Random Classical Fidelity

C. Liverani *, Ph. Marie †, S.Vaienti ‡

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Abstract

We introduce a random perturbed version of the classical fidelity and we show that it converges with the same rate of decay of correlations, but not uniformly in the noise. This makes the classical fidelity unstable in the zero-noise limit.

1 Introduction

A recent series of papers [4, 5, 9], addresses the question of computing the classical fidelity for chaotic systems. They presented qualitative arguments and numerical evidences in favor of the fact that, for some dynamical systems, classical fidelity decays exponentially with the same rate as for correlations functions. The purpose of this note is to provide a rigorous mathematical proof of such a conjecture for a random perturbed version of the classical fidelity.

We first remind that classical fidelity is the classical counterpart of quantum fidelity which is, roughly speaking, a measure of the stability of quantum motion. Let us suppose that $|\psi\rangle$ is an initial quantum state which evolves forward up to time t under the Hamiltonian H_0 and then backward for the same time t under the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon V$ where V is a potential. The overlap of the initial state with its image $e^{iH_\varepsilon t}e^{-iH_0 t}|\psi\rangle$ is quantified by the quantum fidelity defined as:

$$f_q(t) = \left|\left\langle \psi \left| e^{iH_\varepsilon t} e^{-iH_0 t} \right| \psi \right\rangle\right|^2 \tag{1}$$

The accuracy to which the initial quantum state is recovered is also called the Loschmidt echo [5, 11, 17].

A huge physical literature has been devoted to compute the quantum fidelity; in particular it has been shown that, under some restrictions, $f_q(t)$ decays exponentially with a rate given by the classical Lyapunov exponent ([5] and refs. therein). The classical analogue of (1) is simply defined by replacing the Hamiltonian with the

*Departement of Mathematics, University of Roma 2, Rome; liverani@mat.uniroma2.it

†UMR-6207 Centre de Physique Théorique, CNRS, Universités d'Aix-Marseille I, II, Université du Sud, Toulon-Var; pmarie@cpt.univ-mrs.fr

‡UMR-6207 Centre de Physique Théorique, CNRS, Universités d'Aix-Marseille I, II, Université du Sud, Toulon-Var and FRUMAM, Fédéderation de Recherche des Unités de Mathématiques de Marseille; adress: CPT, Luminy Case 907, F-13288 Marseille Cedex 9; vaienti@cpt.univ-mrs.fr

evolution (Koopmann) operator of maps which preserve some invariant measures (see next section). It would be interesting to understand if the asymptotic decay of the classical fidelity takes place with the same exponent or, alternatively, at what time scale the quantum decay shares the same behavior of the classical one. These are among the major motivations to study the classical fidelity.

As we said above, there are numerical evidences that the decay of classical fidelity is ruled by the usual decay of correlations for smooth observables and that this decay takes place after a time t which is of order $\log(\varepsilon^{-1})$, where ε is the strength of the perturbation. Notice that after this transient time, the rate of decay turns out to be independent of ε. The numerical computations have been performed on invertible maps, possibly with singularities, preserving Lebesgue measure. In order to prove rigorously the preceding results for a wide class of dynamical systems, it will be useful to consider, instead of a single perturbed map, a random perturbation of the original system. In fact, in the case of a single perturbed map, the classical fidelity will not generally converge when time goes to infinity, as we will show on a very simple example in the Appendix.[1] The evolution operator will be therefore replaced by a random evolution operator. The advantage of this definition is twofold. First we can prove that the random fidelity has a limit when $t \to +\infty$ and the limit value is correctly identified. Second, the rate of decay towards this limit value is the same as for correlations but it is modulated by a factor of type $\varepsilon^{-\alpha}$ (so far it was simply ε^{-1}), where α depends on the class of observable under consideration. Under the additional assumption that the random dynamical system is stochastically stable, we will show an additional result, namely that the limit value of the fidelity performed by first taking $t \to +\infty$ followed by $\varepsilon \to 0$ is not the same if we interchange the order of the limits. We call this effect *stochastic instability of classical fidelity*, since it shows that the irreversibility which is present for $\varepsilon > 0$ (echo effect), still persist in the zero-noise limit. We finally point out that the computation of classical fidelity in presence of noise has been studied (notably in [5]) and it shares the same properties as for a single perturbed map. We will present in this paper two different ways to perturb a dynamical system: the first consists in perturbing the evolution operator by replacing the dynamics with a suitable Markov chain, while in the other method the orbit of a point is replaced by the random composition of maps close to the unperturbed one. We will show in the last section how to establish the equivalence of the two approaches in a few settings.

2 Random fidelity

In order to mimic the physical situations where the classical fidelity has been studied, even in connection with its quantum counterpart, we will restrict ourselves to dynamical systems defined on compact Riemannian manifolds X equipped with the Riemannian volume (Lebesgue measure) m. We consider then a measurable (with respect to the Borel σ-algebra β) map $T : X \to X$. For the moment we do not quote in detail the regularity properties of T; instead we list the requirements the

[1] Our counterexample concerns the algebraic automorphism of the torus, perturbed with an additive noise. Still for the same map, but perturbed in a different manner, the limit defining the classical fidelity seems to exist, at least numerically [5]; see also our examples 4 in Sect. 3

system should verify for our results to hold. Later on we will provide several explicit examples satisfying our assumptions.

The classical version of (1) is stated in [5, 14], as:

$$f_c(n) = \int_X U_0^n \rho(x)\, U_\varepsilon^n \rho(x)\, dm(x) \tag{2}$$

where U_0 and U_ε denote respectively the evolution (Koopmann) operator associated to T and to T_ε, where T_ε is close to T (in some topology):

$$U_0\rho(x) = \rho(Tx) \quad ; \quad U_\varepsilon \rho(x) = \rho(T_\varepsilon x). \tag{3}$$

Indeed, suppose at first that the map T is invertible. Then the classical analogue of (1) would be: $\int_X \rho(x)(U_0^{-n} U_\varepsilon^n)\rho(x)\, dm(x)$, that is one first evolves ρ for a time n by the dynamics T_ε and then backward by the dynamics T. If m is T invariant, the preceding integral reduces immediately to (2). In general the Lebesgue measure is not invariant, but it converges to some invariant measure in the limit $n \to \infty$, see assumption (H1), thus the two definition coincide asymptotically. Of course, U_0^{-1} does not have much sense in the non invertible case, yet formula (2) makes perfect sense so it is natural to use it a definition of classical fidelity.

We will choose the density ρ as a $\mathcal{C}^1$ function on X; moreover we will take m normalized on X: $m(X) = 1$. As mentioned in the introduction, instead of a single perturbed map T_ε, we will consider a random perturbation of T, that is a family $(\mathcal{X}_n^\varepsilon)_{n\geq 0}$ of Markov chains whose transition probabilities $\{P^\varepsilon(\cdot|x), x \in X\}$ converge uniformly to $\delta_{T(x)}$ as $\varepsilon \to 0$. We now state carefully our assumptions:

(**H1**) The map T admits an invariant measure μ which is the weak*-limit of the Lebesgue measure m, which means,

$$\int_X \varphi\, d\mu = \lim_{n\to+\infty} \int_X \varphi(T^n x) dm \qquad \text{for all} \;\; \varphi \in \mathcal{C}^0(X)$$

We assume that the measure μ is ergodic and exponentially mixing on the space of $\mathcal{C}^1$ function on X and with respect to two norms which for the moment we simply denote with $\|\cdot\|_1$ and $\|\cdot\|_2$

$$\left| \int_X \psi_1(T^n x)\psi_2(x)\, dm(x) - \int_X \psi_1\, d\mu \int_X \psi_2\, dm \right| \leq C\lambda^{-n}\, \|\psi_1\|_1\, \|\psi_2\|_2 \tag{4}$$

where $C > 0$ and $\lambda > 1$ are determined only by the map T. We stress the fact that the 1-norm $\|\cdot\|_1$ refers to the function ψ_1 which is composed with T, while the 2-norm is computed on ψ_2. One could choose stronger *equal* norms, of course, but this would yield weaker estimates, with respect to the noise parameter ε, of the 2-norm of the function (of z) $q_\varepsilon(Ty, z)$ and of the 2-norm of the function (of x) $q_\varepsilon(Tx, y)$, $q_\varepsilon(x, y)$ being defined in the next assumption. We will see in the proof of Theorem 1 that the upper bounds of the 2-norms of q_ε will give the dependence over ε which modulates the exponential decay of the fidelity. This will lead us to use norms as optimal as possible in the examples that we will present later on.

Let us consider on the measure space (X, β) a family of Markov chains $(\mathcal{X}_n^\varepsilon)_{n\geq 0}$ with transition probabilities:

$$P(\mathcal{X}_{n+1}^\varepsilon \in A | \mathcal{X}_n^\varepsilon = z) = \int_A q_\varepsilon(Tz, y)\, dm(y) \tag{5}$$

where $A \in \beta$, $\mathcal{X}_0^\varepsilon$ can have any probability distribution and the measurable function $q_\varepsilon : X \times X \to \mathbb{R}^+$, $\varepsilon \in (0, 1]$ is chosen in such a way that:

$$q_\varepsilon(x, y) = 0 \quad \text{if} \;\; d(x, y) > \varepsilon \tag{6}$$

$$\int_X q_\varepsilon(x, y)\, dm(y) = 1 \quad \text{for all} \;\; x \in X \tag{7}$$

A typical example of such a kernel [7], which we will adopt in the following, is

$$q_\varepsilon(x, y) = \varepsilon^{-d} \bar{q}(\varepsilon^{-1}(y - x)) \tag{8}$$

where $\bar{q}$ is nonnegative and continuously differentiable, $\mathrm{supp}(\bar{q}) \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$, $\int \bar{q}(\xi)\, d\xi = 1$, $\inf\{\bar{q}(\xi) : |\xi| \leq \frac{1}{2}\} > 0$ and finally d is the dimension of the manifold X.

(**H2**) There exists a norm (identified with $\|\cdot\|_2$), a constant c (depending eventually on the function $\bar{q}$) and a real positive exponent α (depending only on the map T) such that:

$$\sup_{y \in X} \|q_\varepsilon(Ty, \cdot)\|_2 \leq c\varepsilon^{-\alpha} \; ; \sup_{z \in X} \|q_\varepsilon(T(\cdot), z)\|_2 \leq c\varepsilon^{-\alpha}. \tag{9}$$

Let us now choose T continuous; then by the compactness of X and the choice (6) of the transition probabilities, our Markov chain admits an absolutely continuous *stationary* measure μ_ε namely a probability measure over (X, β) which verifies for all $g \in \mathcal{C}^0(X)$;[2]

$$\int_X g(x) d\mu_\varepsilon(x) = \int_X \int_X q_\varepsilon(Tx, y) g(y)\, d\mu_\varepsilon(x) dm(y) \tag{10}$$

Notice that (11) can be equivalently written as:

$$\mu_\varepsilon(A) = \int_X (T_\varepsilon \mathbb{1}_A)(x) d\mu_\varepsilon(x) \tag{11}$$

where $A \in \beta$, $\mathbb{1}_A$ is the indicator function of the set A and T_ε is the operator defined on $\mathcal{L}^\infty(X)$ by:

$$(T_\varepsilon g)(x) = \int_X q_\varepsilon(Tx, y) g(y) dm(y) \tag{12}$$

This operator will play an important role in the following: it is the random version of the Koopmann operator $(Ug)(x) = g(Tx)$. It simply replaces the deterministic value $g(Tx)$ with the averaged value of g on a small ball of radius ε around Tx. We are now ready to state our third assumption:

(**H3**) The map T admits a kernel family q_ε for which μ_ε is the only absolutely continuous stationary measure (eventually for ε small) and we have the following rate of decay of correlations for the Markov process $(\mathcal{X}_n^\varepsilon, \mu_\varepsilon)$:

$$\left| \int_X T_\varepsilon^n \psi_1(x) \psi_2(x)\, dm(x) - \int_X \psi_1\, d\mu_\varepsilon \int_X \psi_2\, dm \right| \leq C\lambda^{-n} \|\psi_1\|_1 \|\psi_2\|_2 \tag{13}$$

[2]We would like to point out that in the examples 3 and 4 of Section 3, the map T is not anymore continuous; nevertheless the existence of an absolutely continuous stationary measure can be proved with other arguments [3, 10]

where C, λ and the norms 1 and 2 are the same as in (5).
We also consider another stronger property of the process $(\mathcal{X}_n^\varepsilon, \mu_\varepsilon)$ namely its weak convergence towards (T, μ).
(**H4**) We suppose that the system (T, μ) is stochastically stable, in the sense that μ_ε tends to μ weakly as $\varepsilon \to 0$.

We finally introduce our definition of the classical fidelity which replaces (2) and is given in terms of the prescriptions (**H1**) to (**H3**).

Definition 1 (Random classical fidelity). *Let us suppose T is a Borel measurable map from the compact Riemannian manifold X into itself, and let m be the probability Riemannian measure on X. Let T_ε be the random evolution operator defined in (13) and $\rho \in \mathcal{C}^1(X)$. We define the classical fidelity as:*

$$F_c^\varepsilon(n) = \int_X \rho(T^n x)(T_\varepsilon^n \rho)(x)\, dm(x) \tag{14}$$

We will say that a system enjoy classical fidelity *if $F_n^\varepsilon(n) - \int \rho d\mu \int \rho d\mu_\varepsilon$ tend to zero as n tends to infinity.*

We now state our main result:

Theorem 1. *Let us suppose that the map T introduced in the preceding definition verifies the assumptions (**H1**) to (**H3**). Then there exists $C > 0$:*

$$\left| F_c^\varepsilon(n) - \int_X \rho\, d\mu \int_X \rho\, d\mu_\varepsilon \right| \leq C\varepsilon^{-\alpha}\lambda^{-n} \, \|\rho\|_1 \, \|\rho\|_{\mathcal{C}^0} \, . \tag{15}$$

Remark 1.

1. *The theorem shows that the limit value of the classical fidelity involves the stationary measure μ_ε. Moreover the error term is not uniform in ε and this error begins to be negligible after a time n of order $\log \varepsilon^{-\alpha} / \log \lambda$. This effect has been effectively observed in [5], see also the appendix.*[3]

2. *The presence of ε in the limit value of $F_c^\varepsilon(n)$ when $n \to \infty$, or equivalently, the non-uniformity in ε of the error term, have an interesting consequence if we assume (**H4**), namely the stochastic stability of (T, μ). We first observe that in the absence of noise the correlation integral $F_c^0(n)$ converges towards $\int_X \rho^2\, d\mu$:*

$$\lim_{n\to\infty} \lim_{\varepsilon \to 0} F_c^\varepsilon(n) = \int_X \rho^2 \, d\mu \tag{16}$$

Instead:

$$\lim_{\varepsilon\to 0} \lim_{n \to \infty} F_c^\varepsilon(n) = \left(\int_X \rho \, d\mu \right)^2 \tag{17}$$

and the two limits (16) and (17) in general will differ. We could interpret this fact by saying that the classical fidelity is not stochastically stable. The zero-noise situation: $\lim_{n\to\infty} F_c^0(n)$ is not recovered if we first play the dynamics for $n \to \infty$ and then we send the perturbation to zero. The memory is not destroyed when the noise is turned off after the evolution of the system and this is a sort of irreversibility of our random version of the classical fidelity.

[3]Of course that above estimate is relevant only for times longer than $\log \varepsilon^{-\alpha} / \log \lambda$, since for shorter time it gives a rather large bound while the quantity under consideration is trivially bounded by $2\|\rho\|_\infty^2$.

Proof of the theorem. Let us define $g_{\varepsilon,y}(x) = q_\varepsilon(Tx, y)$ and $\phi_{\varepsilon,y}(z) = q_\varepsilon(Ty, z)$ and notice that (we will equivalently denote the Lebesgue measure with $dm(x)$ and dx):

$$\begin{aligned}\int_X (\rho \circ T^n(x)) T_\varepsilon^n \rho(x)\, dx &= \int_X \rho \circ T^n(x)(T_\varepsilon \circ T_\varepsilon \circ T_\varepsilon^{n-2})\rho(x)\, dm(x) \\ &= \int_{X^3} \rho \circ T^n(x) q_\varepsilon(Tx, y) q_\varepsilon(Ty, z) T_\varepsilon^{n-2}\rho(z)\, dxdydz \\ &= \int_X dy \left[\int_X \rho \circ T^n(x) g_{\varepsilon,y}(x)\, dx\right] \left[\int_X \phi_{\varepsilon,y}(z) T_\varepsilon^{n-2}\rho(z)\, dz\right]\end{aligned}$$

Let's then define for $n \geq 1$:

$$\begin{aligned}\Delta_n &= \left\| \int_X \rho \circ T^n(x) g_{\varepsilon,y}(x)\, dx - \int_X g_{\varepsilon,y}(x)\, dx \int_X \rho\, d\mu \right\|_{\mathcal{C}^0} \\ \Delta_{\varepsilon,n} &= \left\| \int_X \phi_{\varepsilon,y}(z) T_\varepsilon^{n-2}\rho(z)\, dz - \int_X \phi_{\varepsilon,y}(x)\, dx \int_X \rho\, d\mu_\varepsilon \right\|_{\mathcal{C}^0}\end{aligned}$$

The mixing properties of μ with respect to T and of μ_ε with respect to T_ε as we stated in assumptions (**H1**) and (**H3**), imply the following decay of correlations functions:

$$\begin{aligned}\Delta_n &\leq C\lambda^{-n} \|\rho\|_1 \|g_{\varepsilon,y}\|_2 \leq C\lambda^{-n} \|\rho\|_1\ \varepsilon^{-\alpha} \\ \Delta_{\varepsilon,n} &\leq C\lambda^{-n} \|\rho\|_1 \|\phi_{\varepsilon,y}\|_2 \leq C\lambda^{-n} \|\rho\|_1\ \varepsilon^{-\alpha}\end{aligned}$$

where we use the same symbol C to denote possibly different constants depending solely on the map T. Moreover (using $\int_M \phi_{\varepsilon,y}(z)dz = 1$), we can compute:

$$\begin{aligned}&\left| \int_X \rho \circ T^n(x)\, T_\varepsilon^n \rho(x)\, dx - \int_X \rho\, d\mu \int_X \rho\, d\mu_\varepsilon \right| \\ &\leq \int_X dy \Delta_n \left| \int_X \phi_{\varepsilon,y}(z) T_\varepsilon^{n-2}\rho(z)\, dz \right| + \int_X dy \Delta_{\varepsilon,n-2} \left| \int_X \rho\, d\mu \int_X g_{\varepsilon,y}(z)\, dz \right| \\ &\leq C\lambda^{-n+2}\varepsilon^{-\alpha} \|\rho\|_1 \|\rho\|_{\mathcal{C}^0} + C\lambda^{-n+2}\varepsilon^{-\alpha} \|\rho\|_1 \|\rho\|_{\mathcal{C}^0}\end{aligned}$$

since $\int_X g_{\varepsilon,y}(z)\, dz \leq C$, $\int_X \phi_{\varepsilon,y}(z)\, dz \leq C$ and $\|T_\varepsilon^n \rho\|_{\mathcal{C}^0} \leq C \|\rho\|_{\mathcal{C}^0}$ for all n. □

3 Examples

In this section we quote some dynamical systems of physical interest which fit our assumptions and to which we can apply our theorem on the decay of classical fidelity. We remark that the measure μ which we consider as the weak*-limit of the Lebesgue measure, is usually called the SRB or physical measure. For diffeomorphisms, eventually with singularities, it has also two additional properties: first, it has absolutely continuous conditional measures along the unstable foliations; second it can be reconstructed by Birkhoff sums starting from initial points chosen in a basin of positive Lebesgue measure.

For locally (eventually non-uniformly) expanding maps, μ is absolutely continuous with respect to the Lebesgue measure.

1. **Anosov diffeomorphisms**
The exponential decay of correlations for the SRB measure is a classical result; see for instance [8]. Yet, the optimal choice of the norms is a subtle matter. For simplicity we adopt the choice made in [7] where one can also find the decay of correlations for the stationary measure μ_ε constructed with the kernel family q_ε. With such a choice, a direct computation from formula (2.1.7) of [7] yields

$$\|h\|_2 \le \|h\|_{L^1} + \|D^u h\|_{L^1}$$
$$\|h\|_1 \le \|h\|_{\mathcal{C}^1},$$

where D^u is the differential restricted to the unstable directions. Accordingly, $\|g_\varepsilon\|_2 + \|\phi_\varepsilon\|_2 \le C\varepsilon^{-1}$. Hence, in this case, $\alpha = 1$.

2. **Uniformly hyperbolic attractors**
In this case we can refer to the work of Viana [18] although the norms are probably not the optimal ones. Better estimates could probably be obtained by using the recent work of [2] in conjunction with the perturbation theory of [12]. Here we content ourselves with the bound $\|h\|_1 + \|h\|_2 \le C\|h\|_{\mathcal{C}^1}$ which follows from formula (4.29) of [18]. Accordingly we have the (unsatisfactory and almost certainly non optimal) bound $\alpha \le d + 1$.

3. **Piecewise expanding maps of the interval**
For continuous piecewise $\mathcal{C}^2$ expanding maps of the interval without periodic turning points, Baladi and Young [3] have proved the exponential decay of correlations and the stochastic stability of absolutely continuous stationary measures constructed with the convolution kernel:

$$q_\varepsilon(x,y) = \theta_\varepsilon(y - Tx) \qquad \theta_\varepsilon \ge 0\ ;\ \operatorname{supp}\theta_\varepsilon \subset [-\varepsilon; +\varepsilon] \quad \text{and} \quad \int \theta_\varepsilon dm = 1$$

In this case $\|h\|_2 = \|h\|_{BV}$, $\|h\|_1 = \|h\|_{L^1}$, thus $\alpha = 1$.

Similar results can be obtained for piecewise expanding $\mathcal{C}^2$ maps with derivative uniformly larger than 2, see [16].

4. **Uniformly hyperbolic maps with singularities (in two dimensions)**
This is an interesting situation since it covers the numerical simulation produced by Casati and al. [5]. In fact the latter authors perturbed the linear automorphism of the torus by keeping the Lebesgue measure invariant. In this way the perturbations become singular in the sense that the perturbed maps are discontinuous.

This case has been rigorously investigated by [10]. Unfortunately, the norms are a bit unusual, here we will just state the minimum and refer to [10] for more details.

The basic object is a collection Σ of smooth curves close to the stable direction. Given such curves we have (see [10] equations (2.3), (2.4)):

$$\|h\|_2 \le \sup_{W\in\Sigma} \int_W |Dh| + \int_W |h|.$$

Thus we have[4]

$$\|g_\varepsilon\|_2 + \|\phi_\varepsilon\|_2 \le C\varepsilon^{-2}.$$

That is $\alpha \le 2$. Note that the numerically predicted bound is $\alpha = 1$. In view of the smooth multidimensional case and of the one dimensional piecewise smooth case above we conjecture that $\alpha = 1$ is indeed the correct value and the present one ($\alpha = 2$) is a byproduct of our method of proof.

4 Perturbation with random maps

The random process that we used in Sect. 2 could be realized, in contrast to (6), as the composition of random maps (see [1, 15, 13] for more details). In this section we will briefly comment on the relation between this two different ways to realize a random perturbation. We start the discussion by a precise description of the random maps alternative.

Let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in the interval $\Omega_\varepsilon = (-\varepsilon, \varepsilon)$ and with distribution θ_ε. We then associate continuously to each $\omega \in \Omega_\varepsilon$ a map T_ω with $T_0 = T$ and we define the transition probabilities (6) $P(\cdot|z)$ on the σ-algebra β in such a way that :

$$P(A|z) = \theta_\varepsilon(\omega; T_\omega z \in A) \tag{18}$$

for $A \in \beta$. The closeness of the T_ω to T will be made explicit in the concrete examples that we are giving below. In this setting the random evolution operator is replaced by the following one:

$$(U_\varepsilon g)(x) = \int_{\Omega_\varepsilon} g(T_\omega x) d\theta_\varepsilon(\omega) \tag{19}$$

for any $g \in \mathcal{L}^\infty(X)$, and the stationary measure μ_ε should verify for each $h \in \mathcal{C}^0(X)$

$$\int h d\mu_\varepsilon = \int_X \int_{\Omega_\varepsilon} (h \circ T_\omega)(x) d\theta_\varepsilon(\omega) d\mu_\varepsilon(x) = \int_X (U_\varepsilon h)(x) d\mu_\varepsilon(x) \tag{20}$$

The iterations of the unperturbed map, $T^n(x), x \in X$, are thus replaced by the composition of random maps $T_{\omega_n} \circ \cdots \circ T_{\omega_1}$, $\omega_i \in \Omega_\varepsilon; x \in X$. One is therefore tempted to define a random version of the fidelity by setting (compare with (2)):

$$\int \rho(T^n x) \rho(T_{\omega_n} \circ \cdots \circ T_{\omega_1})(x) dm(x)$$

[4] Let us compute, for example, $\|g_\varepsilon\|_2$. For $W \in \Sigma$

$$\int_W |g_\varepsilon| = \int_W q_\varepsilon(Tx, y) dx = \int_{T^{-1}W} |q_\varepsilon(x, y)| J_x T dx \le C\varepsilon^{-1}$$

where JT is the Jacobian of the change of coordinates and the last inequality follows from the fact hat we integrate along a curve instead than on all the space. Analogously,

$$\int_W |Dg_\varepsilon| = \varepsilon^{-1} \int_W |DT\nabla q_\varepsilon(Tx, y)| dx = \varepsilon^{-1} \int_{T^{-1}W} |DT\nabla q_\varepsilon(x, y)| J_x T dx \le C\varepsilon^{-2}.$$

From which the estimate in example 4 follows.

for a given realization $\overline{\omega} = (\omega_1, \omega_2, \cdots) \in \Omega_\varepsilon^{\mathbb{N}}$. Instead of doing that, we will take the average over all realizations, which produces the *annealed* version of the preceding correlation integral and we set the new version of the classical fidelity as:

$$\tilde{F}_c^\varepsilon(n) = \int_X \int_{\Omega_\varepsilon} \rho(T^n x)\rho(T_{\omega_n} \circ \cdots \circ T_{\omega_1})(x) d\theta_\varepsilon^{\mathbb{N}}(\overline{\omega}) dm(x)$$

In terms of the random evolution operator U_ε the above integral can be simply written as

$$\tilde{F}_c^\varepsilon(n) = \int_X \rho(T^n x)(U_\varepsilon^n \rho)(x) dm(x)$$

which is formally similar to the random classical fidelity defined in (13). The advantage of this formula is that it is physically simpler to perturb the map T by randomly composing sequence of maps close to T. A very established theory exists for this kind of random perturbations and all the examples of Sect. 3 fit as well with it. We now show that in many cases the random evolution operator T_ε can be obtained from U_ε by a suitable choice of the probability measure θ_ε and of the random maps T_ω. To make the argument as simple as possible, let us suppose that $X = \mathbb{T}^m$, the m-dimensional torus, and define the additive noise: $T_\omega = T(x) - \omega$ mod $\mathbb{T}^m$, where $\omega \in \mathbb{T}^m$ and then take θ_ε absolutely continuous with respect to the Lebesgue measure $d\omega$ over $\mathbb{T}^m$ and with a continuously differentiable density h_ε with support contained in the square $[-\varepsilon, \varepsilon]^m$: $\int d\theta_\varepsilon = \int h_\varepsilon(\omega) d\omega = 1$. A simple change of variables on the m-dimensional torus immediately gives, for any $g \in \mathcal{L}^\infty(d\omega)$:

$$(U_\varepsilon g)(x) = \int g(T_\omega x) h_\varepsilon(\omega) d\omega = \int g(Tx - \omega) h_\varepsilon(\omega) d\omega = \int g(y) h_\varepsilon(Tx - y) dy$$

from which it follows that $q_\varepsilon(Tx, y) = h_\varepsilon(Tx - y)$.

Appendix

A counterexample to classical fidelity

Here is a simple example of a systems not having classical fidelity (see Definition 1). We consider the algebraic automorphisms T_L of the torus $X = \mathbb{T}^2$ defined by

$$T_L(x_1, x_2) = (x_1 + x_2, x_1 + 2x_2) \text{ mod} 1$$

where $x = (x_1, x_2)$ is a point on the torus, and its perturbed map $T_\omega(x) = T(x) + \omega$ mod1 and we compute for all $\rho \in \mathcal{C}^1(X)$:

$$\rho_\omega(n) = \int_X \rho(T^n x)\rho(T_\omega^n x) dm(x)$$

where m denotes the normalized Lebesgue (Haar) measure over X. By using the Fourier' transform technique, denoting with k an element of $\mathbb{Z}^2$ and finally by posing $\langle \cdot, \cdot \rangle$ the euclidean scalar product, we have: $\rho(x) = \sum_{k \in \mathbb{Z}^2} c_k e^{2i\pi \langle k, x \rangle}$ where c_k are the Fourier coefficients of ρ. It easily follows that:

$$\rho(T^n x) = \sum_{k \in \mathbb{Z}^2} c_k e^{2i\pi \langle k, L^n x \rangle} \qquad \text{and} \qquad \rho(T_\omega^n x) = \sum_{k \in \mathbb{Z}^2} c_k e^{2i\pi \langle k, L^n x + \sum_{j=0}^{n-1} L^j \omega \rangle}$$

where $L = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then:

$$\rho_\omega(n) = \sum_{k \in \mathbb{Z}^2} |c_k|^2 e^{2i\pi\langle k, \sum_{j=0}^{n-1} L^j \omega\rangle} = \sum_{k \in \mathbb{Z}^2} |c_k|^2 e^{2i\pi\langle k, (\mathrm{Id} - L^n)(\mathrm{Id} - L)^{-1}\omega\rangle}$$

We notice that $\rho_\omega(n)$ is the value at the point $(\mathrm{Id} - T_L^n)(\mathrm{Id} - T_L)^{-1}\omega$ mod1 of the function with Fourier expansion:

$$g(x) = \sum_{k \in \mathbb{Z}^2} |c_k|^2 e^{2i\pi\langle k, x\rangle}$$

But the operator $(\mathrm{Id} - T_L)^{-1}$ has the matrix representation

$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

and therefore it applies X onto itself. Moreover, by the ergodicity of T_L, for Lebesgue almost all $\omega \in \mathbb{Z}^2$, the orbit $(\mathrm{Id} - T_L^n)\omega$, $n \geq 0$, is dense in X. It follows that $\rho_\omega(n)$ cannot have any limit unless ρ is constant everywhere.

Sharpness of the bounds

We conclude by showing that the bound $\alpha = 1$ is sharp: in general one cannot expect any decay of the fidelity before the time predicted by the $\alpha = 1$ bound. We consider again a toral automorphism and assume $q_\varepsilon(x, y) = \varepsilon^{-2}\bar{q}(\varepsilon^{-1}(x - y))$,[5] then $\mu_0 = \mu_\varepsilon = m$ and we can compute

$$\begin{aligned}
\int q_\varepsilon(x,y) f(y) dy &= \sum_{k \in \mathbb{Z}^2} f_k \int_{\mathbb{T}^2} q_\varepsilon(x,y) e^{2\pi i\langle k, y\rangle} dy = \sum_{k \in \mathbb{Z}^2} f_k \int_{\mathbb{R}^2} q_\varepsilon(x,y) e^{2\pi i\langle k, y\rangle} dy \\
&= \sum_{k \in \mathbb{Z}^2} f_k \int_{\mathbb{R}^2} \varepsilon^{-2} q(\varepsilon^{-1}\xi) e^{2\pi i\langle k, x - \xi\rangle} d\xi \\
&= \sum_{k \in \mathbb{Z}^2} f_k e^{2\pi i\langle k, x\rangle} \int_{\mathbb{R}^2} q(\xi) e^{-2\pi i\langle \varepsilon k, \xi\rangle} d\xi = \sum_{k \in \mathbb{Z}^2} f_k e^{2\pi i\langle k, x\rangle} \hat{q}(\varepsilon k),
\end{aligned}$$

where $\hat{q}$ is the Fourier transform of $q : \mathbb{R}^2 \to \mathbb{R}_+$. That is $(T_\varepsilon f)_k = f_{L^{-1}k}\hat{q}(\varepsilon L^{-1}k)$.

Using these facts we can compute $F_c^\varepsilon(n)$ to be

$$F_c^\varepsilon(n) = \sum_{k \in \mathbb{Z}^2} \prod_{i=0}^{n-1} |\hat{q}(\varepsilon L^i k)| \cdot |c_k|^2.$$

Now we can chose, for example, $\rho(x) = e^{2\pi i\langle (1,1), x\rangle}$ and $q(x) = \frac{1}{2\pi} e^{-\|x\|^2/2}$, hence $c_k = \delta_{k,(1,1)}$ and $\hat{q}(k) = e^{-2\pi^2\|k\|^2}$. This means the following lower bound on the Fidelity

$$|F_c^\varepsilon(n) - |c_0|^2| \geq e^{-C\varepsilon^2\lambda^{2n}}.$$

[5] Where by $x - y$ we means that x and y are lifted on the universal cover of $\mathbb{T}^2$, that is $\mathbb{R}^2$, then one rescale the variables and then takes the mod 1 to bring it back to the torus.

Since we have chosen an entire function the decay is super exponential, nevertheless it takes place only after a time n such that $\varepsilon\lambda^n \geq 1$. This corresponds exactly to the behavior in which $\alpha = 1$. It is then clear that one cannot hope for an α better than one in the estimates of Section 3.

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