}(\widetilde{B}_{\mathbf{i},t})$

Let us fix ζ', t, $n \in \mathcal{N}(t)$, $\mathbf{i} \in \mathcal{I}(n, t)$ and e as above, until the end of the proof of (4.11). All the objects we shall now introduce shall depend on these choices, although we shall not make this dependence explicit to simplify the notations. Let i, j, ℓ be such that $\mathcal{Z}^e \subset \mathcal{Z}_{i,j,\ell}(R)$, and let $\mathcal{J} = \{m \mid (i, j, \ell, m) \in \mathcal{Z}^e\}$. Since the points in $\mathcal{J}$ are distant of at least $C(C_0)$, Lemma C.2 will apply.

Increasing R, we can ensure that the map

$$\mathcal{T} := \kappa_\zeta^R \circ T_{\mathbf{i},t} \circ (\kappa_{\zeta'}^R)^{-1}$$

is arbitrarily close to its differential $\mathcal{A} = D\mathcal{T}(\ell)$ at $\ell := \kappa_{\zeta'}(w_{\zeta'})$, i.e., the map $(\mathcal{T}^{-1}[\cdot + \mathcal{T}(\ell)] - \ell) \circ \mathcal{A}$ is close to the identity in the C^2 topology, say on the ball $B(0, 2d)$, and that $n(\cdot, t)$ is constant on $(\kappa_{\zeta'}^R)^{-1} B(0, 2d)$. Moreover, the matrix $\mathcal{A}$ sends $\mathcal{C}_{\zeta'}$ to $\mathcal{C}_\zeta$ compactly (recall (2.13), and note that t_{00} can be fixed small while we may require $t \ge t_0$ with t_0 depending on t_{00}), and

$$C_\# \ge \lambda_u(\mathcal{A}, \mathcal{C}_{\zeta'}, \mathcal{C}_\zeta)/\lambda_u^{(n)}(w_{\zeta'}) \ge C_\#^{-1} , \tag{4.13}$$

with similar inequalities for λ_s and Λ_u. In addition $\mathcal{A}(0, 0, v^0) = (0, 0, v^0)$ for any $v^0 \in \mathbb{R}$. Since T is uniformly hyperbolic and satisfies the bunching condition (1.5), we can ensure up to taking larger t (and thus n, and also R, in view of the requirements in the beginning of the paragraph) that $\mathcal{A}$ satisfies the bunching condition (C.1) for the constant $\epsilon = \epsilon(C_0, C_1)$ constructed in Lemma C.2.

Let us make explicit the dependence of t on $C_1 > 1$ and of R on t (and therefore on C_1). Let $\lambda_\beta < 1$ be the supremum in the left-hand-side of the bunching condition (1.5) (λ_β only depends on the dynamics). From (C.17) in the proof of Lemma C.2, there is a constant $C_\#$ depending only on (the extended cones and) C_0 so that we may take $\epsilon(C_0, C_1) = C_\# C_1^{-1}$. Therefore, there is a constant $C_\#$ depending only on C_0, so that if $n \ge \ln(\epsilon^{-1})/\ln(\lambda_\beta^{-1}) = C_\# \ln(C_1)$ then the bunching condition (C.1) holds for $\epsilon(C_0, C_1)$. Since $n \ge t/\widetilde{\tau}_0 - C_\#$, the condition on n transforms to $t \ge t_0(C_1) = C_\# \ln(C_1)$, for a different constant $C_\#$ depending only on the dynamics. Finally, there are $\Lambda > 1$ depending only on the dynamics, and $C_\#$ depending only on the dynamics and on C_0, so that, for $t \ge t_0(C_1)$, if $R \ge R(t) = C_\# \Lambda^t$ with $\Lambda^t \ge C_\# C_1^{C_\# \ln \Lambda}$ then the requirements in the beginning of the previous paragraph (and those in the first step regarding (4.6)) hold for t.

Applying [18] Lemma C.2, we obtain a block diagonal matrix $D = D_{\zeta'}$, a chart $\phi'_\ell = \phi'_{\ell,\zeta'}$ around ℓ, time-shifts $\Delta_m = \Delta_{m,\zeta'}$ and diffeomorphisms $\Psi_m = \Psi_{m,\zeta'}$, and

[18] In order to apply Lemma C.2, we need to extend $\mathcal{T}$ to a diffeomorphism of $\mathbb{R}^d$, which can be done exactly as in the third step of the proof of [6, Lemma 5.1].

$\Psi_{\zeta'}$, such that, for any m in the set $\mathcal{J}'$ of those elements in $\mathcal{J}$ for which $\rho_\zeta \cdot \rho_{\zeta'} \circ T_{\mathbf{i},-t}$ is nonzero,

$$\mathcal{T}^{-1} \circ \phi_\zeta = \phi'_{\ell,\zeta'} \circ \Psi_{\zeta'} \circ D^{-1}_{\zeta'} \circ \Psi_{m,\zeta'} \circ \Delta_{m,\zeta'} \tag{4.14}$$

on the set where $(\rho_\zeta \cdot \rho_{\zeta'} \circ T_{\mathbf{i},t}) \circ \Phi_\zeta$ is nonzero.

Writing $\psi_{\zeta'} = (\rho_{\zeta'} \cdot \psi) \circ (\kappa^R_{\zeta'})^{-1}$, we have (recall that (i, j, ℓ) is fixed so that $\mathcal{Z}^e \subset \mathcal{Z}_{i,j}(R)$)

$$\begin{aligned}
&\sum_{\zeta \in \mathcal{Z}^e} \left\| (\rho_\zeta (\rho_{\zeta'} \cdot \psi) \circ T_{\mathbf{i},-t}) \circ \Phi_\zeta \right\|^p_{H^{r,s,q}_p} \\
&\qquad = \sum_{m \in \mathcal{J}'} \left\| (\rho_m \circ \phi_{i,j,\ell,m}) \cdot (\psi_{\zeta'} \circ \mathcal{T}^{-1} \circ \phi_{i,j,\ell,m}) \right\|^p_{H^{r,s,q}_p} \\
&\qquad = \sum_{m \in \mathcal{J}'} \left\| ((\rho_m \circ \phi_{i,j,\ell,m} \circ \Delta_m^{-1}) \cdot (\psi_{\zeta'} \circ \phi'_\ell \circ \Psi \circ D^{-1} \circ \Psi_m)) \circ \Delta_m \right\|^p_{H^{r,s,q}_p} .
\end{aligned} \tag{4.15}$$

Lemma B.8 bounds the previous expression by

$$C_\# \sum_{m \in \mathcal{J}'} \left\| ((\rho_m \circ \phi_{i,j,\ell,m} \circ \Delta_m^{-1} \circ \Psi_m^{-1}) \cdot (\psi_{\zeta'} \circ \phi'_\ell \circ \Psi \circ D^{-1})) \circ \Psi_m) \right\|^p_{H^{r,s,q+r-s}_p} . \tag{4.16}$$

Using the notations and results of Lemma C.2, the terms in this last equation are of the form $v \circ \Psi_m$, where v is a distribution supported in $\Psi_m(\phi^{-1}_{i,j,\ell,m}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)$. Since the range of Ψ_m contains $B(\Pi m^u, C_0^{1/2})$, if $q \ge 0$ is small enough so that $q+(r-s) < (1-r+s)/(r-s)$, Lemma B.6 gives $\|v \circ \Psi_m\|_{H^{r,s,q+r-s}_p} \le C_\# \|v\|_{H^{r,s,q+r-s}_p}$. Therefore (4.16) is bounded by

$$C_\# \sum_{m \in \mathcal{J}'} \left\| (\rho_m \circ \phi_{i,j,\ell,m} \circ \Delta_m^{-1} \circ \Psi_m^{-1}) \cdot (\psi_{\zeta'} \circ \phi'_\ell \circ \Psi \circ D^{-1}) \right\|^p_{H^{r,s,q+r-s}_p} . \tag{4.17}$$

The functions $\rho_m \circ \phi_{i,j,\ell,m} \circ \Delta_m^{-1} \circ \Psi_m^{-1}$ have a bounded C^1 norm and are supported in the balls $B(\Pi m, C_0^{1/2}/2)$, whose centers are distant by at least C_0, by Lemma C.2 (a). Therefore, by the localisation Lemma B.3 (using that (3.2) is (B.3)), the sum in (4.17) (and thus also the left-hand-side of (4.15)), is bounded by

$$C_\# \left\| \psi_{\zeta'} \circ \phi'_\ell \circ \Psi \circ D^{-1} \right\|^p_{H^{r,s,q+r-s}_p} . \tag{4.18}$$

Similarly, but giving up the glueing in Step 2 of Lemma C.2 (and therefore the need to work with Δ_m and the regularity loss in the flow direction), up to the cost of exponential growth, we get, applying the second estimate in Lemma 4.1 (recall that $r' \ge 0$) to the composition with D^{-1},

$$\begin{aligned}
\sum_{\zeta \in \mathcal{Z}^e} \left\| (\rho_\zeta (\rho_{\zeta'} \cdot \psi) \circ T_{\mathbf{i},-t}) \circ \Phi_\zeta \right\|^p_{H^{r',s',q}_p} &\le C_\# \lambda_{u,n} \left\| \psi_{\zeta'} \circ \phi'_\ell \circ \Psi \circ D^{-1} \right\|^p_{H^{r',s',q}_p} \\
&\le C_\# \lambda_{u,n} \left\| \psi_{\zeta'} \circ \phi'_\ell \right\|^p_{H^{r',s',q}_p} .
\end{aligned} \tag{4.19}$$

We next apply the first estimate in Lemma 4.1 to the composition with D^{-1} in (4.18), in order to obtain a contraction in the $H^{r,s,q+r-s}_p$ norm, up to a bounded term in the $H^{r',s',q+2r-r'-s}_p$ norm for $r' < r$. Since $\psi_{\zeta'}$ is supported in $B(\ell, C_0^{1/2}/2)$ while the range of Ψ contains $B(\ell, C_0^{1/2})$ (by Lemma C.2), Lemma B.6 implies that the composition with Ψ is bounded. Summing up, we obtain

$$\begin{aligned}&\sum_{\zeta\in\mathcal{Z}^e} \|(\rho_\zeta(\rho_{\zeta'}\cdot\psi)\circ T_{\mathbf{i},-t})\circ\Phi_\zeta\|^p_{H_p^{r,s,q+r-s}}\\
&\qquad\le C_\#\big(\chi_n^{(0)}(w_{\zeta'})\,\big\|(\rho_{\zeta'}\cdot\psi)\circ(\kappa^R_{\zeta'})^{-1}\circ\phi'_\ell\big\|^p_{H_p^{r,s,q+r-s}}\\
&\qquad\qquad+\big\|(\rho_{\zeta'}\cdot\psi)\circ(\kappa^R_{\zeta'})^{-1}\circ\phi'_\ell\big\|^p_{H_p^{r',s',q+2r-r'-s}}\big)\,,\end{aligned}\tag{4.20}$$

where

$$\chi_n^{(0)}(w_{\zeta'})=(\max(\lambda_{u,n}^{-r},\lambda_{s,n}^{-(s+r)})^p)(w_{\zeta'})\le\chi_n\,,$$

concluding the proof of (4.11). Summing over all possible values of ζ', n, $\mathbf{i}$, and e, we obtain

$$\begin{aligned}\|\mathcal{L}_t\psi\|^p_\Phi&\le C_\#\cdot\chi_{[t/\widetilde\tau_0]}\sum_{n,\zeta',\mathbf{i}}(C_\# N^p)^{n/N}(D_n^e)^{p-1}\sum_{e=1}^E\big\|(\rho_{\zeta'}\psi)\circ\Phi'_{\zeta',\mathbf{i},e}\big\|^p_{H_p^{r,s,q+r-s}}\\
&+C_\#\widetilde\Lambda^{2[t/\widetilde\tau_0]}\cdot R^{Y/2}\sum_{n,\zeta',\mathbf{i}}(C_\# N^p)^{n/N}(D_n^e)^{p-1}\sum_{e=1}^E\big\|(\rho_{\zeta'}\psi)\circ\Phi'_{\zeta',\mathbf{i},e}\big\|^p_{H_p^{r',s',q+2r-r'-s}}\,.\end{aligned}\tag{4.21}$$

for some $\widetilde\Lambda>1$ depending only on the dynamics.

Fourth step: Complexity at the end/ conclusion of the proof of (3.5)*.* The right hand side of (4.21) is of the form $\|\psi\|^p_{\Phi'}$ for some family of admissible charts Φ', except that several admissible charts may be assigned to a point $w_{\zeta'}$ for $\zeta'\in\mathcal{Z}(R)$. Since E is independent of n, the number of those charts around $w_{\zeta'}$ is at most $C_\#\cdot\mathrm{Card}\{\mathbf{i}\mid\widetilde B_{\mathbf{i},t}\cap A(\zeta')\ne\emptyset\}$. If R is large enough, we can ensure that this quantity is bounded by the intersection multiplicity of the sets $\widetilde B_{\mathbf{i},t}$, which is at most $D^b_{[t/\widetilde\tau_1]}$ by construction (using again that there is no refinement in the flow direction). Therefore, Lemma B.4 gives

$$\begin{aligned}\|\mathcal{L}_t\psi\|^p_\Phi&\le C_\#\max_{n\in\mathcal{N}(t)}\big\{(C_\# N^p)^{n/N}(D_n^e)^{p-1}D_n^b\chi_n\big\}\,\|\psi\|^p_{\mathbf{H}_p^{r,s,q+r-s}(R)}\\
&+C_\#\widetilde\Lambda^{2[t/\widetilde\tau_0]}\cdot R^Y\max_{n\in\mathcal{N}(t)}\big\{(C_\# N^p)^{n/N}(D_n^e)^{p-1}D_n^b\big\}\,\|\psi\|^p_{\mathbf{H}_p^{r',s',q+2r-r'-s}(R)}\,.\end{aligned}$$

This concludes the proof of (4.3) and therefore of (3.5).

Proof of (3.4)*.* To show (3.4), we revisit the steps of the proof of (4.3), without attempting to get an exponential contraction in the first term, or to obtain "compactness" in the second term. In the estimate (B.6) in the first step, we replace $H^{r-1,s,q}$ by $H^{r,s,q}$. We must explain how to avoid the loss from the $H_p^{r,s,0}$ norm to the $H_p^{r,s,r-s}$-norm. Since we may use the second bound of Lemma 4.1 instead of the first, this loss could occur only through the introduction of Δ_m in Step 1 of Lemma C.2 (see Lemma C.4), invoked in the third step of the present proof . In Step 1 of Lemma C.2, we got rid both of the x^s and x^u dependence of $\widetilde F_m^{(1)}$. The x^s dependence was a problem in Step 2 of Lemma C.2 (glueing). If we give up this glueing step in Lemma C.2 then $\widetilde F_m^{(1)}$ can be allowed to depend on x^s, to the cost of exponential growth (in $\Lambda^{\widetilde t}$, for $\Lambda>1$ related to the dynamics). The x^u dependence was a problem in Step 3 of Lemma C.2, where $\widetilde F^{(2)}(Ax^u,Bx^s,x^0)$ could create exponential growth in the norm. Again, if we are willing to deal with an exponential factor, we can allow $\widetilde F_m^{(1)}$ to depend on x^u. In other words, we can get rid of Δ_m in Lemma C.2. This ends the proof of (3.4), and of Lemma 3.1. □

Using our transversality assumption and a simplification of the application of Strichartz' result in Step 2 in the proof of Lemma 3.1, and recalling Lemma 3.2, we obtain the following bound, which will be useful to exploit mollifying and averaging operators to be introduced in the next section:

Corollary 4.2. *For any* $1 < p < \infty$ *and every* $\max(-\beta, -1 + 1/p) < \sigma < 1/p$, *there is* $C_\# > 0$ *so that for any* φ,

$$\|1\!\!1_{X_0}\varphi\|_{H^\sigma_p(M)} \le C_\# \|1\!\!1_{X_0}\varphi\|_{H^\sigma_p(X_0)} \le C^2_\# \|\varphi\|_{H^\sigma_p(M)} .$$

It remains to show the Lasota-Yorke estimates for $\mathcal{R}(z)$:

Proof of Lemma 3.8. We start with a premilinary bound which is also useful elsewhere. Its proof will use interpolation, and we refer to Section 3.1 of [5] for reminders and references (such as [10] and [43]) about complex interpolation.

Clearly, $\|\mathcal{L}_t(\psi)\|_{H^0_p(X_0)} \le \|\psi\|_{H^0_p(X_0)}$ for all $1 \le p \le \infty$. Fix $1 < p < \infty$ and $0 < s_0 < 1/p' = 1 - 1/p$. A toy-model version of the proof of Lemma 3.1 (working on isotropic spaces, and using the original result of Strichartz [41]) easily gives that

$$\|\mathcal{L}_t(\psi)\|_{H^{s_0}_p(X_0)} \le \widetilde\Lambda e^{\widetilde\Lambda t} \|\psi\|_{H^{s_0}_{p'}(X_0)} .$$

Here, $\widetilde\Lambda$ may depend on R, and s_0, and p' which are fixed. One can see that $\widetilde\Lambda$ is uniform in p' as $p' \to 1$. By duality $\|\mathcal{L}_t\psi\|_{H^{-s_0}_p(X_0)} \le \widetilde\Lambda e^{\widetilde\Lambda t}\|\psi\|_{H^{-s_0}_p(X_0)}$. Therefore, using complex interpolation for $s_0 < s < 0$ and Lemma 3.2 we get Λ (which may depend on R, p, and s_0, which are fixed) so that for any $-s_0 < s < 0$

$$\|\mathcal{L}_t\psi\|_{\mathbf{H}^{s,0,0}_p} \le \Lambda \|\mathcal{L}_t\psi\|_{H^s_p(X_0)} \le \Lambda e^{\Lambda|s|t}\|\psi\|_{H^s_p(X_0)} \le \Lambda e^{\Lambda|s|t}\|\psi\|_{\mathbf{H}^{s,0,0}_p} . \tag{4.22}$$

Recall (3.17). For large integer N, applying (3.8) twice, and exploiting Lemma 3.4, we obtain a constant $C_\#$ so that for all $a > A$, $n \ge 0$, and $\psi \in \widetilde{\mathbf{H}}$, writing $z = a + ib$

$$\begin{aligned}
\|\mathcal{R}(z)^{n+1}(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,0}_p} &\le C_\# \int_0^\infty \frac{t^{n-1}}{(n-1)!}\big(e^{-at-t\ln(\lambda^{-1})/N}(\|\mathcal{L}_{t-t/N}\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,r-s}_p} \\
&\qquad + e^{-at+At/N}\|\mathcal{L}_{t-t/N}\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{s,0,2(r-s)}_p}\big)\, dt \\
&\le C_\# \int_0^\infty \frac{t^{n-1}}{(n-1)!}\Big[e^{-at-2t\ln(\lambda^{-1})/N)}\|\mathcal{L}_{t-2t/N}\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,2(r-s)}_p} \\
&\qquad + e^{-t\ln(\lambda^{-1})/N}e^{-at+At/N}\|\mathcal{L}_{t-2t/N}\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{s,0,3(r-s)}_p} \\
&\qquad + C_\#\big(1+\frac{|z|}{R}\big)^{2(r-s)}\big(\frac{1}{a-A}+1\big)e^{-at+At/N}\|\mathcal{L}_{t-t/N}(\psi)\|_{\widetilde{\mathbf{H}}^{s,0,0}_p}\Big]\, dt .
\end{aligned}$$

Exploiting (4.22), and applying (3.8) $N-2$ more times, we get

$$\begin{aligned}
\|\mathcal{R}(z)^{n+1}(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,0}_p} &\le C_\# \int_0^\infty \frac{t^{n-1}}{(n-1)!}\Big[e^{-t(a-\ln(\lambda^{-1}))}\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}^{r,s,N(r-s)}_p} \\
&\quad + [e^{-(N-2)t\ln(\lambda^{-1})/N}e^{-(N-1)\Lambda t|s|/N} + \cdots \\
&\qquad + e^{-t\ln(\lambda^{-1})/N}\big(1+\frac{|z|}{R}\big)^{(3-N)(r-s)}\frac{e^{-2\Lambda t|s|/N}}{C^{N-1}_\#} + \big(1+\frac{|z|}{R}\big)^{(2-N)(r-s)}\frac{e^{-\frac{\Lambda t|s|}{N}}}{C^{N-2}_\#}] \\
&\qquad \cdot C^{N-1}_\#\big(1+\frac{|z|}{R}\big)^{N(r-s)}\big(\frac{1}{a-A}+1\big)e^{-t(a-\Lambda|s|-\frac{A}{N})}\|\psi\|_{\widetilde{\mathbf{H}}^{s,0,0}_p}\Big]\, dt \\
&\le C_\#\big(\frac{1}{a-A}+1\big)\big(1+\frac{|z|}{R}\big)^{N(r-s)}\Big(\frac{1}{(a+\ln(1/\lambda))^n} + \frac{C^{N-1}_\#}{(a-\Lambda|s|-A/N)^n}\Big)\|\psi\|_{\widetilde{\mathbf{H}}^{r,s,0}_p}.
\end{aligned}$$

This is (3.27). We have applied (3.8) successively to $q_j = j(r-s)$, for $j = 0, \dots, N$. We need conditions (3.2) and (3.3) to hold for each q_j. It suffices to check both inequalities on q_N, and they are satisfied if $|s| \le 2r$ and (3.26) hold. A slight modification of the above argument gives (3.28) (the condition on N does not depend on s'). □

5. Mollifiers and stable-averaging operators

It will be convenient in Section 7 to mollify distributions, replacing them by nearby distributions in C^1. As is often the case, our mollification operators $\mathbb{M}_\epsilon$ are obtained through convolution. In this section, we also introduce a key tool in the Dolgopyat estimate of Section 6, the stable averaging operators $\mathbb{A}_\delta$.

Remark 5.1 (Minkowski-type integral inequalities). We note for further use that for any real r and any $1 < p < \infty$, there exists $C > 0$ so that for any integrable $\widetilde{\eta} : \mathbb{R}^d \to \mathbb{R}_+$ and any family $\omega_y \in H^r_p(\mathbb{R}^d)$, uniformly bounded in y,

$$\left\| \int_{\mathbb{R}^d} \widetilde{\eta}(y) \omega_y(\cdot)\, dy \right\|_{H^r_p(\mathbb{R}^d)} \le C \int_{\mathbb{R}^d} \widetilde{\eta}(y) \left\| \omega_y \right\|_{H^r_p(\mathbb{R}^d)} dy \le C \left\| \widetilde{\eta} \right\|_{L^1(\mathbb{R}^d)} \sup_y \left\| \omega_y \right\|_{H^r_p(\mathbb{R}^d)} . \tag{5.1}$$

Indeed, if $r = 0$ the above estimate is the classical Minkowski integral inequality, see e.g. [38, App.A]. The case $|r| \le 1$ may be proved by considering first $r = 1$, recalling that $\|\psi\|_{H^1_p} \sim \|\psi\|_{L^p} + \|D\psi\|_{L^p}$ (see e.g. [37, Prop 2.1.2 (iv)+(vii)]), then $r = -1$, using duality ($H^{r,0,0}_p = H^{-r,0,0}_{p'}$ with $p' = 1/(1-1/p)$) and d-dimensional Fubini (see e.g. [5, Beginning of §4]), and finally using complex interpolation (see [42], and [10] in particular §2.4, §4.1, and Theorem 5.1.2, which says that $[L_1(\mathcal{B}_1, \nu), L_1(\mathcal{B}_2, \nu)]_\theta = L_1([\mathcal{B}_1, \mathcal{B}_2]_\theta, \nu)$, applied here to $\nu = \eta\, dx$, this is Theorem 1.18.4 in [42]). The cases $|r| > 1$ are handled similarly by considering higher order derivatives.

Remark 5.2 (Variants of complex interpolation). We shall use two easy variants of complex interpolation. Let $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ be linear operators acting on $\mathcal{B}_1$, $\mathcal{B}_2$, an interpolation pair of Banach spaces. Then, interpolating at $\theta \in [0, 1]$ between

$$\|\mathcal{P}_1 \omega\|_{\mathcal{B}_1} \le C_1 \|\mathcal{P}_2 \omega\|_{\mathcal{B}_1} \text{ and } \|\mathcal{P}_1 \omega\|_{\mathcal{B}_2} \le C_2 \|\mathcal{P}_2 \omega\|_{\mathcal{B}_1} \tag{5.2}$$

gives

$$\|\mathcal{P}_1(\omega)\|_{[\mathcal{B}_1, \mathcal{B}_2]_\theta} \le C_1^{1-\theta} C_2^\theta \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} ,$$

while interpolating at θ between

$$\|\mathcal{P}_1(\omega)\|_{\mathcal{B}_1} \le \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} \text{ and } \|\mathcal{P}_1(\omega)\|_{\mathcal{B}_2} \le C_1 \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} + C_2 \|\mathcal{P}_3(\omega)\|_{\mathcal{B}_1} , \tag{5.3}$$

gives

$$\|\mathcal{P}_1(\omega)\|_{[\mathcal{B}_1, \mathcal{B}_2]_\theta} \le \max(C_1^\theta, C_2^\theta)(\|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} + \|\mathcal{P}_3 \omega\|_{\mathcal{B}_1}) .$$

For (5.2), one can easily adapt the argument in the last paragraph of [10, Theorem 4.1.2]. Using the notation there (in particular $\mathcal{F}_{\mathcal{B}_1, \mathcal{B}_2}$), let $\omega \in \mathcal{B}_1$. Put $g(z) = C_1^{z-1} C_2^{-z} \mathcal{P}_1(\omega)$. Then $g \in \mathcal{F}_{\mathcal{B}_1, \mathcal{B}_2}$ with $\|g\|_{\mathcal{F}} \le \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1}$. In addition, $g(\theta) = C_1^{\theta-1} C_2^{-\theta} \mathcal{P}_1(\omega)$ so that

$$\|\mathcal{P}_1(\omega)\|_{[\mathcal{B}_1, \mathcal{B}_2]_\theta} \le C_1^{1-\theta} C_2^\theta \|g\|_{\mathcal{F}} \le C_1^{1-\theta} C_2^\theta \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} .$$

For (5.3), put $g(z) = (\max(C_1, C_2))^{-z} \mathcal{P}_1(\omega)$. Then $g \in \mathcal{F}_{\mathcal{B}_1, \mathcal{B}_2}$ with $\|g\|_{\mathcal{F}} \le \max(\|\mathcal{P}_2 \omega\|_{\mathcal{B}_1}, \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} + \|\mathcal{P}_3(\omega)\|_{\mathcal{B}_1}) \|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} + \|\mathcal{P}_3(\omega)\|_{\mathcal{B}_1}$. In addition, $g(\theta) = (\max(C_1, C_2))^{-\theta} \mathcal{P}_1(\omega)$ so that

$$\|\mathcal{P}_1(\omega)\|_{[\mathcal{B}_1, \mathcal{B}_2]_\theta} \le (\max(C_1, C_2))^\theta \|g\|_{\mathcal{F}} \le \max(C_1^\theta, C_2^\theta)(\|\mathcal{P}_2(\omega)\|_{\mathcal{B}_1} + \|\mathcal{P}_3(\omega)\|_{\mathcal{B}_1}) .$$

5.1. **Mollification $\mathbb{M}_\epsilon$ of distributions on M.** Fix ϵ_0 small. Let $\eta : \mathbb{R}^d \to [0,\infty)$ be a bounded and compactly supported C^∞ function, supported in $|x| \le 1$ and bounded away from zero on $|x| \le 1/3$, with $\int \eta(x)\,dx = 1$, and set, for $0 < \epsilon < \epsilon_0$,

$$\eta_\epsilon(y) = \frac{1}{\epsilon^d}\eta\left(\frac{y}{\epsilon}\right).$$

Recall the family of C^2 charts $\kappa_{i,j,\ell} : U_{i,j,\ell,0} \to \mathbb{R}^d$ from Section 2.2 and the standard contact form α_0 from (2.11). We have $(0, x^s, 0) \in \operatorname{Ker} \alpha_0$ for all x^s, and we shall use this as a local fake stable foliation to define the averaging operator $\mathbb{A}^s_\delta$ in Section 5.2 and in the proof of the Dolgopyat estimate in Section 6. Fix C^∞ functions $\bar\theta_{i,j,\ell} : M \to [0,1]$, supported in the set $U_{i,j,\ell,1}$ (recall that $\overline{U}_{i,j,\ell,1} \subset U_{i,j,\ell,0}$) and so that

$$\sum_{i,j,\ell} \bar\theta_{i,j,\ell}(q) = 1 \quad \forall q \in Y_0 := \cup_{i,j,\ell} \bar U_{i,j,\ell,2}\,.$$

(Recall that $U_{i,j,\ell,2}$ was introduced in Step 1 of the proof of Lemma 3.1, with $\overline{U}_{i,j,\ell,2} \subset U_{i,j,\ell,1}$, and that the definition ensures that X_0 is contained in the interior of Y_0.) As before, we write $\zeta = (i,j,\ell)$ to simplify notation. (Note that the parameter $m \in \mathbb{Z}^d$ does not appear in this section.)

The mollifier operator $\mathbb{M}_\epsilon$ is defined for $0 < \epsilon < \epsilon_0$ [19] by first setting for $\psi \in L^\infty(X_0)$ and $x \in \kappa_{i,j,\ell}(U_{i,j,\ell,1})$,

$$(\mathbb{M}_\epsilon(\psi))_\zeta(x) = \int_{\mathbb{R}^d} \eta_\epsilon(x-y)\psi(\kappa_\zeta^{-1}(y))\,dy = [\eta_\epsilon * (\psi \circ \kappa_\zeta^{-1})](x)\,.$$

Then, we set for $\psi \in L^\infty(X_0)$

$$\mathbb{M}_\epsilon(\psi) = \sum_\zeta \bar\theta_\zeta(((\mathbb{M}_\epsilon(\psi))_\zeta \circ \kappa_\zeta)\,. \tag{5.4}$$

Since ψ is supported in X_0, then $\mathbb{M}_\epsilon(\psi)$ is supported in Y_0 for small enough ϵ. We have the following bounds for $\mathbb{M}_\epsilon$:

Lemma 5.3. *For each $p \in (1,\infty)$, and all $-1 < r' \le 0$ and $r' \le r < 2 + r'$ in $\mathbb{R}$, there exists $C_\#$ so that for all small enough $\epsilon > 0$ and every $\psi \in H_p^{r'}(M)$, supported in X_0,*

$$\|\mathbb{M}_\epsilon(\psi)\|_{H_p^r(M)} \le C_\# \epsilon^{r'-r} \|\psi\|_{H_p^{r'}(M)}\,. \tag{5.5}$$

(Smoothness of η is important in the proof of (5.5).)

Lemma 5.4. *Let $p \in (1,\infty)$ and $-1 < r' < s < 0$. There exists $C_\# > 0$ so that for any small enough $\epsilon > 0$, and all $\psi \in H_p^s(X_0)$*

$$\|\mathbb{M}_\epsilon(\psi) - \psi\|_{H_p^{r'}(M)} \le C_\# \epsilon^{s-r'}\, \|\psi\|_{H_p^s(X_0)}\,. \tag{5.6}$$

Proof of Lemma 5.3. We assume that ϵ is small enough so that $\mathbb{M}_\epsilon(\psi)$ is supported in the interior of Y_0. Hence, we can implicitly replace $H_p^\sigma(M)$ by $H_p^\sigma(Y_0)$ in the argument.

Since the charts and partition of unity are C^2, the bound (5.5) on classical Sobolev spaces is standard, using Remark 5.1 and interpolating (interpolation is allowed for Triebel spaces on $\mathbb{R}^d$ or a manifold, beware it cannot be used directly for our spaces $\mathbf{H}$). We provide details for the convenience of the reader: We shall use that for any real $-1 < r' < 1$ and any $1 < p < \infty$

$$\|\psi\|_{H_p^{1+r'}(M)} \le C_\#(\|D\psi\|_{H_p^{r'}(M)} + \|\psi\|_{H_p^{r'}(M)})\,. \tag{5.7}$$

[19] In the application, we shall take ϵ_0 small enough as a function of $n = \lceil c \ln |b| \rceil$, in particular much smaller than $1/R$ and than the "gap" between Y_0 and X_0.

To prove (5.7), combine $\|\omega\|_{H^{1+r'}_p} \sim \|\omega\|_{H^{r'}_p} + \|D\omega\|_{H^{r'}_p}$ (this can be proved by interpolation and duality from the corresponding results for integer $r' \ge 0$ available e.g. in [37, Prop 2.1.2 (iv)+(vii)]), with (5.2), using that $\bar\theta_\zeta$ and κ_ζ are C^2.

The Minkowski integral inequality (5.1) implies that for each $r' \in \mathbb{R}$ and $1 < p < \infty$, there exists $C_\#$ so that $\|\eta_\epsilon * \omega\|_{H^{r'}_p} \le C_\# \|\omega\|_{H^{r'}_p}$ for all ϵ. Next, $|D^\ell(\eta_\epsilon)(x)| \le C_\# \epsilon^{-\ell} |(D^\ell \eta)_\epsilon|$, which implies $\int |D^\ell(\eta_\epsilon)(x)|\, dx \le C_\# \epsilon^{-\ell}$ for all integers $\ell \ge 1$. Thus, by (5.1), for any integer $\ell \ge 1$,

$$\left\|(D^\ell(\eta_\epsilon * \omega)\right\|_{H^{r'}_p} = \left\|(D^\ell(\eta_\epsilon)) * \omega\right\|_{H^{r'}_p} \le C_\# \epsilon^{-\ell} \|\omega\|_{H^{r'}_p} .$$

Therefore, applying (5.1) again, this time with (5.7), and since

$$D(\mathbb{M}_\epsilon \psi) = \sum_\zeta (D\bar\theta_\zeta)(\mathbb{M}_\epsilon(\psi))_\zeta \circ \kappa_\zeta + \sum_\zeta \bar\theta_\zeta D(\mathbb{M}_\epsilon(\psi))_\zeta \circ \kappa_\zeta \, D\kappa_\zeta$$

with $D(\mathbb{M}_\epsilon(\psi))_\zeta(x) = D(\eta_\epsilon) * (\psi \circ \kappa_\zeta^{-1})(x)$, and similarly for the second derivative, and also, putting $G_{\zeta'\zeta} = \kappa_{\zeta'} \circ \kappa_\zeta^{-1}$, if the domain of the map is nonempty,

$$\begin{aligned}&[((\bar\theta_\zeta \bar\theta_{\zeta'}) \circ \kappa_\zeta^{-1})((\mathbb{M}_\epsilon(\psi))_{\zeta'} \circ G_{\zeta'\zeta}](x)\\ &\quad = \frac{(\bar\theta_\zeta \bar\theta_{\zeta'})(\kappa_\zeta^{-1}(x))}{\epsilon^d} \int_{\mathbb{R}^d} \eta\big(\frac{G_{\zeta'\zeta}(x) - y}{\epsilon}\big) \psi(\kappa_\zeta^{-1} \circ G_{\zeta'\zeta}^{-1}(y))\, dy ,\end{aligned}$$

we find for $\ell = 0, 1, 2$, and $-1 < r' \le \ell + r' < 2$

$$\|\mathbb{M}_\epsilon(\psi)\|_{H^{\ell+r'}_p(M)} \le C_\# \epsilon^{-\ell} \|\psi\|_{H^{r'}_p(M)} . \tag{5.8}$$

Our assumptions imply that there exists $0 \le \ell \le 1$ so that $\ell + r' \le r < \ell + 1 + r'$. Interpolating between the corresponding inequalities (5.8) (at $(r - \ell - r')/r'$), we obtain (5.5). □

Proof of Lemma 5.4. Lemma 5.3 implies that

$$\|\mathbb{M}_\epsilon(\psi) - \psi\|_{H^{r'}_p(M)} \le C_\# \|\psi\|_{H^{r'}_p(M)} . \tag{5.9}$$

We shall next prove that there exists a constant $C_\#$ so that for all $-1 \le r' \le 0$

$$\|\mathbb{M}_\epsilon(\psi) - \psi\|_{H^{r'}_p(M)} \le C_\# \epsilon \|\psi\|_{H^{r'+1}_p(M)} . \tag{5.10}$$

Interpolating at $\theta = 1 - (s - r') \in (0, 1)$ between (5.9) and (5.10) gives the result since $\|\psi\|_{H^\sigma_p(M)} = \|\psi\|_{H^\sigma_p(X_0)}$ for $\psi \in H^\sigma_p(X_0)$, by our definition.

Putting $G_{\zeta'\zeta} = \kappa_{\zeta'} \circ \kappa_\zeta^{-1}$, if the domain of the map is nonempty, we have

$$\begin{aligned}&[(\bar\theta_\zeta \bar\theta_{\zeta'}) \circ \kappa_\zeta^{-1}((\mathbb{M}_\epsilon(\psi))_{\zeta'} \circ G_{\zeta'\zeta}](x)\\ &\quad = \frac{(\bar\theta_\zeta \bar\theta_{\zeta'})(\kappa_\zeta^{-1}(x))}{\epsilon^d} \int_{\mathbb{R}^d} \eta\big(\frac{y}{\epsilon}\big) \psi(\kappa_\zeta^{-1} \circ G_{\zeta'\zeta}^{-1}(G_{\zeta'\zeta}(x) - y))\, dy .\end{aligned} \tag{5.11}$$

Therefore, to prove (5.10), we must bound the $H^{r'}_p$ norm of

$$\begin{aligned}&\frac{(\bar\theta_\zeta \bar\theta_{\zeta'})(\kappa_\zeta^{-1}(x))}{\epsilon^d} \int_{\mathbb{R}^d} \eta(y/\epsilon)\\ &\qquad \cdot [\psi(\kappa_\zeta^{-1} \circ G_{\zeta'\zeta}^{-1}(G_{\zeta'\zeta}(x) - y)) - \psi(\kappa_\zeta^{-1} \circ G_{\zeta'\zeta}^{-1}(G_{\zeta'\zeta}(x))]\, dy.\end{aligned} \tag{5.12}$$

Setting $u = G_{\zeta'\zeta}(x)$ and $\omega = \psi \circ \kappa_\zeta^{-1} \circ G_{\zeta'\zeta}^{-1}$, the integral remainder term in the order-zero Taylor expansion gives

$$\omega(u - y) - \omega(u) = -\sum_{\ell=1}^d \int_0^1 y_\ell \cdot \partial_\ell \omega(u - ty)\, dt . \tag{5.13}$$

Therefore, using the Minkowski integral inequality (5.1), and setting $\bar\theta_{\zeta\zeta'} = (\bar\theta_\zeta \bar\theta_{\zeta'}) \circ \kappa_\zeta^{-1}$, we get that the $H^{r'}_p$ norm of (5.12) is bounded by

$$C_\# \sup_{\zeta,\zeta', y\in 2\epsilon \operatorname{supp}(\eta),\ell} \|\bar\theta_{\zeta\zeta'}(x)[\partial_\ell \psi(\kappa_\zeta^{-1}\circ G^{-1}_{\zeta'\zeta})](G_{\zeta'\zeta}(x)-y))\|_{H^{r'}_p} .$$

To conclude the proof of (5.10), use that for all $\ell = 1,\dots,d$, we have $|y_\ell| \le C_\#\epsilon$ for y_ℓ in (5.13) and

$$\bar\theta_{\zeta\zeta'} \cdot \partial_\ell[\psi\circ\kappa_\zeta^{-1}] = \partial_\ell[\bar\theta_{\zeta\zeta'}(\psi\circ\kappa_\zeta^{-1})] - \partial_\ell[\bar\theta_{\zeta\zeta'}]\psi\circ\kappa_\zeta^{-1} ,$$

and that $\|\partial_\ell\omega\|_{H^{r'}_p} \le C_\# \|\omega\|_{H^{r'+1}_p}$. □

5.2. **The stable-averaging operator.** We now discuss the main technical idea used to exploit Dolgopyat's method [20]. It is borrowed from [30] and consists in replacing ψ by its average over a piece of (fake) stable [20] manifold. [21] Put

$$\eta_s : \mathbb{R}^{d_s} \to [0,1] , \quad \eta_s(x) = 1\!\!1_{\{\|x^s\|\le 1\}} , \qquad \eta_{s,\delta}(x^s) = \frac{1}{S_{d_s}\delta^{d_s}} \eta_s(x^s/\delta) ,$$

where S_{d_s} is the volume of the d_s-dimensional unit ball. For fixed small $\delta > 0$ (so that [22] the δ-neighbourhood of $\kappa_\zeta(U_{\zeta,1})$ is included in the domain of κ_ζ^{-1} for each $\zeta = (i,j,\ell)$) we set for each ζ, $\psi \in L^\infty(X_0)$, and $x \in \kappa_\zeta(U_{\zeta,1})$

$$(\mathbb{A}^s_\delta(\psi))_\zeta(x^u,x^s,x^0) = \int_{\mathbb{R}^{d_s}} \eta_{s,\delta}(x^s-y^s)\psi(\kappa_\zeta^{-1}(x^u,y^s,x^0))\, dy^s .$$

(In the above, we implicitly extend ψ by zero outside of its support X_0.)

We set for $\psi \in L^\infty(X_0)$

$$\mathbb{A}^s_\delta(\psi) = \sum_\zeta \bar\theta_\zeta \cdot (\mathbb{A}^s_\delta(\psi))_\zeta \circ \kappa_\zeta . \tag{5.14}$$

We shall assume that δ is small enough so that $\mathbb{A}^s_\delta(\psi)$ is supported in the interior of Y_0. The key estimate on $\mathbb{A}^s_\delta$ is contained in the next lemma.

Lemma 5.5. *Let $1 < p < \infty$ and $-1+1/p < r' < s \le 0$. There exists $C_\# > 0$ so that for every small enough $\delta > 0$, and every bounded function ψ supported in X_0*

$$\|\mathbb{A}^s_\delta(\psi) - \psi\|_{H^{r'}_p(M)} \le C_\# \delta^{s-r'} \|\psi\|_{H^s_p(M)} ,$$

and also $\|1\!\!1_{X_0}\mathbb{A}^s_\delta(\psi) - \psi\|_{H^{r'}_p(X_0)} \le C_\#\delta^{s-r'}\|\psi\|_{H^s_p(X_0)}$.

There is no equivalent to Lemma 5.3 ($\mathbb{A}^s_\delta$ is not a mollifier!).

Proof. We proceed as in Lemma 5.4 to prove the first bound. Note in particular that the argument there only used that $\eta \in L^1$ was a probability density (no smoothness of η was required).

The analogue of (5.11) is

$$(\bar\theta_\zeta\bar\theta_{\zeta'})(\kappa_\zeta^{-1}(x)) \int_{\mathbb{R}^{d_s}} \frac{\eta_s\big(\frac{y^s}{\delta}\big)}{S_{d_s}\delta^{d_s}} \psi(\kappa_\zeta^{-1}(G_{\zeta'\zeta})^{-1}(G_{\zeta'\zeta}(x)-y^s))\, dy^s . \tag{5.15}$$

To prove the analogue of (5.10), we put $\omega = \psi\circ\kappa_\zeta^{-1}\circ G^{-1}_{\zeta'\zeta}$ and consider

$$(\bar\theta_\zeta\bar\theta_{\zeta'})(\kappa_\zeta^{-1}(x)) \int_{\mathbb{R}^{d_s}} \frac{\eta_s\big(\frac{y^s}{\delta}\big)}{S_{d_s}\delta^{d_s}} [\omega(G_{\zeta'\zeta}(x)-y^s) - \omega(G_{\zeta'\zeta}(x))]\, dy^s .$$

[20] In [30], unstable manifolds were used because of the dual nature of the argument there.

[21] In particular, since we do not require the equivalent of [30, Sublemma 3.1] but only the analogue of [30, Sublemma 4.1], we need less smoothness: Liverani [30] required C^4, although he points out that $C^{2+\epsilon}$ should suffice in [30, footnote 6].

[22] In Section 7, we shall need to choose δ small enough as a function of $n = \lceil c\ln|b|\rceil$.

Then, (5.13) is replaced by the following zero-order Taylor expansion with integral remainder term

$$\omega(u-y^s)-\omega(u)=-\sum_{\ell=d_u+1}^{d_u+d_s}\int_0^1 y^s_\ell\cdot\partial_\ell\omega(u-ty^s)\,dt\,. \tag{5.16}$$

Note that, even if ℓ corresponds to a stable coordinate, $\partial_\ell(\psi\circ\kappa_\zeta^{-1}\circ G_{\zeta'\zeta})^{-1})$ can involve all partial derivatives of ψ, since $G_{\zeta'\zeta}$ does not preserve stable leaves in general. We obtain the claimed $C_\#\delta^{s-r'}$ factor by interpolation, since $|y_\ell|\le C_\#\delta$.

The second claim follows from the first one, Corollary 4.2 and the identity $(\mathbb{1}_{X_0}\mathbb{A}^s_\delta(\psi))-\psi=\mathbb{1}_{X_0}(\mathbb{A}^s_\delta(\psi)-\psi)$ for any ψ supported in X_0. □

5.3. **End of the proof of Lemma 3.4 on $\mathcal{R}(z)$.**

End of the proof of Lemma 3.4 on $\mathcal{R}(z)$. Let us deduce Lemma 3.4 from (3.16). The proof is by interpolation, but this interpolation must be done at the level of Triebel spaces. Also, the special form of the admissible charts must be used.

First note that $\mathcal{L}_t\mathcal{R}(z)=\mathcal{R}(z)\mathcal{L}_t$ so that

$$\|\mathcal{R}(z)(\psi)\|_{\widetilde{\mathbf{H}}_p^{r,s,q}(R)}=\sup_{\tau\in[0,t_0]}\|\mathcal{R}(z)\mathcal{L}_\tau(\psi)\|_{\mathbf{H}_p^{r,s,q}(R)}\ .$$

Let us first take $q'=1$, and $q=0$, setting $\widetilde{\psi}=\mathcal{R}(z)(\mathcal{L}_\tau(\psi))$ for $0\le\tau\le t_0$. By Definition 2.5, we must consider

$$\left(\sum_{\zeta=(i,j,\ell,m)\in\mathcal{Z}(R)}\left\|[\rho_m\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|^p_{H_p^{r,s,1}}\right)^{1/p}\ .$$

Now,

$$\begin{aligned}\left\|[\rho_m\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,1}}&\le C_\#\left\|[\rho_m\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\\&\quad+C_\#\left\|\partial_{x^0}[\rho_m\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})\circ\phi_\zeta\ ]\right\|_{H_p^{r,s,0}}\ .\end{aligned}\tag{5.17}$$

We have

$$\begin{aligned}\left\|\partial_{x^0}[\rho_m\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})\circ\phi_\zeta]\right\|_{H_p^{r,s,0}}&\le C_\#\left\|[(\partial_{x_0}\rho_m)\cdot(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\\&\quad+C_\#\left\|[\rho_m\cdot\partial_{x_0}(\widetilde{\psi}\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\ .\end{aligned}\tag{5.18}$$

Recall that $\phi_\zeta(x^u,x^s,x^0)=(F(x^u,x^s),x^s,x^0+f(x^u,x^s))$. If $t>0$ is small enough, since the charts $\kappa_\zeta=\kappa_{i,j,\ell}$ preserve the flow direction and time units, we find

$$\begin{aligned}&\mathcal{R}(z)\mathcal{L}_\tau(\psi((\kappa^R_\zeta)^{-1}(F(x^u,x^s),x^s,x^0+f(x^u,x^s)))\\&\qquad-\mathcal{R}(z)(\mathcal{L}_\tau\psi)((\kappa^R_\zeta)^{-1}(F(x^u,x^s),x^s,x^0-t+f(x^u,x^s))\\&\qquad=(\mathcal{R}(z)\mathcal{L}_\tau(\psi)-\mathcal{R}(z)\mathcal{L}_{\tau+t/R}(\psi))((\kappa^R_\zeta)^{-1}\circ\phi_\zeta)(x)\,.\end{aligned}\tag{5.19}$$

Dividing by t and letting $t\to0$, it follows from (3.16) at $t_1=\tau$ and from (5.17-5.18-5.19) that

$$\begin{aligned}&\left\|[\rho_m\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi))\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,1}}\\&\quad\le C_\#\big(1+\frac{|z|}{R}\big)\left\|[\rho_m\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi))\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\\&\qquad+C_\#\left\|[(\partial_{x_0}\rho_m)\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi)))\circ(\kappa^R_\zeta)^{-1}]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\\&\qquad+C_\#\left\|[\rho_m\cdot(\mathcal{L}_\tau(\psi)\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\right\|_{H_p^{r,s,0}}\ .\end{aligned}\tag{5.20}$$

Since $\sum_{m'\in\mathcal{Z}_{i,j,\ell}(R)} \rho_{i,j,\ell,m'} \equiv 1$, we may use Lemma B.1 (for $\widetilde\beta=\infty$) and (B.4) in Lemma B.3 (using (3.2)=(B.3)) to replace $\partial_{x^0}\rho_m$ by a finite sum of $\rho_{m'}\partial_{x^0}\rho_m$ for neighbours m' of m (the number of neighbours is uniformly bounded, which gives rise to a bounded overcounting). For $q>0$ and $q'\in(q,q+1)$, setting $q''=q(1-(q-q'))$ and applying complex interpolation (see (5.3)) at $\theta=q/q''$ between (5.20) and

$$\begin{aligned}\big\|[\rho_m\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi))\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\big\|_{H^{r,s,q''}_p}&\\ \le \big\|[\rho_m\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi)))\circ(\kappa^R_\zeta)^{-1}]\circ\phi_\zeta\big\|_{H^{r,s,q''}_p}&\,,\end{aligned}$$

we get, since $1\le(\frac{|z|}{R}+1)^{q'-q}$, summing all terms,

$$\begin{aligned}&\Bigg(\sum_{\zeta=(i,j,\ell,m)\in\mathcal{Z}(R)}\Big\|[\rho_m\cdot((\widetilde\psi\circ(\kappa^R_\zeta)^{-1}]\circ\phi_\zeta)\Big\|^p_{H^{r,s,q'}_p}\Bigg)^{1/p}\\
&\le C_\#\big(\frac{|z|}{R}+1\big)^{q'-q}\big[\big(\sum_{\zeta=(i,j,\ell,m)\in\mathcal{Z}(R)}\big\|[\rho_m\cdot(\mathcal{R}(z)(\mathcal{L}_\tau(\psi))\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\big\|^p_{H^{r,s,0}_p}\big)^{1/p}\\
&\qquad+\big(\sum_{\zeta=(i,j,\ell,m)\in\mathcal{Z}(R)}\big\|[\rho_m\cdot(\mathcal{L}_\tau(\psi)\circ(\kappa^R_\zeta)^{-1})]\circ\phi_\zeta\big\|^p_{H^{r,s,q}_p}\big)^{1/p}\big]\,.\end{aligned}$$

In view of (3.15), this ends the proof of Lemma 3.4. □

6. The Dolgopyat estimate

The purpose of this section is to prove the following lemma.

Lemma 6.1. *Assume $d=3$. There exist $C_\#>0$, $\bar\lambda>1$ and $\gamma_0>0$ so that for all $a>1$, $b>1$, $\gamma\ge\gamma_0$, $m\ge C_\# a\gamma\ln b$, and $\widetilde\psi\in H^1_\infty(M)$*

$$\|\mathbb{A}^s_\delta(\mathcal{R}(a+ib)^{2m}(\widetilde\psi))\|_{L^\infty(X_0)}\le C_\# a^{-2m}b^{-\gamma_0}(\|\widetilde\psi\|_{L^\infty(M)}+\nu_a^{-m}\|\widetilde\psi\|_{H^1_\infty(M)})\,,$$

where $\delta=b^{-\gamma}$ and $\nu_a=(1+a^{-1}\ln\bar\lambda)^{-1}$.

In the present section, $C_\#$, γ_0, and $\bar\lambda$ denote constants which depend only on the dynamics and not on r, s, or p.

Note that, since $\overline{\mathcal{R}(a+ib)}=\mathcal{R}(a-ib)$ the obvious counterpart of the above lemma holds true for $b<0$.

The rest of the section consists in the proof of Lemma 6.1 and is a direct, but lengthy, computation.

In the present section, it is more convenient to work directly with the flow rather than with the Poincaré sections, and we will often look at the dynamics at different time steps $\tau_+>\tau_0\gg\tau_-$. Let us be more precise: Remember from (4.5) the choice of $\widetilde\tau_1\le\inf_{i,j,z}\tau_{i,j}(z)$ and $\widetilde\tau_0\ge\sup_{i,j,z}\tau_{i,j}(z)$. Fix once and for all $\tau_0\le\frac14\widetilde\tau_1$ and let $\mu_0=\lceil\frac{\widetilde\tau_0}{\tau_0}\rceil$, $\tau_+=\mu_0k_0\tau_0\ge k_0\widetilde\tau_0$. The size of k_0 will be chosen large enough, but fixed, during the proof of Lemma 6.2. Also, we introduce the following rough bounds for the minimal average expansion and contraction: Let $\bar\lambda_u=\inf_z[\lambda_{u,k_0}(z)]^{\frac{1}{\tau_+}}>1$, $\bar\lambda_s=\sup_z[\lambda_{s,k_0}(z)]^{\frac{1}{\tau_+}}<1$ and set $\bar\lambda=\min\{\bar\lambda_u,\bar\lambda_s^{-1}\}$. Thus the minimal expansion and contraction in a time $t\ge\tau_+$ will be bounded by $\bar\lambda_{u,t}=\bar\lambda^{t-\tau_+}$, $\bar\lambda_{s,t}=\bar\lambda^{-t+\tau_+}$.

This said, we start to compute. First of all it is convenient to localise in time: Consider a smooth function $\widetilde p:\mathbb{R}\to\mathbb{R}$ such that $\operatorname{supp}\widetilde p\subset(-1,1)$, $\widetilde p(s)=\widetilde p(-s)$, and $\sum_{\ell\in\mathbb{Z}}\widetilde p(t-\ell)=1$ for all $t\in\mathbb{R}$. In the following it is convenient to proceed by very small time steps $\tau_-=\frac{\tau_+}{k_1}$, $k_1\in\mathbb{N}$. Let $p(t)=\widetilde p(\frac{t}{\tau_-})$.

For each $f \in L^\infty(M, \mathrm{vol})$ we write

$$
\begin{aligned}
\mathcal{R}(z)^m(f) &= \sum_{\ell\in\mathbb{Z}} \int_0^\infty p(t-\ell\tau_-)\frac{t^{m-1}}{(m-1)!}e^{-zt}\mathcal{L}_t f\, dt\\
&= \sum_{\ell\in\mathbb{N}^*}\int_{-\tau_-}^{\tau_-} p(s)\frac{(s+\ell\tau_-)^{m-1}}{(m-1)!}e^{-z\ell\tau_- - zs}\mathcal{L}_{\tau_-\ell}\mathcal{L}_s f\, ds\\
&\quad+\int_0^{\tau_-} p(s)\frac{s^{m-1}}{(m-1)!}e^{-zs}\mathcal{L}_s f\, ds\,.
\end{aligned}
\tag{6.1}
$$

Next we recall the stable average introduced in Section 5.2. Setting $p_{m,\ell,z}(s) = p(s)\frac{(s+\ell\tau_-)^{m-1}}{(m-1)!}e^{-z\ell\tau_- - zs}$, for each $w \in X_0$ we can write[23]

$$
\mathbb{A}^s_\delta(\mathcal{R}(z)^m(f))(w) = \sum_{\ell\in\mathbb{N}^*,\zeta}\int_{-\tau_-}^{\tau_-} p_{m,\ell,z}(s)\bar\theta_\zeta(w)\int_{W^s_{\delta,\zeta}(w)} \widetilde{\mathfrak{p}}_{\delta,\zeta}(w,\xi)\cdot\mathcal{L}_{\ell\tau_-}\mathcal{L}_s f(\xi)\,,
\tag{6.2}
$$

where (see Section 5.2) $W^s_{\delta,\zeta}(w) = \{\kappa_\zeta^{-1}(\widetilde\kappa_\zeta(w)^u, y^s, \widetilde\kappa_\zeta(w)^0)\}_{y^s\in[-\delta,\delta]} \cap X_0$. The integral is meant with respect to the volume form determined by the Riemannian metric restricted to $W^s_{\delta,\zeta}$ and $\widetilde{\mathfrak{p}}_{\delta,\zeta}(w,\xi) = \eta_{s,\delta}(\kappa_\zeta(w)^s - \kappa_\zeta(\xi)^s)J_\zeta(w,\xi)$, where J_ζ is the Jacobian of the change of coordinates κ_ζ restricted to the manifold $W^s_{\delta,\zeta}$. Note that if w is extremely close to a corner of X_0, then $W^s_{\delta,\zeta}(w)$ could be extremely short, yet in such a case the integral will be trivially small. We can rewrite (6.2) as

$$
\begin{aligned}
\mathbb{A}^s_\delta(\mathcal{R}(z)^m(f))(w) &= \sum_{\ell\in\mathbb{N}^*,\zeta}\int_{-\tau_-}^{\tau_-} p_{m,\ell,z}(s)\bar\theta_\zeta(w)\int_{T_{-\ell\tau_-}W^s_{\delta,\zeta}} \widetilde{\mathfrak{p}}_{\delta,\zeta}\circ T_{\ell\tau_-}\cdot J^s_{\ell\tau_-}\cdot\mathcal{L}_s f\\
&= \sum_{\ell\in\mathbb{N}^*,\zeta}\bar\theta_\zeta(w)\sum_{W\in\mathcal{W}_\ell(w)}\int_{-\tau_-}^{\tau_-} p_{m,\ell,z}(s)\int_W \widetilde{\mathfrak{p}}_{\delta,\zeta}\circ T_{\ell\tau_-}\cdot J^s_{\ell\tau_-}\mathcal{L}_s f\,,
\end{aligned}
\tag{6.3}
$$

where $J^s_{\ell\tau_-}$ is the Jacobian of the change of variable, $\int_{W^s_{\delta,\zeta}(w)}\widetilde{\mathfrak{p}}_{\delta,\zeta}(w,z) \le 1$, and $\mathcal{W}_\ell(w) := \{W_\alpha\}_{\alpha\in A_\ell(w)}$ is a decomposition of $T_{-\ell\tau_-}W^s_{\delta,\zeta}$ in *regular* connected pieces. By *regular* we mean that there exists $t \in [0,\tau_-]$ such that $T_{-t}W_\alpha$ is a C^2 manifold. [24]

The decomposition $\mathcal{W}_\ell$ is performed as follows. We choose $L_0 > 0$ such that a curve in the stable cone of length $L_0\|\lambda_{s,\mu_0 k_0}\|_{L^\infty}$ intersecting an O_i must lie entirely in its $\tau_0/4$ neighbourhood and then we define $\mathcal{W}_{k_1\kappa}$, $\kappa \in \mathbb{N}$, recursively: [25] First, let $\mathcal{W}_0$ be the collection of the connected components of $W^s_{\delta,\zeta} \setminus \cup_{i,j}\partial B_{i,j}$. Given $\mathcal{W}_{k_1\kappa}$ define first $\widetilde{\mathcal{W}}_{k_1(\kappa+1)}$ to be the union of the connected regular pieces of the curves $T_{-\tau_+}W$ for $W \in \mathcal{W}_{k_1\kappa}$. Next, if a curve $W \in \widetilde{\mathcal{W}}_{k_1(\kappa+1)}$ is longer than L_0, we decompose it in curves of length $\frac12 L_0$ apart from the last piece that will have length in the interval $[\frac12 L_0, L_0)$. The set of curves so obtained is $\mathcal{W}_{k_1(\kappa+1)}$. Finally, for the $\ell \in \{k_1\kappa+1, \dots, k_1(\kappa+1)-1\}$ we define $\mathcal{W}_\ell$ as the collection of the connected regular pieces of $T_{(k_1\kappa-\ell)\tau_-}W$, $W \in \mathcal{W}_{k_1\kappa}$. We will call the curves shorter than $L_0/2$ *short,* and the others *long.* Note that, by construction, for each $\alpha \in A_\ell$ there

[23] Here and in the following, when we write $\sum_{\ell\in\mathbb{N}^*}$ we mean to include implicitly also the last term in (6.1). In any case, we will see in (6.12) that the total contributions of the first terms in the sum is negligible.

[24] The issue here is that if W_α intersects one O_i, then it may be discontinuous. Yet, such a lack of smoothness is only superficial since once the manifold is flowed past the section, it becomes smooth. More precisely, if W_α does not intersect any O_i, then it has uniformly bounded curvature (i.e., if g is its parametrisation by arc-length, then $\|g''\|_{L^\infty} \le C_\#$). This can be proved exactly as one proves the same bound on the curvature of the stable leaves of an Anosov map, see [28] for details.

[25] For simplicity we suppress the dependence on w, ζ when this does not create confusion.

exists $t_\alpha \in [0,\tau_0]$ such that $T_{-t_\alpha}W_\alpha$ is a C^2 curve with uniform C^2 norm and $T_{\ell\tau_+-t_\alpha}$, restricted to $T_{-t_\alpha}W_\alpha$, is a C^2 map.

The density $\widetilde{\mathfrak{p}}_{\delta,\zeta}$ has the property $|\nabla \log \widetilde{\mathfrak{p}}_{\delta,\zeta}|_\infty \le C_\#$, for some fixed constant $C_\#$. Then, setting

$$\mathfrak{p}_{\ell,\alpha} = \frac{\widetilde{\mathfrak{p}}_{\delta,\zeta}\circ T_{\ell\tau_-}\cdot J^s_{\ell\tau_-}}{Z_{\ell,\alpha}} \quad ; \quad Z_{\ell,\alpha} = \int_{W_\alpha} \widetilde{\mathfrak{p}}_{\delta,\zeta}\circ T_{\ell\tau_-}\cdot J^s_{\ell\tau_-}\,, \tag{6.4}$$

we have $|\nabla \log \mathfrak{p}_{\ell,\alpha}|_\infty \le C_\#$, provided $C_\#$ is chosen large enough.[26] In addition, note that $\sum_\alpha Z_{\ell,\alpha} \le 1$.[27]

Next, it is convenient to define the r-boundary $\partial_r(\mathcal{W}_\ell)$ of the family $\mathcal{W}_\ell$:

$$\partial_r(\mathcal{W}_\ell) = \cup_{\alpha\in A_\ell}\{x \in W_\alpha \ : \ d(x,\partial W_\alpha) \le r\}\,,$$

where, given any two sets A, B, we define $d(A,B) = \inf_{x\in A, y\in B} d(x,y)$. Not surprisingly, we will call $|\partial_r(\mathcal{W}_\ell)| := \sum_{\alpha\in A_\ell} Z_{\ell,\alpha}\int_{\{d(x,\partial W_\alpha)\le r\}} \mathfrak{p}_{\ell,\alpha}$ the measure of the r-boundary of $\mathcal{W}_\ell$.[28]

Lemma 6.2. *There exists $\sigma \in (0,1)$ such that, for all δ small enough, $|\partial_r(\mathcal{W}_\ell)| \le C_\# \max\{r, \sigma^{\frac{\ell}{k_1}} r\delta^{-1}\}$. In addition, for each $\upsilon \in (0,1)$ and $\sigma^\ell < \delta^{k_1}$, there exists $C_\upsilon > 0$ such that the measure of $\cup_{j\le m} T_{j\tau_-}\partial_{\upsilon^{j\tau_-}r}(\mathcal{W}_j)$ is bounded by $C_\upsilon r$.*

Proof. If $x \in \partial_r(\mathcal{W}_\ell)$ then we have the following possibilities

(1) $T_{\ell\tau_-}x$ belongs to a $r\bar\lambda^{-\tau_-\ell}$-neighbourhood of $\partial\left(W^s_{\delta,\zeta}\setminus \cup_{i,j}O_{i,j}\right)$.
(2) there exists $n \in \mathbb{N}$, $n \le \ell/k_1$, such that $T_{n\tau_+-t}x$, $t \in [0,\tau_+]$, intersects the $\bar\lambda^{-\tau_-(\ell-nk_1)}r$ neighbourhood of the *lateral* boundaries of the flow boxes, i.e., $\cup_{i,j}\partial B_{i,j}\setminus(O_{i,j}\cup T_{\tau_{i,j}}(O_{i,j}))$.

Let us call $\partial^1_r(\mathcal{W}_\ell)$ and $\partial^2_r(\mathcal{W}_\ell)$, respectively, the parts of $\partial_r(\mathcal{W}_\ell)$ that satisfy the above two conditions. Clearly, $|\partial^1_r(\mathcal{W}_\ell)| \le C_\#\bar\lambda^{-\tau_-\ell}\delta^{-1}r$.

To analyze the second case, we must follow the creation of the W_α. By the complexity assumption 1.5, for each $\nu \in (0,1/2)$ we can chose $k_0 \in \mathbb{N}$ and $L_0 > 0$ so that for each curve W in the stable direction and of size smaller than L_0, the number of smooth connected pieces of $T_{-\tau_+}W$ is smaller than $\bar\lambda^{\tau_+}\nu$.

Remark that, by construction, there exists $c_1 > 1$ such that, for each $j \in \{0,\dots,k_1\}$ and $r' > 0$, $T_{j\tau_-}\partial_{r'}(\mathcal{W}_{\kappa k_1+j}) \subset T_{k_1\tau_-}\partial_{c_1 r'}(\mathcal{W}_{(\kappa+1)k_1})$. Accordingly, it suffices to study $\partial_r(\mathcal{W}_{k_1\ell'})$, $\ell' \in \mathbb{N}$. Let $\overline{\mathcal{W}}_j = \mathcal{W}_{k_1 j}$.

For each $W' \in \overline{\mathcal{W}}_\kappa$ we say that $W'' \in \overline{\mathcal{W}}_j$, $j \le \kappa$, is its *ancestor* if W'' is long and, for each $l \in \{0,\dots,\kappa-j-1\}$, $T_{l\tau_+}W'$ never belongs to a long element of $\overline{\mathcal{W}}_{\kappa-l}$. Now for each ancestor $W'' \in \overline{\mathcal{W}}_\kappa$, we can consider the short pieces that are generated[29] in $\overline{\mathcal{W}}_{\ell'}$, $\ell' = \frac{\ell}{k_1}$, by the complexity bound their number is less than $\bar\lambda^{\tau_+(\ell'-\kappa)}\nu^{\ell'-\kappa}$. Note that the image of a curve of length r in $W \in \overline{\mathcal{W}}_{\ell'}$ under $T_{\tau_+(\ell'-\kappa)}$ will be of length smaller than $\bar\lambda^{-\tau_+(\ell'-\kappa)}r$. Thus, the union of the images, call it $P_{\ell',\kappa,r}$, of the r-boundary of the short pieces of $\overline{\mathcal{W}}_{\ell'}$ in an ancestor belonging to $\overline{\mathcal{W}}_\kappa$ will have total length bounded by $\nu^{\ell'-\kappa}r$. By the usual distortion estimate

[26] Indeed, the flow induces one-dimensional maps between $T_{\kappa\tau_0-t_\kappa}W_\alpha$ and $T_{(\kappa+1)\tau_0-t_{\kappa+1}}W_\alpha$ ($t_k \in [0,\tau_0]$ properly chosen), which, by parametrising the curves by arc-length, are uniformly C^2. So the claim follows by the usual distortion results on one-dimensional maps, see [28] for details.

[27] In fact, the sum is exactly equal to one if no manifold is cut by the boundary of X_0.

[28] Note that $|\partial_r\mathcal{W}_\ell| = \int_{T_{\ell\tau_-}[\partial_r(\mathcal{W}_\ell)]}\widetilde{\mathfrak{p}}_{\delta,\zeta}$ and hence the measure of $T_{\ell\tau_-}[\partial_r(\mathcal{W}_\ell)]$ is bounded, above and below, by $C_\#\delta|\partial_r(W_\ell)|$.

[29] That is the set of short pieces in $\overline{\mathcal{W}}_{\ell'}$ whose ancestor is the given curve.

(see footnote 26) this implies that, calling m the induced Riemannian measure on $W^s_{\delta,\zeta}$,

$$\frac{m(T_{\kappa\tau_+}P_{\ell',\kappa,r})}{m(T_{\kappa\tau_+}(W''))} \leq C_\# L_0^{-1}\nu^{\ell'-\kappa} r\,.$$

Thus the measure of the image, in $W^s_{\delta,\zeta}$, of the r-boundary belonging to short pieces with an ancestor in $\overline{\mathcal{W}}_\kappa$ will have measure bounded by $C_\#\delta r\nu^{\ell'-\kappa}$. Hence, the total measure of such pieces will be bounded by $C_\#\delta r$, while the number of pieces that do not have any ancestor must be less than $\lambda^{-\tau_+\ell'}\nu^{\ell'}$ and thus their total measure will be less than $\nu^{\ell'}r$. The first statement follows then by footnote 28 choosing[30]

$$\sigma = \max\{\bar\lambda^{-\tau_+},\nu\}^{\frac{1}{2}}\,. \tag{6.5}$$

The last statement follows by applying the previous results together with footnote 28 again. □

Next, it is convenient to localise in space as well. To this end we need to define a sequence of smooth partitions of unity.

Given a parameter $\theta \in (0,1)$ to be chosen later, there exists $C_\# > 0$ such that, for each $r \in (0,1)$, there exists a C^∞ partition of unity $\{\phi_{r,i}\}_{i=1}^{q(r)}$ enjoying the following properties [31]

(i) for each $i \in \{1,\dots,q(r)\}$, there exists $x_i \in U_{\zeta_i,1} \subset M$ such that $\phi_{r,i}(z) = 0$ for all $z \notin B_{r^\theta}(x_i)$ (the ball, in the sup norm of the chart κ_{ζ_i}, of radius r^θ centered at x_i);
(ii) for each r, i we have $\|\nabla\phi_{r,i}\|_{L^\infty} \leq C_\# r^{-\theta}$;
(iii) $q(r) \leq C_\# r^{-3\theta}$.

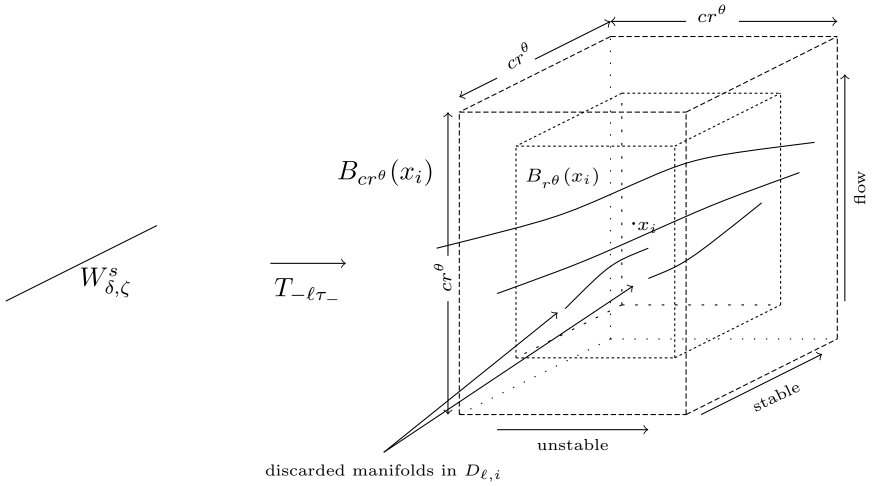

FIGURE 1. The manifolds $\{W_\alpha\}_{\alpha\in A_{\ell,i}}$.

Fix $c > 2$. For each x_i, let $A_{\ell,i} = \{\alpha \in A_\ell \;:\; W_\alpha \cap B_{r^\theta}(x_i) \neq \emptyset\}$, $D_{\ell,i} = \{\alpha \in A_{\ell,i} \;:\; \partial(W_\alpha \cap B_{cr^\theta}(x_i)) \not\subset \partial B_{cr^\theta}(x_i)\}$, $E_{\ell,i} = A_{\ell,i} \setminus D_{\ell,i}$. Call the manifolds with index in $D_{\ell,i}$ the *discarded manifolds*. We choose c large enough so that a manifold intersecting $B_{r^\theta}(x_i)$ can intersect only the front and rear vertical part of the boundary of $B_{cr^\theta}(x_i)$, see figure 1.[32]

---

[30] The square root is for later convenience, see the proof of Lemmata 6.3 and 6.8.

[31] For example, using the function $\widetilde{p}$ introduced to partition in time, one can define, in the charts κ_ζ, $\widetilde{p}(\bar k r^\theta/2 + 2r^{-\theta}\tau_-\eta)\widetilde{p}(\bar j r^\theta/2 + 2r^{-\theta}\tau_-\xi)\widetilde{p}(\bar i r^\theta/2 + 2r^\theta/2\tau_- s)$.

[32] This can be achieved thanks to the fact that the W_α belong to the stable cone.

Set $W_{\alpha,i} = W_\alpha \cap B_{cr^\theta}(x_i)$ and

$$\mathfrak{p}_{\ell,\zeta,\alpha,i} = \frac{\widetilde{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \cdot J^s_{\ell\tau_-}}{Z_{\ell,\alpha,i}} \quad ; \quad Z_{\ell,\alpha,i} = \int_{W_{\alpha,i}} \widetilde{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \cdot J^s_{\ell\tau_-}. \tag{6.6}$$

Moreover it is now natural to chose

$$k_1 = \lceil r^{-\theta}\tau_+\rceil, \text{ hence } \tau_- \cong r^\theta . \tag{6.7}$$

Our next step is to estimate the contribution of the manifolds $W_{\alpha,i}$, $\alpha \in D_{\ell,i}$. Let $K_i \subset D_{\ell,i}$ be the collection of indices for which $W_{\alpha,i} \cap (\cup_j O_j) \neq \emptyset$, then $\cup_{\alpha \in D_{\ell,i}\setminus K_i} W_{\alpha,i} \subset \partial_{cr^\theta}(\mathcal{W}_\ell)$, hence, by Lemma 6.2 the total measure of the elements in $D_{\ell,i} \setminus K_i$, is bounded by $C_\# \max\{r^\theta, \sigma^{\frac{\ell}{k_1}} r^\theta \delta^{-1}\}$.[33] It remains to estimate how many pieces are cut by a manifold O_i. This is done in the following lemma whose proof can be found at the end of Appendix E.

Lemma 6.3. *There exists $\ell_0 \geq 0$ such that for each $\ell \geq k_1\ell_0$, any codimension-one disk*[34] *$\widetilde{O}$ of measure $S > 0$ and $\rho_* > 0$, if we let $D_\ell(\widetilde{O}, \rho_*) = \{W_{\beta,i} \,:\, d(W_{\beta,i}, \widetilde{O}) \leq \rho_*\}$, then*

$$\sum_{\{(\beta,i):W_{\beta,i}\in D_\ell(\widetilde{O},\rho_*)\}} Z_{\beta,i} \leq C_\# \left[\sqrt{S(\rho_* + r^\theta)} + S\sigma^\ell + \delta^{-1}\sigma^\ell\right] .$$

Next, it is convenient to introduce the parameter $c_* \in (0, 1)$ defined by

$$c_* e a \tau_+ = \sigma , \tag{6.8}$$

where $a = \Re(z)$, and assume that m is such that

$$\sigma^{c_* m} \leq \delta r^{\frac{\theta}{2}}. \tag{6.9}$$

Applying Lemma 6.3 with $\rho_* = r^\theta$, $\widetilde{O} = O_l$, we have that

$$\sum_i \sum_{\beta\in K_i} Z_{\beta,i} \leq C_\# r^{\frac{\theta}{2}} .$$

Thus,[35]

$$\left| \sum_{\ell\geq c_* k_1 m} \sum_{k,i} \sum_{\alpha\in D_{\ell,i}} \int_{-\tau_-}^{\tau_-} p_{m,\ell,z}(s) \int_{W_{\alpha,i}} \widetilde{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \cdot J^s_{\ell\tau_-} \cdot \mathcal{L}_s f \right| \leq C_\# \frac{r^{\frac{\theta}{2}}}{a^m} |f|_\infty . \tag{6.10}$$

We are left with the small ℓ and the elements of $E_{\ell,i}$. To treat these cases, it is convenient to introduce extra notation. For each $\alpha \in A_\ell$, define $W^c_\alpha = \cup_{t\in[-\tau_-,\tau_-]} W_\alpha$. For each $W_\alpha \subset U_{\zeta',0}$, the manifold $\kappa_{\zeta'}(W^c_\alpha)$ can be seen as the graph of $\mathbb{F}_\alpha(\xi, s) := (F_\alpha(\xi), y_\alpha + \xi, N_\alpha(\xi) + s)$, $\xi, s \in \mathbb{R}$, where F_α, N_α are uniformly C^2 functions and $\mathbb{F}_\alpha(\xi, 0)$ is the graph of W_α.

Given the above discussion, to estimate the integrals in equation (6.2) it suffices to estimate the integrals

$$\int_{-\tau_-}^{\tau_-} p_{m,\ell,z}(s) \int_{W_\alpha} \widetilde{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \cdot J^s_{\ell\tau_-} \cdot \mathcal{L}_s f = \int_{W^c_\alpha} \bar{p}_{m,\ell,z,\alpha} \cdot \bar{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \cdot \bar{J}^s_{\ell\tau_-} \cdot f , \tag{6.11}$$

where $\bar{p}_{m,\ell,z,\alpha} \circ \kappa^{-1}_{\zeta'} \circ \mathbb{F}_\alpha(\xi, s) = -p_{m,\ell,z}(-s)$, $\bar{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \circ \kappa^{-1}_{\zeta'} \circ \mathbb{F}_\alpha(\xi, s) = \widetilde{\mathfrak{p}}_{\delta,\zeta} \circ T_{\ell\tau_-} \circ \kappa^{-1}_{\zeta'} \circ \mathbb{F}_\alpha(\xi, 0)$, the same for $\bar{J}^s_{\ell,\nu}$ apart from a factor taking into account the speed

[33]Remember that the $W_{\alpha,i}$ inherit from the $B_{cr^\theta}(x_i)$ the property of having a uniformly bounded number of overlaps.

[34]Here we assume the curvature of the $\widetilde{O}$ to be bounded by some fixed constant.

[35]Note that if a box $B_{cr^\theta}(x_i)$ is cut by a manifold O_i or by the boundary of X_0, then it is possible that $E_{\ell,i} = \emptyset$. In particular, the present estimate bounds the contributions of all the $W_{\alpha,i}$ for such a "bad" box.

of the flow, and the integrals are taken with respect to the volume form induced by the Riemannian metric.

Substituting (6.11) in (6.2), setting $\bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i} = \frac{\bar{\mathfrak{p}}_{\delta,\zeta}\circ T_{\ell\tau_-}\cdot \bar{J}^s_{\ell\tau_-}}{Z_{\ell,\alpha,i}}$ and remembering (6.10),[36]

$$
\begin{aligned}
\mathbb{A}^s_\delta(\mathcal{R}(z)^m(f)) &= \sum_{\ell\in\mathbb{N}^*}\sum_{\zeta,i}\sum_{\alpha\in A_{\ell,i}} Z_{\ell,\alpha,i}\bar{\theta}_\zeta \int_{W^c_{\alpha,i}} \bar{p}_{m,\ell,z,\alpha}(s)\phi_{r,i}\cdot\bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i}\cdot f\\
&= \sum_{\ell\geq c_*mk_1}\sum_{\zeta,i}\sum_{\alpha\in E_{\ell,i}} Z_{\ell,\alpha,i}\bar{\theta}_\zeta \int_{W^c_{\alpha,i}} \bar{p}_{m,\ell,z,\alpha}(s)\phi_{r,i}\cdot\bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i}\cdot f \qquad (6.12)\\
&\quad + \mathcal{O}(|f|_\infty([c_*ea\tau_+]^m + r^{\frac{\theta}{2}})a^{-m},
\end{aligned}
$$

Note that (6.9) implies $[c_*ea\tau_+]^m = \sigma^m \leq \delta r^{\frac{\theta}{2}}$.

To continue, following Dolgopyat, we must show that the sum over the manifolds in $E_{\ell,i}$ contains a lot of cancellations and this leads to the wanted estimate. In Dolgopyat scheme such cancellations take place when summing together manifolds that are at a distance larger than r^ϑ, for some properly chosen $\vartheta > \theta$. To make the cancellations evident one must compare different leaves via the unstable holonomy (using the fact that it is C^1). In the present case, due to the discontinuities, the unstable holonomy is defined only on a Cantor set. To overcome this problem, we construct in Appendix D an approximate holonomy which is Lipschitz. Next (following [30]), we use the fact that the flow is contact to show that Lipschitz suffices to have the wanted cancellations, see Appendix E. Unfortunately, the approximate holonomy is efficient only when its fibers are very short, in particular one cannot hope to use it effectively to compare leaves that are at a distance r^θ. We need then to collect our weak leaves into groups that are at a distance smaller than r and require that $\vartheta > 1$.

To start with, we consider the line [37] $x_i+(u,0,0)$, $u\in[-r^\theta, r^\theta]$, and we partition it in intervals of length $r/3$. To each such interval I we associate a point $x_{i,j}\in\cup_{\alpha\in E_{\ell,i}}W^c_\alpha\cap I$, if the intersection is not empty. Next, we associate to each point $x_{i,j}$ Reeb coordinates $\widetilde{\kappa}_{x_{i,j}}$. More precisely, let $x_{i,j}\in W^c_\alpha$, we ask that $x_{i,j}$ is at the origin in the $\widetilde{\kappa}_{x_{i,j}}$ coordinates, that $\widetilde{\kappa}_{x_{i,j}}(W^c_\alpha)\subset\{(0,y,z)\}$, $y,z\in\mathbb{R}$, and that the vector $D_{x_{i,j}}T_{-\ell\tau_-}(1,0,0)$ belongs to the unstable cone. Such changes of coordinates exist and are all uniformly smooth by Lemma A.4.

For each $x_{i,j}$, let us consider (in the coordinates $\widetilde{\kappa}_{x_{i,j}}$) the box $\mathcal{B}_r = \{(\eta,\xi,s) : |\eta|\leq r,\ |\xi|\leq r^\theta,\ |s|\leq r^\theta\}$. We set $\mathcal{B}_{r,i,j} = \widetilde{\kappa}^{-1}_{x_{i,j}}(\mathcal{B}_r)$. The next lemma ensures that Figure 2 is an accurate representation of the manifolds intersecting $\mathcal{B}_r$.

Lemma 6.4. *There exists $c>0$ such that, if a manifold W_α, $\alpha\in E_{\ell,i}$, intersects $\mathcal{B}_{r,i,j}$, then $\widetilde{\kappa}_{x_{i,j}}(W^c_\alpha)\cap\partial\mathcal{B}_{cr,i,j}$ is contained in the unstable boundary of $\mathcal{B}_{cr,i,j}$, provided $\theta\in(\frac{1}{2},1)$ and $\bar{\lambda}^{-c_*m} < r^{\frac{1-\theta}{2}}$.*

Proof. By construction, for each $\mathcal{B}_{r,i,j}$ there is a manifold W^c_α going through its center and perpendicular to $(1,0,0)$ (in the $\widetilde{\kappa}_{x_{i,j}}$ coordinates). Let $\tau_\alpha\in[-\tau_-,\tau_-]$ be such that $\widetilde{W}_\alpha = T_{\tau_\alpha}W_\alpha\cap I\neq\emptyset$.[38]

[36] The sum over the first c_*mk_1 elements is estimated by $|f|_\infty$ times the integral

$$
C_\# \frac{a^{-m+1}}{(m-1)!}\int_0^{c_*m\tau_+} e^{-ax}(xa)^{m-1}dx \leq C_\# a^{-m}\frac{(c_*ma\tau_+)^{m-1}}{(m-1)!} \leq C_\# a^{-m}(c_*ae\tau_+)^m,
$$

where we have used the Stirling formula.

[37] In the κ_{ζ_i} coordinates.

[38] When no confusion arises, to ease notation, we will identify $\widetilde{\kappa}_{x_{i,j}}(W^c_\alpha)$ and W^c_α.

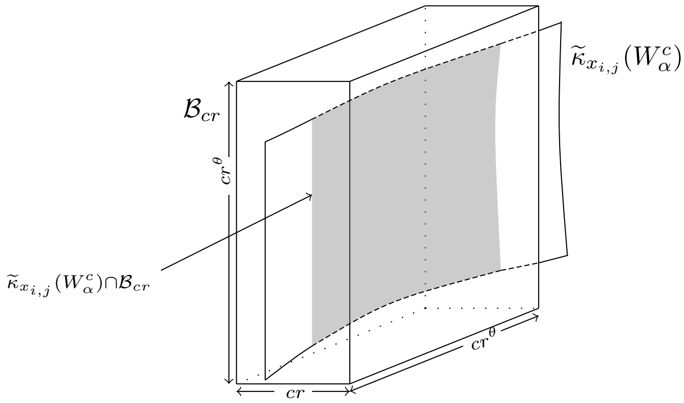

FIGURE 2. A manifold intersecting $\mathcal{B}_r$.

Next, consider another W_β^c, $\beta \in E_{\ell,i}$, intersecting $\mathcal{B}_{r,i,j}$ and let u be the intersection point with I. Again, let $\widetilde{W}_\beta = T_{\tau_\beta} W_\beta$. Consider the segment $J = [0, u]$ and its trajectory $T_t J$, $t \in [0, \ell\tau_-]$. Let t_0 be the first time t for which $T_{t-\tau_-} J \cap (\cup_i \partial O_i) \neq \emptyset$. Since $T_{t_0}\widetilde{W}_\alpha, T_{t_0}\widetilde{W}_\beta$ are uniformly transversal to $\cup_{t\in[0,2\tau_-]} T_t \partial O_i = \partial^c O_i$ and do not intersect it, it follows that their length must be smaller than $C_\#|T_{t_0} J|$, see figure 3.

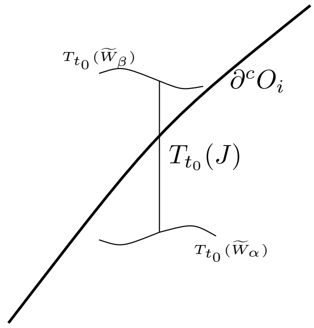

FIGURE 3. Meeting ∂O_i.

Hence, setting $\Lambda = \frac{|\widetilde{W}_{\alpha,i}|}{|T_{t_0}\widetilde{W}_{\alpha,i}|}$, by the invariance of the contact form it follows that

$$C_\#^{-1}\Lambda \leq \frac{|T_{t_0}(J)|}{|J|} \leq C_\#\Lambda\,.$$

Thus,

$$cr^\theta \leq C_\#\Lambda|T_{t_0}\widetilde{W}_{\alpha,i}| \leq C_\#\Lambda|T_{t_0}(J)| \leq C_\#\Lambda^2|J| \leq C_\#\Lambda^2 r\,. \tag{6.13}$$

It follows that the forward dynamics in a neighbourhood of J behaves like in the smooth case until it experiences a hyperbolicity Λ of order at least $C_\# r^{-\frac{1-\theta}{2}}$. Moreover our conditions imply that this amount of hyperbolicity will be achieved in the time we are considering. In turn, this means that the tangent spaces of $\widetilde{W}_\alpha$ and $\widetilde{W}_\beta$, at zero and u respectively, differ at most of $C_\# r^{1-\theta}$.[39] Thus the tangents to

[39]Since hyperbolicity Λ implies that the image of the cone (which contains the tangent to the manifolds) has size, in the horizontal plane, Λ^2 while the axes of two cones at a distance r can differ at most by $C_\# r$, which can be proved in the same manner in which is proven the C^1 regularity of the foliation in the case of a smooth map, see [28].

the manifolds at the above points can at most be at a distance $C_\# r$ inside the box. By the uniform C^2 bounds of the manifolds it follows that the distance between the two manifolds must be bounded by $C_\#(r + r^{2\theta})$. From this, Lemma 6.4 easily follows by choosing c large enough. □

Remark 6.5. Note that the estimates on the tangent plane to the manifolds contained in the previous lemma imply that the "angle" between two nearby boxes $\mathcal{B}_{r,i,j}$ is of the order $r^{1-\theta}$, hence the maximal distance in the unstable direction is of order r. This implies that the covering $\{\mathcal{B}_{r,i,j}\}$ has a uniformly bounded number of overlaps.

Returning to the proof of Lemma 6.1, in view of the previous result it is convenient decompose $E_{\ell,i}$ as $\cup_j E_{\ell,i,j}$ where if $\alpha \in E_{\ell,i,j}$, then $\widetilde{\kappa}_{x_{i,j}}(W_{\alpha,i}) \cap \mathcal{B}_r \neq \emptyset$. We can then rewrite (6.12) as

(6.14)
$$\begin{aligned}\mathbb{A}^s_\delta(\mathcal{R}(z)^m(f)) = \sum_{\ell \geq c_* m k_1} \sum_{\zeta,i,j} \sum_{\alpha \in E_{\ell,i,j}} Z_{\ell,\alpha,i} \bar\theta_\zeta \int_{W^c_{\alpha,i}} \bar p_{m,\ell,z,\alpha}(s) \phi_{r,i} \cdot \bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i} \cdot f \\ + \mathcal{O}(|f|_\infty r^{\frac{\theta}{2}}) a^{-m},\end{aligned}$$

To elucidate the cancellation mechanism it is best to fix ℓ, i, j and use the above mentioned charts (in fact from now on, we will call the quantities in such a coordinate charts with the same names of the corresponding ones in the manifold). For each $\alpha \in E_{\ell,i,j}$, $W_\alpha \cap \mathcal{B}_{cr}$ can be seen as the graph of $\mathbb{F}_\alpha(\xi) := (F_\alpha(\xi), \xi, N_\alpha(\xi))$ for $\|\xi\| \leq cr$, where F_α, N_α are uniformly C^2 functions, and, by Lemma 6.4, $|F'_\alpha| \leq C_\# r^{1-\theta}$. Note that, since both α and $d\alpha$ are invariant under the flow, and the manifolds are the image of manifolds with tangent space in the kernel of both forms, it follows that $N'_\alpha(\xi) = \xi F'_\alpha(\xi)$.

To simplify notations, let us introduce the functions[40]

(6.15)
$$\begin{aligned}&\widetilde{\mathbb{F}}_\alpha(\xi, s) = (F_\alpha(\xi), \xi, N_\alpha(\xi) - s)\\ &\Xi_{\ell,r,i,\zeta,\alpha} = \phi_{r,i} \cdot \bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i}\\ &\mathbf{F}_{\ell,m,i,\alpha}(\xi, s) = p(s) \frac{(\ell\tau_- - s)^{m-1}}{(m-1)!} e^{-z\ell\tau_- + as} \cdot \Xi_{\ell,r,i,\zeta,\alpha} \circ \mathbb{F}_\alpha(\xi, s) \cdot \Omega_\alpha(\xi),\end{aligned}$$

where $\Omega_\alpha ds \wedge d\xi$ is the volume form on W^c_α in the coordinates determined by $\mathbb{F}_\alpha$. Note that[41]

(6.16)
$$\begin{aligned}&|\mathbf{F}_{\ell,m,i,\alpha}|_\infty \leq C_\# r^{-\theta} \frac{(\ell\tau_-)^{m-1}}{(m-1)!} e^{-a\ell\tau_-}\\ &\|\mathbf{F}_{\ell,m,i,\alpha}\|_{\text{Lip}} \leq C_\# r^{-2\theta} \frac{(\ell\tau_-)^{m-1}}{(m-1)!} e^{-a\ell\tau_-}.\end{aligned}$$

We can then write,

(6.17)
$$\begin{aligned}&\int_{W^c_{\alpha,i}} \bar p_{m,\ell,z,\alpha} \phi_{r,i} \cdot \bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i} \cdot f\\ &= \int_{-\tau_-}^{\tau_-} ds \int_{\|\xi\| \leq cr^\theta} d\xi \, e^{ibs} \mathbf{F}_{\ell,m,i,\alpha}(\xi, s) f(F_\alpha(\xi), \xi, N_\alpha(\xi) + s).\end{aligned}$$

At this point, we would like to compare different manifolds by sliding them along an approximate unstable direction. To this end, we use the approximate unstable

[40] To ease notation we suppress some indices.

[41] By tracing the definition of $\bar{\mathfrak{p}}_{\delta,\zeta,i}$ just after (6.11), of $Z_{\ell,\alpha,i}$ and $\mathfrak{p}_{\ell,\zeta,\alpha,i}$ in (6.6), and of $\widetilde{\mathfrak{p}}_{\delta,\zeta}$ after (6.2), it follows that the only large contribution to the Lipschitz norm comes from $\phi_{r,i}$ and p. In particular (6.6) implies that $|\mathfrak{p}_{\ell,\zeta,\alpha,i}| \leq C_\# r^{-\theta}$, the other estimate follows then by property (ii) of the partition.

fibers $\Gamma^{\varkappa}_{i,j,r}$ constructed in Appendix D. In short, for each coordinates $\widetilde{\kappa}_{x_{i,j}}$, we can construct a Lipschitz foliation in $\mathcal{B}_{cr}$ in a $\rho = r^{\varsigma}$ neighbourhood of the "stable" fiber (which is of length $\varrho = r^{\theta}$). In order to have the foliation defined in all $\mathcal{B}_{cr}$ we need $\varsigma < 1$, while for the foliation to have large part where it can be smoothly iterated backward as needed it is necessary that $\varsigma > \theta$, we thus impose

$$\theta < \varsigma < 1. \tag{6.18}$$

The foliation can be locally described by a coordinate change $\mathbb{G}_{i,j,\varkappa}(\eta, \xi, s) = (\eta, G_{i,j,\varkappa}(\eta, \xi), H_{i,j,\varkappa}(\eta, \xi)+s)$ so that the fiber $\Gamma^{\varkappa}_{i,j,r}(\xi, s)$ is the graph of $\mathbb{G}_{i,j,\varkappa}(\cdot, \xi, s)$. Note that, by construction, $\|G'_{i,j,\varkappa}\|$ is small. We consider the holonomy $\Theta_{i,j,\alpha,\varkappa} : W_\alpha \to W_* = \{x^u = 0\}$ defined by $\{z\} = \Gamma^{\varkappa}_{i,j,r}(\Theta_{i,j,\alpha,\varkappa}(\{z\})) \cap W_\alpha$. Note that

$$\Theta_{i,j,\alpha,\varkappa}(F_\alpha(\xi), \xi, N_\alpha(\xi)) = (0, h_\alpha(\xi), \bar{\omega}_\alpha(\xi)).$$

Accordingly, $\mathbb{G}_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi), \bar{\omega}_\alpha(\xi)) = \mathbb{F}_\alpha(\xi)$, that is

$$\begin{aligned} &G_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi)) = \xi \\ &H_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi)) + \bar{\omega}_\alpha(\xi) = N_\alpha(\xi)\,. \end{aligned} \tag{6.19}$$

Lemma 6.6. *There exists $C_\#, \varpi_0 > 0$ such that for each i, j, ℓ, $\varpi \in [0, \varpi_0]$ and $\alpha \in E_{\ell,i,j}$ the following holds true*

$$\begin{aligned} &|h_\alpha(\xi) - \xi| + |h_\alpha^{-1}(\xi) - \xi| \le C_\# r^{1-\varsigma}\,; \quad |1 - h'_\alpha| \le C_\# r^{1-\varsigma} \\ &|\bar{\omega}_\alpha|_{C^{1+\varpi}} + |h_\alpha|_{C^{1+\varpi}} \le C_\#\,, \end{aligned}$$

provided

$$\varsigma(1 + \varpi) \le 1\,. \tag{6.20}$$

Proof. By Lemma D.2 and Remark D.3, both $h_\alpha, \bar{\omega}_\alpha$ are uniformly Lipschitz functions. Indeed,

$$\begin{aligned} \xi =& G_{i,j,\varkappa}(0, h_\alpha(\xi)) + \int_0^{F_\alpha(\xi)} dz\, \partial_z G_{i,j,\varkappa}(z, h_\alpha(\xi)) \\ =& h_\alpha(\xi) + \int_0^{F_\alpha(\xi)} dz \int_0^{h_\alpha(\xi)} dw\, \partial_w \partial_z G_{i,j,\varkappa}(z, w) \end{aligned} \tag{6.21}$$

that is $h_\alpha(\xi) = \xi(1 + \mathcal{O}(r^{1-\varsigma}))$. Moreover, differentiating the first of (6.19),

$$h'_\alpha(\xi) = \frac{1 - \partial_\eta G_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi)) F'_\alpha(\xi)}{\partial_\xi G_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi))}, \tag{6.22}$$

and

$$\begin{aligned} \partial_\xi G_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi)) &= \partial_\xi G_{i,j,\varkappa}(0, h_\alpha(\xi)) + \int_0^{F_\alpha(\xi)} \partial_z \partial_\xi G_{i,j,\varkappa}(z, h_\alpha(\xi)) dz \\ &= 1 + \mathcal{O}(r^{1-\varsigma}) \\ \partial_\eta G_{i,j,\varkappa}(F_\alpha(\xi), h_\alpha(\xi)) &= \partial_\eta G_{i,j,\varkappa}(F_\alpha(\xi), 0) + \int_0^{h_\alpha(\xi)} \partial_z \partial_\eta G_{i,j,\varkappa}(F_\alpha(\xi), z) dz \\ &= \partial_\eta G_{i,j,\varkappa}(F_\alpha(\xi), 0) + \mathcal{O}(r^{\theta-\varsigma})\,, \end{aligned}$$

where we have used property (1) of the foliation representation (see Appendix D). Since $\partial_\eta \mathbb{G}_{i,j,\varkappa}$ belongs to the unstable cone, then $\partial_\eta G_{i,j,\varkappa}$ is uniformly bounded. Also remember the estimate $|F'| \le C_\# r^{1-\theta}$ obtained in the proof of Lemma 6.4. Hence, taking into account (6.18), $|1 - h'_\alpha| \le C_\# r^{1-\varsigma}$, and h_α is invertible with uniform Lipschitz constant.

To estimate the Hölder norm of h' we use the above equations together with property (6) of the foliation:[42]

$$
\begin{aligned}
|h_\alpha|_{C^{1+\varpi}} &\le C_\# \{1 + |F'_\alpha|_\infty |\partial_\eta G_{i,j,\varkappa}|_{C^\varpi} + |1 - \partial_\xi G_{i,j,\varkappa}|_{C^\varpi}\} \\
&\le C_\# \left\{1 + r^{1-\theta}|\partial_\xi\partial_\eta G_{i,j,\varkappa}|_\infty^\varpi + \int_0^r |\partial_\xi\partial_\eta G_{i,j,\varkappa}|_{C^\varpi}\right\} \\
&\le C_\# \left\{1 + r^{1-\varsigma(1+\varpi)}\right\}.
\end{aligned}
$$

Analogously,

$$
|\bar\omega_\alpha|_{C^{1+\varpi}} \le C|h_\alpha|_{C^{1+\varpi}} + |F'_\alpha|_\infty |\partial_\eta H_{i,j,\varkappa}|_{C^\varpi} + |\partial_\xi H_{i,j,\varkappa}|_{C^\varpi} \le C_\# \left\{1 + r^{1-\varsigma(1+\varpi)}\right\}.
$$

This concludes the proof of Lemma 6.6. □

Next, remember that the fibers of $\Gamma^\varkappa_{i,j,r}$ in the domain $\Delta_\varkappa$ can be iterated backward a time $\varkappa\tau_-$ and still remain in the unstable cone. In the following we will use the notation

$$
\partial_{\varkappa,i} f = \sup_{(\xi,s)\in\Delta_\varkappa} \operatorname*{ess-sup}_{|\eta|\le r} |\langle \partial_\eta \mathbb{G}_\varkappa(\eta,\xi,s), (\nabla f)\circ \mathbb{G}_\varkappa(\eta,\xi,s)\rangle|.
$$

With the above construction and notations, and using Lemma D.2, we can continue our estimate left at (6.17)

$$
\begin{aligned}
&\int_{W^c_{\alpha,i}} \bar p_{m\ell,z,\alpha}\phi_{r,i}\cdot \bar{\mathfrak{p}}_{\ell,\zeta,\alpha,i}\cdot f \\
&= \int_{-\tau_-}^{\tau_-} ds \int_{\|\xi\|\le cr^\theta} d\xi\; e^{ibs}\mathbf{F}_{\ell,m,i,\alpha}(\xi,s) f\circ\Theta_{i,j,\alpha,\varkappa}(F_\alpha(\xi),\xi,N_\alpha(\xi)+s) \\
&\quad + \mathcal{O}(r\partial_{\varkappa,i}f + r^\varsigma|f|_\infty)\frac{(\ell\tau_-)^{m-1}}{(m-1)!}e^{-a\tau_-\ell} \\
&= -\int_{-2\tau_-}^{2\tau_-} ds \int_{\|\xi\|\le cr^\theta} d\xi\; e^{ib(\omega_\alpha(\xi)-s)}\mathbf{F}^*_{\ell,m,i,\alpha}(\xi,s) f(0,\xi,s) \\
&\quad + \mathcal{O}(r^{1-\theta}\partial_{\varkappa,i}f + r^{\varsigma-\theta}|f|_\infty)\frac{(\ell\tau_-)^{m-1}}{(m-1)!}e^{-a\tau_-\ell}\tau_-,
\end{aligned}
\tag{6.23}
$$

where we have set $\omega_\alpha(\xi) = \bar\omega_\alpha\circ h_\alpha^{-1}(\xi)$,

$$
\mathbf{F}^*_{\ell,m,i,\alpha}(\xi,s) = \mathbf{F}_{\ell,m,i,\alpha}(h_\alpha^{-1}(\xi),\omega_\alpha(\xi)-s)|h'_\alpha\circ h_\alpha^{-1}(\xi)|^{-1}.
\tag{6.24}
$$

[42] We use also the fact that $|f|_{C^\varpi} \le C_\# |f|_{C^0}^{1-\varpi}|f'|_{C^0}^\varpi$.

At last we can substitute (6.23) into (6.14) and use the Schwartz inequality (first with respect to the integrals and then with respect to the sum on i and j) to obtain

$$
\begin{aligned}
|\mathbb{A}_\delta^s \mathcal{R}(z)^m f| &\leq C_\# \sum_{\ell \geq c_* m k_1} e^{-a\ell\tau_-} \sum_{\zeta,i,j} |f|_\infty r^\theta \\
&\times \Big[\sum_{\alpha,\beta \in E_{\ell,i,j}} Z_{\alpha,i} Z_{\beta,i} \int_{-2\tau_-}^{2\tau_-} ds \int_{\|\xi\|\leq cr^\theta} d\xi \, e^{ib(\omega_\alpha(\xi)-\omega_\beta(\xi))} \mathbf{F}^*_{\ell,m,i,\alpha} \overline{\mathbf{F}^*_{\ell,m,i,\beta}} \Big]^{\frac{1}{2}} \\
&+ C_\# (r^{1-\theta} \sup_i \partial_{\varkappa,i} f + (r^{\frac{\theta}{2}} + r^{\varsigma-\theta}) |f|_\infty) a^{-m} \\
&\leq C_\# \sum_{\ell \geq c_* m k_1} e^{-a\ell\tau_-} \sum_{\zeta} |f|_\infty r^{-\frac{1}{2}} \\
&\times \Big[\sum_{i,j} \sum_{\alpha,\beta \in E_{\ell,i,j}} Z_{\alpha,i} Z_{\beta,i} \int_{-2\tau_-}^{2\tau_-} ds \int_{\|\xi\|\leq cr^\theta} d\xi \, e^{ib(\omega_\alpha(\xi)-\omega_\beta(\xi))} \mathbf{F}^*_{\ell,m,i,\alpha} \overline{\mathbf{F}^*_{\ell,m,i,\beta}} \Big]^{\frac{1}{2}} \\
&+ C (r^{1-\theta} \sup_i \partial_{\varkappa,i} f + (r^{\frac{\theta}{2}} + r^{\varsigma-\theta}) |f|_\infty) a^{-m} \, .
\end{aligned}
\tag{6.25}
$$

To conclude the proof of Lemma 6.1, we need two fundamental, but technical, results whose proofs can be found in Appendix E. The first allows to estimate the contribution to the sum of manifolds that are enough far apart, the other shows that manifolds that are too close are few. For each two sets A, B such that $A_{r,i,j} = A \cap \mathcal{B}_{r,i,j} \neq \emptyset$ and $B_{r,i,j} = B \cap \mathcal{B}_{r,i,j} \neq \emptyset$, let $d_{i,j}(A,B) = d(A_{r,i,j}, B_{r,i,j})$.

Lemma 6.7. *We have*

$$
|\partial_\xi \left[\omega_\alpha - \omega_\beta\right]| \geq C_\# d_{i,j}(W_\alpha, W_\beta) \, . \tag{6.26}
$$

In addition, if $\sigma^{c_* m} \leq C_\# r^{\frac{\vartheta-\theta}{2}}$, *then*

$$
\begin{aligned}
\left| \int_{\|\xi\|\leq cr^\theta} d\xi \, e^{ib(\omega_\alpha(\xi)-\omega_\beta(\xi))} \mathbf{F}^*_{\ell,m,i,\alpha} \overline{\mathbf{F}^*_{\ell,m,i,\beta}} \right|_\infty &\leq C_\# \frac{(\ell\tau_-)^{2m-2}}{[(m-1)!]^2} \\
&\times r^{-\theta} \left[\frac{1}{d_{i,j}(W_\alpha, W_\beta)^{1+\varpi} b^\varpi} + \frac{1}{r^\theta d_{i,j}(W_\alpha, W_\beta) b} \right] \, .
\end{aligned}
\tag{6.27}
$$

Lemma 6.8. *There exists* $\ell_0 \in \mathbb{N}$ *such that for all* $\ell \geq \ell_0 k_1$, $\vartheta > 0$, $\alpha \in E_{\ell,i}$ *and each* $r > 0$

$$
\sum_{\beta \in D^\vartheta_{\ell,i,j,\alpha}} Z_{\beta,i} \leq C_\# [r^{\frac{\vartheta+\theta}{2}} + \delta^{-1} \sigma^{\frac{\ell}{k_1}}] \, ,
$$

where $D^\vartheta_{\ell,i,j,\alpha} = \{\beta \in E_{\ell,i,j} \; : \; d(W_\alpha, W_\beta) \leq r^\vartheta\}$.

To end the proof of Lemma 6.1, it is then convenient to assume that

$$
\sigma^{c_* m} \leq r^{\frac{\vartheta+\theta}{2}} \delta \, . \tag{6.28}
$$

Applying Lemmata 6.7 and 6.8 to (6.25) with $\varkappa = m$ and $f = \mathcal{R}(z)^m\widetilde{\psi}$, we obtain[43]

$$
\begin{aligned}
&|\mathbb{A}^s_\delta \mathcal{R}(z)^{2m}\widetilde{\psi}| \\
&\quad \le C_\# \sum_{\ell \ge c_* m} |\widetilde{\psi}|_{L^\infty} \frac{a^{-2m} e^{-a\ell\tau_-}(\ell\tau_-)^{m-1}}{(m-1)!\, r^{\frac{1}{2}}} \left[\frac{r^{-\vartheta(1+\varpi)-\theta}}{b^{\varpi}} + \frac{r^{-2\theta-\vartheta}}{b} + r^{\frac{\vartheta+\theta}{2}} \right]^{\frac{1}{2}} \\
&\qquad + C_\#(r^{1-\theta}\nu_a^m \|\widetilde{\psi}\|_{H^1_\infty} + (r^{\frac{\theta}{2}} + r^{\varsigma-\theta})|\widetilde{\psi}|_{L^\infty}) a^{-2m} \\
&\quad \le C_\# a^{-2m} \left(\frac{r^{-\frac{1+(1+\varpi)\vartheta+4\theta}{2}}}{b^{\frac{\varpi}{2}}} + r^{\frac{\vartheta-3\theta-2}{4}} + r^{\varsigma-\theta} + r^{\frac{\theta}{2}} \right) |\widetilde{\psi}|_{L^\infty} \\
&\qquad + C_\# a^{-2m} r^{1-\theta} \nu_a^m \|\widetilde{\psi}\|_{H^1_\infty},
\end{aligned}
$$

where $\nu_a = (1 + a^{-1}\ln\bar{\lambda})^{-1}$. For the reader's convenience, we collect all the conditions imposed along the computation

$$
c_* e a \tau_+ = \sigma < 1\,;\;\; \sigma^{c_* m} \le r^{\frac{\vartheta+\theta}{2}}\delta\,;\;\; \frac{1}{2} < \theta < \varsigma < (1+\varpi)^{-1}.
$$

We choose $\varsigma = \frac{3\theta}{2}$, $\vartheta = 2 + 5\theta$, $b = r^{-\frac{4+8\vartheta+18\theta}{\varpi}}$, $\gamma_0 = \frac{\varpi\theta}{2(4+8\vartheta+18\theta)}$. Finally, if we choose $\theta = \frac{5}{8}$, $\varpi < \frac{1}{15}$ and $m \ge C_\# a\gamma \ln b$, for some appropriate fixed constant $C_\#$, we satisfy all the remaining conditions and Lemma 6.1 follows. (Note that the best possible γ_0 given by this argument is $\frac{\varpi}{155} \le \frac{1}{2325}$.) □

7. Finishing the proof

In this section, we prove Proposition 3.9, using the key bound from Lemma 6.1.

Proof of Proposition 3.9. Recall that $\|\cdot\| = \|\cdot\|_{\widetilde{\mathbf{H}}^{r,s,0}_p(R)}$ with $-1+1/p < s < -r < 0 < r < 1/p$ for $p \in (0,1)$. It will be convenient to assume

$$
p > 3\,, |s| \le 2r\,.
$$

Let A be as in (3.7) and (3.8), and let $\gamma_0 > 0$ be given by Lemma 6.1. (They do not depend on r, s, or p.) Let us consider $|b| > b_0$, $a > 10A$, and decompose $n = Lm + m + 2$, with m even for even n, or $n = Lm + m + 3$, with m even for odd n (for simplicity, we consider even n), where $L \ge 1$ and b_0 will be chosen later. We will collect at the end of the argument the upper and lower bounds relating m and $a\ln|b|$ which will have appeared along the way, and check that they are consistent.

Take r' with $\max(-1+1/p, r-\beta) < r' < s < 0$ and so that $r' > -\frac{\gamma_0}{2p}$. We view r' as fixed, but we may need to choose larger p and smaller r, $|s|$, this may affect some constants noted $C_\#$, but will not create any problems, since we may take larger b_0 (without affecting $C_\#$).

Our starting point is the Lasota-Yorke estimate (3.28) from Lemma 3.8: For any integer $N \ge 1$ such that $(1+3N)r < 1/p$ (take the largest such integer), setting

$$
\eta := \Lambda|s| + \frac{A}{N}\,,
$$

for some constant Λ independent of p, r, s and r', and also on p for large enough p (see the beginning of the proof of Lemma 3.8) we have

$$
\begin{aligned}
(7.1)\quad \|\mathcal{R}(z)^{Lm+1+m+1}(\psi)\| &\le C_\#(a + \ln(1/\lambda))^{-Lm}|b|^{N(r-s)}\|\mathcal{R}(z)^{m+1}(\psi)\| \\
&\quad + C_\#^N (a-\eta)^{-Lm}|b|^{N(r-s)-r'}\|\mathcal{R}(z)^{m+1}(\psi)\|_{H^{r'}_p(X_0)},
\end{aligned}
$$

[43] Recall that $\|f\|_{L^\infty} \le a^{-m}\|\widetilde{\psi}\|_{L^\infty}$ while $\partial_{m,i} f \le C_\#(a+\ln\bar{\lambda})^{-m}\|\widetilde{\psi}\|_{H^1_\infty}$ since H^1_∞ functions are (or, better, have a representative) Lipschitz (see [22, Section 4.2.3, Theorem 5]), with Lipschitz constant given by $\|\widetilde{\psi}\|_{H^1_\infty}$. Lipschitz functions in $\mathbb{R}^3$ are Lipschitz when restricted to a C^2 curve and Lipschitz functions are almost surely differentiable by Rademacher's Theorem.

where we used that our assumptions ensure $(1+|z|)^q(1+1/(a-A)) < C_\#|b|^q$. Recall that $\lambda \in (0,1)$ depends on r and s. (This is why the consequence (3.18) of the Lasota-Yorke Lemma 3.1 combined with Lemma 3.4 would not suffice here.) In fact, if t_0 is chosen large enough then there exists $\widehat c > 0$ so that $\ln(1/\lambda) \ge r\widehat c$, recalling our assumption $-s < 2r$.

Using now (3.27) from Lemma 3.8, we get for the same N and η,

$$\|\mathcal{R}(z)^{\ell+1}\| \le C_\#^N \frac{|b|^{N(r-s)}}{(a-\eta)^\ell}\,, \quad \forall \ell \ge 0\,.$$

Thus, the first term of (7.1) is bounded by

$$C_\#^N \frac{|b|^{N(r-s)}}{(a+\ln(1/\lambda))^{Lm}} \frac{|b|^{N(r-s)}}{(a-\eta)^m}\|\psi\|\,. \tag{7.2}$$

Now, if L is large enough so that the strict inequality below holds

$$\eta = \Lambda|s| + \frac{A}{N} \le r(2\Lambda + 3pA) < L\frac{r\widehat c}{4} \le L\frac{\ln(1/\lambda)}{4}\,, \tag{7.3}$$

then (7.2) is not larger than

$$C_\#^N |b|^{2N(r-s)} \left(\frac{1}{a+\ln(1/\lambda)/2}\right)^{Lm+m+2} \|\psi\|\,.$$

Therefore, if

$$Lm+m+2 \ge \frac{2N(r-s)\ln|b| + 2N\ln C_\#}{\ln(1+(\ln(1/\lambda)/(2a)) - \ln(1+(\ln(1/\lambda)/(4a))}\,, \tag{7.4}$$

the first term of (7.1) is not larger than

$$\left(\frac{1}{a+\ln(1/\lambda)/8}\right)^{Lm+m+2} \|\psi\|\,.$$

We may thus concentrate on the second term of (7.1), that is, the weak norm contribution. Taking $\epsilon = b^{-\sigma}$ (for $\sigma > 2$ to be determined later), and using (3.27) from the proof of Lemma 3.8 (which gives $\|\mathcal{R}(z)^m\|_{H_p^{r'}(X_0)} \le C_\#\Lambda(a-\Lambda|r'|)^{-m}$), we see that

$$\begin{aligned} (7.5)\quad & C_\#^N (a-\eta)^{-Lm}|b|^{N(r-s)-r'}\|\mathcal{R}(z)^{m+1}(\psi)\|_{H_p^{r'}(X_0)} \\ &\le C_\#^N \frac{|b|^{N(r-s)-r'}}{(a-\eta)^{Lm}}\big[\|\mathcal{R}(z)^{m+1}(\mathbb{M}_\epsilon(\psi))\|_{\widetilde{\mathbf{H}}_p^{r',0,0}} \\ &\qquad + C_\#(a-\Lambda|r'|)^{-m}\|\psi - \mathbb{M}_\epsilon(\psi)\|_{H_p^{r'}(X_0)}\big]\,. \end{aligned}$$

By Lemma 5.4 and Corollary 4.2 (using also $\widetilde{\mathbf{H}}_p^{r,s,0} \subset H_p^s(X_0)$), the second term in (7.5) is bounded by

$$C_\#^N (a-\eta)^{-Lm}(a-\Lambda|r'|)^{-m}|b|^{N(r-s)-r'}\epsilon^{s-r'}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,0}}\,. \tag{7.6}$$

Next, we get

$$C_\#^N \frac{|b|^{N(r-s)-r'-\sigma(s-r')}}{(a-\eta)^{Lm}(a-\Lambda|r'|)^{-m}} \le \left(\frac{1}{a+\eta}\right)^{Lm+m+2}$$

if σ and $|b|$ are large enough and

(7.7)
$$Lm+m+2 \le \frac{(\sigma(s-r') - N(r-s) + r')\ln|b| - N\ln C_\#}{\ln(1+\eta/a) - (1+1/L)^{-1}\ln(1-\eta/a) - (L+1)^{-1}\ln(1-\Lambda|r'|/a)}\,.$$

(The right-hand-side above is positive for large enough σ.)

The first term in (7.5) will be more tricky to handle. Lemma 5.5 implies that this term is bounded by $C_\#^N(a-\eta)^{-Lm}|b|^{N(r-s)-r'}$ multiplied by

$$\begin{aligned}
&\|\mathcal{R}(z)(\mathbb{1}_{X_0}\mathbb{A}_\delta(\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi))))\|_{\widetilde{\mathbf{H}}_p^{r',0,0}} \\
&\qquad\qquad + \|\mathcal{R}(z)((\mathrm{id}-\mathbb{1}_{X_0}\mathbb{A}_\delta)(\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi))))\|_{\widetilde{\mathbf{H}}_p^{r',0,0}} \\
&\le C_\#\|\mathbb{1}_{X_0}\mathbb{A}_\delta(\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi)))\|_{L^p(X_0)} \qquad (7.8)\\
&\qquad + C_\#(a-A)^{-1}\delta^{s-r'}\|\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi))\|_{H_p^s(X_0)}\,. \qquad (7.9)
\end{aligned}$$

(We used that $(\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi))$ is supported in X_0, although $\mathbb{M}_\epsilon(\psi)$ is not necessarily supported in X_0, and the bounded inclusion $L^p(X_0)\subset\widetilde{\mathbf{H}}_p^{r',0,0}(R)$ for $r'\le 0$.)

Since $d=3$, by the Dolgopyat bound (Lemma 6.1), there exist $C_\#>0$, $\gamma_0>0$, $\bar\lambda>1$, all independent of p, r, and s, so that for $\gamma>\gamma_0$ (we shall take $\gamma>1$), if

$$m\ge 2C_\# a\gamma\ln|b| \qquad (7.10)$$

then (7.8) times $C_\#^N(a-\eta)^{-Lm}|b|^{N(r-s)-r'}$ is bounded by

$$\frac{C_\#^N|b|^{N(r-s)-r'-\gamma_0}}{(a-\eta)^{Lm}a^m}\left(\|\mathbb{M}_\epsilon(\psi)\|_{L^\infty(M)}+\frac{\|\mathbb{M}_\epsilon(\psi)\|_{H^1_\infty(M)}}{(1+(\ln\bar\lambda)/a)^{m/2}}\right). \qquad (7.11)$$

Now, Lemma 5.3, Lemma 3.2, Corollary 4.2, and the Sobolev embeddings give

$$\begin{aligned}
\|\mathbb{M}_\epsilon(\psi)\|_{L^\infty(M)} &\le C_\#\|\mathbb{M}_\epsilon(\psi)\|_{H_p^{d/p}(M)}\le C_\#\epsilon^{s-d/p}\|\psi\|_{H_p^s(M)} \qquad (7.12)\\
&= C_\#\epsilon^{s-d/p}\|\psi\|_{H_p^s(X_0)}\le C_\#\epsilon^{s-d/p}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,0}}\,,
\end{aligned}$$

(using $\psi=\mathbb{1}_{X_0}\psi$) and

$$\begin{aligned}
\|\mathbb{M}_\epsilon(\psi)\|_{H^1_\infty(M)} &\le C_\#\|\mathbb{M}_\epsilon(\psi)\|_{H_p^{1+d/p}(M)}\le C\epsilon^{s-1-d/p}\|\psi\|_{H_p^s(M)} \qquad (7.13)\\
&= C\epsilon^{s-1-d/p}\|\psi\|_{H_p^s(X_0)}\le C_\#\epsilon^{s-1-d/p}\|\psi\|_{\widetilde{\mathbf{H}}_p^{r,s,0}}\,.
\end{aligned}$$

On the one hand, the estimate (7.12) then gives the following bound for the first term of (7.11)

$$C_\#^N\frac{|b|^{N(r-s)+r'-\gamma_0+\sigma(d/p-s)}}{(a-\eta)^{Lm}a^m}\le\left(\frac{1}{a+\eta}\right)^{Lm+m+2},$$

if $\gamma_0-N(r-s)+r'-\sigma(d/p-s)>\gamma_0/2$ (taking r and $|s|$ smaller and p larger if necessary), $|b|$ is large enough, and

$$Lm+m+2\le\frac{(\gamma_0-N(r-s)+r'-\sigma(d/p-s))\ln|b|-N\ln C_\#}{\ln(1+\eta/a)-\ln(1-\eta/a)}\,. \qquad (7.14)$$

On the other hand, (7.13) gives (recall that $\gamma_0<1/2$) that the second term of (7.11) is bounded by

$$C_\#^N\frac{b^{N(r-s)-r'-\gamma_0+\sigma(d/p+1-s)}}{(a-\eta)^{Lm}a^m(1+(\ln\bar\lambda)/a)^{m/2}}\le\left(\frac{1}{a+(\ln\bar\lambda)/(4L)}\right)^{Lm+m+2}$$

if

$$m\ge 2\frac{(\sigma(d/p+1-s)-\gamma_0+N(r-s)-r')\ln|b|+N\ln C_\#}{\ln(1+(\ln\bar\lambda)/a)+2(L+1)[\ln(1-\eta/a)-\ln(1+(\ln\bar\lambda)/(4La))]}\,. \qquad (7.15)$$

We may assume that $\ln(1+(\ln\bar\lambda)/a)>2(L+1)[\ln(1+(\ln\bar\lambda)/(4La))-\ln(1+\eta/a)]$.

We must still estimate (7.9) times $C_\#^N(a-\eta)^{-Lm}|b|^{N(r-s)-r'}$. Take $\delta=b^{-\gamma}$. By (4.22) in the proof of Lemma 3.8 (as for (7.5)), and by Corollary 4.2 followed by

Lemma 5.3 applied to $r = r' = s$ (which does not give us any gain), we get, using also the bounded inclusion $H^s_p(X_0) \subset \widetilde{\mathbf{H}}^{r,s,0}_p$,

$$C^N_\#(a-\eta)^{-Lm}|b|^{N(r-s)-r'}\delta^{s-r'}\|\mathcal{R}(z)^m(\mathbb{M}_\epsilon(\psi))\|_{H^s_p(X_0)} \le C^N_\#\Lambda(a-\eta)^{-Lm}\frac{|b|^{N(r-s)-r'}}{(a-\Lambda|s|)^m}|b|^{-\gamma(s-r')}\|\psi\|_{\widetilde{\mathbf{H}}^{r,s,0}_p}\,. \tag{7.16}$$

Then, we have

$$C^N_\#(a-\eta)^{-Lm}(a-\Lambda|s|)^{-m}|b|^{N(r-s)-r'-\gamma(s-r')} \le \left(\frac{1}{a+\eta}\right)^{m+Lm+2}$$

if (recall (7.3) and note that $\Lambda|s| < \eta$)

$$m + Lm + 2 \le \frac{(-N(r-s)+r'+\gamma(s-r'))\ln|b| - N\ln C_\#}{\ln(1+\eta/a)-\ln(1-\eta/a)}\,. \tag{7.17}$$

(The right-hand side above is positive if, $r' < 0$ and $\gamma > \gamma_0$ being fixed, we take $|s|$ and r close enough to 0 and $p > 3$ large enough, recalling $(1+3N)r < 1/p$.) This takes care of (7.16).

Along the way, we have collected the lower bounds (7.4), (7.10), and (7.15). Taking b_0 large enough (depending possibly on p, r, s, in particular through L and N) and $|b| \ge b_0$, they are all implied by

$$m \ge \widetilde{c}_1 a \ln|b|\,, \tag{7.18}$$

where $\widetilde{c}_1$ is a possibly large constant, which is independent of L, a, b, p, r, and s, but grows linearly like $\gamma > 1$.

Up to taking larger σ, the upper bounds (7.7), (7.14), and (7.17) are compatible with the lower bound (7.18) (this determines $\widetilde{c}_2 > \widetilde{c}_1$) if p is large enough, r, $|s|$ are small enough, $\eta = \Lambda|s| + A/N$ is small enough, and b_0 is large enough. Finally, we take $\nu = \min(\frac{\ln(1/\lambda)}{8}, \eta, \frac{\ln(\bar\lambda)}{4L})$. (Note that ν depends on r, s, p, through η, L, and λ.) □

Appendix A. Geometry of contact flows

In this appendix, we recall some facts about the geometry of contact flows that may not be obvious to all readers.

Lemma A.1. *All ergodic cone hyperbolic contact flows on a compact manifold are Reeb flows.*

Proof. Let v^u be an unstable vector then, by the invariance of the contact form α,

$$\alpha(v^u) = \lim_{t\to\infty} T^*_{-t}\alpha(v_u) = \lim_{t\to\infty}\alpha((T_{-t})_* v^u) = 0\,.$$

Analogously, $\alpha(v^s) = 0$. Since $T^*_t d\alpha = d(T^*_t\alpha)$, we have that also $d\alpha$ is invariant.

Let V be the vector field generating the flow, then for each tangent vector ξ we can decompose it as $\xi = aV + v^s + v^u$. But $d\alpha(V,V) = 0$, $d\alpha(V,v^s) = \lim_{t\to\infty} d\alpha(V,(T_t)_*v^s) = 0$ and analogously $d\alpha(V,v^u) = 0$, thus V is the kernel of $d\alpha$.

Let us define $v(w) = \alpha_w(V(w))$, $w \in M$. By the invariance of α we have, for almost all w,

$$v(w) = \alpha_{T_t w}(T_{*t}V(w)) = \alpha_{T_t w}(V(T_t w)) = v(T_t(w))\,.$$

Since the flow is ergodic it follows that, almost surely, $v(w) = \bar v^{-1} \in \mathbb{R}$. Let us define the new one form $\beta = \bar v\alpha$. Then $d\beta = \bar v d\alpha$, and $\beta \wedge d\beta = \bar v^2 \alpha d\alpha$ is still a volume form, hence β is still a contact form, and is invariant with respect to the flow. Moreover $\beta(V) = 1$, hence the flow is Reeb. □

Remark A.2. In the smooth case ergodicity is not necessary since contact implies automatically mixing, [27].

Lemma A.3. *All diffeomorphisms preserving the standard contact form can be written as*

$$K(x,y,z) = (A(x,y), B(x,y), z + C(x,y)),$$

where

$$\det\begin{pmatrix}\partial_x A & \partial_y A\\ \partial_x B & \partial_y B\end{pmatrix} = 1\,,$$

$\partial_x C = B\partial_x A - y$ *and* $\partial_y C = B\partial_y A$.

Proof. Let us consider the general change of coordinates defined by

$$K(x,y,z) = (A(x,y,z), B(x,y,z), C(x,y,z))\,.$$

The condition that α is left invariant can be written as

$$dC - BdA = dz - ydx$$

that is $d(C-z) = BdA - ydx$. This is possible only if $BdA - ydx$ is a closed form, i.e.,

$$\begin{aligned}\partial_z B\partial_y A - \partial_z A\partial_y B &= 0\\ \partial_z B\partial_x A - \partial_z A\partial_x B &= 0\\ \partial_y B\partial_x A - \partial_y A\partial_x B &= 1\,.\end{aligned} \tag{A.1}$$

Multiplying the first by $\partial_x A$ and subtracting it to the second multiplied by $\partial_y A$, we obtain $\partial_z A\{\partial_x A\partial_y B - \partial_y A\partial_x B\} = 0$. Which, by the last of the above, implies $\partial_z A = 0$. This, in turn, implies $\partial_z B = 0$. From which it follows that $C - z$ is independent of z. This implies the lemma. □

Using the above fact we can prove the following lemma:

Lemma A.4. *Consider the leaf* $(F(x^s), x^s, N(x^s))$ *with tangent space in the stable cone, the point* $(\bar x, \bar y, \bar z) = (F(\bar y), \bar y, N(\bar y))$*, and the vector* v *in the unstable cone and in the kernel of* α*. Then there exist Reeb coordinates in which the selected point is the origin of the coordinates, the leaf reads* $(0, x^s, 0)$ *and the vector* $(1,0,0)$.

Proof. We can consider the foliation

$$(\mathcal{W}(x^u,x^s), x^s, w(x^u,x^s,x^0)) = (x^u + F(x^s), x^s, x^0 + N(x^s))\,.$$

Let us consider a change of coordinates that preserves α. By Lemma A.3 the change of coordinates reads $(A(x^u,x^s), B(x^u,x^s), x^0 + C(x^u,x^s))$. The fact that the foliation is sent into horizontal leaves means that $A(\mathcal{W}(x^u,x^s),x^s)$ must be independent of x^s. Hence, A must be a solution of the first order PDE

$$\begin{aligned}0 &= (\partial_{x^u}A)(\mathcal{W}(x^u,x^s),x^s)\partial_{x^u}\mathcal{W}(x^u,x^s) + (\partial_{x^s}A)(\mathcal{W}(x^u,x^s),\xi)\\ &= [(\partial_{x^u}A)\cdot\Gamma + (\partial_{x^s}A)]\circ\mathbb{W}(x^u,x^s)\,,\end{aligned}$$

where $\mathbb{W}(x^u,x^s) = (\mathcal{W}(x^u,x^s),x^s)$ and $\Gamma(x^u,x^s) = (\partial_{x^s}\mathcal{W})\circ\mathbb{W}^{-1}(x^u,x^s)$.[44] In other words, we have shown that there exist coordinates in which the foliation reads $(W(x^u), x^s, H(x^u,x^s,x^0))$. At last, since the leaves are in the kernel of α, $\partial_{x^s}H = 0$, the foliation is made of lines parallel to the x^s coordinates, as required.

Let $(\widetilde x, \widetilde y, \widetilde z)$ be the selected point in the new coordinates and perform the change of coordinates

$$\begin{aligned}x &= \xi + \widetilde x\,, \qquad y = \eta + \widetilde y\\ z &= \zeta + \widetilde z + \widetilde y\xi\,,\end{aligned}$$

which sends the selected point to $(0,0,0)$ and hence the manifold in $(0,x^s,0)$.

[44] For example choose $A(x,y) = x - F(y) + F(\bar y)$.

Next, consider the changes of coordinates given by

$$\eta = ax + by\,, \qquad \xi = cx + dy$$
$$\zeta = z + \frac{ac}{2}x^2 + adxy + \frac{bd}{2}y^2\,,$$

with $ad - cb = 1$. Let the vector have coordinates (u, s, t). Then $b = 0$ ensures that the horizontal is sent to the horizontal, $a = u^{-1}, d = u, c = -s$, imply that in the new coordinates the vector reads $(1, 0, \widetilde{t})$. But since the vector belongs to the kernel of the contact form and is at the point zero, we must have $\widetilde{t} = 0$. □

Appendix B. Basic facts on the local spaces $H_p^{r,s,q}$

We adapt the bounds of [6, §4.1], state a result about interpolation between Lasota-Yorke inequalities due to S. Gouëzel (Lemma B.7), and prove Lemma B.8, necessary in view of the "glueing" procedure in Step 2 of Lemma C.2 used in Step 3 of the proof of the Lasota-Yorke claim Lemma 3.1. We start with a Leibniz bound:

Lemma B.1. *Fix $\widetilde{\beta} \in (0, 1)$. Let $s \le 0 \le r$, $q \ge 0$ be real numbers with*

$$(1 + q/r)(r - s) < \widetilde{\beta}\,. \tag{B.1}$$

For any $p \in (1, \infty)$, there exists a constant $C_\#$ such that for any $C^{\widetilde{\beta}}$ function $g : \mathbb{R}^d \to \mathbb{C}$,

$$\|g \cdot \omega\|_{H_p^{r,s,q}} \le C_\# \|g\|_{C^{\tilde{\beta}}} \|\omega\|_{H_p^{r,s,q}}\,.$$

Proof. Note that $(r + q)(1 - s/r) > \max(r + q, r - s)$.

If $q = 0$ then the proof of [5, Lemma 22] gives the statement if $r - s < \widetilde{\beta}$. If $s = r = 0$ and $0 < q < \widetilde{\beta}$, we can use Fubini. If $q > 0$ and $r - s > 0$, it suffices to observe that $H_p^{r,s,q}$ can be obtained by interpolating between $H_p^{r_0,s_0,0}$ and $H_p^{0,0,q_0}$ where $r_0 = r + q > 0$, $s_0 = r_0 s/r < 0$, $q_0 = r + q$, at $r/r_0 \in (0, 1)$, and that our conditions ensure $r_0 - s_0 < \widetilde{\beta}$ and $q_0 < \widetilde{\beta}$. □

The following extension of a classical result of Strichartz [41] is an adaptation of [6, Lemma 4.2].

Lemma B.2 (Strichartz bound)**.** *Let $1 < p < \infty$, $s \le 0 \le r$ and $q \ge 0$ be real numbers so that*

$$1/p - 1 < s(1 + \frac{q}{r}) \le 0 \le r(1 + \frac{q}{r}) < 1/p\,. \tag{B.2}$$

Let $e_1, \dots, e_d$ be a basis of $\mathbb{R}^d$, such that $e_{d_u+1}, \dots, e_{d-1}$ form a basis of $\{0\} \times \mathbb{R}^{d_s} \times \{0\}$ and e_d forms a basis of $\{0\} \times \mathbb{R}$. There exists a constant $C_\#$ (depending only on p, s, r, q and the norm of the matrix change of coordinate between $e_1, \dots, e_d$ and the canonical basis of $\mathbb{R}^d$) so that, for any subset U of $\mathbb{R}^d$ whose intersection with almost every line directed by a vector e_i has at most M_{cc} connected components,

$$\|\mathbb{1}_U \omega\|_{H_p^{r,s,q}} \le C_\# M_{cc} \|\omega\|_{H_p^{r,s,q}}\,.$$

Proof. Note that $(1 + |q|/r)s \le s \le 0$. The conditions on r, s, q are obtained by interpolating as in the proof of Lemma B.1, and using the condition $-1 + 1/p < t < 1/p$ for $H_p^t(\mathbb{R}^d)$ coming from [41] (see the proof of [5, Lemma 23]). One finishes by a linear change of coordinates preserving the $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_0} = \mathbb{R}$ directions. □

The following is essentially [6, Lemma 4.3], itself based on [5, Lemma 28].

Lemma B.3 (Localisation principle)**.** *Let $\mathbb{K}$ be a compact subset of $\mathbb{R}^d$. For each $m \in \mathbb{Z}^d$, consider a function η_m supported in $m + \mathbb{K}$, with uniformly bounded C^1 norm. For any $p \in (1, \infty)$ and $s \le 0 \le r$, $q \ge 0$ with*

$$(1 + q/r)(r - s) < 1\,, \tag{B.3}$$

there exists $C_\# > 0$ so that

$$\left(\sum_{m \in \mathbb{Z}^d} \|\eta_m \omega\|^p_{H_p^{r,s,q}} \right)^{1/p} \le C_\# \|\omega\|_{H_p^{r,s,q}} \, .$$

If, in addition, we assume that $\sum_{m\in\mathbb{Z}^d} \eta_m(x) = 1$ for all x, then

$$\|\omega\|_{H_p^{r,s,q}} \le C_\# \left(\sum_{m \in \mathbb{Z}^d} \|\eta_m \omega\|^p_{H_p^{r,s,q}} \right)^{1/p} \, . \tag{B.4}$$

Proof. For the first bound, our condition on r, s, q ensures we can apply Lemma B.1 to $\widetilde{\beta} = 1$ so that $C_\#$ only depends on the C^1 norm of η_m (see the proof of [6, Lemma 4.3]). For the second bound, we refer to [5, Remark 29] and [44, Theorem 2.4.7(i)]. □

The following lemma on partitions of unity is a modification of [5, Lemma 32]:

Lemma B.4 (Partition of unity)**.** *Let r, s, q be arbitrary real numbers. There exists a constant $C_\#$ such that, for any distributions $v_1, \dots, v_l$ with compact support in $\mathbb{R}^d$, belonging to $H_p^{r,s,q}$, there exists a constant C with*

$$\left\| \sum_{i=1}^{l} v_i \right\|^p_{H_p^{r,s,q}} \le C_\# m^{p-1} \sum_{i=1}^{l} \|v_i\|^p_{H_p^{r,s,q}} + C \sum_{i=1}^{l} \|v_i\|^p_{H_p^{r-1,s,q}} \, , \tag{B.5}$$

where m is the intersection multiplicity of the supports K_i of the v_i's, i.e., $m = \sup_{x\in\mathbb{R}^d} \operatorname{Card}\{i \mid x \in K_i\}$, and letting K_i' be neighbourhoods of the K_i having the same intersection multiplicity as the K_i, and choosing C^∞ functions Ψ_i so that Ψ_i is supported in K_i' and $\equiv 1$ on K_i, we have

$$C \le C_\# m^{p-1} \sup_i \|\Psi_i\|_{C^1} \, . \tag{B.6}$$

Replacing $H_p^{r-1,s,q}$ by $H_p^{r',s,q}$ with $r-1 < r' \le r$ in the right-hand-side of (B.5), *we can replace the C^1 norm by the $C^{r-r'}$ norm* (B.6).

Proof. Apply the proof of [5, Lemma 32], noting that [3, Lemma 2.7] also holds for our symbol $a_{r,s,q}$. The bound (B.6), including the claim about $r-1 < r' \le r$, comes from the term in the integration by parts in the proof of [3, Lemma 2.7], and interpolation if $r - r' \notin \mathbb{Z}$. □

The following class of local diffeomorphisms (adapted from [6]) will be useful:

Definition B.5. For $C > 0$ let $D_2^1(C)$ denote the set of C^1 diffeomorphisms Ψ defined on a subset of $\mathbb{R}^d$, sending stable leaves to stable leaves, flow directions to flow directions, and such that

$$\max\Big(\sup |D\Psi(x^u, x^s, x^0)|, \sup |D\Psi^{-1}(x^u, x^s, x^0)|, \\ \sup_{x^u, x^s, \widetilde{x}^0, x^0} \frac{|D\Psi(x^u, x^s, x^0) - D\Psi(x^s, x^s, \widetilde{x}^0)|}{|x^0 - \widetilde{x}^0|} \\ \sup_{x^u, x^s, \widetilde{x}^s, x^0} \frac{|D\Psi(x^u, x^s, x^0) - D\Psi(x^s, \widetilde{x}^s, x^0)|}{|x^s - \widetilde{x}^s|} \Big) \le C \, .$$

Adapting the proof of [6, Lemma 4.7] gives:

Lemma B.6. *Let $C > 0$, and let $s \le 0 \le r$ and $q \ge 0$ be so that* (B.3) *holds. There exists a constant $C' > 0$ so that for any $\Psi \in D_2^1(C)$ whose range contains a* *ball* $B(z, C_0^{1/2})$*, and for any distribution* $\omega \in H_p^{r,s,q}$ *supported in* $B(z, C_0^{1/2}/2)$*, the composition* $\omega \circ \Psi$ *is well defined, and*

$$\|\omega \circ \Psi\|_{H_p^{r,s,q}} \le C' \, \|\omega\|_{H_p^{r,s,q}} \; . \tag{B.7}$$

Proof. Without loss of generality, we may assume $z = \Psi^{-1}(z) = 0$. Let γ be a C^∞ function equal to 1 on $B(0, C_0^{1/2}/2)$ and vanishing outside of $B(0, C_0^{1/2})$. We want to show that the operator $\mathcal{M} : \omega \mapsto (\gamma\omega) \circ \Psi$ is bounded by C' as an operator from $H_p^{r,s,q}$ to itself. By interpolation (see e.g. the proof of Lemma B.1), it is sufficient to prove this statement for for L^p, for $H_p^{1,0,0}$, for $H_p^{0,-1,0}$ and for $H_p^{0,0,1}$, $H_p^{0,0,-1}$. This can be done by adapting the second step of the proof of Lemma 25 in [6]. The result there is formulated for C^2 diffeomorphisms, but a glance at the proof there indicates that the C^2 regularity is only used in the sense of Lipschitz regularity of the Jacobian along the stable leaves, in the argument for $H_p^{0,-1}$. In our setting, we shall also need Lipschitz regularity of the Jacobian along the flow directions, in the argument for $H_p^{0,0,-1}$. The definition of $D_2^1(C)$ ensures that the Jacobian is indeed Lipschitz along stable leaves and along flow directions. □

The idea for the following lemma was explained to us by Sébastien Gouëzel:

Lemma B.7 (Interpolation of Lasota-Yorke-type inequalities)**.** *Let* $1 < p < \infty$ *and* $s \le 0 \le r$*,* $q \ge 0$*,* $q', q'' \ge 0$*, and* $s' \le s$*,* $r' \le r$*. Let* L *be an operator for which there exist constants* c_u*,* c_s *and* $C_u \ge 0$*,* $C_s \ge 0$ *so that*

$$\|Lw\|_{H_p^{r,0,0}} < c_u \, \|w\|_{H_p^{r,0,0}} + C_u \, \|w\|_{H_p^{r',0,q'}} \;, \forall w \in H_p^{r,0,0} \tag{B.8}$$

$$\|Lw\|_{H_p^{0,s,0}} < c_s \, \|w\|_{H_p^{0,s,0}} + C_s \, \|w\|_{H_p^{0,s',q''}} \;, \forall w \in H_p^{0,s,0} \,, \tag{B.9}$$

then for each $\theta \in [0,1]$ *and every* $\omega \in H_p^{\theta r,(1-\theta)s,0}$

$$\begin{aligned}\|L\omega\|_{H_p^{\theta r,(1-\theta)s,0}} &< c_u^\theta c_s^{1-\theta} \, \|\omega\|_{H_p^{\theta r,(1-\theta)s,0}} + C_u^\theta C_s^{1-\theta} \, \|\omega\|_{H_p^{\theta r',(1-\theta)s',\theta q'+(1-\theta)q''}} \\ &\quad + c_u^\theta C_s^{1-\theta} \, \|\omega\|_{H_p^{\theta r,(1-\theta)s',(1-\theta)q''}} + C_u^\theta c_s^{1-\theta} \, \|\omega\|_{H_p^{\theta r',(1-\theta)s,\theta q'}} \; .\end{aligned}$$

Proof of Lemma B.7. For a Triebel-type symbol $a(\xi^u, \xi^s, \xi^0)$ we write $|\omega|_a$ for the Triebel norm $\|\mathcal{F}^{-1}(a\mathcal{F}\omega)\|_{L^p}$. By classical multiplier theorems (see [38]), the norm $c_u \, \|\omega\|_{H_p^{r,0,0}} + C_r \, \|\omega\|_{H_p^{r',0,q}}$ is equivalent to the Triebel norm with symbol $a_u(\xi^u, \xi^s, \xi^0)$ equal to

$$c_u(1 + |\xi^u|^2 + |\xi^s|^2 + |\xi^0|^2)^{r/2} + C_r(1 + |\xi^u|^2 + |\xi^s|^2 + |\xi^0|^2)^{r'/2}(1 + |\xi^0|^2)^{q/2} \, .$$

Therefore, (B.8) reads

$$\|L(\omega)\|_{H_p^{r,0,0}} < |\omega|_{a_u} \, .$$

Similarly, defining

$$a_s(\xi^u, \xi^s, \xi^0) = c_s(1 + |\xi^s|^2)^{s/2} + C_s(1 + |\xi^s|^2)^{s'/2}(1 + |\xi^0|^2)^{q/2} \, ,$$

(B.9) becomes $\|L(\omega)\|_{H_p^{0,s,0}} < |\omega|_{a_s}$. One may thus apply standard interpolation. A result of Triebel gives that the interpolation of $|.|_{a_u}$ and $|.|_{a_s}$ is equivalent to $|.|_{a_u^\theta a_s^{1-\theta}}$, with uniform equivalence constants over the norms we are considering. □

Finally, we shall need a new result, specific to our flow situation (and which is proved with the help of Lemma B.7):

Lemma B.8 (Composing with a perturbation in the flow direction)**.** *Let* Δ *be defined by* $\Delta(x^u, x^s, x^0) = (x^u, x^s, x^0 + \delta(x^s, x^u))$*,* $x^s \in \mathbb{R}^{d_s}$*,* $x^u \in \mathbb{R}^{d_u}$*,* $x^0 \in \mathbb{R}$*, where* $\delta : \mathbb{R}^{d_s+d_u} \to \mathbb{R}$ *is a* C^1 *map whose range contains a ball* $B(z, C_0^{1/2})$*. Then,* *for any $s < 0 < r$ and $q \ge 0$ so that* (B.3) *holds, there is C so that for any distribution ω supported in $B(z, C_0^{1/2}/2)$,*

$$\|\omega \circ \Delta\|_{H_p^{r,s,q}} \le C\|\omega\|_{H_p^{r,s,q+r-s}} .$$

Proof. As usual, we proceed by interpolation. The Jacobian of Δ and of Δ^{-1} is equal to 1. Therefore

$$\|\omega \circ \Delta\|_{H_p^{0,0,0}} \le \|\omega\|_{H_p^{0,0,0}} . \tag{B.10}$$

Since $\partial_{x^0}(\omega \circ \Delta) = (\partial_{x^0}\omega) \circ \Delta$ and $\partial_{x^s}(v \circ \Delta^{-1}) = (\partial_{x^s} v) \circ \Delta^{-1} + (\partial_{x^0} v) \circ \Delta^{-1} \partial_{x^s}\delta$, we have

$$\|\omega \circ \Delta\|_{H_p^{0,-1,0}} \le \|\omega\|_{H_p^{0,-1,0}} + C\|D\delta\|_{L^\infty}\|\omega\|_{H_p^{0,-1,1}} . \tag{B.11}$$

Indeed, letting $\Omega_s \in H_p^{0,0,0}$ be such that $\partial_{x^s}\Omega_s = \omega$, and writing $D\delta$ for $\partial_{x^s}\delta$,

$$\begin{aligned}
&\sup_{\{\psi:|\psi|_{H_{p'}^{0,1,0}} \le 1\}} \int (\omega \circ \Delta) \cdot \psi \, dx = \sup_\psi \int \omega \cdot (\psi \circ \Delta^{-1}) \, dx \\
&\quad= \sup_\psi \Big(-\int \Omega_s \cdot ((\partial_{x^s}\psi) \circ \Delta^{-1}) - \int \Omega_s \cdot ((\partial_{x^0}\psi) \circ \Delta^{-1}) \cdot D\delta \Big) \\
&\quad= \sup_\psi \Big(-\int (\Omega_s \circ \Delta)\partial_{x^s} \cdot \psi - \int (\Omega_s \circ \Delta) \cdot (\partial_{x^0}\psi) \cdot (D\delta \circ \Delta) \Big) \\
&\quad\le \int |\Omega_s \circ \Delta|^p + \sup_\psi \int ((\partial_{x^0}\Omega_{x^s}) \circ \Delta) \cdot \psi \cdot (D\delta \circ \Delta) \\
&\quad\le (\int |\Omega_{x^s}|^p)^{1/p} + \sup_\psi \int ([(\partial_{x^0}\Omega_{x^s})D\delta] \circ \Delta) \cdot \psi \\
&\quad\le |\omega|_{H_p^{0,-1,0}} + \sup|D\delta|(\int |(\partial_{x^0}\Omega_{x^s}) \circ \Delta|^p)^{1/p} \le |\omega|_{H_p^{0,-1,0}} + \sup|D\delta||\omega|_{H_p^{0,-1,1}} .
\end{aligned}$$

Using also $\partial_{x^u}(\omega \circ \Delta) = (\partial_{x^u}\omega) \circ \Delta + (\partial_{x^0}\omega \circ \Delta)\partial_{x^u}\delta$, $\partial_{x^s}(\omega \circ \Delta) = \partial_{x^s}(\omega) \circ \Delta + (\partial_{x^0}\omega \circ \Delta)\partial_{x^s}\delta$, we get

$$\|\omega \circ \Delta\|_{H_p^{1,0,0}} \le \|\omega\|_{H_p^{1,0,0}} + C\|D\delta\|_{L^\infty}\|\omega\|_{H_p^{0,0,1}} . \tag{B.12}$$

Beware that interpolating between Lasota-Yorke inequalities is not licit in general. However, for the simple symbols in presence, we may use Lemma B.7 (for $C_u = r = r' = q' = 0$ and $s = -1$, $s' = -2$), and we deduce from (B.10)–(B.11) that

$$\|\omega \circ \Delta\|_{H_p^{0,-(1-\theta''),0}} \le \|\omega\|_{H_p^{0,-(1-\theta''),0}} + C\|D\delta\|_{L^\infty}^{1-\theta''}\|\omega\|_{H_p^{0,-2(1-\theta''),1-\theta''}} , \ \forall \theta'' \in [0,1] .$$

and from Lemma B.7 (for $C_s = s = s' = q'' = 0$, $r = 1$, $r' = 0$) and (B.10)–(B.12) that

$$\|\omega \circ \Delta\|_{H_p^{\theta',0,0}} \le \|\omega\|_{H_p^{\theta',0,0}} + C\|D\delta\|_{L^\infty}^{\theta'}\|\omega\|_{H_p^{0,0,\theta'}} , \forall \theta' \in [0,1] .$$

Since $\|w\|_{H_p^{\theta\theta',-2(1-\theta)(1-\theta''),(1-\theta)(1-\theta'')}}$ and $\|w\|_{H_p^{0,-(1-\theta)(1-\theta''),\theta\theta'}}$, are dominated by $\|w\|_{H_p^{r,s,r-s}}$, using again Lemma B.7 to interpolate at $(r,s,0) = \theta(\theta',0,0) + (1-\theta)(0,-(1-\theta''),0)$, we get

(B.13)

$$\begin{aligned}
\|\omega \circ \Delta\|_{H_p^{r,s,0}} &\le C\big(\|\omega\|_{H_p^{r,s,0}} + \max(\|D\delta\|_{L^\infty}^{r-s}, \|D\delta\|_{L^\infty}^{r}, \|D\delta\|_{L^\infty}^{|s|})\|\omega\|_{H_p^{r,s,r-s}}\big) \\
&\le C\|\omega\|_{H_p^{r,s,r-s}} .
\end{aligned}$$

Finally, since $\partial_{x^0}(\omega \circ \Delta) = (\partial_{x^0}\omega) \circ \Delta$, we have

$$\|\omega \circ \Delta\|_{H_p^{0,0,1}} \le \|\omega\|_{H_p^{0,0,1}} . \tag{B.14}$$

Therefore, interpolating with (B.13), we get the lemma. □

Appendix C. Admissible charts are invariant under composition

Lemma C.2 below is the analogue of [6, Lemma 3.3]. (There will be a nontrivial difference, the presence of the map Δ_m.) We need one more notation:

Definition C.1. Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be extended cones (Definition 2.4). If an invertible matrix $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$ sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly, let $\lambda_u(\mathcal{A}) = \lambda_u(\mathcal{A}, \mathcal{C}, \widetilde{\mathcal{C}})$ be the least expansion under $\mathcal{A}$ of vectors in $\mathcal{C}^u$, and $\lambda_s(\mathcal{A}) = \lambda_s(\mathcal{A}, \mathcal{C}, \widetilde{\mathcal{C}})$ be the inverse of the least expansion under $\mathcal{A}^{-1}$ of vectors in $\widetilde{\mathcal{C}}^s$. Denote by $\Lambda_u(\mathcal{A}) = \Lambda_u(\mathcal{A}, \mathcal{C}, \widetilde{\mathcal{C}})$ and $\Lambda_s(\mathcal{A}) = \Lambda_s(\mathcal{A}, \mathcal{C}, \widetilde{\mathcal{C}})$ the strongest expansion and contraction coefficients of M on the same cones.

Lemma C.2. *Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be extended cones and let $\beta \in (0,1)$. For any large enough C_0 (depending on $\mathcal{C}$ and $\widetilde{\mathcal{C}}$) and any $C_1 > 2C_0$, there exist constants C (depending on $\mathcal{C}$, $\widetilde{\mathcal{C}}$ and C_0) and ϵ (depending on $\mathcal{C}$, $\widetilde{\mathcal{C}}$, C_0 and C_1) satisfying the following properties:*

Let $\mathcal{T}$ be a C^2 diffeomorphism of $\mathbb{R}^d = \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \times \mathbb{R}$ with $\mathcal{T}(0) = 0$, so that, $D\mathcal{T}(z)(0,0,v^0) = (0,0,v^0)$ for all $z \in \mathbb{R}^d$ and $v^0 \in \mathbb{R}$, and, setting $\mathcal{A} = D\mathcal{T}(0)$,

$$\left\|\mathcal{T}^{-1} \circ \mathcal{A} - \mathrm{id}\right\|_{C^2} \le \epsilon\,, \quad \mathcal{A} \text{ sends } \mathcal{C} \text{ to } \widetilde{\mathcal{C}} \text{ compactly,}$$

$$\lambda_s(\mathcal{A})^{1-\beta}\Lambda_u(\mathcal{A})^{1+\beta}\lambda_u(\mathcal{A})^{-1} < \epsilon\,, \qquad \lambda_u(\mathcal{A}) > \epsilon^{-1}\,,\ \lambda_s(\mathcal{A})^{-1} > \epsilon^{-1}\,. \tag{C.1}$$

Let $\mathcal{J} \subset \mathbb{R}^d$ be a finite set such that $|m - m'| \ge C$ for all $m \neq m' \in \mathcal{J}$, and consider a family of charts $\{\phi_m \in \mathcal{F}(m, \widetilde{\mathcal{C}}^s, \beta, C_0, C_1) \mid m \in \mathcal{J}\}$. Then, defining

$$\mathcal{J}' = \{m \in \mathcal{J} \mid B(m,d) \cap \mathcal{T}(B(0,d)) \neq \emptyset\}\,,$$

and setting $\Pi(x^u, x^s, x^0) = (x^u, 0, 0)$, we have:

(a) $|\Pi m - \Pi m'| \ge C_0$ for all $m \neq m'$ in $\mathcal{J}'$.

(b) There exist $\phi' \in \mathcal{F}(0, \mathcal{C}^s, \beta, C_0, C_1)$, and diffeomorphisms $\mathbb{T}_m$, for $m \in \mathcal{J}'$, such that

$$\mathcal{T}^{-1} \circ \phi_m = \phi' \circ \mathbb{T}_m \quad \text{ on } \phi_m^{-1}(B(m,d) \cap \mathcal{T}(B(0,d)))\,,\ \forall m \in \mathcal{J}'\,. \tag{C.2}$$

(c) For each $m \in \mathcal{J}'$, we can write $\mathbb{T}_m = \Psi \circ D^{-1} \circ \Psi_m \circ \Delta_m$, where

- *The diffeomorphism $\Psi_m(x^u, x^s, x^0) = (\widetilde{\psi}_m(x^u, x^s), x^0)$ is in $D^1_2(C)$, its range contains $B(\Pi(m^u, 0), C_0^{1/2})$, and*

$$\Psi_m(\Delta_m \phi_m^{-1}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)\,.$$

- *$\Delta_m(x^u, x^s, x^0) = (x^u, x^s, x^0 + \delta_m(x^u, x^s))$, where δ_m is a C^1 function defined on $B(m^u, 0), C_0^{2/3})$ with $\|D\delta_m\|_{L^\infty} \le C$.*
- *The matrix D is block diagonal, of the form $D = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with*

$$|Av| \ge C^{-1}\lambda_u(A)|v| \text{ and } |Bv| \le C\lambda_s(A)|v|\,.$$

- *The diffeomorphism Ψ is in $D^1_2(C)$, its range contains $B(0, C_0^{1/2})$.*

Note that (c) implies in particular that each $\mathbb{T}_m$ sends stable leaves to stable leaves. Note also that if C_0 is large enough, then $\phi' \in \mathcal{F}(0, \mathcal{C}^s, \beta, C_0, C_1)$ implies $(\phi')^{-1}(B(0,d)) \subset B(0, C_0^{1/2}/2)$ (because $\|(\phi')^{-1}\|_{C^1} \le C_\#$ by Lemma 2.3).

Remark C.3. Composing with translations, we deduce a more general result from Lemma C.2, replacing 0 by $\ell \in \mathbb{R}^d$, and allowing $\mathcal{T}(\ell) \neq \ell$: Just replace $\mathcal{A}$ by $D\mathcal{T}(\ell)$, the projection Π by $\Pi(x^u, x^s, x^0) = (x^u, x^s_{\mathcal{T}(\ell)}, x^0_{\mathcal{T}(\ell)})$, where $\mathcal{T}(\ell) = (x^s_{\mathcal{T}(\ell)}, x^u_{\mathcal{T}(\ell)}, x^0_{\mathcal{T}(\ell)})$, and assume that

$$\left\|(\mathcal{T}^{-1}[\cdot + \mathcal{T}(\ell)] - \ell) \circ D\mathcal{T}(\ell) - \mathrm{id}\right\|_{C^2} \le \epsilon$$

and that $D\mathcal{T}(\ell)$ sends $\mathcal{C}$ to $\widetilde{\mathcal{C}}$ compactly. One then uses the condition $B(m,d)\cap\mathcal{T}(B(\ell,d))\neq\emptyset$ to define $\mathcal{J}'$. Of course, ϕ' is then in $\mathcal{F}(\ell,\mathcal{C}^s,\beta,C_0,C_1)$, equality (C.2) holds on $\phi_m^{-1}(B(m,d)\cap\mathcal{T}(B(\ell,d)))$, and the range of Ψ contains $B(\ell,C_0^{1/2})$. Finally, we have $(\phi')^{-1}(B(\ell,d))\subset B(\ell,C_0^{1/2}/2)$.

The proof below is an adaptation of the proof of [6, Lemma 3.3].

Proof of Lemma C.2. We shall write π_1, π_2, and π_3 for, respectively, the first, second and third projection in $\mathbb{R}^d=\mathbb{R}^{d_u}\times\mathbb{R}^{d_s}\times\mathbb{R}$.

Step zero: Preparations. We shall write $C_\#$ and $\epsilon_\#$ for a large, respectively small, constant, depending only on $\mathcal{C},\widetilde{\mathcal{C}}$, that may vary from line to line. For the other parameters, we will always specify if they depend on C_0 or C_1.

The set $\mathcal{A}(\mathbb{R}^{d_u}\times\{0\}\times\{0\})$ is contained in $\widetilde{\mathcal{C}}^u$, hence uniformly transversal to $\{0\}\times\mathbb{R}^{d_s+1}$. Therefore, it can be written as a graph $\{(x^u,P(x^u))\}$ for some matrix P with norm depending only on $\widetilde{\mathcal{C}}$. Let $Q(x^u,x^s,x^0)=(x^u,(x^s,x^0)-P(x^u))$, so that $Q\mathcal{A}$ sends $\mathbb{R}^{d_u}\times\{0\}\times\{0\}$ and $\{0\}\times\{0\}\times\mathbb{R}$ to itself. In the same way, $\mathcal{A}^{-1}(\{0\}\times\mathbb{R}^{d_s}\times\{0\})$ is contained in $\mathcal{C}^s$, hence it is a graph $\{(P_u'(x^s),x^s,P_0'(x^s))\}$. Letting $Q'(x^u,x^s,x^0)(x^u-P_u'(x^s),x^s,x^0-P_0'(x^s))$, the matrix $D=Q\mathcal{A}(Q')^{-1}$ leaves $\{0\}\times\{0\}\times\mathbb{R}$, $\mathbb{R}^{d_u}\times\{0\}\times\{0\}$, and $\{0\}\times\mathbb{R}^{d_s}\times\mathbb{R}$ invariant, i.e., it is block-diagonal, of the form $\left(\begin{smallmatrix}A&0&0\\0&B&0\\0&0&\sigma\end{smallmatrix}\right)$, and moreover $|Av|\geq C_\#^{-1}\lambda_u|v|$ and $|Bv|\leq C_\#\lambda_s|v|$ (since the matrices Q and Q', as well as their inverses, are uniformly bounded in terms of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$) and the scalar matrix $\sigma=1$.

We can readily prove assertion (a) of the lemma. Let $m\in\mathcal{J}'$, there exists $z\in B(m,d)\cap\mathcal{T}(B(0,d))$. The set $Q\mathcal{T}(B(0,d))=DQ'(\mathcal{T}^{-1}\mathcal{A})^{-1}(B(0,d))$ is included in $\{(x^u,x^s,x^0)\mid|(x^s,x^0)|\leq C_\#\}$ for some constant $C_\#$ (the role of Q is important here). Since $Qz\in Q\mathcal{T}(B(0,d))$, we obtain $|\pi_2(Qz)|\leq C_\#$ and $|\pi_3(Qz)|\leq C_\#$. Since $|z-m|\leq d$, we also have $|Qz-Qm|\leq C_\#$, hence $|\pi_2(Qm)|\leq C_\#$, $|\pi_3(Qm)|\leq C_\#$ (for a different constant $C_\#$). Since

$$Qm-\Pi m(m^u,\pi_2(Qm),\pi_3(Qm))-(m^u,0,0)(0,\pi_2(Qm),\pi_3(Qm))\,,$$

we obtain

$$|Qm-\Pi m|\leq C_\#\,. \tag{C.3}$$

Since the points $m\in\mathcal{J}'$ are far apart by assumption, the points Qm for $m\in\mathcal{J}'$ are also far apart, and it follows that the points Πm are also far apart. Increasing the distance between points in $\mathcal{J}'$, we can in particular ensure that $|\Pi m-\Pi m'|\geq C_0$ for any $m\neq m'\in\mathcal{J}'$, proving (a).

The strategy of the rest of the proof is the following: We write

$$\mathcal{T}^{-1}=\mathcal{T}^{-1}\mathcal{A}\cdot(Q')^{-1}\cdot D^{-1}\cdot Q\,. \tag{C.4}$$

We shall start from the partial foliation given by the maps ϕ_m for $m\in\mathcal{J}$, apply Q (Step 1) to obtain a new partial foliation at Qm, modify it via glueing (Step 2) to obtain a global foliation, and then push this foliation successively with D^{-1} (Step 3), $(Q')^{-1}$ (Step 4), and $\mathcal{T}^{-1}\mathcal{A}$ (last step).

We next define spaces of functions which will be used in the proof. For $C_\#>0$, let us denote by $\mathcal{D}(C_\#)$ the class of C^1 maps Ψ defined on an open subset of $\mathbb{R}^j$ (the value of j will be clear from the context), taking their values in $\mathbb{R}^j$ and satisfying

$$C_\#^{-1}|z-z'|\leq|\Psi(z)-\Psi(z')|\leq C_\#|z-z'|, \tag{C.5}$$

for any z,z' in the domain of definition of Ψ. It follows that any such Ψ is a local diffeomorphism, and that $\|D\Psi\|\leq C_\#$, $\|(D\Psi)^{-1}\|\leq C_\#$.

For $\beta \in (0,1)$, we denote by $\mathcal{K} = \mathcal{K}^\beta(C)$ the class of matrix-valued functions K on $\mathbb{R}^{d-1}$ such that, for all $x^u, y^u \in \mathbb{R}^{d_u}$, for all $x^s, y^s \in \mathbb{R}^{d_s}$,

$$|K(x^u, x^s)| \le C\,,$$
$$|K(x^u, x^s) - K(y^u, x^s)| \le C|x^u - y^u|^\beta\,, |K(x^u, x^s) - K(x^u, y^s)| \le C|x^s - y^s|\,,$$
$$|K(x^u, x^s) - K(y^u, x^s) - K(x^u, y^s) + K(y^u, y^s)| \le C|x^u - y^u|^\beta |x^s - y^s|^{1-\beta}\,.$$

The spaces of local diffeomorphisms $\mathcal{D}(C_\#)$ and of matrix-valued functions $\mathcal{K}(C_\#) = \mathcal{K}^{1,\beta}(C_\#)$ were introduced in [6]. As in Remark A.6 of Appendix A of [6], we will write $\mathcal{K}(C_\#, A)$ for the functions defined on a set A and satisfying the inequalities defining $\mathcal{K}(C_\#)$ (A will sometimes be omitted when the domain of definition is obvious). The map $\phi_m = (F_m(x^u, x^s), x^s, x^0 + \widetilde{f}_m(x^s, x^u))$ belongs to $\mathcal{D}(C_\#)$ (see the proof of Lemma 2.3), the matrix-valued function DF_m belongs to $\mathcal{K}(C_\#, B((m^u, m^s), C_0))$ (boundedness of DF_m is proved in Lemma 2.3, while the Hölder-like properties are given by (2.3)–(2.5)).

First step: Pushing the foliations with Q. We formulate in detail the construction in this first step (a version of Lemma C.4 will be used also in the last step, replacing Q by $\mathcal{T}^{-1}\mathcal{A}$, while steps 2-3-4 are much simpler). The statement below is the analogue in our setting of [6, Lemma 3.5]:

Lemma C.4. *(Notation as in Lemma C.2 and Step 0 of its proof.) There exists a constant $C_\#$ such that, if C_0 is large enough and $C_1 > 2C_0$, for any $m = (m^u, m^s, m^0) \in \mathcal{J}'$ there exist maps $\Psi_m : B((m^u, m^s), C_0^{2/3}) \times \mathbb{R} \to \mathbb{R}^d$, $\phi_m^{(1)} : B(\Pi m, C_0^{1/2}) \to \mathbb{R}^d$, and a map Δ_m from $B((m^u, m^s), C_0^{2/3}) \times \mathbb{R}$ to itself such that*

$$\phi_m^{(1)} \circ \Psi_m \circ \Delta_m = Q \circ \phi_m \text{ on } \phi_m^{-1}(B(m,d))\,.$$

Moreover, Ψ_m is a diffeomorphism in $D_2^1(C_\#)$ whose range contains $B(\Pi m, C_0^{1/2})$, and $\Psi_m(\Delta_m \phi_m^{-1}(B(m,d))) \subset B(\Pi m, C_0^{1/2}/2)$, while $\Delta_m(x^u, x^s, x^0)(x^u, x^s, x^0 + \delta_m(x^u, x^s))$ with $\|\delta_m\|_{C^1} \le C$. Finally,

$$\phi_m^{(1)}(x^u, x^s, x^0) = (F_m^{(1)}(x^u, x^s), x^s, x^0)$$

on $B(\Pi m, C_0^{1/2})$, with $F_m^{(1)}$ a C^1 map with $DF_m^{(1)}$ in $\mathcal{K}(C_\#, B((m^u, 0), C_0^{1/2}))$.

If $\mathcal{E}$ is the foliation given by $\phi_m(x^u, x^s, x^0) = (F_m(x^u, x^s), x^s, \widetilde{F}_m(x^u, x^s, x^0))$, then by definition $\phi_m^{(1)}$ sends the stable leaves of $\mathbb{R}^d$ to the foliation $Q(\mathcal{E})$, i.e., $\phi_m^{(1)}$ is the standard parametrisation of the foliation $Q(\mathcal{E})$.

Proof of Lemma C.4. Fix $m = (m^u, m^s, m^0) \in \mathcal{J}'$. The map $Q \circ \phi_m$ does not qualify as $\phi_m^{(1)}$ for three reasons. First, $\pi_2 \circ Q \circ \phi_m(x^u, x^s, x^0)$ is not equal to x^s in general. Second, $\pi_3 \circ Q \circ \phi_m(x^u, x^s, x^0)$ is not equal to x^0 in general. Thirdly, $\pi_1 \circ Q \circ \phi_m(x^u, 0, x^0)$ is not equal to x^u in general. We shall use two maps $\Gamma^{(0)}$ and $\Gamma^{(1)}$ (sending stable leaves to stable leaves, and flow directions to flow directions) and a perturbation of the flow direction Δ_m to solve these three problems. The map $\Gamma^{(0)}$ will have the form $\Gamma^{(0)}(x^u, x^s, x^0) = (x^u, G(x^u, x^s), x^0)$, where, for fixed x^0 and x^u, the map $x^s \mapsto G(x^u, x^s)$ is a diffeomorphism of the stable leaf $\{x^u\} \times \mathbb{R}^{d_s} \times \{x^0\}$, so that $\pi_2 \circ Q \circ \phi_m \circ \Gamma^{(0)}(x^u, x^s, x^0) = x^s$, while the map $\Delta_m(x^u, x^s, x^0)(x^u, x^s, x^0 + \delta_m(x^u, x^s))$ is so that $\pi_3 \circ Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)}(x^u, x^s, x^0) \equiv x^0$ (note that $\pi_2 \circ Q \circ \phi_m \circ \Gamma^{(0)}(x^u, x^s, x^0) = \pi_2 \circ Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)}(x^u, x^s, x^0)$). In particular, $Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)}(x^u, 0, x^0)$ is of the form $(L^{(1)}(x^u), 0, x^0)$ for some map $L^{(1)}$. Choosing

$$\Gamma^{(1)}(x^u, x^s, x^0) = ((L^{(1)})^{-1}(x^u), x^s, x^0)$$

solves our last problem: the map

$$\phi_m^{(1)} = Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$$

satisfies $\pi_2 \circ \phi_m^{(1)}(x^u, x^s, x^0) = x^s$, $\pi_3 \circ \phi_m^{(1)}(x^u, x^s, x^0) = x^0$, and $\pi_1 \circ \phi_m^{(1)}(x^u, 0, x^0) = x^u$, as desired. (And it still has the property that $\partial_{x^0} \pi_1 \circ \phi_m^{(1)}(x^u, x^s, x^0) \equiv 0$.) Then, the map $\Psi_m = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1}$ sends stable leaves to stable leaves and flow directions to flow directions, and $Q \circ \phi_m \phi_m^{(1)} \circ \Psi_m \circ \Delta_m$.

We shall now be more precise, justifying the existence of the maps mentioned above, and estimating their domain of definition, their range and their smoothness.

The maps $\Gamma^{(0)}$ *and* Δ_m. For fixed x^u the map $(x^u, x^s) \mapsto G(x^u, x^s)$ should satisfy $\pi_2 \circ Q \circ \phi_m(x^u, G(x^u, x^s), x^0) = x^s$, i.e., it should be the inverse to the map

$$L_{x^u} : x^s \mapsto \pi_2 \circ Q \circ \phi_m(x^u, x^s, x^0) x^s - \pi_2(P F_m(x^u, x^s)) \,, \tag{C.6}$$

where we denote $\phi_m(x^u, x^s, x^0) = (F_m(x^u, x^s), x^s, \widetilde{F}_m(x^s, x^0))$. We claim that the map L_{x^u} is invertible onto its image, and that there exists $\epsilon^0_\# > 0$ such that

$$\begin{aligned} &|L_{x^u}(y^s) - L_{x^u}(x^s)| \ge \epsilon^0_\# |y^s - x^s| \,, \\ &\qquad \forall x^u \in B(m^u, C_0) \,, \quad \forall x^s, y^s \in B(m^s, C_0) \,. \end{aligned} \tag{C.7}$$

Indeed, fix $x^u \in B(m^u, C_0)$, let $w = y^s - x^s$, write $F(x^s) = F_m(x^u, x^s)$. We have

$$L_{x^u}(y^s) - L_{x^u}(x^s) w - \pi_2 \biggl(P \int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma \biggr) \,. \tag{C.8}$$

Define

$$\mathcal{C}^{ws} = \{ v + (0,0,x^0) \mid v \in \mathcal{C}^s, x^0 \in \mathbb{R} \} \,.$$

Each vector $(\partial_{x^s} F(x^s + \sigma w) w, w, x^0)$ belongs to $\widetilde{\mathcal{C}}^{ws}$. Since this cone is transversal to $\mathbb{R}^{d_u} \times \{0\} \times \{0\}$ and defined through a linear map (see (1.1) and the corresponding footnote), the set $\widetilde{\mathcal{C}}^{ws} \cap (\mathbb{R}^{d_u} \times \{w\} \times \{x^0\})$ is convex, hence

$$v_1 \left(\int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma, w, x^0 \right) \in \widetilde{\mathcal{C}}^{ws} \,. \tag{C.9}$$

On the other hand, since the graph of P is included in $\widetilde{\mathcal{C}}^u$,

$$v_2 = (\int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma, P \int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma)$$

belongs to $\widetilde{\mathcal{C}}^u$. Let $\epsilon^0_\# > 0$ be such that $B(v, \epsilon^0_\# |v|) \cap \widetilde{\mathcal{C}}^u = \emptyset$ for any $v \in \widetilde{\mathcal{C}}^{ws} - \{0\}$. Since $v_1 \in \widetilde{\mathcal{C}}^{ws}$ and $v_2 \in \widetilde{\mathcal{C}}^u$, we get $|v_1 - v_2| \ge \epsilon^0_\# |v_1|$. As v_1 and v_2 have the same first and third components if $x^0 := \pi_3(P \int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma)$, this gives $|\pi_2(v_1) - \pi_2(v_2)| \ge \epsilon^0_\# |v_1|$, i.e.,

$$\left| w - \pi_2(P \int_{\sigma=0}^1 \partial_{x^s} F(x^s + \sigma w) w \, d\sigma) \right| \ge \epsilon^0_\# |w| \,,$$

which implies (C.7) by (C.8).

For each x^s the map $(x^u, x^s) \mapsto \delta_m(x^u, x^s)$ should satisfy

$$\pi_3 \circ Q \circ \phi_m(x^u, G(x^u, x^s), x^0 - \delta_m(x^u, G(x^u, x^s))) \equiv x^0 \,,$$

i.e.,

$$\widetilde{F}_m(x^u, G(x^u, x^s), x^0 - \delta_m(x^u, G(x^u, x^s))) - \pi_3(P F_m(x^u, G(x^u, x^s))) = x^0 \,. \tag{C.10}$$

Letting $P_3 = \pi_3 P$, and recalling that $\partial_{x^0} \widetilde{F}_m(x^s, x^0) \equiv 1$, the condition reads, for $y^s = G(x^u, x^s)$,

$$\widetilde{f}_m(x^u, y^s) - \pi_3(P F_m(x^u, y^s)) = \delta_m(x^u, y^s) \,. \tag{C.11}$$

The map $\Lambda^{(0)} : (x^u, x^s) \mapsto (x^u, L_{x^u}(x^s))$ is well defined on $B((m^u, m^s), C_0)$, its derivative is bounded by a constant $C_\#$, and its second component satisfies (C.7). Then, [6, Lemma A.1] (with x^u and x^s exchanged) shows that $\Lambda^{(0)} \in \mathcal{D}(C_\#, B((m^u, m^s), C_0))$ for some constant $C_\#$. In particular, $\Lambda^{(0)}$ admits an inverse which also belongs to $\mathcal{D}(C_\#)$. By [6, Lemma A.2], the range of $\Lambda^{(0)}$ contains the ball $B(\Lambda^{(0)}(m), C_0/C_\#)$. Moreover, $\Lambda^{(0)}(m^u, m^s) = (m^u, \pi^2(Qm))$. By (C.3), we have $|Qm - \Pi m| \le C_\#$, hence the domain of definition of $(\Lambda^{(0)}))^{-1}(x^u, x^s) = (x^u, G(x^u, x^s))$ contains $B(\Pi m, C_0/C_\# - C_\#)$. If C_0 is large enough, this contains $B(\Pi m, C_0^{2/3})$.

Finally, one gets from (C.11) and Lemma 2.3 that $\delta_m(x^u, x^s)$ is defined on $B((m^u, 0), C_0^{2/3})$ with $\|\delta_m\|_{C^1} \le C_\#$ (in particular, $\Delta_m \in \mathcal{D}(C_\#)$), and we put

$$\Gamma^{(0)}(x^u, x^s, x^0) = (x^u, G(x^u, x^s), x^0)((\Lambda^{(0)}))^{-1}(x^u, x^s), x^0)\,,$$
$$\Delta_m(x^u, x^s, x^0) = x^0 + \delta_m(x^u, x^s)\,.$$

The map $\Gamma^{(1)}$. Consider $\phi_m^{(0)} Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)}$. It is a composition of maps in $\mathcal{D}(C_\#)$, hence it also belongs to $\mathcal{D}(C_\#)$. Moreover, its restriction to $\mathbb{R}^{d_u} \times \{0\} \times \mathbb{R}$ has the form $(x^u, 0, x^0) \mapsto (L^{(1)}(x^u), 0, x^0)$. It follows that the map $L^{(1)}$ (defined on a subset of $\mathbb{R}^{d_u}$) also satisfies the inequalities defining $\mathcal{D}(C_\#)$. In particular, this map is invertible, and we may define $\Gamma^{(1)}(x^u, x^s, x^0) = ((L^{(1)})^{-1}(x^u), x^s, x^0)$. This map belongs to $\mathcal{D}(C_\#)$. By construction, $\phi_m^{(1)} = Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$ can be written as

$$(F_m^{(1)}(x^u, x^s), x^s, x^0)$$

with $F_m^{(1)}(x^u, 0) = x^u$.

We have $\phi_m^{(0)}(Qm) = Qm$. Since $|\Pi m - Qm| \le C_\#$ by (C.3), and $\phi_m^{(0)}$ is Lipschitz, we obtain $|\phi_m^{(0)}(\Pi m) - \Pi m| \le C_\#$, i.e., $|L^{(1)}(m^u) - m^u| \le C_\#$. Since $L^{(1)} \in \mathcal{D}(C_\#)$, [6, Lemma A.2] shows that $L^{(1)}(B(m^u, C_0^{2/3}))$ contains the ball $B(m^u, C_0^{2/3}/C_\# - C_\#)$. Therefore, it contains the ball $B(m^u, C_0^{1/2})$ if C_0 is large enough. Hence, the domain of definition of the map $\Gamma^{(1)}$ contains $B(\Pi m, C_0^{1/2})$. This shows that $\phi_m^{(1)}$ is defined on $B(\Pi m, C_0^{1/2})$.

The map Ψ_m. We can now define

$$\Psi_m = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1} = (L^{(1)}(x^u), L_{x^u}(x^s), x^0)\,,$$

so that $Q \circ \phi_m = \phi_m^{(1)} \circ \Psi_m \circ \Delta_m$. We have seen that $\Psi_m \in \mathcal{D}(C_\#)$, hence $D\Psi_m$ and $D\Psi_m^{-1}$ are uniformly bounded. To show that $\Psi_m \in D_2^1(C_\#)$, we should check that $|D\Psi_m(x^u, x^s, x^0) - D\Psi_m(x^u, y^s, x^0)| \le C_\# |x^s - y^s|$. This follows directly from the construction and the inequality (2.3) for DF_m. Finally, since $\Psi_m \in \mathcal{D}(C_\#)$ and $\Delta_m \in \mathcal{D}(C_\#)$,

$$\Psi_m(\Delta_m \phi_m^{-1}(B(m, d))) \subset \Psi_m(B(m, C_\#)) \subset B(\Psi_m(m), C_\#)\,.$$

Since $Qm = \phi_m^{(1)}(\Psi_m(m))$ and $\Pi m = \phi_m^{(1)}(\Pi m)$, we get $|\Psi_m(m) - \Pi m| \le C_\# |Qm - \Pi m| \le C_\#$ by (C.3). Therefore, $\Psi_m(\phi_m^{-1}(B(m, d))) \subset B(\Pi m, C_\#)$, and this last set is included in $B(\Pi m, C_0^{1/2}/2)$ if C_0 is large enough.

The regularity of $DF_m^{(1)}$. To finish the proof, we should prove that $DF_m^{(1)}$ satisfies the bounds defining $\mathcal{K}(C_\#)$ for some constant $C_\#$ independent of C_0. Since $\phi_m^{(1)} = Q \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$, we have

$$D\phi_m^{(1)}(DQ \circ \phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\phi_m \circ \Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\Delta_m^{-1} \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\Gamma^{(0)} \circ \Gamma^{(1)}) \cdot D\Gamma^{(1)}\,. \tag{C.12}$$

Since $\mathcal{K}$ is invariant under multiplication ([6, Proposition A.4]), and under composition by Lipschitz maps sending stable leaves to stable leaves ([6, Proposition A.5]), it is sufficient to show that $D\phi_m$, $D\Delta_m^{-1}$, $D\Gamma^{(0)}$, and $D\Gamma^{(1)}$ all satisfy the bounds defining $\mathcal{K}(C_\#)$ (note that this is where (2.4)–(2.5) will be used) For $D\phi_m$, this follows from our assumptions, and for $D\Delta_m^{-1}$ from our assumptions and (C.11).

Since $\Gamma^{(0)} = ((\Lambda^{(0)})^{-1}, H)$, we have $D\Gamma^{(0)} = (D\Lambda^{(0)})^{-1} \circ \Gamma^{(0)}$. Since $D\Lambda^{(0)}$ is expressed in terms of DF_m, it belongs to $\mathcal{K}$. As $\mathcal{K}$ is invariant under inversion ([6, Proposition A.4]) and composition, we obtain $D\Gamma^{(0)} \in \mathcal{K}(C_\#)$.

Since $D\phi_m^{(1)}(x^u, 0, x^0) = \mathrm{id}$, it follows from (C.12) that, on the set $\{(x^u, 0, x^0)\}$, $D\Gamma^{(1)}$ is the inverse of the restriction of a function in $\mathcal{K}$, and in particular

$$D\Gamma^{(1)}(x^u, 0, x^0)$$

is a β-Hölder continuous function of x^u, by [6, (A.7)] and a Lipschitz function of x^0 by construction. Since $D\Gamma^{(1)}(x^u, x^s, x^0)$ only depends on x^u and x^0, it follows that $D\Gamma^{(1)}$ belongs to $\mathcal{K}$. This concludes the proof of Lemma C.4. □

We return to the proof of Lemma C.2:

Second step: Glueing the foliations $\phi_m^{(1)}$ together.

Just as in [6], a glueing step is necessary (see Step 3 of the proof of the Lasota-Yorke Lemma 3.1: the localisation lemma gives $\sum_{m\in\mathbb{Z}^d} \|\eta_m\omega\|^p_{H_p^{r,s,q}} \le C_\# \|\omega\|^p_{H_p^{r,s,q}}$ but *not* $\sum_{m\in\mathbb{Z}^d} \|\eta_m\omega_m\|^p_{H_p^{r,s,q}} \le \sup_m C_\# \|\omega_m\|^p_{H_p^{r,s,q}}$).

Let $\gamma(x^u, x^s)$ be a C^∞ function equal to 1 on the ball $B(C_0^{1/2}/2)$, vanishing outside of $B(C_0^{1/2})$. Let $\phi_m^{(1)}(x^u, x^s, x^0) = (F_m^{(1)}(x^u, x^s), x^s, x^0)$ be a foliation defined by Lemma C.4, and put

$$\phi_m^{(2)}(x^u, x^s, x^0) = (\gamma(x^u - m^u, x^s)(F_m^{(1)}(x^s, x^u) - x^u) + x^u, x^s, x^0)\,.$$

By construction,

$$\phi_m^{(2)}(x^u, x^s, x^0) = (F_m^{(2)}(x^u, x^s), x^s, x^0)\,,$$

with $F_m^{(2)}(x^u, 0) = x^u$. In addition $\phi_m^{(2)}$ defines a foliation on the ball of radius $C_0^{1/2}$ around Πm, coinciding with $\phi_m^{(1)}$ on $B(\Pi m, C_0^{1/2}/2)$, with $F_m^{(2)}$ equal to x^u on the (x^u, x^s)-boundary of $B(\Pi m, C_0^{1/2})$. Moreover, $DF_m^{(2)}$ is expressed in terms of γ, $D\gamma$, $F_m^{(1)}$ and $DF_m^{(1)}$. All those functions belong to $\mathcal{K}(C_\#)$ (the first three functions are Lipschitz and bounded, hence in $\mathcal{K}(C_\#)$, while we proved in Lemma C.4 that $DF_m^{(1)} \in \mathcal{K}(C_\#)$). Therefore, $DF_m^{(2)} \in \mathcal{K}(C_\#)$ by a modification of [6, Proposition A.4]. We proved in (a) that the balls $B(\Pi m, C_0^{1/2})$ for $m \in \mathcal{J}'$ are disjoint, therefore all the foliations $\phi_m^{(2)}$ can be glued together (with the trivial stable foliation outside of $\bigcup_{m\in\mathcal{J}'} B(\Pi m, C_0^{1/2})$), to get a single foliation parameterised by $\phi^{(2)} : \mathbb{R}^d \to \mathbb{R}^d$. We emphasize that this new foliation is not necessarily contained in the cone $Q(\widetilde{\mathcal{C}}^s)$, since the function γ contributes to the derivative of $\phi^{(2)}$. Nevertheless, it is uniformly transversal to the direction $\mathbb{R}^{d_u} \times \{0\} \times \mathbb{R}$, and this will be sufficient for our purposes. Write

$$\phi^{(2)}(x^u, x^s, x^0)(F^{(2)}(x^u, x^s), x^s, x^0)$$

where $F^{(2)}$ coincides everywhere with a function $F_m^{(2)}$ or with the function $(x^u, x^s) \mapsto x^u$. Since all the derivatives of those functions belong to $\mathcal{K}(C_\#)$, it follows that $DF^{(2)} \in \mathcal{K}(C_\#)$ (for some other constant $C_\#$, worse than the previous one due to the glueing). Since we will need to reuse this last constant, let us denote it by $C_\#^{(0)}$.

Third step: Pushing the foliation $\phi^{(2)}$ with D^{-1}. This very simple step is the heart of the argument and this is where (C.1) is needed: Define a new foliation by

$$F^{(3)}(x^u, x^s) = A^{-1} F^{(2)}(Ax^u, Bx^s)\,,\ \phi^{(3)}(x) = (F^{(3)}(x^u, x^s), x^s, x^0)\,, \tag{C.13}$$

so that $D^{-1}\phi^{(2)} = \phi^{(3)} D^{-1}$. We have $F^{(3)}(x^u, 0) = x^u$. Moreover

$$\partial_{x^u} F^{(3)}(x^u, x^s) = A^{-1}(\partial_{x^u} F^{(2)})(Ax^u, Bx^s)A\,,$$
$$\partial_{x^s} F^{(3)}(x^u, x^s) = A^{-1}(\partial_{x^s} F^{(2)})(Ax^u, Bx^s)B\,.$$

In particular, if $|A^{-1}|$ and $|B|$ are small enough (which can be ensured by decreasing ϵ in (C.1)), we can make $\partial_{x^s} F^{(3)}$ arbitrarily small. Since $|B| \le 1 \le |A|$, it also follows that (see [6, (3.12)])

$$|DF^{(3)}(x^u, x^s) - DF^{(3)}(x^u, y^s)| \le |A^{-1}||A|C_{\#}^{(0)}|B||x^s - y^s|\,. \tag{C.14}$$

In the same way as (C.14) (see [6, (3.13)]),

$$\begin{aligned}&|DF^{(3)}(x^u, x^s) - DF^{(3)}(x^u, y^s) - DF^{(3)}(y^u, x^s) + DF^{(3)}(y^u, y^s))|\\ &\qquad\le |A^{-1}||A|C_{\#}^{(0)}|A|^{\beta}|B|^{1-\beta}|x^u - y^u|^{\beta}|x^s - y^s|^{1-\beta}\,.\end{aligned} \tag{C.15}$$

If the bunching constant ϵ in (C.1) is small enough (depending on C_1), we can ensure that the two last equations are bounded, respectively, by $|x^s - y^s|/(2C_1)$ and $|x^u - y^u|^{\beta}|x^s - y^s|^{1-\beta}/(4C_0^2 C_1)$, i.e., the map $F^{(3)}$ satisfies the requirements (2.3) and (2.5) for admissible foliations, with better constants.

Taking $y^s = 0$ in (C.15), we obtain

$$\begin{aligned}&|DF^{(3)}(x^u, x^s) - DF^{(3)}(y^u, x^s)|\\ &\qquad\le |x^u - y^u|^{\beta}|x^s|^{1-\beta}/(4C_0^2 C_1) + |DF^{(3)}(x^u, 0) - DF^{(3)}(y^u, 0)|\,.\end{aligned}$$

Moreover, $\partial_{x^u} F^{(3)}(x^u, 0) = \partial_{x^u} F^{(3)}(y^u, 0) = \mathrm{id}$, so that (see the computation in the lines above [6, (3.14)]),

$$|DF^{(3)}(x^u, 0) - DF^{(3)}(y^u, 0)| \le |A^{-1}||B|C_{\#}^{(0)}|A|^{\beta}|x^u - y^u|^{\beta}\,.$$

The quantity $|A^{-1}||B||A|^{\beta}$ is bounded by $C_{\#}\lambda_u^{-1}\lambda_s\Lambda_u^{\beta}$. Choosing ϵ small enough in (C.1), it can be made arbitrarily small. For $|x^s| \le C_0^2$, this yields

$$|DF^{(3)}(x^u, x^s) - DF^{(3)}(y^u, x^s)| \le |x^u - y^u|^{\beta}/(2C_1)\,, \tag{C.16}$$

which is a small reinforcement of (2.4).

We see that for fixed $\mathcal{C}$, $\widetilde{C}$, there is a constant C depending only on C_0 so that the smallness condition on ϵ is of the form

$$\epsilon \le \frac{C}{C_1}\,. \tag{C.17}$$

Fourth step: Pushing the foliation $\phi^{(3)}$ with $(Q')^{-1}$. Define maps

$$F^{(4)}(x^u, x^s) = F^{(3)}(x^u, x^s) + P'_u(x^s)\,,\ \ \widetilde{F}^{(4)}(x^s, x^0) = x^0 + P'_0(x^s)$$

and let $\phi^{(4)}(x^u, x^s, x^0) = (F^{(4)}(x^u, x^s), x^s, \widetilde{F}^{(4)}(x^s, x^0))$. The corresponding foliation is the image of $\phi^{(3)}$ under $(Q')^{-1}$. Let us fix a cone $\mathcal{C}_1^s$ which sits compactly between $\mathcal{C}_0^s$ and $\mathcal{C}^s$. Since the graph $\{(P'_u(x^s), x^s, P'_0(x^s)\}$ is contained in $\mathcal{C}_0^s$, the foliation $F^{(4)}$ is contained in $\mathcal{C}_1^s$ if $\partial_{x^s} F^{(3)}$ is everywhere small enough. Moreover, the bounds of the previous step concerning $DF^{(3)}$ directly translate into the following bounds for $DF^{(4)}$ for all $x^u, y^u \in \mathbb{R}^{d_u}$ and all $x^s, y^s \in B(0, C_0^2)$:

$$|DF^{(4)}(x^u, x^s) - DF^{(4)}(x^u, y^s)| \le |x^s - y^s|/(2C_1)\,, \tag{C.18}$$

$$|DF^{(4)}(x^u, x^s) - DF^{(4)}(y^u, x^s)| \le |x^u - y^u|^\beta/(4C_0^2C_1)\,, \tag{C.19}$$

$$\begin{aligned}&|DF^{(4)}(x^u, x^s) - DF^{(4)}(x^u, y^s) - DF^{(4)}(y^u, x^s) + DF^{(4)}(y^u, y^s)| \\ &\qquad\qquad \le |x^u - y^u|^\beta |x^s - y^s|^{1-\beta}/(2C_1)\,,\end{aligned} \tag{C.20}$$

In particular, since $\partial_x F^{(4)}(x^u, 0) = \mathrm{id}$, the bound (C.18) implies that $\partial_{x^u} F^{(4)}$ is bounded and has a bounded inverse on a ball of radius $C_1 \ge 2C_0$.

In addition, linearity of P_0' implies that

$$|D\widetilde{F}^{(4)}(x^s, x^0) - D\widetilde{F}^{(4)}(y^s, y^0)| = 0\,. \tag{C.21}$$

Last step: Pushing the foliation $\phi^{(4)}$ with $\mathcal{T}^{-1}\mathcal{A}$. Let $\mathcal{U} = \mathcal{T}^{-1}\mathcal{A}$, and consider ϕ' the foliation obtained by pushing $\phi^{(4)}$ with $\mathcal{U}$. We claim that ϕ' belongs to $\mathcal{F}(0, \mathcal{C}^s, C_0, C_1)$, and that we can write $\mathcal{U} \circ \phi = \phi' \circ \Psi'$ for some $\Psi' \in D_1^2(C_\#)$.

To prove this, we follow the arguments in the proof of Lemma C.4 (with simplifications here since $\mathcal{U}$ is close to the identity, noting also that $\mathcal{U}$ preserves the flow direction so that $\mathcal{U}(x^u, x^s, x^0)(U_u(x^u, x^s), U_s(x^u, x^s), U_0(x^u, x^s, x^0)$ with $U_0(x^u, x^s, x^0) = x^0 + u_0(x^u, x^s)$, and in particular the property $\partial_{x^0} F' \equiv 0$ will be given for free). First, fix x^u, and consider the map $L_{x^u} : x^s \mapsto \pi_2 \circ \mathcal{U} \circ \phi^{(4)}(x^u, x^s, x^0)$. Writing $\mathcal{U} = \mathrm{id} + \mathcal{V}$ where $\|\mathcal{V}\|_{C^2} \le \epsilon$, we have $L_{x^u}(x^s) = x^s + \pi_2 \circ \mathcal{V}(F^{(4)}(x^u, x^s), x^s, \widetilde{F}^{(4)}(x^s, x^0))$. (Note that L_{x^u} does not depend on x^0 because U_s doesn't.) Since $F^{(4)}$ is bounded in C^1 on the ball $B(0, 2C_1)$, it follows that, if ϵ is small enough, then the restriction of L_{x^u} to the ball $B(0, 2C_1)$ (in $\mathbb{R}^{d_s}$) is arbitrarily close to the identity. Therefore, its inverse is well defined, and we can set $\Gamma^{(0)}(x^u, x^s, x^0) = (x^u, L_{x^u}^{-1}(x^s), x^0)$. By construction, the map $\mathcal{U} \circ \phi^{(4)} \circ \Gamma^{(0)}(x^u, 0, x^0)$ has the form $(L^{(1)}(x^u), 0, x^0 + L_{x^u}^{(2)})$ for some function $L^{(1)}$, which is bounded in $C^{1+\beta}$ and arbitrarily close to the identity in C^1 if ϵ is small, and some function $L^{(2)}(x^u)$ which is bounded in $C^{1+\beta}$ and arbitrarily close to zero in C^1, if ϵ is small. Let $\Gamma^{(1)}(x^u, x^s, x^0) = ((L^{(1)})^{-1}(x^u), x^s, x^0 - L^{(2)}(x^u))$, then the map $\phi' = \mathcal{U} \circ \phi^{(4)} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$ is defined on the set $\{(x^u, x^s, x^0) \mid |x^s| \le C_1\}$ (which contains $B(0, C_0)$), and it takes the form $\phi'(x^u, x^s, x^0) = (F'(x^u, x^s), x^s, x^0 + \widetilde{f}'(x^u, x^s))$ for some functions F', $\widetilde{f}'$, with $F'(x^u, 0) = x^u$ and $\widetilde{f}'(x^u, 0) = 0$.

Since ϕ' is obtained by composing $\phi^{(4)}$ with diffeomorphisms arbitrarily close to the identity, it follows from (C.18)–(C.20) that F' satisfies (2.3)–(2.5) and from (C.18) (recall that $\widetilde{F}^4$ does not depend on x^u) that $\widetilde{f}'$ satisfies (2.7). Indeed, the present analogue of (C.12) is

$$D\phi'(D\mathcal{U} \circ \phi^{(4)} \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\phi^{(4)} \circ \Gamma^{(0)} \circ \Gamma^{(1)}) \cdot (D\Gamma^{(0)} \circ \Gamma^{(1)}) \cdot D\Gamma^{(1)}\,,$$

where $\Gamma^{(0)}$ and $\Gamma^{(1)}$ here satisfy the same properties as the maps with the same names in the proof of Lemma C.4, and where $D\mathcal{U} \circ \phi^{(4)} \circ \Gamma^{(0)} \circ \Gamma^{(1)}$ is bounded and Lipschitz and thus belongs to $\mathcal{K}$. We may thus argue exactly as in the last step of the proof of Lemma C.4 for F', while the case of $\widetilde{f}'$ is easier.

Moreover, since $(\partial_{x^s} F^{(4)}(z)w, w, \partial_{x^s} \widetilde{F}^{(4)} w)$ takes its values in the cone $\mathcal{C}_1^s$, it follows that $(\partial_{x^s} F'(z)w, w, \partial_{x^s} \widetilde{f}'(z))$ lies in the cone $\mathcal{C}^s$ if $\mathcal{U}$ is close enough to the identity. Hence, the foliation defined by ϕ' is contained in $\mathcal{C}^s$. This shows that ϕ' belongs to $\mathcal{F}(0, \mathcal{C}^s, C_0, C_1)$.

Finally, the function $\Psi = (\Gamma^{(0)} \circ \Gamma^{(1)})^{-1}$ belongs to $D_1^2(C_\#)$. This concludes the proof of Lemma C.2. □

Appendix D. Approximate unstable foliations

Let us consider $\bar{x} \in U_{i,j,\ell,1}$ and ϱ so small that $B(\bar{x}, \varrho) \subset U_{i,j,\ell,1} \cap B_{i,j}$.[45] First we describe all the objects in $B(\bar{x}, \varrho)$ by using the chart $\kappa_{i,j,\ell}$.

Let W be a surface with curvature bounded by some fixed constant $C_\#$ such that $\bar{x} \in W$, $\partial(W \cup B(\bar{x}, \varrho)) \subset \partial B(\bar{x}, \varrho)$ and with tangent space, at each point, given by the span of the flow direction and a vector, in the kernel of α, contained in the stable cone. Recall that in the present coordinates the contact form has the expression $\alpha_0 = dx^0 - x^s dx^u$. Note that almost every point in W has a well-defined unstable direction. We can thus assume without loss of generality that the unstable direction is well defined at $\bar{x}$. By Lemma A.4 we can then change coordinates so that $\bar{x} = 0$, $W = \{(0, \xi, s)\}_{\xi, s \in \mathbb{R}}$ and the unstable direction at $\bar{x}$ is given by $(1, 0, 0)$. From now on we will work in such coordinates without further mention.

Our goal is to define smooth approximate strong unstable foliations in a $c\rho < \varrho$ neighbourhood of W.

More precisely, we look for C^{1+Lip} foliations Γ_m, described by the triangular change of coordinates $\mathbb{G}_m(\eta, \xi, s) = (\eta, G_m(\eta, \xi), H_m(\eta, \xi) + s)$, with domains $\Delta_m \subset \{\xi \in \mathbb{R}^2 \,;\, \|\xi\| \le \varrho\}$ and constants $c, \varpi > 0$, $\sigma \in (0,1)$, $m_0 \in \mathbb{N}$ such that

1. $G_m(0, \xi) = \xi$, $H_m(0, \xi) = 0$;
2. $\partial_\eta H_m = G_m$ (i.e., $\alpha_0(\partial_\eta \mathbb{G}) = 0$);
3. for all $m_0 \ge m' \ge m$, $\Delta_{m'} \subset \Delta_m$;
4. for all $m_0 \ge m' \ge m$ and $\xi \in \Delta_{m'}$, $\|G_m(\xi, \cdot) - G_{m'}(\xi, \cdot)\|_{C^1} \le c\sigma^m$, $\|H_m(\xi, \cdot) - H_{m'}(\xi, \cdot)\|_{C^1} \le c\sigma^m$;
5. $\|\partial_\xi \partial_\eta \mathbb{G}_m\|_\infty + \|\partial_s \partial_\eta \mathbb{G}_m\|_\infty \le c\rho^{-1}$;
6. $\|\partial_\xi \partial_\eta \mathbb{G}_m\|_{C^\varpi} + \|\partial_s \partial_\eta \mathbb{G}_m\|_{C^\varpi} \le c\rho^{-1-\varpi}$;
7. if $\xi \in \Delta_m$, then $\partial_\eta T_{-\tau_+ n} \mathbb{G}_m(\eta, \xi)$ is well defined and belongs to the unstable cone for all $n \le m \le m_0$ and $\|\eta\| \le c\rho$;
8. $m(\Delta_m^c \cap W) \le c\rho$, for all $m \le m_0$.

Remark D.1. Note that the above properties are not all independent, we spelled them out in unnecessary details for the reader's convenience. In particular, (1) is just a condition on the parametrisation used to describe the foliation and can be assumed without loss of generality; (3, 7) imply (4) due the the usual cone contraction of hyperbolic dynamics.

Lemma D.2. *There exists $\varpi_0 > 0$ such that, provided $\varrho > \rho > 0$, $\varpi \in [0, \varpi_0]$,[46] for each $m_0 \in \mathbb{N}$, there exists at least one set of foliations Γ_m, $m \in \{0, \dots, m_0\}$, in a ρ-neighbourhood of W, which satisfies the properties (1-8).*

Proof. First of all, notice that once we construct the foliation over $W_0 = \{(0, \xi, 0)\}_{\xi \in \mathbb{R}}$ we can obtain the foliation on W by simply flowing it with the dynamics and G, H will have automatically the wanted s dependence. Contrary to the notation in section 6, we will call $\mathcal{W}_m$ the set of regular manifolds obtained by W_0 under backward iteration by T_{τ_+}.

Second, for each $\varpi \in (0,1)$, $K > 0$, there exists $c > 0$ and $\upsilon, \upsilon_1 \in (0,1)$, $\upsilon_1 < \upsilon$, close enough to one, with the following properties. Consider any foliation in a $\upsilon^m \rho$ neighbourhood of $\mathcal{W}_m$ aligned with the unstable cone, with fibers in the kernel of the contact form and with C^1 and $C^{1+\varpi}$ norm bounded by $Kc\upsilon^{-m}\rho^{-1}$ and $Kc\upsilon^{-m(1+\varpi)}\rho^{-1-\varpi}$ respectively (in the sense of conditions (5, 6) above). Then the image under any $T_{k\tau_+}$ provides a foliation in the $\upsilon^{m-k}\rho$ neighbourhood of $\mathcal{W}_{m-k}$ satisfying the same conditions of $\mathcal{W}_m$ but with bounds $c\upsilon^{-m}\upsilon_1^k \rho^{-1}$ and

[45] By $B(\bar{x}, \varrho)$ we mean the ball of radius ϱ centered at $\bar{x}$.

[46] In fact, it should be possible to have the result for each $\varpi \in [0,1)$, but this is not necessary for our purposes.

$c\upsilon^{-m(1+\varpi)}\upsilon_1^{k(1+\varpi)}\rho^{-1-\varpi}$, provided ρ is chosen small enough and c large enough. This follows by standard distortion estimates as in the construction of stable manifolds for a smooth Anosov map, see [28]. We will call such foliations *allowed.*

Our strategy will be as follows: For each given allowed foliation Γ_m^* defined in a $\upsilon^m\rho$ neighbourhood of $\mathcal{W}_m$ we will show how to construct foliations Γ_n represented by coordinates $\mathbb{G}_n$, $n \leq m$, satisfying conditions (1-8). The problem, of course, is what to do with the fibers that are cut by a singularity.

As a first, very rough, approximation of the unstable foliation, let us choose in each $V_{i,j,\ell,0} = \kappa_{i,j,\ell}(U_{i,j,\ell,0})$ the foliation given by the leaves $\{\eta, \xi, s + \xi\eta\}_{\eta\in\mathbb{R}}$. We call Γ the resulting set of foliations, note that the tangent space to the leaves belongs the unstable cone and to the kernel of the contact form. [47]

Next, we proceed by induction. Given Γ_0^* we set $\Delta_0 = W_0$ and chose Γ_0^* itself as foliation. All the non vacuous conditions are then satisfied. Next, suppose we have defined a construction of the foliation for all $n < m$ and we are given an allowed foliation Γ_m^* in the neighbourhood of $\mathcal{W}_m$. Note that by the transversality condition on the singularities, there exists $c_* > 1$ such that any fiber of $T_{\tau_+}\Gamma_m^*$ which has been cut by $\partial B_{i,j}$ and intersects $\mathcal{W}_{m-1}$ must intersect $T_{\tau_+}(\partial_{c_*\upsilon^m\rho}\mathcal{W}_m)$. We define $\overline{W}_m = \cup_{W\in\mathcal{W}_m} W$ and S_m to be the union of the elements of $\mathcal{W}_m$ shorter than $2c_*\upsilon^m\rho$.

We are now ready to define an allowed foliation Γ_{m-1}^*. On the set $\overline{W}_{m-1} \setminus (T_{\tau_+}(\partial_{2c_*\upsilon^m\rho}\mathcal{W}_m))$ it is simply given by $T_{\tau_+}\Gamma_m^*$. On $(T_{\tau_+}(\partial_{c_*\upsilon^m\rho}\mathcal{W}_m)) \cup T_{\tau_+}S_m$ we define it to be Γ.[48] Inside small intervals at whose boundaries the foliation has now been defined, we must interpolate. To do so precisely, it is best to write explicitly the objects of interest.

Let $(f(\xi), \xi, g(\xi))_{\xi\in[a',b']}$ be the graph of an element of $\mathcal{W}_{m-1}$ and $[a, b]$, $a' < a < b < b'$, the interval on which we want to define the interpolating foliation.[49] By construction there exist fixed constants $c_-, c_+ > 0$ such that $c_+\rho\upsilon^{m-1} \geq |b - a| \geq c_-\rho\upsilon^{m-1}$. Let $F(\xi) = (f(\xi), \xi, g(\xi))$ and $\gamma_a(\eta, \xi) = F(\xi) + (\eta, \sigma_a(\eta, \xi), \zeta_a(\eta, \xi))$, $\gamma_b(\eta, \xi) = F(\xi) + (\eta, \sigma_b(\eta, \xi), \zeta_b(\eta, \xi))$, $|\eta| \leq \upsilon^{m-1}\rho$, a parametrisation of the two foliations we must interpolate. Clearly, we can assume without loss of generality $\sigma_a(0, \xi) = \sigma_b(0, \xi) = \zeta_a(0, \xi) = \zeta_b(0, \xi) = 0$. By construction the above curves are in the kernel of the contact form, that is $\partial_\eta\zeta_a(\eta, \xi) = \xi + \sigma_a(\eta, \xi)$, $\partial_\eta\zeta_b(\eta, \xi) = \xi + \sigma_b(\eta, \xi)$, and in the unstable cone, that is $|\partial_\eta\sigma_a| + |\partial_\eta\zeta_a| \leq \upsilon K_0$ and $|\partial_\eta\sigma_b| + |\partial_\eta\zeta_b| \leq \upsilon K_0$.[50] In addition, since Γ_m^* is allowed,

$$
\begin{aligned}
&|\partial_\xi\partial_\eta\sigma_a|_{C^0} \leq c\upsilon^{-m}\upsilon_1\rho^{-1}, \qquad &|\partial_\xi\partial_\eta\sigma_a|_{C^\varpi} \leq c[\upsilon^{-m}\upsilon_1\rho^{-1}]^{1+\varpi}\\
&|\partial_\xi\partial_\eta\sigma_b|_{C^0} \leq c\upsilon^{-m}\upsilon_1\rho^{-1}, \qquad &|\partial_\xi\partial_\eta\sigma_b|_{C^\varpi} \leq c[\upsilon^{-m}\upsilon_1\rho^{-1}]^{1+\varpi}.
\end{aligned}
$$

Next, fix once and for all $\bar\varphi, \bar\psi \in C^2(\mathbb{R}, [0, 1])$, such that $\bar\varphi(0) = \partial_\xi\bar\varphi(0) = \partial_\xi\bar\varphi(1) = \bar\psi(0) = \bar\psi(1) = 0$, $\bar\varphi(1) = 1$, $\int_0^1 \bar\psi(\xi) = 1 - c_1$, c_1 to be chosen later small enough. Define then $\varphi(\xi) = \bar\varphi(\frac{\xi-a}{b-a})$ and $\psi(\xi) = \bar\psi(\frac{\xi-a}{b-a})$. Clearly, $\varphi(a) =$

[47]By, if necessary, restricting the chart, we can assume without loss of generality that the unstable cone is given by the condition $\|\xi\| + \|s\| \leq K_0\|\eta\|$ for some K_0 small.

[48]Note that, by choosing L_0 in Lemma 6.2, we can always assume that each element of $\mathcal{W}_m$ is contained in some $U_{i,j,\ell,1}$, thus one can use the corresponding element of Γ.

[49]We can assume without loss of generality that the foliation has been already defined in $[a', a] \cup [b, b']$ and satisfies the wanted properties.

[50]The υ comes from the fact that the foliation is either an image of a foliation already in the unstable cone or is a fixed foliation well inside the cone.

$\partial_\xi\varphi(a)=\partial_\xi\varphi(b)=\psi(a)=\psi(b)=0$, $\varphi(b)=1$, $\int_a^b\psi(\xi)=(1-c_1)(b-a)$. Define

$$\theta_0(\eta,\xi)=\left(\partial_\xi\sigma_b(\eta,b)\frac{\xi-a}{b-a}+\partial_\xi\sigma_a(\eta,a)\frac{b-\xi}{b-a}\right)(1-\psi(\xi))$$
$$\theta(\eta,\xi)=\theta_0(\eta,\xi)-\frac{1}{(1-c_1)(b-a)}\psi(\xi)\int_a^b\theta_0(\eta,z)dz$$
$$\sigma(\eta,\xi)=\sigma_b(\eta,b)\varphi(\xi)+\sigma_a(\eta,a)(1-\varphi(\xi))+\int_a^\xi\theta(\eta,z)dz\,,$$

for all $\|\eta\|\le\upsilon^{m-1}\rho$ and $\xi\in(a,b)$. Next, define $\zeta(\eta,\xi)=\xi\eta+\int_0^\eta\sigma(z,\xi)dz$. We can then define Γ^*_{m-1} for $\xi\in[a',b']$ as the foliation with fibers

$$\gamma(\eta,\xi)=F(\xi)+\begin{cases}\gamma_a(\eta,\xi) & \text{for } \xi\in[a',a]\\ (\eta,\sigma(\eta,\xi),\zeta(\eta,\xi)) & \text{for } \xi\in(a,b)\\ \gamma_b(\eta,\xi) & \text{for } \xi\in[b,b']\,.\end{cases}$$

The definition of ζ implies that the leaves of γ are in the kernel of the contact form. Moreover, note that $\sigma(\eta,a)=\sigma_a(\eta,a)$ and

$$\sigma(\eta,b)=\sigma_b(\eta,b)+\int_a^b dz\theta_0(\eta,z)\left[1-\frac{1}{(1-c_1)(b-a)}\int_a^b dw\psi(w)\right]=\sigma_b(\eta,b)\,.$$

From this follows $\gamma\in C^0$. In addition, $\partial_\eta\sigma(\eta,a)=\partial_\eta\sigma_a(\eta,a)$ and, since

$$\partial_\eta\theta_0(\eta,\xi)=\left(\partial_\eta\partial_\xi\sigma_b(\eta,b)\frac{\xi-a}{b-a}+\partial_\eta\partial_\xi\sigma_a(\eta,a)\frac{b-\xi}{b-a}\right)(1-\psi(\xi))$$
$$\partial_\eta\theta(\eta,\xi)=\partial_\eta\theta_0(\eta,\xi)-\frac{1}{(1-c_1)(b-a)}\psi(\xi)\int_a^b\partial_\eta\theta_0(\eta,z)dz$$
$$\partial_\eta\sigma(\eta,\xi)=\partial_\eta\sigma_b(\eta,b)\varphi(\xi)+\partial_\eta\sigma_a(\eta,a)(1-\varphi(\xi))+\int_a^\xi\partial_\eta\theta(\eta,z)dz\,.$$

It follows that $\partial_\eta\gamma\in C^0$ and a similar computation shows $\partial_\xi\gamma\in C^0$. Since all the quantities are piecewise C^2, it follows that $\gamma\in C^{1+\mathrm{Lip}}$. In addition,

$$\begin{aligned}\|\partial_\eta\sigma\|_{L^\infty}&\le\upsilon K_0+c\upsilon^{-m+1}\rho^{-1}\left[\int_a^b(1-\psi)+\int_a^b\frac{\psi}{(1-c_1)(b-a)}\int_a^b(1-\psi)\right]\\&\le\upsilon K_0+2cc_+c_1\,.\end{aligned}$$

Which, by choosing c_1 small enough, ensures that the fibers of γ belong to the unstable cone. Finally, we have,

$$\partial_\xi\partial_\eta\sigma(\eta,\xi)=(\partial_\eta\sigma_b(\eta,b)-\partial_\eta\sigma_a(\eta,a))\,\varphi'(\xi)+\partial_\eta\theta(\eta,\xi)\,,$$

which implies $\partial_\xi\partial_\eta\sigma(\eta,a)=\partial_\xi\partial_\eta\sigma_a(\eta,a)$ and $\partial_\xi\partial_\eta\sigma(\eta,b)=\partial_\xi\partial_\eta\sigma_b(\eta,b)$ and also $\partial_\eta\partial_\xi\gamma\in C^0$. The last estimate is

$$\begin{aligned}\|\partial_\xi\partial_\eta\sigma\|_{L^\infty}&\le 2K_0\|\varphi'\|_{L^\infty}+\|\partial_\eta\theta(\eta,\xi)\|_{L^\infty}\\&\le 2K_0\|\bar\varphi'\|_{L^\infty}c_-^{-1}\rho^{-1}\upsilon^{-m+1}+c\upsilon^{-m}\upsilon_1\rho^{-1}\,,\end{aligned}$$

which yields $\|\partial_\xi\partial_\eta\sigma\|_{L^\infty}\le c\upsilon^{-m+1}\rho^{-1}$, provided c is large enough. Similar computations verify the C^ϖ bounds, provided $\varpi>0$ is chosen small enough.

In other words γ is an allowed foliation.

By the inductive hypotheses we can then take $\Gamma^*_{m-1}=\gamma$ as the starting point to construct foliations $\widetilde{\mathbb{G}}^m_n$ and domains Δ_n, for $n<m-1$, satisfying hypotheses (1-8). We then define the domain $\Delta_m=T_{(m-1)\tau_+}\left[T^{-(m-1)\tau_+}\Delta_{m-1}\setminus T_{\tau_+}(\partial_{2c_*\upsilon^m\rho}\mathcal{W}_m)\right]$.

To conclude we define $\mathbb{G}_m$, $m\le m_0$, to be the foliations $\widetilde{\mathbb{G}}^{m_0}_n$ obtained by the above procedure when starting from the initial allowed foliation Γ. It is immediate to check that the above construction satisfies properties (1-7). Property (8) follows by noticing that $\Delta_m^c \subset \cup_{n\leq m} T_{n\tau_+}(\partial_{2c_*\upsilon^n\rho}\mathcal{W}_n)$ and by applying Lemma 6.2 with $\delta = \varrho$, $r = 2c_*\rho$ and $\vartheta = \upsilon$. □

Remark D.3. Note that, given two manifolds of size ϱ uniformly C^2 and uniformly transversal to the unstable direction having a distance less than ρ, one can use the above lemma to construct a uniformly Lipschitz holonomy, approximating the unstable one, between the two manifolds.[51]

Appendix E. Cancellations estimates

In this appendix we detail the basic, but technical, cancellations estimates in the Dolgopyat argument.

Proof of Lemma *6.7.* To start with, let us obtain a formula for ω_α, this is the analogous of the formulae in [30, 27] adapted to the present context. By translating W_α along the flow direction we can assume, without loss of generality, that the leaf of $\Gamma^{\varkappa}_{i,j,r}$ starting from $\bar x$ intersects W_α. Let $\widetilde{W}_\alpha = \Theta_{i,\alpha,\varkappa}W_\alpha \subset W_*^i$ and consider the path $\gamma(\xi)$ running from $\bar x$ along the leaf of $\Gamma^{\varkappa}_{i,j,r}$ up to W_α, then along W_α up to $\bar x + (F_\alpha(\xi), \xi, N_\alpha(\xi))$ and back to W_*^i along the leaf of $\Gamma^{\varkappa}_{i,j,r}$ again, then to the axis $(\eta, \xi, s) = (\bar x^u, \xi, \bar x^0)$ along the flow direction and finally back to $\bar x$ along the axis, (see Figure 4 for a pictorial explanation). By construction,

$$
\begin{aligned}
\bar\omega_\alpha(\xi) &= \int_{\gamma(\xi)} \alpha = \int_{\Sigma_\alpha(\xi)} d\alpha + \int_{\Omega_\alpha(\xi)} d\alpha = \int_{\Sigma_\alpha(\xi)} d\alpha \\
&= \int_0^\xi dz \int_0^{F_\alpha(z)} d\eta\, \partial_\xi G_{i,j,\varkappa}(\eta, h_\alpha(z)) h'_\alpha(z) \\
&= \int_0^{h_\alpha(\xi)} dz \int_0^{F_\alpha\circ h_\alpha^{-1}(z)} d\eta\, \partial_\xi G_{i,j,\varkappa}(\eta, z)\,,
\end{aligned}
$$

where we have used the fact that all the curve $\gamma(\xi)$, apart for the piece in the flow direction, is in the kernel of α and Stokes' Theorem. Moreover, $\Sigma_\alpha(\xi)$ is the surface traced by the fibers of $\Gamma^{\varkappa}_{i,r}$ while moving along W_α up to ξ and $\Omega_\alpha(\xi)$ the portion of W_*^i between the ξ axes and the curve $\Theta_{i,\alpha,\varkappa}(W_\alpha)$ up to $h_\alpha(\xi)$.

By Lemma A.4 we can assume without loss of generality that $\bar x = 0$. Using properties (1) and (5) of the approximate foliations in appendix D, it follows that

$$
\begin{aligned}
\partial_\xi\omega_\beta(\xi) - \partial_\xi\omega_\alpha(\xi) &= \int_{F_\alpha\circ h_\alpha^{-1}(\xi)}^{F_\beta\circ h_\beta^{-1}(\xi)} d\eta\ \partial_\xi G_{i,j,\varkappa}(\eta,\xi) \\
&= \int_{F_\alpha\circ h_\alpha^{-1}(\xi)}^{F_\beta\circ h_\beta^{-1}(\xi)} d\eta \left[1 + \int_0^\eta dz \partial_\xi\partial_z G_{i,j,\varkappa}(z,\xi)\right] \\
&= \left[F_\beta\circ h_\beta^{-1}(\xi) - F_\alpha\circ h_\alpha^{-1}(\xi)\right](1 + \mathcal{O}(r^{1-\varsigma}))\,.
\end{aligned}
\tag{E.1}
$$

To continue note that, in analogy with (6.21) we have

$$
\begin{aligned}
\left|h_\beta^{-1}(\xi) - h_\alpha^{-1}(\xi)\right| &= \left|\int_{F_\alpha\circ h_\alpha^{-1}(\xi)}^{F_\beta\circ h_\beta^{-1}(\xi)} dz \int_0^\xi dw\ \partial_z\partial_w G_{i,\varkappa}(z,w)\right| \\
&\leq C_\# r^{\theta-\varsigma}\left|F_\beta\circ h_\alpha^{-1}(\xi) - F_\alpha\circ h_\beta^{-1}(\xi)\right|\,.
\end{aligned}
\tag{E.2}
$$

[51] Indeed $\partial_\eta\mathbb{G}$ provides a Lipschitz vector field, hence one can consider the associated flow, the holonomy map is nothing else than the Poincaré map between the two manifolds. It follows then by standard computations that the the time taken by the flow to go between the two surfaces is Lipschitz and bounded by a fixed constant times ρ. Finally, (5) readily implies that the holonomy is Lipschitz as well.

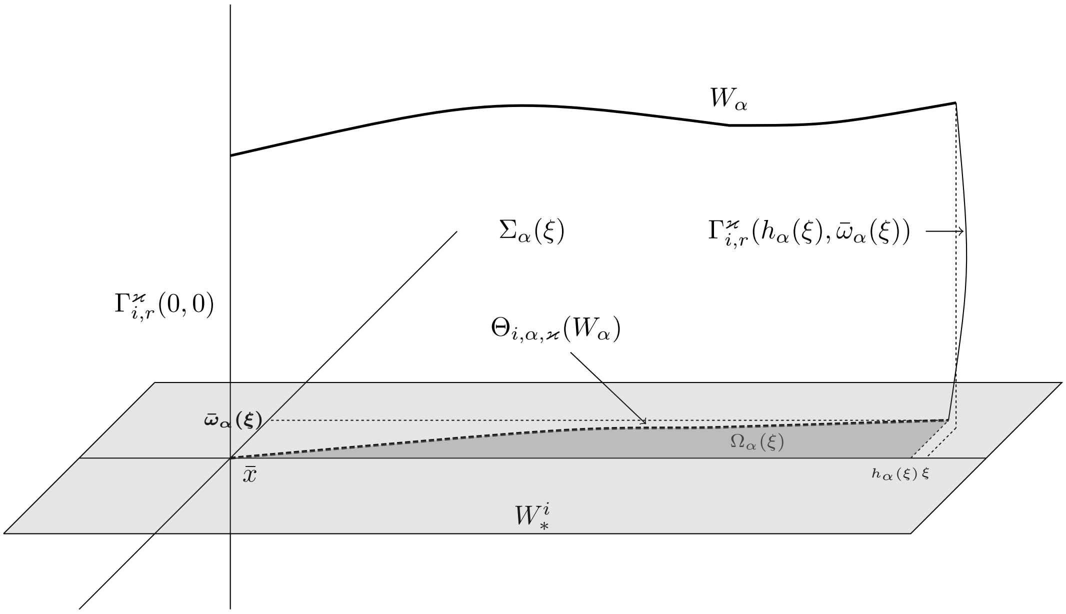

FIGURE 4. Definition of $\bar\omega_\alpha$.

To conclude recall that we are working in coordinates in which $|F'_\alpha| \le C_\# r^{1-\theta}$, cf. the proof of Lemma 6.4, hence

$$\begin{aligned}\left|F_\beta \circ h_\beta^{-1} - F_\alpha \circ h_\alpha^{-1}\right| &\ge \left|F_\beta \circ h_\beta^{-1} - F_\alpha \circ h_\beta^{-1}\right| - C_\# r^{1-\theta}|h_\beta^{-1} - h_\alpha^{-1}| \\ &\ge \left|F_\beta \circ h_\beta^{-1} - F_\alpha \circ h_\beta^{-1}\right| - C_\# r^{1-\varsigma}\left|F_\beta \circ h_\beta^{-1} - F_\alpha \circ h_\alpha^{-1}\right|\end{aligned}$$

which, together with (E.1), proves (6.26) provided we can show that the distance between the manifolds is comparable with $|F_\beta - F_\alpha|$ at any point. To prove such a fact recall that in (6.13), and following lines, we have seen that

$$|F'_\beta(\xi) - F'_\alpha(\xi)| \le C_\# c^{-1} r^{-\theta} |F_\beta(\xi) - F_\alpha(\xi)|\,,$$

provided that

$$\bar\lambda^{2\ell} \ge C_\# r^\theta |F_\beta(\xi) - F_\alpha(\xi)|^{-1} \ge C_\# r^{\theta-\vartheta}\,,$$

which is ensured by our assumptions. Thus setting $d_{\alpha,\beta}(\xi) = |F_\alpha(\xi) - F_\beta(\xi)|$ we have

$$\frac{d}{d\xi} d_{\alpha,\beta}(\xi) \le C_\# c^{-1} d_{\alpha,\beta}(\xi) r^{-\theta}\,,$$

Hence, by Gronwall's lemma, $|d_{\alpha,\beta}|_\infty \le C_\# d_{\alpha,\beta}(0)$ and

$$|d_{\alpha,\beta}(\xi) - d_{\alpha,\beta}(0)| \le \frac{1}{2} d_{\alpha,\beta}(0)\,, \tag{E.3}$$

for all $|\xi| \le r^\theta$, provided c has been chosen large enough. Next, remember that $H'_{i,j,\varkappa} = G_{i,j,\varkappa}$, thus $|H'_{i,j,\varkappa}|_\infty \le C_\# r$ and

$$|N_\alpha(0) - N_\beta(0)| = |H_{i,j,\varkappa}(F_\alpha(0), 0) - H_{i,j,\varkappa}(F_\beta(0), 0)| \le C_\# r |F_\alpha(0) - F_\beta(0)|\,.$$

In addition, since $N'_\alpha(\xi) = \xi F_\alpha(\xi)$, it follows that $|N_\alpha - N_\beta|_\infty \le C_\# r|F_\alpha - F_\beta|_\infty$. Which proves

$$\partial_\xi \omega_\beta(\xi) - \partial_\xi \omega_\alpha(\xi) \ge C_\# d_{i,j}(W_\alpha, W_\beta)\,. \tag{E.4}$$

To prove the second statement, let us introduce $\omega_{\alpha,\beta}(\xi) = \omega_\alpha(\xi) - \omega_\beta(\xi)$ and $A_{\alpha,\beta} = \frac{[(m-1)!]^2}{(\ell\tau_-)^{2m-2}} \mathbf{F}^*_{\ell,m,i,\alpha} \overline{\mathbf{F}^*_{\ell,m,i,\beta}}$. Next we introduce a sequence a_i, $a_0 = -cr^\theta$, such that $\partial_\xi \omega_{\alpha,\beta}(a_i)(a_{i+1}-a_i) = 2\pi b^{-1}$ and let $M \in \mathbb{N}$ be such that $a_M \leq cr^\theta$ and $a_{M+1} > cr^\theta$. Also, we establish the following notation: $\delta_i = a_{i+1} - a_i$. By Lemma 6.6 it follows that

$$|\omega_{\alpha,\beta}(\xi) - \omega_{\alpha,\beta}(a_i) - \partial_\xi \omega_{\alpha,\beta}(a_i)(\xi - a_i)| \leq C_\# \delta_i^{1+\varpi}.$$

In addition, equations (6.16), (6.24), and Lemma 6.6 imply

$$|A_{\alpha,\beta}(\xi,s) - A_{\alpha,\beta}(a_i,s)| \leq C_\# \{\delta_i^\varpi r^{-2\theta} + \delta_i r^{-3\theta}\}.$$

Then, remembering E.3 and using the first part of the lemma,

$$\begin{aligned}
&\left|\int_{a_i}^{a_{i+1}} e^{-ib\omega_{\alpha,\beta}(\xi)} A_{\alpha,\beta}(\xi,s) d\xi\right| \\
&= \left|\int_{a_i}^{a_{i+1}} e^{-ib[\partial_\xi \omega_{\alpha,\beta}(a_i)\xi + \mathcal{O}(\delta_i^{1+\varpi})]}[A_{\alpha,\beta}(a_i) + \mathcal{O}((\delta_i^\varpi r^{-2\theta} + r^{-3\theta}\delta_i)] d\xi\right| \\
&\leq C_\# \left(b\delta_i^{1+\varpi} r^{-2\theta} + \delta_i^\varpi r^{-2\theta} + r^{-3\theta}\delta_i\right)\delta_i \\
&\leq C_\# \left(\frac{r^{-2\theta}}{d_{i,j}(W_\alpha,W_\beta)^{1+\varpi} b^\varpi} + \frac{r^{-3\theta}}{d_{i,j}(W_\alpha,W_\beta)b}\right)\delta_i .
\end{aligned}$$

We may be left with the integral over the interval $[a_M, cr^\theta]$ which is trivially bounded by $C_\# r^{-2\theta}\delta_M \leq C_\#[r^{2\theta} b d_{i,j}(W_\alpha,W_\beta)]^{-1}$. The statement follows since the manifolds we are considering have length at most cr^θ, hence $\sum_{i=0}^{M-1}\delta_i \leq cr^\theta$. □

Proof of Lemma 6.3. We start by introducing a function $\bar{R} : W^s_{\delta,\zeta} \to \mathbb{N}$ such that $\bar{R}(\xi)$ is the first $t \in \mathbb{N}$ at which $T_{-t\tau_-}\xi$ belongs to a regular component of $T_{-t\tau_-}W^s_{\delta,\zeta}$ of size larger than $L_0/2$. We define then $R(t) = \min\{\bar{R}(t), \ell\}$. Let $\mathcal{P} = \{J_i\}$ be the coarser partition of $W^s_{\delta,\zeta}$ in intervals on which R is constant. Note that, for each $W_{\beta,i}$, $T_{\ell\tau_-}W_{\beta,i} \subset J_j$ for some $J_j \in \mathcal{P}$.

Let $\Sigma_{\ell,j} = \{(\beta,i) : W_{\beta,i} \in D_\ell(\widetilde{O},\rho_*),\ T_{\ell\tau_-}W_{\beta,i} \subset J_j\}$. Then, by the usual distortion estimates, for each $(\beta,i) \in \Sigma_{\ell,j}$

$$\begin{aligned}
Z_{\beta,i} &\leq C_\# \int_{W_{\beta,i}} \delta^{-1} J^s T_{\ell\tau_-} \leq C_\# \delta^{-1}\frac{|J_j|}{|\bar{W}_j|}\int_{W_{\beta,i}} J^s T_{(\ell-R_j)\tau_-} \\
&\leq C_\# \delta^{-1}\frac{|J_j|}{|\bar{W}_j|}|T_{(\ell-R_j)\tau_-}W_{\beta,i}|,
\end{aligned}$$

where $R_j = R(J_j)$ and $\bar{W}_j = T_{-R_j\tau_-}J_j$. Note that, by construction, either $|\bar{W}_j| \geq L_0/2$ or $R_j = \ell$.

Let us analyze first the case in which $R_j < \ell$. We apply Lemma D.2 to $\bar{W}_j$, with $\varrho = |\bar{W}_j| \geq L_0/2$, in order to obtain a foliation Γ transversal to $\bar{W}_j$ with leaves of size $\rho < \frac{L_0}{2c}$.[52] Let $\Omega_{\beta,i}$ be the set of leaves that intersect $\Delta_{\ell-R_j} \cap T_{(\ell-R_j)\tau_-}W_{\beta,i}$. By the construction of the covering $B^\theta_{cr}(x_i)$, the $\Omega_{\beta,i}$ have at most $C_\#$ overlaps. In addition, by the uniform transversality between stable and unstable direction,

$$\begin{aligned}
\sum_{(\beta,i)\in\Sigma_{\ell,j}} m(\Omega_{\beta,i}) &\geq C_\# \sum_{(\beta,i)\in\Sigma_{\ell,j}} |\Delta_{\ell-R_j} \cap T_{(\ell-R_j)\tau_-}W_{\beta,i}|\rho \\
&\geq C_\# \sum_{(\beta,i)\in\Sigma_{\ell,j}} |T_{(\ell-R_j)\tau_-}W_{\beta,i}|\rho - C_\#\rho^2,
\end{aligned}$$

where we have used the estimate on the complement of $\Delta_{\ell-R_j}$ given by property (8) of the foliation.

[52]We work in coordinates in which $\bar{W}_j$ is flat, this can always be achieved by Lemma A.4.

Accordingly, for each j such that $R_j < \ell$,

$$\sum_{(\beta,i)\in\Sigma_{\ell,j}} Z_{\beta,i} \le C_\# \delta^{-1}|J_j| \left[\rho^{-1}\sum_{(\beta,i)\in\Sigma_{\ell,j}} m(T_{-(\ell-R_j)\tau_-}\Omega_\beta) + \rho\right], \tag{E.5}$$

where we have used the invariance of the volume associated to the contact form. Remembering that the $T_{-(\ell-R_j)\tau_-}\Omega_\beta$ have a fixed maximal number of overlaps and since they are all contained in a $\rho_* + r^\theta + \bar\lambda^{-(\ell-R_j)\tau_-}\rho$ neighbourhood of $\widetilde{O}$ we have,[53]

$$\begin{aligned}\sum_j \sum_{(\beta,i)\in\Sigma_{\ell,j}} Z_{\beta,i} \le & \sum_{\{j\,:\,R_j\le\frac{\ell}{2}\}} C_\#\delta^{-1}|J_j|\left[\rho^{-1}S(\rho_*+r^\theta) + S\bar\lambda^{-\frac{\ell}{2}\tau_-}+\rho\right]\\ &+\sum_{\{j\,:\,R_j>\frac{\ell}{2}\}}\sum_{(\beta,i)\in\Sigma_{\ell,j}} Z_{\beta,i}\,.\end{aligned} \tag{E.6}$$

By our assumption on complexity (Definition 1.5), it follows that the number of pieces in $T_{-k\tau_-}W^s_\delta$ that have always been shorter than L_0 grows at most sub-exponentially with k. Remember that $\sigma \ge \bar\lambda^{-\frac{\tau_+}{2}}$, see (6.5). Then there exists $\ell_0\in\mathbb{N}$ such that, the number of pieces that are never longer than $L_0/2$ in $k\ge\ell_0 k_1$ time steps are bounded by $(\bar\lambda^{\tau_+}\nu)^{\frac{k}{k_1}}$. Then, remembering again (6.5),

$$\sum_{\{j\,:\,R_j>\frac{\ell}{2}\}}\sum_{(\beta,i)\in\Sigma_{\ell,j}} Z_{\beta,i} \le \sum_{\{j\,:\,R_j>\frac{\ell}{2}\}} C_\#\delta^{-1}|J_j| \le C_\#\sum_{k=\frac{\ell}{2}}^{\ell}\delta^{-1}\nu^k \le C_\#\delta^{-1}\sigma^{\frac{\ell}{k_1}}\,.$$

The result follows by choosing $\rho = S^{\frac12}(\rho_*+r^\theta)^{\frac12}$. □

***Proof of Lemma* 6.8.** We argue exactly like in the proof of Lemma 6.3, where $\widetilde{O}$ is replaced by $W_{\alpha,i}$ and $\rho_* = r^\vartheta$, up to formula (E.5). At this point, we notice that $T_{-(\ell-R_j)\tau_-}\Omega_\beta$ are all contained in a $r^\vartheta + \bar\lambda^{-\tau_-\ell}\rho$ neighbourhood of $W_{\alpha,i}$. Then, arguing as in (E.6)

$$\begin{aligned}\sum_{(\beta,i)\in\Sigma_{\ell,j}} Z_\beta \le &\sum_{\{j\,:\,R_j\le\frac{\ell}{2}\}} C_\#\delta^{-1}|J_j|\left[\rho^{-1}r^{\theta+\vartheta}+r^\theta\bar\lambda^{-\ell\tau_-}+\rho\right] + C_\#\delta^{-1}\sigma^{\frac{\ell}{k_1}}\\ &\le C_\#\left[\rho^{-1}r^{\theta+\vartheta}+\rho\right]+C_\#\delta^{-1}\sigma^{\frac{\ell}{k_1}}\,.\end{aligned} \tag{E.7}$$

The result follows by choosing $\rho = r^{\frac{\theta+\vartheta}{2}}$. □

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[53] Remember that the J_j are all disjoint , $\cup_j J_j = W^s_{\delta,\zeta}$ and $|W^s_{\delta,\zeta}|\le\delta$.

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