A Spectral Gap for a One-dimensional Lattice of Coupled Piecewise Expanding Interval Maps

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Abstract. We study one-dimensional lattices of weakly coupled piecewise expanding interval maps as dynamical systems. Since neither the local maps need to have full branches nor the coupling map needs to be a homeomorphism of the infinite dimensional state space, we cannot use symbolic dynamics or other techniques from statistical mechanics. Instead we prove that the transfer operator of the infinite dimensional system has a spectral gap on suitable Banach spaces generated by measures with marginals that have densities of bounded variation. This implies in particular exponential decay of correlations in time and space.

1 Introduction

Typical dynamical systems have a multitude of invariant probability measures. There are essentially two ways to characterize the "physically relevant" ones among them: in the spirit of statistical mechanics one can look at those measures which satisfy a variational principle with a potential of the type "logarithm of the unstable Jacobian". From a more dynamical perspective one may look at those measures which are absolutely continuous w.r.t. the natural volume measure m on the state space, or at those for which the space averages of regular observables equal the corresponding time averages for a set of initial conditions of positive m-measure. In many cases both approaches lead to the same result. In the case of coupled map lattices, which are infinite-dimensional dynamical systems, one needs some extra care to apply these ideas. For the statistical mechanics approach this is done in other chapters of this book. Here we concentrate on the dynamical systems approach.

Let L be a finite or countable index set, e.g. $L = \mathbb{Z}$ or $L = \mathbb{Z}/d\mathbb{Z}$ and let $I = [0, 1]$. We investigate time-discrete dynamics on the state space $X = I^L$, composed of independent chaotic actions on each component I of X and of some weak interaction between the components that does not destroy the

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chaotic character of the whole system. More specifically, let τ be a piecewise C^2 map from I to I with singularities at $\zeta_1, \dots, \zeta_{N-1} \in (0,1)$ in the sense that τ is monotone and C^2 on each component of $I \setminus \{\zeta_0 = 0, \zeta_1, \dots, \zeta_{N-1}, \zeta_N = 1\}$. We assume that $\tau''/(\tau')^2$ is bounded and that τ satisfies the following combined expansion and regularity assumption:

There is $M \in \mathbb{N}$ such that $\kappa_M := \inf |(\tau^M)'| > 2$ and such that

$$\tau^m(0), \tau^m(\zeta_1\pm), \dots, \tau^m(\zeta_{N-1}\pm), \tau^m(1) \notin \{\zeta_1, \dots, \zeta_{N-1}\} \tag{1}$$

for $m = 0, \dots, M-1$.

If $\inf |\tau'| > 2$ this condition is trivially satisfied for $M = 1$. A simple but prominent example of such a map with $M > 1$ is a symmetric mixing tent map, i.e. a map $\tau_s(x) = s(\frac{1}{2} - |x - \frac{1}{2}|)$ with slope $s \in (\sqrt{2}, 2]$. It satisfies (1) with $N = 2$, $\zeta_1 = \frac{1}{2}$, $M = 2$ and $\kappa_M = s^2$.[1]

Now a map $T_0 : X \to X$ describing the uncoupled dynamics is defined by

$$(T_0 \boldsymbol{x})_i = \tau(x_i) \quad (i \in L) \tag{2}$$

and coupled maps $T_\epsilon := \Phi_\epsilon \circ T_0$ are introduced using appropriate continuous couplings $\Phi_\epsilon : X \to X$ close to the identity on X.[2] One of the most widely used couplings in numerical studies – which despite its simplicity resisted for quite some time a rigorous mathematical treatment – is the diffusive nearest neighbor coupling on $\mathbb{Z}$ or $\mathbb{Z}/d\mathbb{Z}$

$$(\Phi_\epsilon \boldsymbol{x})_i = \frac{\epsilon}{2} x_{i-1} + (1-\epsilon) x_i + \frac{\epsilon}{2} x_{i+1} \quad (i \in L) . \tag{3}$$

It is an example of a class of more general C^2-couplings Φ_ϵ whose C^2 distance to the identity Φ_0 is of order ϵ and is controlled in terms of constants $a_1, a_2 > 0$ – see Sect. 3.1 for details. We say that such a coupling has *finite coupling range* if there is $w > 0$ such that $\partial_j \Phi_{\epsilon,i} = 0$ whenever $|i - j| > w$.

Our main result is:

Theorem 1.1. *Let $L = \mathbb{Z}$. Given a mixing*[3] *local map τ as introduced above and given $a_1, a_2, w > 0$, there exists $\epsilon_{\max} > 0$ such that for each (a_1, a_2)-coupling Φ_ϵ with coupling range w and each $\epsilon \in [0, \epsilon_{\max}]$ holds:*

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[1] An elementary discussion of the basic dynamical properties of these maps can be found in [1]. For the mixing property see [2].

[2] The regularity assumption in (1) seems unavoidable if a weakly coupled system T_ϵ is to behave like a small perturbation of T_0, because weak couplings affect each individual map τ like a small perturbation, and it is known that in the absence of the above assumption arbitrarily small perturbations can change the dynamics of τ completely, see the examples in [3, 4, 5].

[3] Under the assumptions made on τ there exists at least one invariant probability density for τ. We say that τ is *mixing*, if no power of τ has any other invariant probability density. This will be discussed in some detail in Sect. 2.5.

1. *The coupled system T_ϵ has an invariant probability measure μ_ϵ whose finite-dimensional marginals are absolutely continuous w.r.t. Lebesgue measure and have densities of bounded variation. It has finite entropy density, and it is unique among all measures from this class for which the variation of the marginals increases at most subexponentially with the dimension.*
2. *There are constants $\gamma, \gamma', \theta \in (0,1)$ and $C, C' > 0$ such that for bounded observables $\phi, \psi : X \to \mathbb{R}$ which depend only on coordinates $x_{a+1}, \ldots, x_b$,*

$$\left| \int \phi \cdot (\psi \circ T_\epsilon^n)\, d\mu_\epsilon - \int \phi\, d\mu_\epsilon \int \psi\, d\mu_\epsilon \right| \le C\theta^{-(b-a)}\, \gamma^n\, \|\phi\|_{C^1} \|\psi\|_{C^0} \quad (4)$$

and

$$\left| \int \phi \cdot (\psi \circ \sigma^n)\, d\mu_\epsilon - \int \phi\, d\mu_\epsilon \int \psi\, d\mu_\epsilon \right| \le C' \gamma'^{|n|-(b-a)}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \quad (5)$$

where σ is the left shift on $X = I^{\mathbb{Z}}$.
3. *The distance (in a suitable metric) between μ_ϵ and μ_0 is of order $\epsilon \ln \epsilon^{-1}$.*

The proof of this theorem relies on a spectral analysis of the transfer operator associated with T_ϵ. It combines results and ideas from [6, 7, 8, 9] and is developed step by step in this chapter. The existence part and the finiteness of the entropy density are proved in Theorem 4.1 in Sect. 4.4. Uniqueness of μ_ϵ, the exponential decay of correlations, and the estimate on the distance between μ_ϵ and μ_0 are derived in Sects. 4.7, 4.8, and 4.9 from Theorem 4.3, which guarantees the existence of a spectral gap for the transfer operator of T_ϵ on suitable Banach spaces. In Sect. 4.8 we also prove the following strong law of large numbers (compare [10, Theorem 5.1]):

Corollary 1.1. *In the situation of Theorem 1.1, let $\psi : X \to \mathbb{R}$ be a continuous observable. Let $f : I \to \mathbb{R}$ be any probability density of bounded variation, and let (fm) be the corresponding probability measure on I. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(T_\epsilon^k(\boldsymbol{x})) = \int \psi\, d\mu_\epsilon \quad \textit{for } (fm)^{\mathbb{Z}}\textit{-a.e. } \boldsymbol{x} \quad (6)$$

where $(fm)^{\mathbb{Z}}$ is the infinite product measure on $X = I^{\mathbb{Z}}$ with one-dimensional factors (fm).

This result suggests the interpretation of μ_ϵ as the unique physical (or observable) measure of T_ϵ.[4] It is supported by the stability of μ_ϵ under independent random noise discussed (without proof) in Sect. 4.9.

[4] For a discussion of physical measures and related notions in more general settings see e.g. the contribution of L.A. Bunimovich in this volume.

2 Dynamics at a Single Site

The dynamics at each single site of our system is modeled by a mixing piecewise expanding map. We will see that the dynamics of such maps are statistically stable in several respects: there is a unique stationary probability density towards which each initial density converges under the action of the dynamics (*asymptotic stability*), and neither this stationary density nor the rate at which initial densities are attracted by it change much under small perturbations of the map. Therefore we can expect that in weakly coupled systems, where the mutual coupling between the single site maps can also be interpreted as a kind of perturbation, the behavior of the system at single sites does not change drastically under the influence of the coupling.

2.1 Piecewise Expanding Maps

We say that a map $\tau : I \to I$ is *piecewise expanding (p.w.e.)* if

- there are $\zeta_1, \ldots, \zeta_{N-1} \in (0,1)$ which define subintervals $I_i = (\zeta_{i-1}, \zeta_i)$ such that each $\tau|_{I_i}$ is monotone and uniformly C^2, and
- there are $M \in \mathbb{N}$ and $\kappa_M > 2$ such that $|(\tau^M)'| \geq \kappa_M$.[5]

Our assumptions imply that $D_m := \sup |(\frac{1}{(\tau^m)'})'| < \infty$ for all $m \in \mathbb{N}$.

2.2 The Transfer Operator

As the derivatives of p.w.e. maps grow exponentially, the trajectory-wise dynamics are very sensitive to initial conditions. But at the same time this instability is responsible for good asymptotic properties of the *transfer operator*[6] P_τ which describes the evolution of initial densities under the dynamics. This operator associates to each measurable $f : I \to \mathbb{R}$ the function $P_\tau f : I \to \mathbb{R}$,

$$P_\tau f(x) = \sum_{i=1}^{N} \left(\frac{f}{|\tau'|} \right) \circ (\tau|_{I_i})^{-1}(x) \cdot 1_{\tau(I_i)}(x) \tag{7}$$

where $1_{\tau(I_i)}$ denotes the indicator function of the set $\tau(I_i)$. By change of variables it follows that for Lebesgue (m) integrable $f : I \to \mathbb{R}$ and bounded measurable $\psi : I \to \mathbb{R}$

$$\int_I P_\tau f(x)\, \psi(x)\, dx = \int_I f(x)\, \psi(\tau(x))\, dx \, . \tag{8}$$

[5] This means that $|(\tau^M)'(x)| \geq \kappa_M$ at all points x where this derivative is defined, i.e. at all x such that $\tau^i(x) \notin \{\zeta_0, \ldots, \zeta_N\}$ for $i = 0, \ldots, M-1$. In the sequel all expressions involving derivatives of τ should be read in this way. Note that it suffices to have $|(\tau^m)'| > \kappa_m > 1$ for some $m \in \mathbb{N}$ and to choose $M = km$ such that $\kappa_m^k > 2$.

[6] also called *Perron–Frobenius operator*

In particular, if $f = \tilde{f}$ m-a.e., then also $P_\tau f = P_\tau \tilde{f}$ m-a.e. so that P_τ can be interpreted as an operator on L^1_I, the space of equivalence classes of Lebesgue integrable functions from I to $\mathbb{R}$. As an operator on L^1_I, P_τ is unambiguously defined by (8). The following properties of P_τ are elementary consequences of its definition (see e.g. [11, Sect. 4.2]):

$$P_\tau \text{ is linear and positive.} \tag{9}$$

$$\textstyle\int_I P_\tau f\, dm = \int_I f\, dm \text{ and } \int_I |P_\tau f|\, dm \le \int_I |f|\, dm \text{ for all } f \in L^1_I. \tag{10}$$

$$\begin{gathered} P_\tau h = h \text{ if and only if } \mu = hm \text{ is a } \tau\text{-invariant measure, i.e. if} \\ \textstyle\int f \circ \tau\, d\mu = \int f\, d\mu \text{ for each bounded measurable } f : I \to \mathbb{R}. \end{gathered} \tag{11}$$

$$P_{\tau_2 \circ \tau_1} = P_{\tau_2} P_{\tau_1} \text{ whenever the three operators are well defined.} \tag{12}$$

Remark 2.1. The "(pre-)dual" characterization of transfer operators by (8) and its elementary consequences (9)–(12) are not special for 1D maps. They are valid with exactly the same proofs in rather abstract settings, see e.g. [12, Sect. 3.2]. Therefore we will use them in later sections where we study transfer operators for systems of maps without recalling them in detail.

2.3 Functions of Bounded Variation

In order to guarantee the existence of a unique invariant density and its asymptotic stability one needs to study how P_τ acts on spaces of more regular functions. The first space that comes to mind is probably $C^1(I)$, but as $P_\tau f$ may have discontinuities even if f has none (see the explicit formula (7) for $P_\tau f$), this space is not invariant under P_τ. The next natural choice that preserves as much of the flavor of $C^1(I)$ but allows for discontinuities is the space $BV(I)$ of functions of bounded variation.

The variation of a C^1-function $f : I \to \mathbb{R}$ can be defined as

$$V(f) = \int_0^1 |f'(x)|\, dx\ . \tag{13}$$

Approximating this integral by Riemann sums yields the more common expression

$$V(f) = \sup \left\{ \sum_{i=1}^{r} |f(\xi_i) - f(\xi_{i-1})| \right\} \tag{14}$$

where the supremum extends over all finite partitions $0 \le \xi_0 < \xi_1 < \ldots < \xi_r \le 1$ of $[0,1]$. This expression is well-defined for any measurable $f : I \to \mathbb{R}$. A third characterization follows from the first one in view of the integration by parts formula: let

$$\mathcal{T}_{I,0} = \{\varphi \in C^1(I) : |\varphi| \le 1, \varphi(0) = \varphi(1) = 0\} \tag{15}$$

be a set of C^1-test functions on I bounded by 1. Then

$$V(f) = \sup_{\varphi \in \mathcal{T}_{I,0}} \int_0^1 f'(x)\varphi(x)\,dx = \sup_{\varphi \in \mathcal{T}_{I,0}} \int_0^1 f(x)\varphi'(x)\,dx\ . \tag{16}$$

Just as the previous one this characterization can be used to define the variation of any function $f \in L^1_I$ (and not merely that of C^1-functions). Indeed, (14) leads to the definition

$$\mathrm{var}_I(f) = \inf\{V(\tilde{f}) : \tilde{f} = f \ m\text{-a.e.}\} \tag{17}$$

and (16) extends immediately to

$$\mathrm{var}_I(f) = \sup_{\varphi \in \mathcal{T}_{I,0}} \int_I f(x)\varphi'(x)\,dx\ . \tag{18}$$

It is a little extra piece of work to show that the definitions given in (17) and in (18) really coincide.[7] In the sequel we will only use the definition via test functions in (18). Note that

$$\sum_{i=1}^N \mathrm{var}_{\bar{I}_i}(f) \le \mathrm{var}_I(f)\ . \tag{19}$$

This follows because if $\varphi_i \in \mathcal{T}_{\bar{I}_i,0}$ $(i = 1, \ldots, N)$, then $\varphi : I \to \mathbb{R}$, which is (unambiguously!) defined by $\varphi(x) = \varphi_i(x)$ if $x \in \bar{I}_i$, belongs to $\mathcal{T}_{I,0}$. A direct consequence of the definition of variation in (18) is

$$\mathrm{var}_I(f) \le \liminf_{n\to\infty} \mathrm{var}_I(f_n) \tag{20}$$

whenever $f, f_n \in L^1_I$ and $\lim_{n\to\infty} \int_I |f - f_n|\,dm = 0$. Here (and in the sequel) we use $\int |f|\,dm$ as a shorthand notation for $\int |f(x)|\,dx$. We denote

$$BV(I) = \{f \in L^1_I : \mathrm{var}_I(f) < \infty\}\ . \tag{21}$$

All these considerations apply to any compact interval I, not just to $I = [0, 1]$.

For technical reasons we will often prefer to work with the following variant of the notion of variation. For any compact interval J, let

$$\mathcal{T}_J = \{\varphi \in C^1(J) : |\varphi| \le 1\} \tag{22}$$

and define

$$\mathrm{Var}_J(f) = \sup_{\varphi \in \mathcal{T}_J} \int_J f(x)\varphi'(x)\,dx\ . \tag{23}$$

Here is a first observation on "Var".

[7] See e.g. [11, Theorem 2.3.12].

Lemma 2.1. *Let $J = [a, b]$, and suppose that $(a + \delta, b - \delta)$ is a neighborhood of I for some $\delta > 0$. If $f \in C^1(J)$ with $f(x) = 0$ for $x \notin (a + \delta, b - \delta)$, then*

$$\mathrm{Var}_I(f) \leq \int_J |f'| \, dm \, .$$

Proof. Each $\varphi \in \mathcal{T}_I$ can be extended to a function $\tilde{\varphi} : J \to \mathbb{R}$ by linear interpolation between the points $(a|0)$ and $(a+\delta|\varphi(0))$ on the interval $[a, a+\delta]$ and between the points $(b - \delta|\varphi(1))$ and $(b|0)$ on the interval $[b - \delta, b]$, and by the constant values $\varphi(0)$ and $\varphi(1)$ on the intervals $[a + \delta, 0]$ and $[1, b - \delta]$ respectively. In this way, $\tilde{\varphi}$ is continuous, $\sup |\tilde{\varphi}| \leq 1$, $\tilde{\varphi}(a) = \tilde{\varphi}(b) = 0$, and $\tilde{\varphi}$ is differentiable except at possibly four points. Hence

$$\int_I f\varphi' \, dm = \int_{a+\delta}^{b-\delta} f\tilde{\varphi}' \, dm = \int_a^b f\tilde{\varphi}' \, dm = -\int_a^b f'\tilde{\varphi} \, dm \leq \int_a^b |f'| \, dm$$

from which the lemma follows. □

The next lemma is a kind of tool-box for our work with "var" and "Var".

Lemma 2.2. *Let $J = [a, b]$, $f \in L^1_J$, $\dot{\varphi} \in L^\infty_J$, $c \in \mathbb{R}$, and let $\varphi : J \to \mathbb{R}$, $\varphi(x) = c + \int_a^x \dot{\varphi}(\xi) \, d\xi$.*

(a) $\int_J f\dot{\varphi} \, dm \leq \sup |\varphi| \; \mathrm{Var}_J(f)$
(b) $\mathrm{Var}_J(f\varphi) \leq \sup |\varphi| \; \mathrm{Var}_J(f) + \operatorname{ess\,sup} |\dot{\varphi}| \; \int_J |f| \, dm$
(c) $\int_J f\dot{\varphi} \, dm \leq \sup_{u,v \in J} |\varphi(u) - \varphi(v)| \; \mathrm{var}_J(f) + \frac{\varphi(b)-\varphi(a)}{b-a} \int_J f \, dm.$
(d) If $\varphi(a) = \varphi(b) = 0$, *then* $\int_J f\dot{\varphi} \, dm \leq \sup |\varphi| \; \mathrm{var}_J(f)$.
(e) $\mathrm{var}_J(f) \leq \mathrm{Var}_J(f) \leq 2 \, \mathrm{var}_J(f) + \frac{2}{b-a} \left| \int_J f \, dm \right|$.

Before we prove this lemma, we discuss a number of consequences.

Corollary 2.1. $\int_J |f| \, dm \leq \frac{1}{2}|J| \; \mathrm{Var}_J(f)$.

Proof. Let $\dot{\varphi} = 1_{\{f>0\}} - 1_{\{f<0\}}$. Then $\int_J |f| \, dm = \int_J f\dot{\varphi} \, dm \leq \frac{1}{2}|J| \; \mathrm{Var}_J(f)$ by Lemma 2.2a applied to $\varphi(x) = -\int_a^{(a+b)/2} \dot{\varphi}(\xi) \, d\xi + \int_a^x \dot{\varphi}(\xi) \, d\xi$. □

Remark 2.2. It is easy to check that $\mathrm{Var}_I(.)$ and $\mathrm{var}_I(.)$ are seminorms, i.e. subadditive and positively homogeneous. Because of Corollary 2.1, $\|f\|_{BV} = \mathrm{Var}_I(f)$ defines indeed a norm on $BV(I) = \{f \in L^1_I : \mathrm{Var}_I(f) < \infty\}$. It is equivalent to the more common norm $\mathrm{var}_I(f) + \int_I |f| \, dm$ on $BV(I)$, see Lemma 2.2e.

Corollary 2.2. *Let $\varphi : J \to \mathbb{R}$ and suppose that the interval J is partitioned into subintervals $J_1, \ldots, J_r$ such that $\varphi|_{J_k}$ is continuously differentiable for each $k = 1, \ldots, r$. Let $f \in L^1_J$. Then*

$$\int_J f\varphi' \, dm \leq 2 \, \sup |\varphi| \left(\mathrm{var}_J(f) + \frac{1}{\min_k |J_k|} \int_J |f| \, dm \right) . \tag{24}$$

Proof. We apply Lemma 2.2c to each interval J_k separately:

$$\int_{J_k} f\varphi' \, dm \le 2 \sup|\varphi| \left(\mathrm{var}_{J_k}(f) + \frac{1}{|J_k|} \int_{J_k} |f| \, dm \right) .$$

Summing over k and observing (19) this yields (24). $\square$

Proof of Lemma 2.2. Let $\varepsilon > 0$, and fix $s > 1$ such that $\int_{\{|f|>s\}} |f| \, dm < \varepsilon$. There exists $\dot\psi \in C(J)$ such that $\int_J |\dot\varphi - \dot\psi| \, dm < \frac{\varepsilon}{s}$, $\sup|\dot\psi| \le \operatorname{ess\,sup}|\dot\varphi| + 1$, and $\int_J \dot\varphi \, dm = \int_J \dot\psi \, dm$. Hence

$$\int_J f(\dot\varphi - \dot\psi) \, dm \le \varepsilon \operatorname{ess\,sup}|\dot\varphi - \dot\psi| + s \int_J |\dot\varphi - \dot\psi| \, dm \le C\varepsilon \tag{25}$$

where $C = 2 + 2 \operatorname{ess\,sup}|\dot\varphi|$.

Let $\psi(x) = c + \int_a^x \dot\psi(\xi) \, d\xi$. Then $\psi \in C^1(J)$ so that $\tilde\psi := \psi / \sup|\psi| \in \mathcal{T}_J$. Hence $\int_J f\tilde\psi' \, dm \le \mathrm{Var}_J(f)$. As $\sup|\psi| \le \sup|\varphi| + \int_J |\dot\psi - \dot\varphi| \, dm \le \sup|\varphi| + \varepsilon$, it follows from (25) that

$$\int_J f\dot\varphi \, dm \le \int_J f\psi' \, dm + C\varepsilon \le (\sup|\varphi| + \varepsilon) \, \mathrm{Var}_J(f) + C\varepsilon .$$

As $\varepsilon > 0$ is arbitrary, we conclude

$$\int_J f\dot\varphi \, dm \le \sup|\varphi| \ \mathrm{Var}_J(f) \tag{26}$$

which is part (a) of the lemma.

Now let $\psi \in C^1(J)$, $|\psi| \le 1$. Then $(\varphi\psi)(x) = (\varphi\psi)(a) + \int_a^x (\dot\varphi\psi + \varphi\psi')(\xi) \, d\xi$, so we may write $(\dot{\varphi\psi}) = \dot\varphi\psi + \varphi\psi'$. Hence, in view of part (a),

$$\int_J (f\varphi)\psi' \, dm = \int_J f \, (\dot{\varphi\psi}) \, dm - \int_J f\dot\varphi\psi \, dm \le \sup|\varphi| \ \mathrm{Var}_J(f) + \operatorname{ess\,sup}|\dot\varphi| \int_J |f| \, dm .$$

This is part (b) of the lemma.

If $\varphi(a) = \varphi(b) = 0$, then also $\psi(a) = \psi(b) = 0$ so that $\psi \in \mathcal{T}_{J,0}$, and we can estimate by $\mathrm{var}_J(f)$ instead $\mathrm{Var}_J(f)$ on the right hand side of (26). This is part (d) of the lemma.

Next, for $\varphi \in \mathcal{T}_J$, let $\tilde\varphi(x) = \frac{x-a}{b-a}(\varphi(x) - \varphi(b)) + \frac{b-x}{b-a}(\varphi(x) - \varphi(a)) = \int_a^x (\varphi'(\xi) - \frac{\varphi(b)-\varphi(a)}{b-a}) \, d\xi$. As $\tilde\varphi(a) = \tilde\varphi(b) = 0$ we can apply part (d) of the lemma to $\tilde\varphi$. So

$$\int_J f \left(\varphi' - \frac{\varphi(b) - \varphi(a)}{b - a} \right) dm \le \sup|\tilde\varphi| \ \mathrm{var}_J(f) \le 2 \sup|\varphi| \ \mathrm{var}_J(f) . \tag{27}$$

As $\sup|\tilde\varphi| \le \sup_{u,v\in J} |\varphi(u) - \varphi(v)|$, this proves part (c) of the lemma. Finally, (27) also implies

$$\mathrm{Var}_J(f) = \sup_{\varphi\in\mathcal{T}_J} \int_J f\varphi' \, dm \le 2 \ \mathrm{var}_J(f) + \frac{2}{b-a} \left| \int_J f \, dm \right|$$

and $\mathrm{var}_J(f) \le \mathrm{Var}_J(f)$ follows directly from the definition. This proves part (e) of the lemma. $\square$

2.4 The Lasota–Yorke Inequality

Coming back to the transfer operator P_τ we show next that $P_\tau(BV) \subseteq BV$. (Here and in the sequel we write BV instead of $BV(I)$ and $\mathrm{var}(f)$ instead of $\mathrm{var}_I(f)$.) In fact, we prove much more – an inequality which was discovered in this context by Lasota and Yorke [16]. In that paper, as well as in numerous subsequent generalizations of this result, the proof is based on the "elementary" approach (14) to variation. Here we give a proof using test functions as in (18).

Proposition 2.1 (Lasota–Yorke inequality). *Let $\tau : I \to I$ be a p.w.e. map as defined in Sect. 2.1. Let $\ell \in \mathbb{N}$ and recall that $\kappa_\ell := \inf |(\tau^\ell)'| > 0$ and $D_\ell = \sup |(\frac{1}{(\tau^\ell)'})'| < \infty$. Let also $E_\ell := 2/(\kappa_\ell \min_i |I_i^\ell|)$ where the I_i^ℓ's are monotonicity intervals of τ^ℓ which are finitely many. Then, for $f \in L^1_I$,*

$$\int_I |P_\tau f|\, dm \le \int_I |f|\, dm \tag{28}$$

$$\mathrm{Var}(P_\tau^\ell f) \le \frac{2}{\kappa_\ell}\,\mathrm{var}(f) + (D_\ell + E_\ell) \int_I |f|\, dm\ . \tag{29}$$

Proof. Equation (28) is just a restatement of (10). We turn to (29). As τ^ℓ is again a piecewise expanding map, it suffices to prove this estimate for $\ell = 1$. Let $\varphi \in \mathcal{T}_I$. As $(\varphi\circ\tau)'(x) = \varphi'(\tau(x))\,\tau'(x)$ for all $x \in I \setminus \{\zeta_0, \dots, \zeta_N\}$, we have

$$\begin{aligned}\int_I P_\tau f\, \varphi'\, dm &= \int_I f\,(\varphi'\circ\tau)\, dm = \int_I f\,\frac{(\varphi\circ\tau)'}{\tau'}\, dm \\ &= \int_I f\,\left(\frac{\varphi\circ\tau}{\tau'}\right)' dm - \int_I f\,(\varphi\circ\tau)\left(\frac{1}{\tau'}\right)' dm\ .\end{aligned} \tag{30}$$

The second term is bounded by $D_1 \int_I |f|\, dm$. To the first term we apply Corollary 2.2. As $|\tau'| \ge \kappa_1$, this yields (29) (for $\ell = 1$). □

If one applies inequality (29) to $P_\tau^\ell f, P_\tau^{2\ell} f, P_\tau^{3\ell} f, \dots$ and observes (28), one obtains by recurrence for each $k \in \mathbb{N}$

$$\mathrm{Var}(P_\tau^{k\ell} f) \le \left(\frac{2}{\kappa_\ell}\right)^k \mathrm{var}(f) + (D_\ell + E_\ell) \sum_{j=0}^{k-1} \left(\frac{2}{\kappa_\ell}\right)^j \int_I |f|\, dm\ . \tag{31}$$

As $\kappa_M > 2$ by assumption (see Sect. 2.1), it follows at once that

$$\mathrm{Var}(P_\tau^{kM} f) \le \left(\frac{2}{\kappa_M}\right)^k \mathrm{var}(f) + (D_M + E_M)\frac{\kappa_M}{\kappa_M - 2} \int_I |f|\, dm\ . \tag{32}$$

In order to extend this inequality to powers P_τ^n which are not multiples of M we decompose $n = kM + p$ with $0 \le p < M$. Equation (31) yields

$$\mathrm{Var}(P_\tau^p f) \le \left(\frac{2}{\kappa_1}\right)^p \mathrm{var}(f) + (D_1 + E_1)\frac{(2/\kappa_1)^M + 1}{|2/\kappa_1 - 1|} \int_I |f|\, dm \tag{33}$$

and combining this with (32) we arrive at

$$\|P_\tau^n f\|_{BV} = \mathrm{Var}(P_\tau^n f) \le C_1 \alpha^n \,\mathrm{var}(f) + C_2 \int_I |f|\, dm \tag{34}$$

where $0 < \alpha := (2/\kappa_M)^{\frac{1}{M}} < 1$ and C_1, C_2 are constants that depend on τ only through $M, \kappa_1, \kappa_M, D_1, D_M, E_1, E_M$. In particular, P_τ is a bounded linear operator on $(BV, \|.\|_{BV})$.

2.5 Compact Embedding and the Spectral Gap

The usefulness of the space BV is mainly due to the fact that it embeds compactly into L^1_I: the unit ball of BV is compact in L^1_I, that is, each sequence $(f_n)_n$ of L^1_I functions with bounded BV-norm has a subsequence which converges (in L^1_I-norm) to an element of BV. It follows directly that $(BV, \|.\|_{BV})$ is complete, i.e. BV is a Banach space.

In its simplest form this is known as Helly's theorem. For the test function approach to variation that we follow here and that we will extend to multivariate functions in Chap. 3 this is proved e.g. in [13, 14, 15].[8]

A first consequence is the existence of an invariant probability density of bounded variation for the map τ: let $f_n := \frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k 1$. Then $\|f_n\|_{BV} \le \frac{1}{n}\sum_{k=0}^{n-1} \|P_\tau^k 1\|_{BV}$, and (34) implies $\sup_n \|f_n\|_{BV} \le C_2 < \infty$. Hence there are $h \in L^1_m$ and a subsequence $(f_{n_j})_j$ such that $\lim_{j\to\infty} \int_I |h - f_{n_j}|\, dm = 0$. It follows from the elementary properties (9–11) of P_τ that h is a probability density and that the measure $\mu = hm$ is τ-invariant. The bound $\|h\|_{BV} \le C_2$ follows from (20).

But much more is true. The Lasota–Yorke inequality (29), together with the compact embedding property of BV into L^1_I, allows to apply the Ionescu-Tulcea/Marinescu theorem [17]:

Theorem 2.1 (Quasi-compactness of P_τ). *The operator $P_\tau : BV \to BV$ is* quasi-compact*, i.e. its canonical complexification has only finitely many eigenvalues of modulus one which all have finite multiplicities, and the rest of the spectrum is contained in a disc of radius $\rho < 1$ centered at 0. As seen before, 1 is an eigenvalue of P_τ. (We will fix ρ such that the rest of the spectrum is indeed contained in the interior of the disc of radius ρ.)*[9]

[8] It is a simple exercise to derive the compact embedding of BV into L^1_I from Lemma 2.2. *Hint:* Subdivide I into 2^n intervals of length 2^{-n}. For $f \in BV$ let $f_n = \sum_{k=1}^{2^n} 1_{I_k} 2^n \int_{I_k} f\, dm$. Let $\varphi_n = \mathrm{sign}(f - f_n)$. Then $\int_I |f - f_n|\, dm \le \sum_{k=1}^{2^n} 2^{-n} \mathrm{var}_{I_k}(f - f_n) \le 2^{-n} \mathrm{var}_I(f)$ by Lemma 2.2c and (18).

[9] More detailed accounts of this theorem can be found e.g. in [11, Chap. 7] and [18, Sect. 3.2]. See also [19].

From now on we assume that τ is *mixing*. That means that 1 is a simple eigenvalue of P_τ and that there is no other eigenvalue of modulus one. For p.w.e. maps this in fact equivalent to the usual notion of mixing in ergodic theory, see e.g. [11, Corollary 7.2.1]. For mixing τ, $(1-\rho)$ quantifies a *spectral gap*, i.e. the simple eigenvalue 1 is separated from the moduli of all other spectral value by $(1-\rho)$ at least. We have indeed

Corollary 2.3 (Spectral gap of P_τ). *If τ is mixing, then there is a constant $C_3 > 0$ such that*

$$\int_I |P_\tau^n f|\, dm \leq \|P_\tau^n f\|_{BV} \leq C_3\, \rho^n\, \|f\|_{BV} \tag{35}$$

for all $n \in \mathbb{N}$ and all $f \in BV$ with $\int_I f\, dm = 0$.[10]

Remark 2.3. Although we will not use it explicitly we note the following fact: both constants ρ and C_3 do not change much under small perturbations of τ as long as τ and its perturbations satisfy a Lasota–Yorke inequality (34) with the same constants α, C_1, and C_2; see [20].)

3 Finite Systems

As an intermediate step towards infinite coupled systems, this section deals with finite coupled systems of d piecewise expanding maps described by a transformation T_ϵ on the d-dimensional unit cube. We will see below that – for sufficiently small $|\epsilon|$ – the maps T_ϵ are piecewise expanding and that one can develop a spectral theory for their transfer operators P_{T_ϵ} in just the same way as we did it for the 1D map τ in Chap. 2.

3.1 The Coupling

We recall the notation from the Introduction:

- L is a finite set of cardinality $d > 0$: it serves as the set of sites. For notational convenience we work with $L = \{1, \ldots, d\}$ in this section without interpreting L as a subset of the one-dimensional lattice $\mathbb{Z}$.
- $X = I^L$ is the state space of the system: it is a d-dimensional cube.
- $\tau : I \to I$ is a p.w.e. map as defined in Sect. 2.1.

[10] In the spectral theoretic approach the constants C_3 and ρ cannot be determined easily from the "formula" for the map τ. For some maps explicit estimates for ρ with $C_3 = 1$ are derived in [21]. The proof, which is a refined version of the proof of our Lemma 2.2c, bypasses spectral theory completely. In [22] (see also [23, Sect. 8]) it is shown how to obtain explicit estimates on C_3 and ρ using Birkhoff cones. A rigorous numerical approach to estimate these constants is discussed in [23].

- $T_0 : X \to X$ is the d-fold product of the map τ: $(T_0\boldsymbol{x})_i = \tau(x_i)$ $(i \in L)$.
- $\Phi_\epsilon : X \to X$ $(|\epsilon| < \epsilon_0)$ is a family of coupling maps ϵ-close to the identity in a C^2 sense made precise below.
- $T_\epsilon = \Phi_\epsilon \circ T_0 : X \to X$ are the maps describing the coupled systems.

The precise assumptions on Φ_ϵ are: $\Phi_\epsilon(\boldsymbol{x}) = \boldsymbol{x} + A_\epsilon(\boldsymbol{x})$ is a (a_1, a_2)-*coupling*, i.e. there are $L \times L$ matrices A', A'' with $a_1 = \|A'\|_1$, $a_2 = \|A''\|_1$ (maximal column sum norm) such that for all $i, j, k \in L$

$$|(A_\epsilon)_i| \le 2|\epsilon|, \quad |(DA_\epsilon)_{ij}| \le 2|\epsilon|A'_{ij}, \quad |\partial_k(DA_\epsilon)_{ij}| \le 2|\epsilon|A''_{ij} \,. \tag{36}$$

Here ∂_j denotes the partial derivative w.r.t. x_j. The diffusive nearest neighbor coupling (3) is an example of a $(1, 0)$-coupling.

Later we will need the following estimates on $(D\Phi_\epsilon)^{-1}$ derived from (36):

$$|((D\Phi_\epsilon)^{-1})_{ij}| \le ((E - 2|\epsilon|A')^{-1})_{ij} \quad \text{where } E \text{ is the identity matrix,} \tag{37}$$

$$\sum_{i=1}^{d} |((D\Phi_\epsilon)^{-1})_{ij}| \le \frac{1}{1 - 2a_1|\epsilon|} \,, \; \sum_{i=1}^{d} |\partial_i((D\Phi_\epsilon)^{-1})_{ij}| \le \frac{2a_2|\epsilon|}{(1 - 2a_1|\epsilon|)^2} \,. \tag{38}$$

Observe first that $(D\Phi_\epsilon)^{-1} = \sum_{n=0}^{\infty}(-DA_\epsilon)^n$ and that $(-DA_\epsilon)^n$ is dominated coefficient-wise by $|2\epsilon|^n A'^n$ in view of (36). This yields (37). Let $\mathbf{1} = (1, \ldots, 1)$ and let $\boldsymbol{e}_j$ be the j-th unit vector. We interpret both as *matrices*, which plays a role when we evaluate their $\|.\|_1$-norms. Then

$$\sum_{i=1}^{d} |((D\Phi_\epsilon)^{-1})_{ij}| \le \sum_{n=0}^{\infty} |2\epsilon|^n \mathbf{1} A'^n \boldsymbol{e}_j \le \sum_{n=0}^{\infty} |2\epsilon|^n \|\mathbf{1}\|_1 \|A'\|_1^n \|\boldsymbol{e}_j\|_1 = \frac{1}{1 - 2a_1|\epsilon|} \,.$$

This is the first estimate in (38), and the second one is proved along the same lines.

3.2 The Transfer Operator

Recall from Sect. 2.1 that we denote the intervals restricted to which the map τ is C^2 by $I_1, \ldots, I_N$. Let $\mathcal{Q}_d = \{I_{i_1} \times \cdots \times I_{i_d} : \; i_1, \ldots, i_d \in \{1, \ldots, N\}\}$ be the family of rectangular domains restricted to which the product map T_0 is C^2. As in (7) we define the transfer operator P_{T_0} of T_0 acting on measurable $f : X \to \mathbb{R}$ by

$$P_{T_0} f(\boldsymbol{x}) = \sum_{Q \in \mathcal{Q}_d} \frac{f}{|\det(DT_0)|} \circ (T_0|_Q)^{-1}(\boldsymbol{x}) \cdot 1_{T_0(Q)}(\boldsymbol{x}) \,. \tag{39}$$

As in the one-dimensional case, P_{T_0} can be interpreted as a positive linear contraction on the space L^1_X of equivalence classes of Lebesgue integrable functions from X to $\mathbb{R}$, unambiguously defined by

$$\int_X P_{T_0} f(\boldsymbol{x})\, \psi(\boldsymbol{x})\, d\boldsymbol{x} = \int_X f(\boldsymbol{x})\, \psi(T_0(\boldsymbol{x}))\, d\boldsymbol{x} \tag{40}$$

where $d\boldsymbol{x}$ is an abbreviation for $dx_1 \dots dx_d$.

In the same way we define a transfer operator P_{Φ_ϵ} for the coupling map Φ_ϵ. Since Φ_ϵ is injective, its explicit form is particularly simple:

$$P_{\Phi_\epsilon} f(\boldsymbol{x}) = \frac{f}{|\det(D\Phi_\epsilon)|} \circ \Phi_\epsilon^{-1}(\boldsymbol{x}) \cdot 1_{\Phi_\epsilon(X)}(\boldsymbol{x})\ . \tag{41}$$

In view of the elementary properties of general transfer operators discussed in Remark 2.1, we have for the transfer operator P_{T_ϵ} of the coupled map $T_\epsilon = \Phi_\epsilon \circ T_0$

$$P_{T_\epsilon} = P_{\Phi_\epsilon} P_{T_0} \tag{42}$$

and both, P_{Φ_ϵ} and P_{T_ϵ}, have a (pre)-dual characterization as linear L^1_X operators analogous to (40).

3.3 Multivariate Functions of Bounded Variation

As in the 1D case we need a subspace of L^1_X of more regular functions on which the transfer operators just introduced have "good" spectral properties. Multivariate functions of bounded variation turn out to be a suitable choice.

There are many equivalent ways to define the variation of a multivariate function $f : X \to \mathbb{R}$, see e.g. [13, 14, 15]. The most intuitive one is perhaps to define it just in terms of coordinate-wise one-dimensional variation of f. To this end, and also for later use, we introduce the following notation: For $i \in \{1, \dots, d\}$ we identify $\boldsymbol{x}$ and $(x_i, \boldsymbol{x}_{\neq i})$ where $\boldsymbol{x}_{\neq i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$. Since we never permute coordinates, this will not lead to any confusion. We also denote by $X_{\neq i}$ the $(d-1)$-dimensional cube $\{\boldsymbol{x}_{\neq i} : \boldsymbol{x} \in X\}$ and by $f_{\boldsymbol{x}_{\neq i}} : I \to \mathbb{R}$, $f_{\boldsymbol{x}_{\neq i}}(x) = f(x, \boldsymbol{x}_{\neq i})$, the $\boldsymbol{x}_{\neq i}$-section of f. Now we define for $f \in L^1_X$

$$\mathrm{Var}^i_X(f) = \int_{X_{\neq i}} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}})\, d\boldsymbol{x}_{\neq i}, \quad \mathrm{Var}_X(f) = \max_{i=1,\dots,d} \mathrm{Var}^i_X(f)\ . \tag{43}$$

Observe that $f_{\boldsymbol{x}_{\neq i}} \in L^1_I$ for Lebesgue-a.e. $\boldsymbol{x}_{\neq i} \in X_{\neq i}$ by Fubini's theorem. So $\mathrm{Var}(f_{\boldsymbol{x}_{\neq i}}) \in [0, \infty]$ is well defined for Lebesgue-a.e. $\boldsymbol{x}_{\neq i}$. That it depends measurably on $\boldsymbol{x}_{\neq i}$ will be shown in Lemma 3.1b. We note the following immediate consequences of Lemma 2.2e and Corollary 2.1:

$$\mathrm{Var}^i_X(1) = 2 \quad \text{and} \quad \int_X |f|\, dm \le \frac{1}{2} \mathrm{Var}^i_X(f) \quad \text{for each } i = 1, \dots, d, \tag{44}$$

where m denotes Lebesgue measure on X.

Let

$$BV(X) = \{f \in L^1_X : \mathrm{Var}_X(f) < \infty\} \tag{45}$$

be the space of functions of bounded variation on X, and note that $\mathrm{Var}_X(.)$ is a norm on $BV(X)$.

The above definition of $\mathrm{Var}^i_X(f)$ is equivalent to a more direct one generalizing the test function approach (18) we used already in dimension one. Let

$$\mathcal{T}_X = \{\varphi \in C^1(X) : |\varphi| \le 1\} \ . \tag{46}$$

Proposition 3.1. *For each measurable $f : X \to \mathbb{R}$ and each $i = 1, \ldots, d$,*

$$\mathrm{Var}^i_X(f) = \sup_{\varphi \in \mathcal{T}_X} \int_X f(\boldsymbol{x})\, \partial_i \varphi(\boldsymbol{x})\, d\boldsymbol{x} \ . \tag{47}$$

An immediate consequence is that

$$\mathrm{Var}_X(f) \le \liminf_{n\to\infty} \mathrm{Var}_X(f_n) \tag{48}$$

whenever $f, f_n \in L^1_X$ and $\lim_{n\to\infty} \int |f - f_n|\, dm = 0$.

The proof of this proposition requires smoothing of functions and test functions by *mollifiers*: Let $\eta : \mathbb{R} \to [0, \infty)$ be a symmetric (at zero) C^∞ function with $\int_{\mathbb{R}} \eta(t)\, dt = 1$ and $\eta(t) = 0$ if $|t| \ge 1$. For $\delta > 0$ let $\eta_\delta(t) = \delta^{-1}\, \eta(\frac{t}{\delta})$. The *convolution* of a function $u : I \to \mathbb{R}$ with η_δ is defined by $(u * \eta_\delta)(x) = \int_{\mathbb{R}} u(x - t)\eta_\delta(t)\, dt$, where $u(x - t)$ is understood to be zero if $(x - t) \notin I$.

Lemma 3.1. *Let $f_{\boldsymbol{x}_{\neq i},\delta} = f_{\boldsymbol{x}_{\neq i}} * \eta_\delta$.*

(a) $\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}}) = \lim_{\delta\to 0} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta})$ *for every $\boldsymbol{x}_{\neq i} \in X_{\neq i}$.*
(b) $\boldsymbol{x}_{\neq i} \mapsto \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta})$ *and* $\boldsymbol{x}_{\neq i} \mapsto \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}})$ *are nonnegative measurable functions. In particular,* $\mathrm{Var}^i_X(f)$ *is well defined in (43).*
(c) $\int_{X\neq i} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta})\, d\boldsymbol{x}_{\neq i} \le \sup_{\varphi\in\mathcal{T}_X} \int_X f(\boldsymbol{x})\partial_i\varphi(\boldsymbol{x})\, d\boldsymbol{x} + o(\delta)$.

Proof. (a) It is a rather classical result from real analysis [14, Theorem 1.6.1] that for each $\boldsymbol{x}_{\neq i} \in X_{\neq i}$

$$\lim_{\delta\to 0} \int_I |f_{\boldsymbol{x}_{\neq i},\delta}(x_i) - f_{\boldsymbol{x}_{\neq i}}(x_i)|\, dx_i = 0 \ . \tag{49}$$

This implies at once that

$$\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}}) \le \liminf_{\delta\to 0} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta}) \ . \tag{50}$$

We turn to the reverse inequality. Let $\varphi \in \mathcal{T}_I$, $\varepsilon > 0$, and let $\tilde\varphi$ be any C^1 extension of φ to all of $\mathbb{R}$ with $|\tilde\varphi| \le 1 + \varepsilon$. Then $\tilde\varphi * \eta_\delta|_I \in (1 + \varepsilon)\, \mathcal{T}_I$. So

$$\int_I f_{\boldsymbol{x}_{\neq i},\delta}(x_i)\varphi'(x_i)\,dx_i = \int_I (f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)(x_i)\tilde{\varphi}'(x_i)\,dx_i$$
$$= \int_I f_{\boldsymbol{x}_{\neq i}}(x_i)(\tilde{\varphi}' * \eta_\delta)(x_i)\,dx_i$$
$$= \int_I f_{\boldsymbol{x}_{\neq i}}(x_i)(\tilde{\varphi} * \eta_\delta)'(x_i)\,dx_i \leq \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}}) \quad (51)$$

by definition in (23). Hence $\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta}) \leq \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}})$.

(b) $\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta}) = \int_I |f'_{\boldsymbol{x}_{\neq i},\delta}|\,dm$ is a nonnegative measurable function of the argument $\boldsymbol{x}_{\neq i}$, and so is $\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i}})$ in view of part (a) of this lemma.

(c) Let $\psi_{\boldsymbol{x}_{\neq i}}(x_i) = \mathrm{sign}(f'_{\boldsymbol{x}_{\neq i},\delta}(x_i))$, and let $J = [-1, 2]$. By Lemma 2.1,

$$\mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta}) \leq \int_J |f'_{\boldsymbol{x}_{\neq i},\delta}(x_i)|\,dx_i = \int_J (f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)'(x_i)\,\psi_{\boldsymbol{x}_{\neq i}}(x_i)\,dx_i\,. \quad (52)$$

Let $\varepsilon > 0$. As $(f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)'$ is bounded and as $|\psi_{\boldsymbol{x}_{\neq i}}| \leq 1$, there is $\varphi \in C^1(X_{\neq i} \times J)$ with $|\varphi| \leq 1$ such that

$$\int_{X_{\neq i}} \int_J (f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)'(x_i)\,\psi_{\boldsymbol{x}_{\neq i}}(x_i)\,dx_i d\boldsymbol{x}_{\neq i}$$
$$\leq \varepsilon + \int_{X_{\neq i}} \int_J (f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)'(x_i)\,\varphi_{\boldsymbol{x}_{\neq i}}(x_i)\,dx_i d\boldsymbol{x}_{\neq i} \quad (53)$$
$$= \varepsilon + \int_{X_{\neq i}} \int_J f_{\boldsymbol{x}_{\neq i}}(x_i)\,(\varphi_{\boldsymbol{x}_{\neq i}} * \eta_\delta)'(x_i)\,dx_i d\boldsymbol{x}_{\neq i}\,.$$

For the last identity observe that $(f_{\boldsymbol{x}_{\neq i}} * \eta_\delta)(x_i) = 0$ if $x_i \notin (-\delta, 1+\delta)$. As $f_{\boldsymbol{x}_{\neq i}}(x_i) = 0$ for $x_i \notin I$, the integral over J in the last expression can be replaced by an integral over I. Define $\tilde{\varphi} : X \to \mathbb{R}$ by $\tilde{\varphi}(\boldsymbol{x}) = (\varphi_{\boldsymbol{x}_{\neq i}} * \eta_\delta)(x_i)$. Clearly, $\varphi \in \mathcal{T}_X$, and combining (52) with (53) we obtain

$$\int_{X_{\neq i}} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq i},\delta})\,d\boldsymbol{x}_{\neq i} \leq \varepsilon + \int_X f(\boldsymbol{x})\partial_i\tilde{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}\,.$$

As $\varepsilon > 0$ is arbitrary, this finishes the proof. □

Proof of Proposition 3.1. Fix f and i and denote the expression on the right hand side of (47) by $V(f)$. Then $V(f) \leq \mathrm{var}^i_X(f)$ because, for $\varphi \in \mathcal{T}_X$, all $\varphi_{\boldsymbol{x}_{\neq i}}$ ($\boldsymbol{x}_{\neq i} \in X_{\neq i}$) belong to the set $\mathcal{T}_I$ of univariate test functions, see (22).

The reverse inequality follows at once from Lemma 3.1a, Fatou's lemma, and Lemma 3.1c. □

3.4 The Lasota–Yorke Inequality

The Lasota–Yorke inequality (29) for iterates of one-dimensional p.w.e. maps involves three constants on its right hand side which are determined by basic properties of the map τ^ℓ:

- κ_ℓ is the minimal slope of τ^ℓ, i.e. κ_ℓ^{-1} is an upper bound on the contraction rate of each single inverse branch of τ^ℓ,
- D_ℓ is determined essentially by the second derivative of τ, and
- E_ℓ can be controlled in terms of the inverse of the minimal size of intervals of monotonicity of τ^ℓ.[11]

The following proposition shows that these are also the quantities one needs to control for a Lasota–Yorke type inequality for $P^\ell_{T_\epsilon}$.

Proposition 3.2 (Lasota–Yorke inequality for finite coupled systems). *Let $\ell \in \mathbb{N}$. For each $\alpha_\ell > \frac{2}{\kappa_\ell}$ and each $C_{4,\ell} > D_\ell + E_\ell$ there is $\epsilon_1 \in (0, \epsilon_0]$ such that for $|\epsilon| \le \epsilon_1$*

$$\int_X |P_{T_\epsilon} f| \, dm \le \int_X |f| \, dm \tag{54}$$

$$\mathrm{Var}_X(P^\ell_{T_\epsilon} f) \le \alpha_\ell \, \mathrm{Var}_X(f) + C_{4,\ell} \int_X |f| \, dm \, . \tag{55}$$

Given ℓ, α_ℓ and $C_{4,\ell}$, the choice of ϵ_1 depends only on the constants a_1, a_2 which qualify Φ_ϵ as a (a_1, a_2)-coupling, see (36).

Equation (54) follows again from (10), see also the remark thereafter. For (55) we will give a complete proof only when $\ell = 1$. In this case it follows directly from the following separate estimates for P_{T_0} and P_{Φ_ϵ}.

Lemma 3.2.

$$\mathrm{Var}_X(P^\ell_{T_0} f) \le \frac{2}{\kappa_\ell} \mathrm{Var}_X(f) + (D_\ell + E_\ell) \int_X |f| \, dm \, . \tag{56}$$

Lemma 3.3.

$$\mathrm{Var}_X(P_{\Phi_\epsilon} f) \le \frac{1}{1 - 2a_1|\epsilon|} \mathrm{Var}_X(f) + \frac{2a_2|\epsilon|}{(1 - 2a_1|\epsilon|)^2} \int_X |f| \, dm \, . \tag{57}$$

Remark 3.1. As the Lasota–Yorke inequality in Proposition 3.2 is useful only if $\kappa_\ell = \inf |(\tau^\ell)'| > 2$, the restriction to the case $\ell = 1$ means that we assume $\inf |\tau'| > 2$. This was the case dealt with in [6]. It was only in the unpublished thesis [7] that the geometrically much more subtle case of general ℓ was dealt with.[12]

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[11] A closer look at the definition of E_ℓ in Proposition 2.1 reveals that one can do better: It is essentially the minimal size of the *images* of the intervals of monotonicity which determines E_ℓ.

[12] The treatment of this case in [7] is based on an alternative proof of Lemma 3.2 as given in [6, Lemma 3.1]. Instead of using the product structure of T_0^ℓ it suffices to use the fact that the domains restricted to which T_0^ℓ is C^2 and expanding are direct products of intervals (on which τ^ℓ is monotone and C^2). Let us call this the *rectangular domain property.* (The proof of [6, Lemma 3.1] is rather

Proof of Lemma 3.2. The map $T_0 = \tau \times \cdots \times \tau$ is the d-fold direct product of the p.w.e. map τ, so $T_0^\ell = \tau^\ell \times \cdots \times \tau^\ell$ is the d-fold direct product of the p.w.e. map τ^ℓ. Therefore, without loss of generality, we may just treat the case $\ell = 1$ in this lemma.

As $\mathrm{Var}_X(f) = \max_{i=1,\dots,d} \mathrm{Var}_X^i(f)$, it suffices to prove inequality (56) for each Var_X^i separately. We will make as complete use as possible of the product structure of T_0. For notational simplicity we will estimate $\mathrm{Var}_X^1(P_{T_0}f)$ only, the other $\mathrm{Var}_X^i(P_{T_0}f)$ are treated in just the same way.

We write T_0 as $T_0 = S_2 \circ S_1$ where $S_1 = \mathrm{Id}_I \times (\tau \times \cdots \times \tau)$ and $S_2 = \tau \times (\mathrm{Id}_I \times \cdots \times \mathrm{Id}_I)$. Then $P_{T_0} = P_{S_2} P_{S_1}$, and we can do the estimate in two steps.

We start by estimating $\mathrm{Var}_X^1(P_{S_2}f)$ for $f \in L_X^1$. Because of the product structure of S_2, the operator P_{S_2} acts formally like a tensor product operator on L_X^1. More precisely, $(P_{S_2}f)_{\boldsymbol{x}_{\neq 1}}(x_1) = (P_\tau f_{\boldsymbol{x}_{\neq 1}})(x_1)$. Hence

$$
\begin{aligned}
\mathrm{Var}_X^1(P_{S_2}f) &= \int_{X_{\neq 1}} \mathrm{Var}_I((P_{S_2}f)_{\boldsymbol{x}_{\neq 1}})\, d\boldsymbol{x}_{\neq 1} = \int_{X_{\neq 1}} \mathrm{Var}_I(P_\tau f_{\boldsymbol{x}_{\neq 1}})\, d\boldsymbol{x}_{\neq 1} \\
&\leq \frac{2}{\kappa_1} \int_{X_{\neq 1}} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq 1}})\, d\boldsymbol{x}_{\neq 1} + (D_1 + E_1) \int_{X_{\neq 1}} \int_I |f_{\boldsymbol{x}_{\neq 1}}(x_1)|\, dx_1 d\boldsymbol{x}_{\neq 1} \\
&= \frac{2}{\kappa_1} \mathrm{Var}_X^1(f) + (D_1 + E_1) \int_X |f|\, dm\ .
\end{aligned}
$$

Here we used the Lasota–Yorke inequality (29) for 1D maps from Proposition 2.1. Hence,

$$
\mathrm{Var}_X^1(P_{T_0}f) = \mathrm{Var}_X^1(P_{S_2}(P_{S_1}f)) \leq \frac{2}{\kappa_1} \mathrm{Var}_X^1(P_{S_1}f) + (D_1 + E_1) \int_X |P_{S_1}f|\, dm\ .
$$

As $\int_X |P_{S_1}f|\, dm \leq \int_X |f|\, dm$ (compare (10)), the proof of Lemma 3.2 will be finished by showing that $\mathrm{Var}_X^1(P_{S_1}f) \leq \mathrm{Var}_X^1(f)$: let $\varphi \in \mathcal{T}_X$. For $\boldsymbol{x}_{\neq 1} \in X_{\neq 1}$ let $\psi_{\boldsymbol{x}_{\neq 1}}(x_1) = \varphi(S_1(x_1, \boldsymbol{x}_{\neq 1}))$. Then $\psi_{\boldsymbol{x}_{\neq 1}} \in \mathcal{T}_I$, and

---

straightforward analysis.) Now, if one passes to coupled systems, things change. Although $T_\epsilon = \Phi_\epsilon \circ T_0$ still possesses the rectangular domain property, this is no longer true for powers T_ϵ^ℓ, $\ell \geq 2$. In fact, already to make sure that, by passing from T_0 to T_ϵ, no new domains occur one needs the full strength of the regularity assumption (1). But with this assumption one can prove a geometrically much finer result [7]:

There are constants $\tilde{\epsilon} > 0$ and $c > 0$, *independent of the size of* L, such that for $|\epsilon| \leq \tilde{\epsilon}$ and for each domain Z_ϵ on which T_ϵ^M is C^2 and expanding, there is a diffeomorphism Ψ_{Z_ϵ} between Z_ϵ and the corresponding rectangular domain Z_0 of T_0^M which is C^2 close to the identity in the sense of a $(1, c)$-coupling, see (36). This allows to reduce variation estimates of functions $f\, 1_{Z_\epsilon}$ to variation estimates of functions $\tilde{f}\, 1_{Z_0}$, and the latter ones can be dealt with using [6, Lemma 3.1]. See also [24] for a more details.

$$\int_X P_{S_1} f(\boldsymbol{x}) \partial_1 \varphi(\boldsymbol{x})\, d\boldsymbol{x} = \int_X f(\boldsymbol{x}) \partial_1 \varphi(S_1(\boldsymbol{x}))\, d\boldsymbol{x}$$
$$= \int_{X_{\neq 1}} \int_I f_{\boldsymbol{x}_{\neq 1}}(x_1)\, \psi'_{\boldsymbol{x}_{\neq 1}}(x_1)\, dx_1 d\boldsymbol{x}_{\neq 1} \leq \int_{X_{\neq 1}} \mathrm{Var}_I(f_{\boldsymbol{x}_{\neq 1}})\, d\boldsymbol{x}_{\neq 1} = \mathrm{Var}^1_X(f)\ .$$

Now $\mathrm{Var}^1_X(P_{S_1} f) \leq \mathrm{Var}^1_X(f)$ follows from Proposition 3.1. □

Proof of Lemma 3.3. Let $f \in L^1_X$, $\varphi \in \mathcal{T}_X$, and $j \in \{1, \ldots, d\}$. Denote (just for this proof) the matrix $(D\Phi_\epsilon)^{-1}$ by (b_{ij}). Then

$$\begin{aligned}(\partial_j \varphi) \circ \Phi_\epsilon = (D(\varphi \circ \Phi_\epsilon)(D\Phi_\epsilon)^{-1})_j &= \sum_{i=1}^d \partial_i(\varphi \circ \Phi_\epsilon) \cdot b_{ij} \\ &= \sum_{i=1}^d \partial_i(\varphi \circ \Phi_\epsilon \cdot b_{ij}) - \sum_{i=1}^d \varphi \circ \Phi_\epsilon \cdot \partial_i b_{ij}\ .\end{aligned} \tag{58}$$

Let $\psi_{ij} = \varphi \circ \Phi_\epsilon \cdot b_{ij} = \varphi \circ \Phi_\epsilon \cdot ((D\Phi_\epsilon)^{-1})_{ij}$. As all functions ψ_{ij} are in $C^1(X)$, this implies

$$\begin{aligned}\int_X P_{\Phi_\epsilon} f(\boldsymbol{x})\, \partial_j \varphi(\boldsymbol{x})\, d\boldsymbol{x} &= \int_X f(\boldsymbol{x})\, \partial_j \varphi(\Phi_\epsilon \boldsymbol{x})\, d\boldsymbol{x} \\ &\leq \sum_{i=1}^d \sup_{\boldsymbol{x}} |\psi_{ij}(\boldsymbol{x})|\, \mathrm{Var}_X(f) + \sum_{i=1}^d \sup_{\boldsymbol{x}} |\partial_i b_{ij}(\boldsymbol{x})| \int_X |f|\, dm\ .\end{aligned} \tag{59}$$

The two suprema in this estimate are precisely controlled by our assumptions (36) on Φ_ϵ and their consequences (38): for $j = 1, \ldots, d$,

$$\sum_{i=1}^d \sup_{\boldsymbol{x}} |\psi_{ij}(\boldsymbol{x})| \leq \sum_{i=1}^d \sup_{\boldsymbol{x}} |((D\Phi_\epsilon)^{-1})_{ij}(\boldsymbol{x})| \leq \frac{1}{1 - 2a_1|\epsilon|}$$
$$\sum_{i=1}^d \sup_{\boldsymbol{x}} |\partial_i b_{ij}(\boldsymbol{x})| = \sum_{i=1}^d \sup_{\boldsymbol{x}} |\partial_i((D\Phi_\epsilon)^{-1})_{ij}(\boldsymbol{x})| \leq \frac{2a_2|\epsilon|}{(1 - 2a_1|\epsilon|)^2}\ .$$

As φ is an arbitrary test function in $\mathcal{T}_X$, this finishes the proof of Lemma 3.3. □

3.5 Existence of Absolutely Continuous Invariant Measures

Having derived inequality (55) one can proceed as in the one-dimensional case: as in Sect. 2.4 it follows that

$$\mathrm{Var}_X(P^n_{T_\epsilon} f) \leq 2C_1\, \alpha^n\, \mathrm{Var}_X(f) + 2C_2 \int_X |f|\, dm \tag{60}$$

for all $f \in L^1_X$ and all $n \in \mathbb{N}$ provided $|\epsilon| \leq \epsilon_1$. The constants are from (34), and the additional factor 2 accounts for the passage from $\epsilon = 0$ to $|\epsilon| \leq \epsilon_1$.

(Indeed, there is nothing special with the factor 2. Any factor strictly larger than 1 would do as well.)

In perfect analogy with the case of piecewise expanding maps of the interval Proposition 3.2 immediately implies the existence of at least one absolutely continuous invariant measure.

Theorem 3.1. *Let $T_\epsilon : X \to X$ be a coupled map on $X = I^L$ as described in Sect. 3.1. For each $|\epsilon| \leq \epsilon_1$ there exists a T_ϵ-invariant probability measure μ_ϵ which belongs to $BV(X)$.*

Proof. As $Var_X(1) = 2$ inequality (60) implies $\limsup_{n\to\infty} \mathrm{Var}_X(P^\ell_{T_\epsilon} 1) \leq 2C_2$. Let $h_n = \frac{1}{n}\sum_{k=0}^{n-1} P^k_{T_\epsilon} 1$. Then also $\limsup_{n\to\infty} \mathrm{Var}_X(P^k_{T_\epsilon} 1) \leq 2C_2$. As the space $BV(X)$ embeds compactly into L^1_X, see e.g. [13, Theorem 1.19] or [14, Corollary 5.3.4], it follows that $\{h_n\}$ has accumulation points in L^1 which belong to BV and are the density of an invariant measure. □

Although we will not use this observation explicitly, an argument of the same type will guarantee the existence of an invariant measure with marginal densities of bounded variation for the infinite coupled system in Chap. 4.

For sufficiently small $|\epsilon|$ the operator P_{T_ϵ} is again quasi-compact on the Banach space $BV(X)$, and one can show that it has a spectral gap if the single site map τ is mixing.[13] It is not possible, however, to obtain in this way a useful d-dependent control over the constants C_3 and ρ in the spectral gap estimate (35). To achieve this we will apply a more recent technique in Chap. 4. As a by-product we obtain the following d-dependent estimate on the mixing rate for uncoupled systems: for $f \in BV(X)$ with $\int_X f\,dm = 0$,

$$\int_X |P^n_{T_0} f|\,dm \leq (2 + C_2)C_3\,d\,\rho^n\ \mathrm{Var}_X(f) \tag{61}$$

with constants C_2, C_3, and ρ from (34) and Corollary 2.3. This is proved at the end of Sect. 4.5.

4 Infinite Systems over $L = \mathbb{Z}$

The first problem that comes to mind if one attempts to transfer the finite system theory from Chap. 3 to the case $L = \mathbb{Z}$ is certainly: what is a class of measures which can play the role that the absolutely continuous ones play in the finite-dimensional case? These are not the measures absolutely continuous w.r.t. the infinite product Lebesgue measure $m^{\mathbb{Z}}$ on the "infinite-dimensional unit cube" $X = I^{\mathbb{Z}}$. Just look at the uncoupled map T_0: If $\mu = hm$ is an invariant measure for the p.w.e. map τ, then its infinite product $\mu^{\mathbb{Z}}$ should be

[13] For the case when τ is a mixing tent map a proof is published in [25].

the measure to look at. But $\mu^{\mathbb{Z}}$ is absolutely continuous w.r.t. $m^{\mathbb{Z}}$ if and only if μ is the Lebesgue measure on I, i.e. if $h = 1$.[14]

So we are lead to look at measures whose finite-dimensional marginals are absolutely continuous. We will introduce various norms on spaces of such measures, derive Lasota–Yorke inequalities for the transfer operator of T_ϵ on such spaces and prove the existence of spectral gaps.

4.1 Classes of Measures and Distributions

Let us fix some notation:

- $X = I^{\mathbb{Z}}$, and $\mathcal{M}$ is the space of signed Borel measures on X.
- For $\Lambda' \subset \Lambda \subset \mathbb{Z}$, let $\pi_\Lambda : X \to I^\Lambda$ and $\pi^\Lambda_{\Lambda'} : I^\Lambda \to I^{\Lambda'}$ be the canonical coordinate projections.
- $|\Lambda|$ is the cardinality of Λ.
- For $\nu \in \mathcal{M}$ and $\Lambda \subset \mathbb{Z}$, let $\nu\pi_\Lambda^{-1}$ be the projection of ν to I^Λ, i.e. $\nu\pi_\Lambda^{-1}(U) = \nu(\pi_\Lambda^{-1}U)$ for measurable $U \subseteq I^\Lambda$.
- $\mathcal{I}$ is the family of all intervals $\Lambda = [a, b] \subset \mathbb{Z}$ including the empty set.
- $L^1_{\mathbb{Z}} = \{\nu \in \mathcal{M} : \nu\pi_\Lambda^{-1}$ is absolutely continuous w.r.t. m^Λ for all $\Lambda \in \mathcal{I}\}$.
- For $\nu \in L^1_{\mathbb{Z}}$ and $\Lambda \in \mathcal{I}$ we denote by ν_Λ the density of $\nu\pi_\Lambda^{-1}$ w.r.t. m^Λ. If $\Lambda = \emptyset$, then ν_Λ has the constant value $\nu(X)$.
- $BV_{\mathbb{Z}} = \{\nu \in L^1_{\mathbb{Z}} : \mathrm{Var}_{I^\Lambda}(\nu_\Lambda) < \infty$ for all $\Lambda \in \mathcal{I}\}$.

We define two scales of norms on $L^1_{\mathbb{Z}}$ and (subspaces of) $BV_{\mathbb{Z}}$.

Definition 4.1. *For $0 < \theta \le 1$ and $\nu \in L^1_{\mathbb{Z}}$ let*

$$|\nu|_\theta = \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda|} \int |\nu_\Lambda|\, dm \tag{62}$$

$$\|\nu\|_\theta = \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda|}\, \mathrm{Var}(\nu_\Lambda)\ . \tag{63}$$

(Here $\int |\nu_\Lambda|\, dm$ and $\mathrm{Var}(\nu_\Lambda)$ are shorthand notations for $\int_{I^\Lambda} |\nu_\Lambda|\, dm^\Lambda$ and $\mathrm{Var}_{I^\Lambda}(\nu_\Lambda)$, respectively.) Observe that $|\nu|_\theta \le \frac{1}{2}\|\nu\|_\theta$ in view of (44).

L^1_θ and BV_θ are now defined to be the completions of $L^1_{\mathbb{Z}}$ and of the space $\{\nu \in BV_{\mathbb{Z}} : \|\nu\|_\theta < \infty\}$, w.r.t. the norms[15] $|.|_\theta$ and $\|.\|_\theta$, respectively.

[14] For a proof of this let $\psi : I \to \mathbb{R}$ be any bounded measurable function. By the law of large numbers $\frac{1}{n}\sum_{k=0}^{n-1} \psi(x_k)$ converges to $\int_I \psi\, dm$ for $m^{\mathbb{Z}}$-a.e. $\boldsymbol{x}$ and to $\int_I \psi\, d\mu$ for $\mu^{\mathbb{Z}}$-a.e. $\boldsymbol{x}$. It follows that $\mu^{\mathbb{Z}}$ is absolutely continuous w.r.t. $m^{\mathbb{Z}}$ if and only if these two integrals coincide for any such function ψ, i.e. if $\mu = m$. We note for later use that the same argument applies to any two stationary product measures on $I^{\mathbb{Z}}$. In particular, two such measures are singular to each other if they are not identical.

[15] $|.|_\theta$ and $\|.\|_\theta$ are obviously seminorms. To see that $|.|_\theta$ is indeed a norm, suppose that $|\nu|_\theta = 0$ for some $\nu \in L^1_{\mathbb{Z}}$. Then $\nu_\Lambda = 0$ for all $\Lambda \in \mathcal{I}$ so that $\nu(\varphi) = 0$ for each $\varphi \in C(X)$ which depends on only finitely many coordinates. As the space of these functions is dense in $C(X)$, this means that $\nu = 0$ as a signed measure. As $|\nu|_\theta \le \|\nu\|_\theta$ by (44), also $\|.\|_\theta$ is a norm.

Example 4.1. Let f be a probability density on I and consider the infinite stationary product measure ν with one-dimensional factors fm. Observe that $\mathrm{Var}_{I^\Lambda}(f) \geq 2 = \mathrm{Var}_{I^\Lambda}(1)$ for all Λ by (44). Hence $|\nu|_\theta = 1$ and $\|\nu\|_\theta = \mathrm{Var}_I(f)$ for all θ. In particular, $\|m^{\mathbb{Z}}\|_\theta = 2$ for all θ.

In the following lemma we collect a few simple observations.

Lemma 4.1. *(a)* $\mathrm{Var}_{I^{\Lambda'}}(\nu_{\Lambda'}) \leq \mathrm{Var}_{I^\Lambda}(\nu_\Lambda)$ *for* $\nu \in L^1_{\mathbb{Z}}$ *and* $\Lambda' \subset \Lambda \in \mathcal{I}$.
(b) $|\nu|_{\theta=1} = \lim_{n\to\infty} \int |\nu_{[-n,n]}|\, dm$ *and* $\|\nu\|_{\theta=1} = \lim_{n\to\infty} \mathrm{Var}(\nu_{[-n,n]})$.
(c) $|\nu|_\theta \leq |\nu|_{\theta=1}$ *and* $\|\nu\|_\theta \leq \|\nu\|_{\theta=1}$ *for all* $\theta \in (0,1]$ *and all* $\nu \in L^1_{\mathbb{Z}}$.

Proof. (a) Just observe that if $\varphi \in \mathcal{T}_{I^{\Lambda'}}$, then $\varphi \circ \pi^\Lambda_{\Lambda'} \in \mathcal{T}_{I^\Lambda}$, and for each $i \in \Lambda'$

$$\int \nu_{\Lambda'}\, \partial_i \varphi\, dm^{\Lambda'} = \int (\nu_{\Lambda'} \circ \pi^\Lambda_{\Lambda'})\, \partial_i(\varphi \circ \pi^\Lambda_{\Lambda'})\, dm^\Lambda = \int \nu_\Lambda\, \partial_i(\varphi \circ \pi^\Lambda_{\Lambda'})\, dm^\Lambda .$$

Now (b) follows from (a), and (c) is a direct consequence of the definitions. □

As we are going to describe the quantitative dynamical properties of coupled systems in terms of properties of transfer operators acting on the spaces L^1_θ and BV_θ, it is worth to spend some effort to give more concrete models of these spaces which are defined rather abstractly as completions.

Remark 4.1. $L^1_{\theta=1} = L^1_{\mathbb{Z}}$

It is a little exercise in measure theory to see that, for $\nu \in L^1_{\mathbb{Z}}$, $|\nu|_{\theta=1} = \sup_{\Lambda\in\mathcal{I}} \int |\nu_\Lambda|\, dm$ coincides with the total variation norm $|\nu|_1$ of the signed measure ν.[16] Hence $L^1_{\theta=1}$ is just the closed subspace $L^1_{\mathbb{Z}}$ of $(\mathcal{M}, |.|_1)$.

Remark 4.2. $BV_{\theta=1} = \{\nu \in BV_{\mathbb{Z}} : \sup_{\Lambda\in\mathcal{I}} \mathrm{Var}(\nu_\Lambda) < \infty\}$.

This means that in the definition of $BV_{\theta=1}$ the completion was not necessary. The completeness of the space on the right hand side for the norm $\|.\|_{\theta=1}$ follows easily from the completeness of the spaces $(BV(I^\Lambda), \mathrm{Var}_{I^\Lambda})$.[17]

Hence, for $\theta = 1$, we have defined spaces of signed measures with additional regularity properties, and we will show that our coupled T_ϵ always has a unique invariant measure that belongs to $BV_{\theta=1}$. But neither $\|.\|_{\theta=1}$ nor $|.|_{\theta=1}$ is suited to describe the convergence of measures $P^n_{T_\epsilon}\nu$ to the invariant measure – not even for $\epsilon = 0$ – as the following example shows.

[16] Each signed measure ν has a unique decomposition $\nu = \nu^+ - \nu^-$ as a difference of two finite positive measures which are singular to each other. The total variation norm of ν is defined as $|\nu|_1 = \nu^+(X) + \nu^-(X)$. For a proof that $|\nu|_1 = |\nu|_{\theta=1}$ see e.g. [6, Lemma 2.4].

[17] Let (ν_n) be a Cauchy sequence in $BV_{\theta=1}$. As $|.|_{\theta=1} \leq \|.\|_{\theta=1}$, it is a fortiori a Cauchy sequence in $L^1_{\theta=1}$. Let $\nu = L^1_{\theta=1}\text{-}\lim_{n\to\infty} \nu_n$, and let $\varepsilon_n = \sup_{k\geq n} \|\nu_k - \nu_n\|_\theta$. Let $\Lambda \in \mathcal{I}$, $\varphi \in \mathcal{T}_{I^\Lambda}$, and $i \in \Lambda$. Then $\int (\nu - \nu_n)_\Lambda\, \partial_i\varphi\, dm \leq \lim_k \int (\nu - \nu_k)_\Lambda\, \partial_i\varphi\, dm + \sup_{k\geq n} \int (\nu_k - \nu_n)_\Lambda\, \partial_i\varphi\, dm \leq \varepsilon_n$ so that $\|\nu - \nu_n\|_{\theta=1} = \sup_{\Lambda\in\mathcal{I}} \mathrm{Var}((\nu - \nu_n)_\Lambda) \leq \varepsilon_n \to 0$.

Example 4.2. Let h be the unique invariant probability density of the local p.w.e. map τ, and let μ be the infinite product measure $(hm)^{\mathbb{Z}}$. Recall from Footnote 14 at the beginning of this chapter that any two stationary product measures are singular to each other if they are not identical. Hence, as long as $P_\tau^n 1 \neq h$, the measures $m^{\mathbb{Z}} T_0^{-n}$ and μ are mutually singular, which means that $\|m^{\mathbb{Z}} T_0^{-n} - \mu\|_{\theta=1} \geq |m^{\mathbb{Z}} T_0^{-n} - \mu|_1 = 2$.[18]

We conclude that the "$\theta = 1$"-norms are unsuited to describe the convergence of $m^{\mathbb{Z}} T_0^{-n}$ to the invariant limit measure μ. For this purpose we will use the norms $|.|_\theta$ and $\|.\|_\theta$ with $0 < \theta < 1$. As long as this is all we want to do with them we need not bother that the spaces L_θ^1 and BV_θ for $0 < \theta < 1$ are no longer spaces of signed measures. Note, however, that the positive elements in L_θ^1 are finite measures on X.

Example 4.3. Here is an example of a Cauchy sequence in L_θ^1 whose limit cannot be interpreted as a finite signed measure on X. Let $f : I \to \mathbb{R}$, $\int f\, dm = 0$, $\int |f|\, dm = 1$. Denote by ν_k the infinite product signed measure with one-dimensional factor measures fm at sites $i = 1, \ldots, k$ and m at all other sites. Then $|\nu_k|_1 = 1$ and $\nu_k \pi_\Lambda^{-1} = 0$ if $\Lambda \cap \{1, \ldots, k\} \neq \emptyset$.

For $r \in (1, \theta^{-1})$ let $\mu_n = \sum_{k=1}^n \alpha_k \nu_k$ with coefficients $|\alpha_k| \leq \frac{r-1}{r} r^k$. Then $\mu_n \pi_{\{1,\ldots,d\}}^{-1} = \sum_{k=1}^{d \wedge n} \alpha_k \nu_k$ so that, for each $\Lambda \in \mathcal{I}$ which contains $\{1, \ldots, d\}$, $|\mu_n \pi_\Lambda^{-1}|_1 \leq \sum_{k=1}^{d \wedge n} \alpha_k \leq r^d$. Therefore $|\mu_n|_\theta = \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda|} |\mu_n \pi_\Lambda^{-1}|_1 \leq 1$. Similarly one shows that, if $l > n$ and $\Lambda \supseteq \{1, \ldots, d\}$, then $\theta^{|\Lambda|} |(\mu_n - \mu_l)\pi_\Lambda^{-1}|_1 \leq (\theta r)^n$, whence $|\mu_m - \mu_l|_\theta \leq (\theta r)^n$, and $(\mu_n)_n$ is indeed a Cauchy sequence in L_θ^1. Now let $p \in \mathbb{N}$ such that $r^p > \frac{2r}{r-1}$, and let $\alpha_k = r^k$ if k is an integer multiple of p and $\alpha_k = 0$ otherwise. Then similar estimates show that the total variation norm $|\mu_n|_1$ is at least $\frac{r-1}{r} r^n$, so no "reasonable" limit of the sequence $(\mu_n)_n$ can be a finite signed measure.

4.2 The Infinite Coupled Map

From now on we consider exclusively the case $L = \mathbb{Z}$. Our basic assumptions on τ and Φ_ϵ are the same as those made in Sect. 3.1, namely

- $X = I^{\mathbb{Z}}$ is the state space of the system.
- $\tau : I \to I$ is a p.w.e. map as defined in Sect. 2.1.
- $T_0 : X \to X$ is the infinite product of the map τ: $(T_0 \boldsymbol{x})_i = \tau(x_i)$ $(i \in \mathbb{Z})$.
- $\Phi_\epsilon : X \to X$ $(|\epsilon| < \epsilon_0)$ is a family of coupling maps ϵ-close to the identity in a C^2 sense made precise by the notion of (a_1, a_2)-coupling in (36).
- $T_\epsilon = \Phi_\epsilon \circ T_0 : X \to X$ are the maps describing the coupled systems.

We assume additionally

- The Φ_ϵ have *finite coupling range* $w > 0$, i.e. $\partial_j \Phi_{\epsilon,i} = 0$ whenever $|i-j| > w$. So $A'_{ij} = A''_{ij} = 0$ when $|i - j| > w$ for the matrices A', A'' introduced in (36).

[18] If $\nu \in \mathcal{M}$, then $\nu T_\epsilon^{-1}(U) = \nu(T_\epsilon^{-1} U)$, so $\nu T_\epsilon^{-1} \in \mathcal{M}$.

For a proof of the existence of a T_ϵ-invariant measures in $BV_{\theta=1}$ (Sect. 4.4) one knows how to work around this additional assumption, see [7], but we need a finite coupling range for the proof of a spectral gap in Sect. 4.7. So we make our life easier using this assumption throughout this chapter. The diffusive nearest neighbor coupling (3), for example, has coupling range $w = 1$.

In order to reduce estimates on transformed measures $\nu T_\epsilon^{-\ell}$ on the infinite system to estimates on their finite-dimensional marginal densities $(\nu T_\epsilon^{-\ell})_\Lambda$, we must relate projections onto different Λ to each other: there are no isolated finite subsystems in infinite coupled systems. To be more specific, for $\Lambda = [a, b] \in \mathcal{I}$ and $\ell \in \mathbb{N}$ let $\Lambda(\ell) = [a - \ell w, b + \ell w]$. Denote $\iota_\Lambda : I^\Lambda \to X$ the map $(\iota_\Lambda(\boldsymbol{x}))_i = x_i$ if $i \in \Lambda$ and $(\iota_\Lambda(\boldsymbol{x}))_i = 0$ otherwise[19], and let

$$T_{\epsilon,\Lambda} = \Phi_{\epsilon,\Lambda} \circ T_{0,\Lambda} \quad \text{where } T_{0,\Lambda} = \pi_\Lambda \circ T_0 \circ \iota_\Lambda,\ \Phi_{\epsilon,\Lambda} = \pi_\Lambda \circ \Phi_\epsilon \circ \iota_\Lambda\ . \tag{64}$$

Observe that $T_{0,\Lambda}$ is just the uncoupled map on I^Λ and that $\Phi_{\epsilon,\Lambda}$ is a (a_1, a_2)-coupling on the finite-dimensional space I^Λ with the same constants a_1, a_2 as above. Hence all considerations of Chap. 3, in particular the Lasota–Yorke inequality (55), apply to $T_{\epsilon,\Lambda}$. The important link between T_ϵ and $T_{\epsilon,\Lambda}$ is given by

$$\pi_\Lambda \circ T_\epsilon^\ell = \pi_\Lambda^{\Lambda(\ell)} \circ T_{\epsilon,\Lambda(\ell)}^\ell \circ \pi_{\Lambda(\ell)} \quad \text{for all } \Lambda \in \mathcal{I} \text{ and } \ell \in \mathbb{N}. \tag{65}$$

This follows immediately from the finite coupling range property of Φ_ϵ: no influence of a coordinate x_i with $i \in L \setminus \Lambda(\ell)$ can propagate to Λ within ℓ steps of time.

4.3 The Transfer Operator and a Lasota–Yorke Inequality

We are going to define transfer operators P_{T_ϵ} on L^1_θ in terms of the action of T_ϵ on the densities ν_Λ of the finite-dimensional projections of $\nu \in L^1_\mathbb{Z}$. Observe that, for $\nu \in L^1_\mathbb{Z}$ and $\varphi : I^\Lambda \to \mathbb{R}$, (65) implies

$$\begin{aligned}\int_X \varphi \circ \pi_\Lambda \, d(\nu T_\epsilon^{-\ell}) &= \int_X \varphi \circ \pi_\Lambda \circ T_\epsilon^\ell \, d\nu \\ &= \int_{I^{\Lambda(\ell)}} (\varphi \circ \pi_\Lambda^{\Lambda(\ell)}) \circ T_{\epsilon,\Lambda(\ell)}^\ell \cdot \nu_{\Lambda(\ell)} \, dm^{\Lambda(\ell)}\ .\end{aligned} \tag{66}$$

This means that

$$(\nu T_\epsilon^{-\ell})_\Lambda = \left((P_{T_{\epsilon,\Lambda(\ell)}^\ell} \nu_{\Lambda(\ell)}) m^{\Lambda(\ell)}\right)_\Lambda =: (P_{T_{\epsilon,\Lambda(\ell)}^\ell} \nu_{\Lambda(\ell)})_\Lambda \tag{67}$$

where we take the last term just as a short hand for the middle one. Hence,

$$\begin{aligned}|\nu T_\epsilon^{-\ell}|_\theta &= \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda|} \int |(\nu T_\epsilon^{-\ell})_\Lambda| \, dm \le \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda|} \int |P_{T_{\epsilon,\Lambda(\ell)}^\ell} \nu_{\Lambda(\ell)}| \, dm \\ &\le \theta^{-2\ell w} \sup_{\Lambda \in \mathcal{I}} \theta^{|\Lambda(\ell)|} \int |\nu_{\Lambda(\ell)}| \, dm \le \theta^{-2\ell w} \, |\nu|_\theta\ .\end{aligned} \tag{68}$$

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[19] Any other measurable section from I^Λ to X would do as well.

In particular, $\nu \mapsto \nu T_\epsilon^{-1}$ is a linear operator on $L^1_{\mathbb{Z}}$ bounded w.r.t. the norm $|.|_\theta$. Hence it extends to a bounded linear operator P_{T_ϵ} on L^1_θ. That P_{T_ϵ} is also a bounded linear operator on BV_θ is an immediate consequence of the following Lasota–Yorke type inequality.

Proposition 4.1 (Lasota–Yorke inequality). *Let $\ell \in \mathbb{N}$. For each $\alpha_\ell > \frac{2}{\kappa_\ell}$ and each $C_{4,\ell} > D_\ell + E_\ell$ there are $\epsilon_1 \in (0, \epsilon_0]$ and $\theta_1 \in (0,1)$ such that for $|\epsilon| \le \epsilon_1$, $\theta \in [\theta_1, 1]$, and $\nu \in BV_\theta$,*

$$|P^\ell_{T_\epsilon}\nu|_\theta \le \theta^{-2w\ell}|\nu|_\theta \tag{69}$$

$$\|P^\ell_{T_\epsilon}\nu\|_\theta \le \alpha_\ell \,\|\nu\|_\theta + C_{4,\ell}\,|\nu|_\theta \,. \tag{70}$$

Given ℓ, α_ℓ and $C_{4,\ell}$, the choice of θ_1 depends only on the coupling range w, that of ϵ_1 only on the constants a_1, a_2 which qualify Φ_ϵ as a (a_1, a_2)-coupling, see (36).

Observe the difference between (69) and the corresponding inequality (54) for finite systems, where P_{T_ϵ} is a contraction w.r.t. the weak norm.

Proof. Equation (69) is just a restatement of (68). We turn to (70). Let $\tilde\alpha_\ell = (\frac{2}{\kappa_\ell}\alpha_\ell)^{1/2}$ and $\tilde C_{4,\ell} = ((D_\ell + E_\ell)C_{4,\ell})^{1/2}$. In view of (67), Lemma 4.1a and the finite-dimensional Lasota–Yorke inequality (55), we have for each $\nu \in L^1_{\mathbb{Z}}$

$$\begin{aligned}
\|P^\ell_{T_\epsilon}\nu\|_\theta &= \|\nu T_\epsilon^{-\ell}\|_\theta = \sup_{\Lambda\in\mathcal{I}} \theta^{|\Lambda|}\operatorname{Var}((\nu T_\epsilon^{-\ell})_\Lambda) \\
&= \sup_{\Lambda\in\mathcal{I}} \theta^{|\Lambda|}\operatorname{Var}\left((P_{T^\ell_{\epsilon,\Lambda(\ell)}}\nu_{\Lambda(\ell)})_\Lambda\right) \\
&\le \theta^{-2w\ell}\sup_{\Lambda\in\mathcal{I}} \theta^{|\Lambda(\ell)|}\left(\tilde\alpha_\ell \operatorname{Var}(\nu_{\Lambda(\ell)}) + \tilde C_{4,\ell}\int |\nu_{\Lambda(\ell)}|\,dm\right) \\
&\le \theta^{-2w\ell}\tilde\alpha_\ell\,\|\nu\|_\theta + \theta^{-2w\ell}\tilde C_{4,\ell}\,|\nu|_\theta
\end{aligned}$$

for $|\epsilon| \le \epsilon_1$, where the choice of ϵ_1 depends on $\ell, \alpha_\ell, C_{4,\ell}, a_1$, and a_2, see Proposition 3.2. Now choose θ_1 such that $\theta_1^{-2w\ell} \le \min\{\alpha_\ell/\tilde\alpha_\ell, C_{4,\ell}/\tilde C_{4,\ell}\}$. □

As in Sect. 2.4 it follows that there are constants $C_1', C_2' > 0$ such that

$$\|P^n_{T_\epsilon}\nu\|_\theta \le C_1'\,\alpha^n\,\|\nu\|_\theta + C_2'\theta^{-2wn}\,|\nu|_\theta \le \left(C_1' + \frac{1}{2}C_2'\theta^{-2wn}\right)\|\nu\|_\theta \tag{71}$$

for all $\nu \in BV_\theta$ and all $n \in \mathbb{N}$ provided $|\epsilon| \le \epsilon_1$ and $\theta \in [\theta_1, 1]$. The constant α can be any number in $((\frac{2}{\kappa_M})^{1/M}, 1)$. C_1', C_2', ϵ_1, and θ_1 will then be chosen as indicated above and at the end of Sect. 2.4. (Recall that $\kappa_M > 2$ by assumption (1).)

4.4 Existence of Invariant Measures with Absolutely Continuous Finite-Dimensional Marginals

In this section we prove the existence of (at least) one probability measure in $BV_{\theta=1}$ which is invariant under T_ϵ.

Theorem 4.1. *Let $T_\epsilon : X \to X$ be a coupled map on $X = I^{\mathbb{Z}}$ as described in Sect. 4.2. For each $|\epsilon| \leq \epsilon_1$ there exists a T_ϵ-invariant probability measure μ_ϵ which belongs to $BV_{\theta=1}$. Indeed, $\|\mu_\epsilon\|_{\theta=1} \leq C_2'$.*

Proof. Let $\nu = m^{\mathbb{Z}}$. As $|\nu|_{\theta=1} = 1$ and $\|\nu\|_{\theta=1} = 2$ (see Example 4.1), we have $\limsup_{n\to\infty} \|P_{T_\epsilon}^\ell \nu\|_{\theta=1} \leq C_2'$ by (71). Let $\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} P_{T_\epsilon}^k \nu$. Then also $\limsup_{n\to\infty} \|\nu_n\|_{\theta=1} \leq C_2'$. This implies that $\limsup_{n\to\infty} \mathrm{Var}_{I^\Lambda}((\nu_n)_\Lambda) \leq C_2'$ for each $\Lambda \in \mathcal{I}$. As BV_{I^Λ} embeds compactly into $L^1_{I^\Lambda}$ (see Sect. 3.4), there is a subsequence of $((\nu_n)_\Lambda)_{n>0}$ which converges in $L^1_{I^\Lambda}$ to some probability density $h_\Lambda \in L^1_{I^\Lambda}$. By a diagonal procedure one even finds such a subsequence for which this convergence holds for all $\Lambda \in \mathcal{I}$. Observe that the family of densities h_Λ, $\Lambda \in \mathcal{I}$, is consistent in the sense that for any $\Lambda' \subset \Lambda$ holds $(h_\Lambda m^\Lambda)(\pi^\Lambda_{\Lambda'})^{-1} = h_{\Lambda'} m^{\Lambda'}$, because all the $(\nu_n)_\Lambda$ have the same property. Hence, by Kolmogorov's theorem, there is a probability measure $\mu_\epsilon \in \mathcal{M}$ such that $\mu_\epsilon \pi_\Lambda^{-1} = h_\Lambda m^\Lambda$ for all $\Lambda \in \mathcal{I}$. As $\lim_{n\to\infty} \int |(\mu_\epsilon)_\Lambda - (\nu_n)_\Lambda|\, dm^\Lambda = 0$, the estimate $\mathrm{Var}_{I^\Lambda}((\mu_\epsilon)_\Lambda) \leq \liminf_{n\to\infty} \mathrm{Var}_{I^\Lambda}((\nu_n)_\Lambda) \leq C_2'$ follows from (48). Hence $\|\mu_\epsilon\|_{\theta=1} \leq C_2'$.

It remains to show that μ_ϵ is T_ϵ-invariant:

$$
\begin{aligned}
|\mu_\epsilon T_\epsilon^{-1} - \mu_\epsilon|_{\theta=1} &= \lim_{n\to\infty} |\nu_n T_\epsilon^{-1} - \nu_n|_{\theta=1} \\
&\leq \lim_{n\to\infty} \frac{1}{n}(|\nu T_\epsilon^{-n}|_{\theta=1} + |\nu|_{\theta=1}) = 0\,.
\end{aligned}
$$

□

Corollary 4.1 (Finite entropy density). *The T_ϵ-invariant measures μ_ϵ from Theorem 4.1 have an entropy density bounded by $\ln(C_2'/2)$. Indeed, for each $\nu \in BV_{\theta=1}$ and each $\Lambda \in \mathcal{I}$ we have $\int \nu_\Lambda \ln \nu_\Lambda\, dm \leq |\Lambda|\, \ln\left(\frac{1}{2}\|\nu\|_{\theta=1}\right)$.*

Proof. Let $\Lambda = [a,b]$, $\Lambda' = [a, b-1]$. Then

$$
\begin{aligned}
\int_{I^\Lambda} \nu_\Lambda \ln \nu_\Lambda\, dm &= \int_{I^{\Lambda'}} \nu_{\Lambda'} \ln \nu_{\Lambda'} dm + \int_{I^\Lambda} \nu_\Lambda \ln \frac{\nu_\Lambda}{\nu_{\Lambda'}}\, dm \\
&\leq \int_{I^{\Lambda'}} \nu_{\Lambda'} \ln \nu_{\Lambda'} dm + \ln \int_{I^\Lambda} \nu_\Lambda \frac{\nu_\Lambda}{\nu_{\Lambda'}}\, dm
\end{aligned}
\tag{72}
$$

by Jensen's inequality, and

$$
\begin{aligned}
\int_{I^\Lambda} \nu_\Lambda \frac{\nu_\Lambda}{\nu_{\Lambda'}}\, dm &\leq \int_{I^{\Lambda'}} \sup_{x_b} \big((\nu_\Lambda)_{\boldsymbol{x}_{\neq b}}(x_b)\big)\, d\boldsymbol{x}_{\neq b} \\
&\leq \frac{1}{2} \int_{I^{\Lambda'}} \mathrm{Var}_I \left((\nu_\Lambda)_{\boldsymbol{x}_{\neq b}}\right) d\boldsymbol{x}_{\neq b} \leq \frac{1}{2} \mathrm{Var}_{I^\Lambda}(\nu_\Lambda) \leq \frac{1}{2}\|\nu\|_{\theta=1}\,.
\end{aligned}
\tag{73}
$$

Applying the same estimate to smaller and smaller boxes one arrives at

$$
\int_{I^\Lambda} \nu_\Lambda \ln \nu_\Lambda\, dm \leq |\Lambda|\, \ln\left(\frac{1}{2}\|\nu\|_{\theta=1}\right)\,. \tag{74}
$$

4.5 Uniqueness and a Spectral Gap – the Uncoupled Case

The compact embedding of BV_{I^Λ} into $L^1_{I^\Lambda}$ for each $\Lambda \in \mathcal{I}$ which we used in the last section does by no means imply a compact embedding of BV_θ into L^1_θ, which would be needed in order to prove the quasi-compactness of P_{T_0} on BV_θ. In fact, we have already seen in Example 4.2 that one cannot expect this operator to be quasi-compact on $BV_{\theta=1}$ – even when $\epsilon = 0$, and one can argue (differently) that quasi-compactness fails also for $\theta \in (0,1)$.

So we will give a more direct proof that $\lim_{n\to\infty} \|P^n_{T_\epsilon}\nu - \mu_\epsilon\|_\theta = 0$ (with exponential speed) whenever $\theta \in [\theta_1, 1)$ and ν is a probability measure in BV_θ. Observe that this implies in particular the uniqueness of the invariant μ_ϵ, even in BV_θ. (See the proof of Corollary 4.4 for details of this argument.) We start with the uncoupled case $\epsilon = 0$ in this section. The proof is made up such that it can be extended to the coupled case in the next section.

Theorem 4.2 (Spectral gap). *Let $\nu \in L^1_\theta$, and assume that $\nu_\emptyset = \nu(X) = 0$. Then*

$$\|P^n_{T_0}\nu\|_\theta \le \frac{C_6}{1-\theta}\,\hat\rho^n\,\|\nu\|_\theta \tag{75}$$

where $C_6 = C'_1(C'_1 + \frac{1}{2}C'_2) + 2C'_2C_3$ and $\hat\rho = \max\{\alpha, \rho\}^{1/2} \in (0,1)$ are constants derived from (35) and (71).

The proof relies on the following lemma, a variant of which was used for the first time in [8].

Lemma 4.2. *Let $\Lambda', \Lambda \in \mathcal{I}$, $\Lambda = [a,b]$, $\Lambda' = [a, b-1]$, and let $S : I^{\mathbb{Z}\setminus\{b\}} \to I^{\mathbb{Z}\setminus\{b\}}$ be measurable. Suppose that there is some $\tilde\Lambda \in \mathcal{I}$ such that the maps $(S(\boldsymbol{x}_{\neq b}))_j$, $j \in \Lambda'$, depend only on coordinates $i \in \tilde\Lambda$. Consider the map $\tau \times S : I^{\mathbb{Z}} \to I^{\mathbb{Z}}$, $\boldsymbol{x} \mapsto (\tau(x_b), S(\boldsymbol{x}_{\neq b}))$. Then*

$$\int_{I^\Lambda} |(P^\ell_{\tau\times S}\nu)_\Lambda|\,dm \le (2 + C_2)\,C_3\,\rho^\ell\,\mathrm{Var}^b_{I^{\tilde\Lambda}}(\nu_{\tilde\Lambda}) + \int_{I^{\Lambda'}} |(P^\ell_{\tau\times S}\nu)_{\Lambda'}|\,dm \tag{76}$$

with constants C_2, C_3, and ρ from (34) and Corollary 2.3.

Proof. Let $\Lambda \in \mathcal{I}$. As $\int_{I^\Lambda} |(P^\ell_{\tau\times S}\nu)_\Lambda|\,dm = \sup_\psi \int_X \psi\,d(P^\ell_{\tau\times S}\nu)$ where the supremum extends over all continuous $\psi : X \to \mathbb{R}$ that depend only on coordinates x_i with $i \in \Lambda$ and satisfy $|\psi| \le 1$, we start by estimating the integrals under the supremum. Given such a test function ψ, let

$$\Psi(\boldsymbol{x}) = \int_0^{x_b} \psi(\tau^\ell(\xi), S^\ell(\boldsymbol{x}_{\neq b}))\,d\xi - x_b \int_0^1 h(\xi)\psi(\tau^\ell(\xi), S^\ell(\boldsymbol{x}_{\neq b}))\,d\xi \tag{77}$$

where h is the unique invariant density of the p.w.e. map τ, see Sect. 2.5. Then

$$\partial_b\Psi(\boldsymbol{x}) = \psi \circ (\tau\times S)^\ell(\boldsymbol{x}) - \bar\psi \circ (\tau\times S)^\ell(\boldsymbol{x}) \tag{78}$$

where $\bar{\psi}(\boldsymbol{x}) = \int_0^1 h(\xi)\psi(\xi, \boldsymbol{x}_{\neq b})\, d\xi$. Observe that $\bar{\psi}$ depends only on coordinates x_i with $i \in \Lambda'$. In particular $\bar{\psi}$ does in fact not depend on x_b. Furthermore, Ψ depends only on x_b and on coordinates x_i with $i \in \tilde{\Lambda}$. Hence

$$\begin{aligned}
\int_X \psi \, d(P^\ell_{\tau\times S}\nu) &= \int_X \psi \circ (\tau \times S)^\ell \, d\nu = \int_X \partial_b \Psi \, d\nu + \int_X \bar{\psi} \circ (\tau \times S)^\ell \, d\nu \\
&= \int_{I^{\tilde{\Lambda}}} \partial_b \Psi \, \nu_{\tilde{\Lambda}} \, dm + \int_{I^{\Lambda'}} \bar{\psi} \, (P^\ell_{\tau\times S}\nu)_{\Lambda'} \, dm \\
&\leq \sup_{\boldsymbol{x}} |\Psi(\boldsymbol{x})| \, \mathrm{Var}^b_{I^{\tilde{\Lambda}}}(\nu_{\tilde{\Lambda}}) + \int_{I^{\Lambda'}} |(P^\ell_{\tau\times S}\nu)_{\Lambda'}| \, dm
\end{aligned} \tag{79}$$

and we must estimate $\sup_{\boldsymbol{x}} |\Psi(\boldsymbol{x})|$:

$$\Psi(\boldsymbol{x}) = \int_0^1 P^\ell_\tau(1_{[0,x_b]} - x_b\, h)(\xi)\, \psi(\xi, S^\ell(\boldsymbol{x}_{\neq b}))\, d\xi \leq C_3 \rho^\ell \|1_{[0,x_b]} - x_b\, h\|_{BV} \tag{80}$$

by inequality (35). As $\|h\|_{BV} \leq C_2$, (76) follows now from (79) and (80). □

Proof of Theorem 4.2. Let $\Lambda = \tilde{\Lambda} = [a,b] \in \mathcal{I}$, $\Lambda' = [a, b-1]$, and denote by S the uncoupled map on $I^{\mathbb{Z}\setminus\{b\}}$. Then $T_0 = \tau \times S$, and Lemma 4.2 implies

$$\int_{I^\Lambda} |(P^\ell_{T_0}\nu)_\Lambda| \, dm \leq (2 + C_2)\, C_3\, \rho^\ell \, \mathrm{Var}_{I^\Lambda}(\nu_\Lambda) + \int_{I^{\Lambda'}} |(P^\ell_{T_0}\nu)_{\Lambda'}| \, dm \,. \tag{81}$$

Multiplying this inequality by θ^Λ and taking suprema over all $\Lambda \in \mathcal{I}$ this yields

$$|P^\ell_{T_0}\nu|_\theta \leq (2 + C_2)\, C_3\, \rho^\ell \|\nu\|_\theta + \theta\, |P^\ell_{T_0}\nu|_\theta \tag{82}$$

where one has to keep in mind that $(P^\ell_{T_0}\nu)_\emptyset = \nu(T_0^{-\ell}X) = \nu(X) = 0$. Hence

$$|P^\ell_{T_0}\nu|_\theta \leq \frac{(2 + C_2)\, C_3}{1 - \theta}\, \rho^\ell \, \|\nu\|_\theta \,. \tag{83}$$

We combine this with the Lasota–Yorke type estimate (71) for the special case $w = 0$: for all $k, \ell \in \mathbb{N}$,

$$\begin{aligned}
\|P^{k+\ell}_{T_0}\nu\|_\theta &\leq C_1' \alpha^k \|P^\ell_{T_0}\nu\|_\theta + C_2' |P^\ell_{T_0}\nu|_\theta \\
&\leq C_1' \alpha^k (C_1' + \frac{1}{2} C_2') \|\nu\|_\theta + C_2' \frac{(2 + C_2)\, C_3}{1 - \theta}\, \rho^\ell \, \|\nu\|_\theta \,.
\end{aligned} \tag{84}$$

With $k, l = [\frac{n}{2}](+1)$ and $\hat{\rho} = \max\{\rho^{1/2}, \alpha^{1/2}\}$ this yields (75). □

Proof of Equation (61). For $j = 1, \ldots, d$ let $X_j = I^{\{1,\ldots,j\}}$. Let $f \in BV(X_d)$ and define $f_j : X_j \to \mathbb{R}$, $f_j(\boldsymbol{x}_{1:j}) = \int f(\boldsymbol{x})\, dx_{j+1} \ldots dx_d$. Then $\mathrm{Var}_{X_j}(f_j) \leq \mathrm{Var}_{X_d}(f_d)$ for all $j = 1, \ldots, d$, and if $\int f\, dm = 0$, a repeated application of (82) to any measure ν with $\nu_{\{1,\ldots,d\}} = f$ yields

$$\int_{X_d} |P_{T_0}^\ell f| \, dm \le (2+C_2)C_3 \, \rho^\ell \sum_{j=1}^{d} \mathrm{Var}_{X_j}(f_j) \le (2+C_2)C_3 \, \ell \, \rho^\ell \, \mathrm{Var}_{X_d}(f) \, . \quad (85)$$

□

The previous inequality implies in particular that finite uncoupled systems have a unique absolutely continuous invariant probability measure. Indeed, the existence of a P_{T_0}-invariant probability density $h_0 \in BV(X)$ is proved in Theorem 3.1. (Observe that $h_0(\boldsymbol{x}) = h(x_1)\dots h(x_d)$ where h is the unique invariant density for the single site map τ.) Suppose there is another P_{T_0}-invariant probability density $h_0' \in L_X^1$. As $C^1(X)$ is dense in L_X^1 there is, for each $\delta > 0$, some $f_\delta \in C^1(X) \subset BV(X)$ with $\int_X |h_0' - f_\delta| \, dm < \delta$. Then (85) implies, for each $\ell > 0$,

$$\begin{aligned}\int |h_0 - h_0'| \, dm &\le \int |P_{T_0}^\ell (h_0 - f_\delta)| \, dm + \int |P_{T_0}^\ell (f_0 - h_0')| \, dm \\ &\le (2+C_2)C_3 \, \ell \, \rho^\ell \, \mathrm{Var}_{X_d}(f) + \int |f_0 - h_0'| \, dm \, .\end{aligned}$$

In the limit $\ell \to \infty$ this yields $\int |h_0 - h_0'| \, dm \le \delta$. This proves the claim.

Corollary 4.2. *The infinite uncoupled system has a unique invariant probability measure with absolutely continuous finite-dimensional marginals. (This would not be true for coupled systems as shown by the examples in [26]. See also the chapter by E. Jarvenpää in this book.)*

Proof. If ν is a T_0-invariant probability measure with absolutely continuous finite-dimensional marginal densities ν_Λ, then these densities are invariant for the uncoupled system on I^Λ. By the previous observation, $\nu\pi_\Lambda^{-1}$ is therefore the product measure μ^Λ, where μ is the unique absolutely continuous τ-invariant probability measure. Hence $\nu = \mu^{\mathbb{Z}}$ by Kolmogorov's theorem. □

4.6 A Perturbation Result and a Decoupling Estimate

For our treatment of infinite coupled systems we need a procedure to "decouple" a given site b from all other sites. Technically this boils down to compare a coupling Φ_ϵ with a modified one. Following [9] we provide such an estimate in this section.

Proposition 4.2. *Let $F, \tilde{F} : X \to X$ be two Lipschitz maps*[20] *with Lipschitz constant $L > 0$ that are close in the following sense: There are constants $K_0, K_1, K_2 > 0$ such that*

[20] $F : X \to X$ is a "Lipschitz map", if all $F_i(\boldsymbol{x})$ are Lipschitz w.r.t. each coordinate x_j with uniformly bounded Lipschitz constants. This means in particular that all partial derivatives of all F_i exist Lebesgue-a.e., are uniformly bounded and that $F_i(x + se_k) - F_i(x) = \int_0^s \partial_k F_i(x + \xi e_k) d\xi$.

(i) $\sum_{i\in\mathbb{Z}} \sup_{\boldsymbol{x}} |\tilde F_i(\boldsymbol{x}) - F_i(\boldsymbol{x})| \le K_0$,

(ii) $\sum_{i\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \sup_{\boldsymbol{x}} |\partial_j \tilde F_i(\boldsymbol{x}) - \partial_j F_i(\boldsymbol{x})| \le K_1$, *and*

(iii) $\sup\{\mathrm{Var}(P_{F_{t,\Lambda}} f) : 0 \le t \le 1, \Lambda \in \mathcal{I}, f \in BV(I^\Lambda), \mathrm{Var}_{I^\Lambda}(f) \le 1\} \le K_2$ *where* $F_{t,\Lambda} = \pi_\Lambda \circ \left(t\tilde F + (1-t)F\right) \circ \iota_\Lambda$, *compare (64).*

Assume also that $\partial_j F_i = 0$ *and* $\partial_j \tilde F_i = 0$ *if* $|i-j| > w$. *Then*

$$\int_X |(P_{\tilde F}\nu)_\Lambda - (P_F\nu)_\Lambda|\, dm \le K_2 \left(K_0 + \frac{1}{2}K_1\right) \mathrm{Var}_{I^{\Lambda(1)}}(\nu_{\Lambda(1)}) \tag{86}$$

for $\Lambda \in \mathcal{I}$ *and* $\nu \in L^1_{\mathbb{Z}}$.

Proof. The maps $F_{t,\Lambda(1)} : I^{\Lambda(1)} \to I^{\Lambda(1)}$ are Lipschitz with Lipschitz constant L, and for any $\psi \in C^1(I^\Lambda)$ with $|\psi| \le 1$,

$$\begin{aligned}
&\int_X \psi\circ\pi_\Lambda\, d(P_{\tilde F}\nu) - \int_X \psi\circ\pi_\Lambda\, d(P_F\nu) = \int_X (\psi\circ\pi_\Lambda\circ\tilde F - \psi\circ\pi_\Lambda\circ F)\, d\nu\\
&= \int_{I^{\Lambda(1)}} (\psi\circ\pi_\Lambda\circ\tilde F\circ\iota_{\Lambda(1)} - \psi\circ\pi_\Lambda\circ F\circ\iota_{\Lambda(1)})\, \nu_{\Lambda(1)}\, dm\\
&= \int_{I^{\Lambda(1)}} \int_0^1 \frac{\partial}{\partial t}(\psi\circ\pi_\Lambda^{\Lambda(1)}(F_{t,\Lambda(1)}(\boldsymbol{x})))\, dt\, \nu_{\Lambda(1)}(\boldsymbol{x})\, d\boldsymbol{x}\\
&= \int_0^1 \sum_{i\in\Lambda} \left(\int_{I^{\Lambda(1)}} \partial_i\psi(\pi_\Lambda^{\Lambda(1)}(F_{t,\Lambda(1)}(\boldsymbol{x})))\, \frac{\partial}{\partial t}F_{t,\Lambda(1),i}(\boldsymbol{x})\, \nu_{\Lambda(1)}(\boldsymbol{x})\, d\boldsymbol{x}\right) dt\\
&= \int_0^1 \sum_{i\in\Lambda} \left(\int_{I^{\Lambda(1)}} \partial_i\psi_\Lambda(\boldsymbol{x}) \left(P_{F_{t,\Lambda(1)}}\left(\frac{\partial}{\partial t}F_{t,\Lambda(1),i}\ \nu_{\Lambda(1)}\right)\right)(\boldsymbol{x})\, d\boldsymbol{x}\right) dt
\end{aligned} \tag{87}$$

where $\psi_\Lambda := \psi\circ\pi_\Lambda^{\Lambda(1)}$. As $\psi \in C^1(I^\Lambda)$ is an arbitrary test function with $\sup|\psi| \le 1$, this implies

$$\begin{aligned}
\int_{I^\Lambda} |(P_{\tilde F}\nu)_\Lambda - (P_F\nu)_\Lambda|\, dm &\le \int_0^1 \sum_{i\in\Lambda} \mathrm{Var}_{I^{\Lambda(1)}}\left(P_{F_{t,\Lambda(1)}}\left(\frac{\partial}{\partial t}F_{t,\Lambda(1),i}\ \nu_{\Lambda(1)}\right)\right) dt\\
&\le K_2 \sum_{i\in\Lambda} \mathrm{Var}_{I^{\Lambda(1)}}\left((F_{1,\Lambda(1)} - F_{0,\Lambda(1)})_i\ \nu_{\Lambda(1)}\right)\ .
\end{aligned} \tag{88}$$

Next recall that the variation of multivariate functions is defined as the maximum of the variations over individual coordinates, see (43). Hence Lemma 2.2b, which provides an estimate for the variation of a product of functions of one variable, carries over to the present setting, and we conclude (observing also (44))

$$\int_{I^\Lambda} |(P_{\tilde F}\nu)_\Lambda - (P_F\nu)_\Lambda|\, dm \le K_2\, (K_0 + \frac{1}{2}K_1)\ \mathrm{Var}_{I^{\Lambda(1)}}(\nu_{\Lambda(1)})\ . \tag{89}$$

□

The estimate from this proposition does not allow to compare directly Φ_ϵ and $\Phi_0 = \mathrm{Id}$: for any finite lattice L the right hand side of (86) would grow like $|L|$ – because the constants K_0 and K_1 are supposed to bound sums over all lattice sites. On the infinite lattice the right hand side of (86) is thus infinite. Therefore we apply Proposition 4.2 "site by site" and evaluate (86) by aid of the next lemma.

Let $\Lambda \in \mathcal{I}$ and $b \in \Lambda$. In order to decouple site b from all other sites we introduce the following notation: let $\bar{\iota}_b : I^{\mathbb{Z}} \to I^{\mathbb{Z}}$ be the map $(\bar{\iota}_b(\boldsymbol{x}))_i = x_i$ if $i \neq b$ and $(\bar{\iota}_b(\boldsymbol{x}))_b = 0$. Then define $\Phi_{\epsilon,b}, T_{\epsilon,b} : I^{\mathbb{Z}} \to I^{\mathbb{Z}}$,

$$(\Phi_{\epsilon,b}(\boldsymbol{x}))_i = \begin{cases} x_b & \text{if } i = b \\ (\Phi_\epsilon(\bar{\iota}_b(\boldsymbol{x})))_i & \text{if } i \neq b \end{cases} \quad \text{and} \quad T_{\epsilon,b} = \Phi_{\epsilon,b} \circ T_0 \,. \tag{90}$$

Our task is to show that the passage from Φ_ϵ to $\Phi_{\epsilon,b}$ leads to an error (in the sense of Proposition 4.2) of order $|\epsilon|$ independent of the size of Λ (depending heavily on ℓ, though!). Denote by E_b the $\mathbb{Z} \times \mathbb{Z}$ matrix with $(E_b)_{ij} = 1$ if $i = j \neq b$ and $(E_b)_{ij} = 0$ otherwise.

Lemma 4.3. *Let Φ_ϵ be a (a_1, a_2)-coupling.*

(a) $\sum_{i\in\mathbb{Z}} \sup_{\boldsymbol{x}} |(\Phi_{\epsilon,b}(\boldsymbol{x}))_i - (\Phi_\epsilon(\boldsymbol{x}))_i| \le 2|\epsilon|(a_1 + 1)$.
(b) $\sum_{i\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \sup_{\boldsymbol{x}} |\partial_j(\Phi_{\epsilon,b}(\boldsymbol{x}))_i - \partial_j(\Phi_\epsilon(\boldsymbol{x}))_i| \le 2|\epsilon|(2a_1 + a_2)$.
(c) All $F_t := t\Phi_{\epsilon,b} + (1-t)\Phi_\epsilon$ $(0 \le t \le 1)$ are (a_1, a_2)-couplings and satisfy assumption (iii) in Proposition 4.2 for $K_2 = \frac{1-2a_1|\epsilon|+a_2|\epsilon|}{(1-2a_1|\epsilon|)^2}$, and $K_2 \le 2$ as long as $|\epsilon| \le \min\{\frac{1}{6a_1}, \frac{2}{9a_2}\}$.

Proof. The following estimate yields (a):

$$\begin{aligned} &\sum_{i\in\mathbb{Z}} \sup_{\boldsymbol{x}} |(\Phi_{\epsilon,b}(\boldsymbol{x}))_i - (\Phi_\epsilon(\boldsymbol{x}))_i| \\ &\quad \le \sup_{\boldsymbol{x}} |x_b - (\Phi_\epsilon(\boldsymbol{x}))_b| + \sum_{i\neq b} \sum_{j\in\mathbb{Z}} \sup_{\boldsymbol{x}} |(D\Phi_\epsilon(\boldsymbol{x}))_{ij}| \sup_{\boldsymbol{x}} |(\bar{\iota}_b(\boldsymbol{x}))_j - x_j| \\ &\quad \le 2|\epsilon| \left(1 + \sum_{i\neq b} A'_{ib}\right) \le 2|\epsilon|\,(1 + a_1) \cdot \end{aligned}$$

For (b) observe first that

$$(\partial_j(\Phi_{\epsilon,b} - \Phi_\epsilon))_i = \begin{cases} -\partial_j A_{\epsilon,i} & \text{if } i = b \text{ or } j = b \\ (\partial_j \Phi_{\epsilon,i}) \circ \bar{\iota}_b - \partial_j \Phi_{\epsilon,i} & \text{if } i \neq b \text{ and } j \neq b \,. \end{cases}$$

The difference $(\partial_j \Phi_{\epsilon,i}) \circ \bar{\iota}_b - \partial_j \Phi_{\epsilon,i}$ can be bounded by $\sup_{\boldsymbol{x}} |\partial_b(\partial_j \Phi_{\epsilon,i})(\boldsymbol{x})| = \sup_{\boldsymbol{x}} |\partial_j(\partial_b \Phi_{\epsilon,i})(\boldsymbol{x})| \le 2|\epsilon| A''_{ib}$. Hence

$$\begin{aligned} \sum_{i\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} |(\partial_j(\Phi_{\epsilon,b} - \Phi_\epsilon))_i| &\le 2|\epsilon| \left(\sum_{i\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} (A' - E_b A' E_b)_{ij} + \sum_{i\in\mathbb{Z}} (E_b A'' E_b)_{ib} \right) \\ &\le 2|\epsilon|(2a_1 + a_2) \cdot \end{aligned}$$

We turn to (c): That all F_t (and hence also all $F_{t,\Lambda}$) are (a_1, a_2)-couplings is a trivial observation. The K_2-bound then follows from Lemma 3.3. □

Combining Proposition 4.2 and Lemma 4.3 we obtain

Corollary 4.3. *Let Φ_ϵ be a (a_1, a_2)-coupling, $\Lambda \in \mathcal{I}$, and $b \in \Lambda$. Then*

$$\int_X |((P_{\Phi_{\epsilon,b}} - P_{\Phi_\epsilon})\nu)_\Lambda| \, dm \leq |\epsilon| \, (8a_1 + 2a_2 + 4) \, \mathrm{Var}_{I^{\Lambda(1)}}(\nu_{\Lambda(1)}) \tag{91}$$

for $\nu \in L^1_{\mathbb{Z}}$, as long as $|\epsilon| \leq \min\{\frac{1}{6a_1}, \frac{2}{9a_2}\}$.

4.7 Uniqueness and a Spectral Gap – the Coupled Case

We are going to follow essentially the same strategy as in the proof of Theorem 4.2. To this end we decouple the dynamics at a site $b \in \Lambda = [a, b]$ from all other sites. As a result we obtain an estimate like (81) with an additional error term. Finally we use the Lasota–Yorke inequality to control this error term.

Recall that $\epsilon_1 \in (0, \epsilon_0]$ and $\theta_1 \in (0, 1)$ were determined in Proposition 4.1 and $\hat{\rho} = \max\{\rho, \alpha\}^{1/2} < 1$ in Theorem 4.2.

Theorem 4.3 (Spectral gap). *Let $\gamma \in (\hat{\rho}, 1)$. There is $\theta_2 \in [\theta_1, 1)$ such that for each $\theta \in [\theta_2, 1)$ there exist $C_\theta > 0$ and $\epsilon_\theta \in (0, \epsilon_1]$ such that*

$$\|P^n_{T_\epsilon} \nu\|_\theta \leq C_\theta \, \gamma^n \, \|\nu\|_\theta \tag{92}$$

for all $|\epsilon| \leq \epsilon_\theta$, all $\nu \in L^1_\theta$ with $\nu_\emptyset = \nu(X) = 0$, and all $n \in \mathbb{N}$.

In particular, if $m^{\mathbb{Z}}$ denotes the product Lebesgue measure on X, then

$$\|P^n_{T_\epsilon} m^{\mathbb{Z}} - \mu_\epsilon\|_\theta \leq C'_\theta \, \gamma^n := (2 + C'_2) C_\theta \, \gamma^n \, . \tag{93}$$

Before we prove this theorem, we note the following corollary.

Corollary 4.4 (Uniqueness). *Let $\theta \in [\theta_2, 1)$ and $|\epsilon| \leq \epsilon_\theta$. There is a unique T_ϵ-invariant probability measure μ_ϵ in BV_θ, and μ_ϵ belongs in fact to $BV_{\theta=1}$.*

Proof. The existence of $\mu_\epsilon \in BV_{\theta=1}$ was proved in Theorem 4.1. For the uniqueness assume that $\tilde{\mu}_\epsilon \in BV_\theta$ is also T_ϵ-invariant. Let $\nu = \mu_\epsilon - \tilde{\mu}_\epsilon$. Then Theorem 4.3 applies to ν, and as $P_{T_\epsilon}\nu = \nu$, it follows that $\nu = 0$.

Proof of Theorem 4.3. Let $\Lambda = [a, b]$. The proof consists of three steps. In view of Corollary 4.3 we may first replace $P^\ell_{T_\epsilon}\nu$ by $P^\ell_{T_{\epsilon,b}}\nu$ at the expense of an error of size $\mathcal{O}(\ell|\epsilon|)$. Then we can profit from the product structure of $T_{\epsilon,b}$ (the site b is completely decoupled now!) and reduce estimates on the box Λ to estimates on the smaller box $\Lambda' = [a, b-1]$ as we did it in the proof of Theorem 4.2. Finally we must show that it is indeed sufficient to do all this for a large but fixed ℓ.

To begin with,

$$\begin{aligned}
&\left|\int_{I^\Lambda} |(P^\ell_{T_{\epsilon,b}}\nu)_\Lambda|\,dm - \int_{I^\Lambda} |(P^\ell_{T_\epsilon}\nu)_\Lambda|\,dm\right| \le \int_{I^\Lambda} |(P^\ell_{T_{\epsilon,b}}\nu - P^\ell_{T_\epsilon}\nu)_\Lambda|\,dm \\
&\quad\le \sum_{i=0}^{\ell-1}\int_{I^\Lambda}\left|\left(P^i_{T_{\epsilon,b}}(P_{T_{\epsilon,b}}-P_{T_\epsilon})P^{\ell-i-1}_{T_\epsilon}\nu\right)_\Lambda\right|\,dm \\
&\quad\le \sum_{i=0}^{\ell-1}\int_{I^{\Lambda(i)}} |((P_{\Phi_{\epsilon,b}}-P_{\Phi_\epsilon})P_{T_0}P^{\ell-i-1}_{T_\epsilon}\nu)_{\Lambda(i)}|\,dm \\
&\quad\le |\epsilon|(8a_1+2a_2+4)\sum_{i=0}^{\ell-1}\mathrm{Var}_{I^{\Lambda(i+1)}}((P_{T_0}P^{\ell-i-1}_{T_\epsilon}\nu)_{\Lambda(i+1)}) \\
&\quad\le |\epsilon|(8a_1+2a_2+4)\sum_{i=0}^{\ell-1}\theta^{-|\Lambda(i+1)|}\|P_{T_0}P^{\ell-i-1}_{T_\epsilon}\nu\|_\theta \\
&\quad\le \ell\,\theta^{-|\Lambda(\ell)|}|\epsilon|(8a_1+2a_2+4)\cdot(C_1'+\frac{1}{2}C_2)\|\nu\|_\theta =: \ell\,\theta^{-|\Lambda(\ell)|}|\epsilon|\,C_7\,\|\nu\|_\theta
\end{aligned} \tag{94}$$

where we used Corollary 4.3 and (71).

Exactly the same reasoning yields

$$\left|\int_{I^{\Lambda'}} |(P^\ell_{T_{\epsilon,b}}\nu)_{\Lambda'}|\,dm - \int_{I^{\Lambda'}} |(P^\ell_{T_\epsilon}\nu)_{\Lambda'}|\,dm\right| \le \ell\,\theta^{-|\Lambda(\ell)|}|\epsilon|\,C_7\,\|\nu\|_\theta\,. \tag{95}$$

In the next step we will profit from the decoupling: as $T_{\epsilon,b}: X \to X$ is the direct product $\tau\times S$ of τ with a map $S: I^{\mathbb{Z}\setminus\{b\}} \to I^{\mathbb{Z}\setminus\{b\}}$ for which $(S(\boldsymbol{x}_{\neq b}))_j$, $j\in\Lambda'$ depends only on coordinates from $\tilde\Lambda = [a-\ell w, b+\ell w]$, we can apply Lemma 4.2 again and obtain

$$\int_{I^\Lambda} |(P^\ell_{T_{\epsilon,b}}\nu)_\Lambda|\,dm \le (2+C_2)\,C_3\,\rho^\ell\,\mathrm{Var}_{I^{\tilde\Lambda}}(\nu_{\tilde\Lambda}) + \int_{I^{\Lambda'}} |(P^\ell_{T_{\epsilon,b}}\nu)_{\Lambda'}|\,dm\,. \tag{96}$$

Before we put together (94)–(96) we let $\tilde\gamma = (\hat\rho\gamma)^{1/2}$. Then $\hat\rho < \tilde\gamma < \gamma$ and $\hat\rho/\tilde\gamma = \tilde\gamma/\gamma < 1$. So we can fix $\theta_2 \in [\theta_1, 1)\cap((\tilde\gamma/\gamma)^{1/w}, 1)$, and for $\theta\in[\theta_2,1)$ we can first choose ℓ_θ such that

$$(2\,+\,C_2)C_3\rho^{\ell_\theta} \le (1-\theta)\tilde\gamma^{2\ell_\theta}\theta^{2\ell_\theta w} \text{ and } \left(C_1'\left(C_1'+\frac{1}{2}C_2'\right)+C_2'\right)(\tilde\gamma\theta_2^{-w})^{2\ell_\theta} \le \gamma^{2\ell_\theta} \tag{97}$$

and then $\epsilon_\theta \in (0,\epsilon_1]$ so small that $2\ell_\theta\theta^{-2\ell_\theta w}\epsilon_\theta C_7 \le (1-\theta)\tilde\gamma^{2\ell_\theta}$. Then

$$\begin{aligned}
&\theta^{|\Lambda|}\int_{I^\Lambda} |(P^{\ell_\theta}_{T_\epsilon}\nu)_\Lambda|\,dm \\
&\qquad\le (1-\theta)\tilde\gamma^{2\ell_\theta}\left(\|\nu\|_\theta + \theta^{|\tilde\Lambda|}\,\mathrm{Var}_{I^{\tilde\Lambda}}(\nu_{\tilde\Lambda})\right) + \theta^{|\Lambda|}\int_{I^{\Lambda'}} |(P^{\ell_\theta}_{S_\epsilon}f)_{\Lambda'}|\,dm
\end{aligned} \tag{98}$$

for $|\epsilon| \le \epsilon_\theta$. Taking the supremum over all $\Lambda\in\mathcal{I}$ this yields, as in (82),

$$|P_{T_\epsilon}^{\ell_\theta}\nu|_\theta \leq (1-\theta)\tilde{\gamma}^{2\ell_\theta}\|\nu\|_\theta + \theta\,|P_{T_\epsilon}^{\ell_\theta}\nu|_\theta\ . \tag{99}$$

Now the proof of the proposition can be finished along the lines of the proof of Theorem 4.2: invoking the Lasota–Yorke type estimates (69) and (71) one obtains the exponential estimate

$$\begin{aligned}\|P_{T_\epsilon}^{2\ell_\theta}\nu\|_\theta &\leq C_1'\alpha^{\ell_\theta}\|P_{T_\epsilon}^{\ell_\theta}\nu\|_\theta + C_2'\theta^{-2w\ell_\theta}|P_{T_\epsilon}^{\ell_\theta}\nu|_\theta \\ &\leq \left(C_1'\tilde{\gamma}^{2\ell_\theta}(C_1'+\frac{1}{2}C_2'\theta^{-2w\ell_\theta}) + C_2'\theta^{-2w\ell_\theta}\tilde{\gamma}^{2\ell_\theta}\right)\|\nu\|_\theta \\ &\leq \gamma^{2\ell_\theta}\,\|\nu\|_\theta\ .\end{aligned} \tag{100}$$

This yields $\|P_{T_\epsilon}^n\nu\|_\theta \leq \gamma^n\|\nu\|_\theta$ for even multiples $n = 2k\ell_\theta$ of ℓ_θ, valid for $|\epsilon| \leq \epsilon_\theta$. For general $n = 2k\ell_\theta + j$ with $0 \leq j < 2\ell_\theta$ one uses (71) to conclude that $\|P_{T_\epsilon}^n\nu\|_\theta \leq (C_1'+C_2'\theta^{-2w\ell_\theta})\|P_{T_\epsilon}^{2k\ell_\theta}\nu\|_\theta \leq \left((C_1'+C_2'\theta^{-2w\ell_\theta})\gamma^{-2\ell_\theta}\right)\gamma^n\|\nu\|_\theta$. This is (92). □

4.8 Exponential Decay of Correlations

Lemma 4.4 (Exponential decay in time). *Let $\phi,\psi : X \to \mathbb{R}$ be bounded observables that depend only on coordinates in intervals Λ_ϕ and Λ_ψ, respectively. Let ϵ and θ be as in Theorem 4.3, and let $\tilde{\mu}$ be a probability measure from $BV_{\theta=1}$. Then*

$$\left|\int \phi\cdot\psi\circ T_\epsilon^n\,d\tilde{\mu} - \int\phi\,d\tilde{\mu}\cdot\int\psi\,d\mu_\epsilon\right| \leq C_\theta\theta^{-|\Lambda_\psi|}\,\gamma^n\|\tilde{\mu}\|_{\theta=1}\,\|\phi\|_{C^1}\|\psi\|_{C^0}\ . \tag{101}$$

For $\tilde{\mu} = \mu_\epsilon$ this is slightly stronger than (4) of Theorem 1.1.

Proof. It suffices to restrict to the case where $\int\phi\,d\tilde{\mu} = 0$. Let $\nu = \phi\tilde{\mu}$. Then $\nu(X) = \int\phi\,d\tilde{\mu} = 0$ so that

$$\begin{aligned}\left|\int\phi\cdot\psi\circ T_\epsilon^n\,d\tilde{\mu}\right| &\leq \int_{I^{\Lambda_\psi}}|(P_{T_\epsilon}^n\nu)_{\Lambda_\psi}|\,\psi\,dm \leq \|\psi\|_{C^0}\,\theta^{-|\Lambda_\psi|}\|P_{T_\epsilon}^n\nu\|_\theta \\ &\leq \|\psi\|_{C^0}\,\theta^{-|\Lambda_\psi|}C_\theta\gamma^n\|\nu\|_\theta \leq \|\psi\|_{C^0}\,\theta^{-|\Lambda_\psi|}C_\theta\gamma^n\|\nu\|_{\theta=1}\end{aligned} \tag{102}$$

by Theorem 4.3 and Lemma 4.1c when $|\epsilon| \leq \epsilon_\theta$.

It remains to bound $\|\nu\|_{\theta=1}$: Remembering Lemma 4.1b it suffices to consider boxes $\Lambda \in \mathcal{I}$ which include Λ_ϕ. Let $\varphi \in \mathcal{T}_{I^\Lambda}$. Then

$$\begin{aligned}\int_{I^\Lambda}\partial_i\varphi\,\nu_\Lambda\,dm &= \int_{I^\Lambda}\partial_i(\varphi\phi)\,\tilde{\mu}_\Lambda\,dm - \int_{I^\Lambda}\varphi\,\partial_i\phi\,\tilde{\mu}_\Lambda\,dm \\ &\leq \|\phi\|_{C^0}\,\mathrm{Var}(\tilde{\mu}_\Lambda) + \|\partial_i\phi\|_{C^0}\int|\tilde{\mu}_\Lambda|\,dm \leq \|\tilde{\mu}\|_{\theta=1}\,\|\phi\|_{C^1}\ .\end{aligned} \tag{103}$$

Hence $\|\nu\|_{\theta=1} \leq \|\phi\|_{C^1}\|\tilde{\mu}\|_{\theta=1}$. This finishes the proof of (101). □

Lemma 4.5 (Exponential decay in space). *Let $\phi, \psi : X \to \mathbb{R}$ be bounded observables that depend only on coordinates in the interval Λ. Let γ and θ be as in Theorem 4.3, and let θ be sufficiently close to 1 that $\gamma' := \gamma^{\frac{1}{2w}}\theta^{-1} < 1$ (recall that w is the coupling range). Then, for $|\epsilon| \le \epsilon_\theta$,*

$$\left| \int \phi \cdot (\psi \circ \sigma^n)\, d\mu_\epsilon - \int \phi\, d\mu_\epsilon \int \psi\, d\mu_\epsilon \right| \le C' {\gamma'}^{|n|-|\Lambda|}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \tag{104}$$

where σ is the left shift on $X = I^{\mathbb{Z}}$.

Proof. We may assume that $\int \phi\, d\mu_\epsilon = 0$. Let $\tilde\psi = \psi \circ \sigma^n$. As ϕ and $\tilde\psi$ depend on variables at distance $|n| - |\Lambda|$ at least, it follows that $\phi \circ T_\epsilon^k$ and $\tilde\psi \circ T_\epsilon^k$ depend on disjoint sets of variables for $k = [\frac{|n|-|\Lambda|}{2w}]$. Accordingly, by Theorem 4.3,

$$\begin{aligned}
\left| \int \phi \cdot (\psi \circ \sigma^n)\, d\mu_\epsilon \right| &\le \left| \int \phi\, \tilde\psi\, d(P_{T_\epsilon}^k m^{\mathbb{Z}}) \right| + C'_\theta \gamma^k\, \theta^{-|\Lambda|-|n|}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \\
&= \left| \int (\phi \circ T_\epsilon^k)(\tilde\psi \circ T_\epsilon^k)\, dm^{\mathbb{Z}} \right| + C'_\theta \gamma^k\, \theta^{-|\Lambda|-|n|}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \\
&= \left| \int (\phi \circ T_\epsilon^k)\, dm^{\mathbb{Z}} \right| \cdot \left| \int (\tilde\psi \circ T_\epsilon^k)\, dm^{\mathbb{Z}} \right| + C'_\theta \gamma^k\, \theta^{-|\Lambda|-|n|}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \\
&= \left| \int \phi\, d(P_{T_\epsilon}^k m^{\mathbb{Z}}) \right| \cdot \left| \int \tilde\psi\, d(P_{T_\epsilon}^k m^{\mathbb{Z}}) \right| + C'_\theta \gamma^k\, \theta^{-|\Lambda|-|n|}\, \|\phi\|_{C^0} \|\psi\|_{C^0} \\
&\le 2C'_\theta \gamma^k\, \theta^{-|\Lambda|-|n|}\, \|\phi\|_{C^0} \|\psi\|_{C^0}\ .
\end{aligned} \tag{105}$$

As $\gamma' = \gamma^{\frac{1}{2w}}\theta^{-1}$, (104) follows immediately. □

4.9 μ_ϵ as Unique Physical Measure

The invariant measure μ_ϵ has some properties which qualify it as the unique physical (or observable) measure: it governs a strong law of large numbers (Corollary 1.1), and it is stable under small independent random perturbations. Below we prove the first assertion, and we formulate precisely what we mean by the second one (without providing a proof, though).

Proof of Corollary 1.1 (Strong law of large numbers): Let $f : I \to \mathbb{R}$ be a probability density of bounded variation and let $\nu = (fm)^{\mathbb{Z}}$ the infinite product measure of the probability measure with density f. Then $\nu \in BV_{\theta=1}$, and indeed $\|\nu\|_{\theta=1} = \mathrm{Var}_I(f) < \infty$ by Example 4.1. Let ψ be a C^1 observable that depends only on finitely many coordinates. (It clearly suffices to prove the corollary for such observables, because each continuous observable can be uniformly approximated by them.) In view of [10, Theorem 5.1] it suffices to prove the two following properties:[21]

[21] Given (106) and (107), the proof of the law of large numbers is easy. We more or less copy it from [10]. Let $S_n(\boldsymbol{x}) = \sum_{k=0}^{n-1} \big(\psi(T_\epsilon^k \boldsymbol{x}) - \int \psi \circ T_\epsilon^k\, d\nu\big)$. Then

$$\lim_{k\to\infty}\int \psi\circ T_\epsilon^k\,d\nu = \int \psi\,d\mu_\epsilon \tag{106}$$

$$\sup_{k>0}\sum_{j=0}^{\infty}\left|\int(\psi\circ T_\epsilon^j)(\psi\circ T_\epsilon^k)\,d\nu - \int\psi\circ T_\epsilon^j\,d\nu\int\psi\circ T_\epsilon^k\,d\nu\right| \le C_{\theta,\psi} < \infty\ . \tag{107}$$

Equation (106) is contained in the special case $\tilde{\mu} = \nu$ and $\phi = 1$ of (101). Equation (107) follows also from (101): first replace ψ by $\psi - \int\psi\,d\mu_\epsilon$, i.e. assume that $\int\psi\,d\mu_\epsilon = 0$. Then apply (101) with $\tilde{\mu} = P_{T_\epsilon}^\ell \nu$ and $\ell = |j-k|$ to show that

$$\sum_{j=0}^{\infty}\left|\int(\psi\circ T_\epsilon^j)(\psi\circ T_\epsilon^k)\,d\nu\right| \le C'_{\theta,\psi} < \infty$$

and finally apply (101) with $\tilde{\mu} = \nu$ and $\phi = 1$ to show that

$$\sum_{j=0}^{\infty}\left|\int\psi\circ T_\epsilon^j\,d\nu\int\psi\circ T_\epsilon^k\,d\nu\right| \le \|\psi\|_{C^0}\sum_{j=0}^{\infty}\left|\int\psi\circ T_\epsilon^k\,d\nu\right| \le C''_{\theta,\psi} < \infty\ .$$

For both of these estimates one uses the fact that $\sup_k \|P_{T_\epsilon}^k \nu\|_{\theta=1} < \infty$, see (71).

Remark 4.3. Here is another reason why μ_ϵ should be considered as the unique observable invariant measure: it is stable under independent random perturbations. More precisely, consider a smooth family $(S_\omega)_{\omega\in\mathbb{R}}$ of C^2 maps from $I \to I$, $S_0 = \mathrm{Id}_I$. Let $(\nu_\delta)_{\delta>0}$ be a family of probability measures on $\mathbb{R}$ with C^1 densities supported on $[-\delta,\delta]$, and consider the random process defined by the Markov operator[22]

$\int(\frac{1}{n}S_n)^2\,d\nu \le \frac{1}{n^2}nC_{\theta,\psi} = \frac{1}{n}C_{\theta,\psi}$ by (107), so that the subsequence $(\frac{1}{k^2}S_{k^2})_{k\ge1}$ is L^1_ν-summable. In particular, $\frac{1}{k^2}S_{k^2} \to 0$ ν-a.e.. To show the a.e.-convergence of the full sequence, let $m_n = [\sqrt{n}]^2$. The bound $m_n \le n \le m_n + 2\sqrt{n}$ ensures $\frac{m_n}{n} \to 1$ and $|S_n - S_{m_n}| \le 2\sqrt{n}\|\psi\|_{C^0}$. Hence

$$\frac{1}{n}S_n = \frac{m_n}{n}\,\frac{1}{m_n}S_{m_n} + \frac{S_n - S_{m_n}}{n} \to 0 \quad \text{almost surely as } n\to\infty$$

and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\psi(T_\epsilon^k\boldsymbol{x}) = \int\psi\,d\mu_\epsilon$ for ν-a.e. $\boldsymbol{x}$ follows from (106).

[22] To convince oneself that this is quite a general way to write down random perturbations consider the case in which the S_ω are defined on the circle $I = \mathbb{R}/\mathbb{Z}$ by $S_\omega(x) = x + \omega$ and where $\nu_\delta(d\omega) = \delta^{-1}q(\delta^{-1}\omega)\,d\omega$. Then, for measures $\mu(dx) = h(x)\,dx$,

$$\begin{aligned}\tilde{P}_{\nu_\delta}\mu(\varphi) &= \int_{\mathbb{R}}\int_I \varphi(x+\omega)h(x)\delta^{-1}q(\delta^{-1}\omega)\,dx\,d\omega \\ &= \int_I \varphi(y)\left(\int_{\mathbb{R}} h(y-\omega)\delta^{-1}q(\delta^{-1}\omega)\,d\omega\right)\,dy \end{aligned}\tag{108}$$

$$\tilde{P}_{\nu_\delta}\mu(\varphi) := \int_{\mathbb{R}} \int_I \varphi \circ S_\omega \, d\mu \, d\nu_\delta(\omega) \; . \tag{109}$$

We will consider independent random perturbations of this type at each site, that is

$$P_{\nu_\delta}\mu(\varphi) := \int_{\mathbb{R}^{\mathbb{Z}}} \int_I \varphi \circ (\otimes_{i\in\mathbb{Z}} S_{\omega_i}) \, d\mu \, \otimes_{i\in\mathbb{Z}} d\nu_\delta(\omega_i) \; . \tag{110}$$

Define now $P_{\epsilon,\delta} := P_{\nu,\delta} P_{T_\epsilon}$. One can show the following (actually, this is a nice exercise for the reader):

- $P_{\epsilon,\delta}$ has a unique invariant probability measure $\mu_{\epsilon,\delta}$, and this measure belongs to $BV_{\theta=1}$.
- $\|\mu_{\epsilon,\delta} - \mu_\epsilon\|_\theta \le C_\theta \, \delta \, (\ln \delta^{-1})^2$, in particular $\lim_{\delta\to 0} \mu_{\epsilon,\delta} = \mu_\epsilon$ in the weak topology.

We finish by proving assertion 3 of Theorem 1.1.

Lemma 4.6. *There exist $\theta_* \in (0,1)$ such that, for each $\theta \in [\theta_*, 1)$ and $\epsilon \in (0, \epsilon_\theta]$ (where ϵ_θ is as in Theorem 4.3), there exists $C''_\theta > 0$ such that*

$$\|\mu_0 - \mu_\epsilon\|_\theta \le C''_\theta \, \epsilon \ln \epsilon^{-1} \; . \tag{111}$$

Proof. By a repeated application of Corollary 4.3, for each $\Lambda \in \mathcal{I}$ and each probability measure $\nu \in BV_{\theta=1}$,

$$\theta^{|\Lambda|} \int_{I^\Lambda} |(P_{T_0}\nu - P_{T_\epsilon}\nu)_\Lambda| \, dm \le D' \, \epsilon \, \theta^{|\Lambda|} |\Lambda| \, |\nu|_{\theta=1} \le D \, \epsilon \, |\nu|_{\theta=1}$$

for some constants $D', D > 0$. Thus, observing that $|P_{T_0}|_\theta = 1$,

$$|P^n_{T_0}\mu_0 - P^n_{T_\epsilon}\mu_0|_\theta \le \sum_{k=0}^{n-1} |(P_{T_0} - P_{T_\epsilon}) P^{n-k-1}_{T_\epsilon} \mu_0|_\theta \le n C'''_\theta \epsilon$$

for a suitable constant $C'''_\theta > 0$. Hence, in view of (92) and Theorem 4.1,

$$|\mu_\epsilon - \mu_0|_\theta \le |\mu_\epsilon - P^n_{T_\epsilon}\mu_0|_\theta + |P^n_{T_0}\mu_0 - P^n_{T_\epsilon}\mu_0|_\theta \le C''''_\theta (\gamma^n + n\epsilon)$$

and (111) follows by choosing n proportional to $\ln \epsilon^{-1}$. □

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