BIRTH OF AN ELLIPTIC ISLAND IN A CHAOTIC SEA

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Abstract. I consider a one parameter family of area preserving smooth maps that cross a non-uniformly hyperbolic situation into an elliptic one. I prove that exponentially close to such a family there are maps with positive metric entropy.

1. introduction

Although it is expected that generically symplectic maps exhibit mixed behavior (coexistence of integrable and chaotic behavior) almost no examples are available in which such a behavior is present. Noticeable exceptions are the cases where the two behaviors are separates by a homoclinic or etheroclinic invariant manifold and an example, due to Wojtkowski [23, 24], of a continuous (but not $\mathcal{C}^1$) map of the torus where the etheroclinic tangle is shown to be of positive measure. In the first context one can mention the work of Przytycki [19] in which he constructs a $\mathcal{C}^\infty$ one parameter family of area preserving toral diffeomorphisms that crosses the boundary of the set of Anosov diffeomorphisms due to a fixed point that from hyperbolic becomes first parabolic (this is the boundary diffeomorphism) and then elliptic. He shows that, for properly chosen families, when the fixed point becomes elliptic, both an elliptic island and an ergodic component of positive measure (which is Bernoulli) are present. As already mentioned, the two regions are sharply separated by an invariant (etheroclinic) manifold. I will call this type of situations the "Przytycki scenario." Later various authors constructed cases of three dimensional flows (e.g., Donnay's light bulb example [4, 5] of a geodesic flow on the sphere or the two torus, or the case of a particle moving in a special potential, always on the two torus, see [6]). More recently, one can mention the "mushroom billiards" by Bunimovich [3] (closely related to the two ergodic components example by Wojtkowski [25]).[1]

The work of this paper is very much in Przytycki spirit and it hints to the fact that each time a one parameter symplectic family leaves the Anosov class by loosing hyperbolicity at a single periodic orbit (the orbit becomes first parabolic and then elliptic), the Przytycki scenario is automatic, provided one is willing to perform an extremely small deformation of the family.

It should be remarked that the present work has noting to say on the problem of the frequency of positive metric entropy for symplectic maps. In this respect it

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[1] I am sure that the above list is far from exhaustive, my goal was simply to emphasize that there has been a considerable activity in trying to find relevant examples. For a further discussion of the issue consult the review article [21].

is know that the metric entropy is upper semi continuous [18], yet Mañé [13] has argued that the metric entropy is zero for a $\mathcal{C}^1$ dense set (see [2] for a proof). A nice discussion of problems connected to the metric entropy can be found in [7], (see also [17, 20] for related issues).

For the shake of clarity I will discuss a concrete one parameter family but the following could be applied more generally to families that exhibit the "Przytycki scenario" (see footnote 3).

2. THE MODEL

Let us consider the maps[2] $T_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$

$$T_\varepsilon(x,y) = \begin{cases} x_1 = x + y_1 \mod 2\pi \\ y_1 = y + h_\varepsilon(x) \mod 2\pi. \end{cases} \tag{2.1}$$

Where[3]

$$h_\varepsilon(x) = x - (1+\varepsilon)\sin x.$$

On the one hand, if $\varepsilon = -1$, then we have the well known linear automorphism of the torus, the basic example of Anosov maps. On the other hand, for ε very large the map becomes increasingly similar to the classical standard map, whose behavior is known to be very hard to describe. Let us try to follow the changes in the dynamics as ε increases.

Since zero is a fixed point for all ε it is instructive to see what happens to it: for $\varepsilon < 0$ it is hyperbolic, for $\varepsilon = 0$ it is parabolic and for $\varepsilon > 0$ elliptic. This turns out to reflect more global properties of the maps.

Indeed, for $\varepsilon \in (-1, 0)$ the system remains Anosov. For $\varepsilon = 0$ the systems is still hyperbolic (and mixing), but not uniformly so. To see this it suffices to notice that the cone $\mathcal{C}_+ := \{(u,v) \in \mathbb{R}^2 \mid uv \geq 0\}$ is (eventually) strictly invariant for each $\varepsilon < 0$ ($\varepsilon = 0$) and use classical results by Wojtkowski [22] (the mixing follows by [12, 8]).

Here we wish to push our understanding a bit further and investigate small, positive, ε.

The result of the paper is as follows.

Theorem 2.1. *There exists a constant $c > 0$, such that for each $\varepsilon > 0$, sufficiently small, there exists a symplectic map $\widetilde{T}_\varepsilon$ with positive metric entropy (close to the entropy of the map T_0) exponentially close to T_ε, that is*

$$d_{\mathcal{C}^2}(T_\varepsilon, \widetilde{T}_\varepsilon) \leq e^{-c\varepsilon^{-\frac{1}{2}}}.$$

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[2] Note that the following formula is equivalent, by the symplectic change of variable $q = x - y$, $p = y$, to the map

$$\overline{T}_\varepsilon(q,p) = \begin{cases} q + p \\ p + h_\varepsilon(q+p) \end{cases}$$

which belongs to the standard map family. Yet, the functions h_ε considered here differs from the sine function which would correspond to the classical Chirikov-Taylor well known example.

[3] Indeed, all the following would apply as well to a more general "Przytycki scenario". That is an h_ε with the following properties: $h_\varepsilon(x) = -h_\varepsilon(-x)$ for all ε; $h_0'(x) > 0$ for all $x \neq 0$; $h_0'(0) = 0$; $h_0'''(0) > 0$; $\frac{d}{d\varepsilon}h_\varepsilon'(0) < 0$.

In addition, it is possible to choose $\widetilde{T}_\varepsilon \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ *so that*[4]

$$m(\{x \in \mathbb{T}^2 \mid T_\varepsilon(x) \neq \widetilde{T}_\varepsilon(x)\}) \leq e^{-c\varepsilon^{-\frac{1}{2}}}.$$

Remark 2.2. *Although the above theorem is far from proving that the map* T_ε *itself has positive entropy, nevertheless it shows that any attempt to investigate numerically the continuity of the metric entropy is likely to be doomed.*

Remark 2.3. *We will see (Definition 1) that there exists a simple geometric condition to decide if, given a map* T *(exponentially close to* T_ε*), Theorem 2.1 applies or not.*

Remark 2.4. *Here we consider maps close in the* $\mathcal{C}^2$ *topology, it would be equally possible to consider perturbations in* $\mathcal{C}^k$ *or* $\mathcal{C}^\infty$ *topology, yet such a possibility does not seem very relevant in the present context.*[5] *It is instead unclear to me if one can consider analytic perturbations.*

The proof of the above theorem is the content of the next section.

More precisely, section 3.1 describes a flow approximation that will allow a precise control of the dynamics for quite long times. In addition, the perturbations to which Theorem 2.1 applies are defined. Section 3.2 show that such perturbations exist. Section 3.3 uses the results of section 3.1 to gain the needed control on the dynamics. Section 3.4 describes an eventually strictly invariant cone field for the perturbations. Finally section 3.5 concludes the proof.

3. PROOF

The basic problem is to gain a sufficient control on the dynamics near zero. This can be achieved via a flow approximation.

3.1. Blow up.

Let us consider the local change of coordinates $\Xi_\epsilon(x, y) := (q, p)$, $\epsilon := \sqrt{\varepsilon}$,

$$q := \epsilon^{-1}x; \quad p := \epsilon^{-2}y. \tag{3.1}$$

The change of coordinates is not symplectic, yet it has constant Jacobian hence the map $\hat{T}_\epsilon := \Xi_\epsilon T_{\epsilon^2} \Xi_\epsilon^{-1}$ is again symplectic. More precisely, we have

$$\hat{T}_\epsilon(q, p) = \begin{cases} q_1 = q + \epsilon p_1 \\ p_1 = p + \epsilon^{-2} h_{\epsilon^2}(\epsilon q). \end{cases}$$

where

$$\varepsilon^{-2} h_{\epsilon^2}(\epsilon q) = \epsilon g(q) + \epsilon^3 r(\epsilon q)\ ; \quad g(q) := -q + \frac{1}{6}q^3.$$

Thus

$$\hat{T}_\epsilon(q, p) = \begin{cases} q_1 = q + \epsilon p_1 \\ p_1 = p + \epsilon g(q) + \epsilon^3 r(\epsilon q). \end{cases} \tag{3.2}$$

[4]By m we designate the Lebesgue measure.

[5]In fact, in such a case one would have $T_\varepsilon \neq \hat{T}_\varepsilon$ on a much larger set which, from my point of view, would be less interesting, see Remark 3.5. At any rate, one can obtain good estimates in the $\mathcal{C}^\infty$ setting by using the theory of Gevrey classes for the partition of unity in Lemma 3.1.

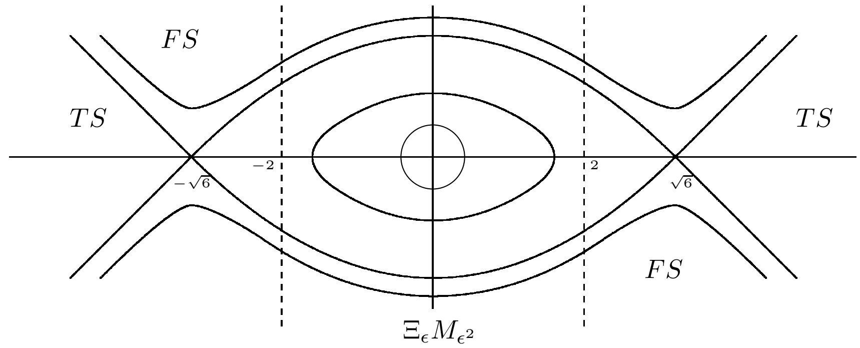

FIGURE 1. Phase portrait

Note that $\hat{T}_\epsilon$ is a perturbation of the identity, accordingly we can apply the results in [1] which state that there exists a local (time independent) Hamiltonian H_ϵ such that the associated time ϵ flow Ψ_ϵ has the property[6]

$$\|\hat{T}_\epsilon - \Psi_\epsilon\|_{\mathcal{C}^k} \leq k! C^{-k} e^{-C\epsilon^{-1}} \quad \forall k \in \mathbb{N}, \tag{3.3}$$

for some $C > 0$. The Hamiltonian can be computed as a power series:

$$H_\epsilon(q,p) = \frac{1}{2}p^2 + V(q) + \epsilon H^1_\epsilon(q,p)\ ; \quad V(q) = \frac{1}{2}q^2 - \frac{1}{4!}q^4. \tag{3.4}$$

The phase portrait of such an Hamiltonian flow is depicted in figure 1. Notice that, by the usual stability theorems, this implies that $\hat{T}_\epsilon$ has three fixed points as well, one elliptic and two hyperbolic, moreover the two hyperbolic fixed points have stable and unstable manifolds exponentially close to the one of the flow [9]. In addition, by KAM theory, there exist invariant tori for the map which are exponentially close to the separatrices.[7] Hence, the true map has an elliptic island exponentially close to the one of the Hamiltonian H_ϵ, accordingly trajectories coming from outside cannot enter in a neighborhood of zero which is exponentially close to the elliptic island of the flow. The only substantial difference between the phase portrait of the map and the one of the flow is that the latter may have a transversal etheroclinic intersection (one has to compute the related Melnikov integral to verify the transversality, yet it seems inevitable by genericity).[8]

The presence or not of the etheroclinic intersection is the dividing wall between the easily tractable cases that are discussed here and the much more difficult situation in which the coexistence between the integrable and ergodic behavior is intertwined in a cantor set like manner. Of course the latter case is the generic one, yet essentially nothing is known about it.

In the following we will then restrict our considerations to the case in which the two behaviors have a chance to be divided in a sharp manner.

[6]In fact, similar results, although in a less explicit form, are already present in [16] and, at the formal level, in [15]. More generally, every symplectic map can be viewed as a time one Hamiltonian flow provided the Hamiltonian is taken to be time dependent [14].

[7]This requires a somewhat careful analysis of the KAM estimates to be verified. I do not indulge in it since it is irrelevant for the task at hand.

[8]This also is an issue that requires quite a bit of work to be deal with. Yet the argument can be patterned after the various study of the splitting in a slow pendulum [11].

Definition 1. *Let $\mathcal{T}_{\varepsilon,c}$ be the set of maps $T \in \mathcal{C}^2(\mathbb{T}^2, \mathbb{T}^2)$ such that*

$$\|T_\varepsilon - T\|_{\mathcal{C}^2} \leq e^{-c\varepsilon^{-\frac{1}{2}}}.$$

and such that the two hyperbolic fixed points are joined by separatrices.

In other words we consider only maps whose phase space is akin to figure 1. Of course one may wonder if such a set is empty or not.

3.2. **Perturbations.** The set $\mathcal{T}_{\varepsilon,c}$ is far from empty and, in particular, contains $\mathcal{C}^\infty$ maps equal to T_ε on a large set.

Lemma 3.1. *For c and ε small enough, there exists $T \in \mathcal{T}_{\varepsilon,c}$ such that $T \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{T}^2)$ and the measure of the set $\{x \in \mathbb{T}^2 \mid T_\varepsilon x \neq Tx\}$ is smaller than $e^{c\varepsilon^{-\frac{1}{2}}}$.*

Proof. To exhibit the wanted perturbation we intend to construct a map that, away from the separatrices of the flow, coincides with the original map while close to them it coincides with the Hamiltonian flow Ψ_ϵ. In particular, $T = T_\varepsilon$ away from zero. We can then restrict our discussion to a neighborhood of zero and use the coordinates (3.1).

To carry out the above program, while maintaining area preserving, it is best to consider the generating functions of the two maps. Let $L(x, y_1) = xy_1 + \frac{\epsilon}{2}y_1^2 - G_\epsilon(x)$, where $G'_\epsilon(x) = \epsilon^{-2}h_{\epsilon^2}(\epsilon x)$, then the map defined by

$$x_1 = \frac{\partial L}{\partial y_1}$$
$$y = \frac{\partial L}{\partial x}$$

is exactly the map $\hat{T}_\varepsilon$.

On the other hand, for ϵ small enough, calling x, x_1 the initial and final (at time ϵ) position of a trajectory of the flow, the trajectory is uniquely determined (e.g., see [10]). Let us call $q(x, x_1, t)$ such a trajectory (clearly $x = q(x, x_1, 0)$ and $x_1 = q(x, x_1, \epsilon)$). Then the function S defined by[9]

$$\begin{aligned} S(x, y_1) &:= y_1 x_1 - \int_0^\epsilon \mathcal{L}(q(x, x_1, s), \dot{q}(x, x_1, s))\, ds \\ y_1 &= \frac{\partial \mathcal{L}}{\partial \dot{q}}(x_1, \dot{q}(x, x_1, \epsilon)), \end{aligned} \tag{3.5}$$

(where the last equation defines x_1 as a function of x, y_1) is the generating function of the map Ψ_ϵ.

Next, let Γ_ϵ be the separatrices of the flow, and let $\tilde{\Gamma}_\epsilon$ be a $\mathcal{C}^\infty$ curve, containing Γ_ϵ in its interior, such that

$$\frac{1}{2}e^{-\frac{C}{4}\epsilon^{-1}} \leq \inf_{x \in \tilde{\Gamma}_\epsilon} d(x, \Gamma_\epsilon) \leq \sup_{x \in \tilde{\Gamma}_\epsilon} d(x, \Gamma_\epsilon) \leq 2e^{-\frac{C}{4}\epsilon^{-1}}.$$

Note that the curve can be constructed so that its curvature is bonded by $c_1 e^{\frac{C}{4}\epsilon^{-1}}$, for some $c_1 > 0$, and the derivative of the curvature is bounded by $c_1 e^{\frac{C}{2}\epsilon^{-1}}$. Let Σ_ϵ be a $c_2 e^{-\frac{C}{4}\epsilon^{-1}}$ neighborhood of $\tilde{\Gamma}_\epsilon$.[10] Clearly the complement of Σ_ϵ in $\mathbb{R}^2$ consists of

[9]Clearly $\mathcal{L}$ is the Lagrangian associated to the Hamiltonian (3.4).

[10]Where $c_2 < \frac{1}{4}$ is taken sufficiently small, with respect to c_1, to insure that in Σ_ϵ is well defined the system of coordinates (s, ρ), where s is the arclenght along $\tilde{\Gamma}_\epsilon$ and ρ is the distance from $\tilde{\Gamma}_\epsilon$.

two connected components, the one containing zero together with the elliptic island, and the unbounded one. Finally, let $\chi_e : \mathbb{R}^2 \to \mathbb{R}_+$ be a smooth function equal to zero in the bounded component and to one in the unbounded one. In addition, we require that $\|\chi_e\|_{\mathcal{C}^3} \leq c_3 e^{\frac{3C}{4}\epsilon^{-1}}$ for some c_3 large enough.[11] Analogously, we can consider a smooth curve $\bar\Gamma_\epsilon$ at the same distance from Γ_ϵ but inside it, and let $\bar\Sigma_\epsilon$ be a $c_2 e^{-\frac{C}{4}\epsilon^{-1}}$ neighborhood of $\bar\Gamma_\epsilon$. We can then consider the smooth function $\chi_i : \mathbb{R}^2 \to \mathbb{R}_+$ equal to one in the bounded component of the complement of $\bar\Sigma_\epsilon$ and to zero in the unbounded one, always with the requirement $\|\nabla\chi_i\|_{\mathcal{C}^3} \leq c_3 e^{\frac{3C}{4}\epsilon^{-1}}$. Let $\chi = \chi_e + \chi_i \geq 0$, by construction χ equals zero in an exponentially small neighborhood of the separatrices and equal one away from it. Define

$$\tilde L_\epsilon := \chi L_\epsilon + (1-\chi) S_\epsilon.$$

The function $\tilde L_\epsilon$ generates the map $\tilde T_\epsilon$ which coincides with $\hat T_\varepsilon$ away from the separatrices and with Ψ_ε in a neighborhood of it. Such a map is the perturbation mentioned in Theorem 2.1 (after the obvious change of coordinates). Notice that, since L_ϵ and S_ϵ must be exponentially close, $\|T_\epsilon - T\|_{\mathcal{C}^2} \leq c_4 e^{\frac{C}{4}\epsilon^{-1}}$ and that the maps differ only in a set of exponentially small measure. □

Now that we verified the existence of many maps of the wanted type, we can go back to the study of their dynamics. In fact, thanks to the results of section 3.1, it is possible to gain a rather precise control on the dynamics near zero.

3.3. **Near separatrix dynamics.** By the results of section 3.1 it follows that, for each $T \in \mathcal{T}_{\varepsilon,c}$,

$$|H_\epsilon(T(q,p)) - H_\epsilon(q,p)|_\infty \leq |H_\epsilon(\Psi_\epsilon(q,p)) - H_\epsilon(T(q,p))|_\infty \leq 2e^{-c\epsilon^{-1}}.$$

Hence

$$|H_\epsilon(T^n(q,p)) - H_\epsilon(q,p)|_\infty \leq 2ne^{-c\epsilon^{-1}}. \tag{3.6}$$

Thus the trajectories remain close to the energy levels of the Hamiltonian for an exponentially long time. Yet, the trajectories of the two maps can diverge much faster. The best estimates available in such a generality are[12]

$$\begin{aligned}|T^n\xi - \Psi_\epsilon^n\xi| &\leq \sum_{k=1}^n |T^k \circ \Psi_\epsilon^{n-k}\xi - T^{k-1}\circ\Psi_\epsilon^{n-k+1}\xi| \leq \sum_{k=1}^n e^{c(n-k)\epsilon}2e^{-c\epsilon^{-1}}\\ &\leq 2\epsilon^{-1}e^{cn\epsilon - c\epsilon^{-1}} \leq e^{-\frac{c}{2}\epsilon^{-1}},\end{aligned} \tag{3.7}$$

provided $n \leq \frac{1}{3}\epsilon^{-2}$ and ϵ is small enough.

Analogous estimates hold for the derivatives

$$\begin{aligned}|D_\xi T^n - D_\xi\Psi_\epsilon^n| &\leq \sum_{k=1}^n |D_{\hat T^{n-k+1}\xi}T^{k-1}[D_{T^{n-k}\xi}T - D_{\Psi_\epsilon^{n-k}\xi}\Psi_\epsilon]D_\xi\Psi_\epsilon^{n-k}|\\ &\leq 4ne^{cn\epsilon}e^{-\frac{c}{2}\epsilon^{-1}} \leq e^{-\frac{c}{4}\epsilon^{-1}},\end{aligned} \tag{3.8}$$

[11] For example, consider the function $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ defined by $g(x) = 0$ for $x \leq 0$ and $g(x) = e^{-\frac{1}{x}}$ for $x > 0$. Then define $\chi_e(s,\rho) = \frac{g(\rho+\delta)}{g(\rho+\delta)+g(\delta-\rho)}$, $\delta := c_2 e^{-\frac{c}{4}\epsilon^{-1}}$, where I have used the coordinates introduced in footnote 10.

[12] Note that here and in the following we abuse notations by using Ψ_ϵ both to designate the flow in the (x,y) and the (q,p) coordinates. This should create no confusion since the coordinate systems is always clear form the context.

provided $n \leq \frac{1}{6}\epsilon^{-2}$.

Lemma 3.2. *Let* $\{x, Tx, \dots, T^n x\}$ *be a trajectory from entering to exiting the neighborhood* $[-\sqrt{\varepsilon}, \sqrt{\varepsilon}] \times [-\varepsilon, \varepsilon]$. *There exists* $L > 0$ *such that if the trajectory enters at a distance larger than* $Le^{-c\epsilon^{-1}}$ *from the stable manifolds of the hyperbolic fixed points, then*

$$\|T^n - \Psi_\epsilon^n\|_{\mathcal{C}^1} \leq e^{-\frac{c}{4}\epsilon^{-1}}.$$

Proof. Note that if the trajectory keep always a distance $\mathcal{O}(\epsilon^3)$ from the x-axis, then $n = \mathcal{O}(\epsilon^{-2})$ and the result follows from (3.7),(3.8).

Let us consider closer encounters for the flow first. Let E_* be the energy of the two hyperbolic fixed points, if the energy of the entering trajectory is smaller than E_*, then the trajectory remains all the time on the same side of the hyperbolic fixed point. Suppose instead the energy larger than $E_* + Le^{-C\epsilon^{-1}}$. Let $p_*(q) \geq 0$ be the equation of the stable manifolds of the hyperbolic points for the flow, the separatrix and the unstable one (the analysis for $p < 0$ is completely similar). Note that p_* is analytic but at the hyperbolic fixed points. Clearly $H_\epsilon(q, p_*(q)) = E_*$.

Let $\delta_E := E - E_* \geq Le^{-C\epsilon^{-1}}$ and $\delta p(q)$ be defined by

$$\delta_E = H_\epsilon(q, p_*(q) + \delta p(q)) - E_*.$$

Then

$$\delta p(q) = \frac{-\frac{\partial H_\epsilon}{\partial p}(q, p_*) + \sqrt{\left(\frac{\partial H_\epsilon}{\partial p}(q, p_*)\right)^2 + 2\delta_E \frac{\partial^2 H_\epsilon}{\partial p^2}(q, \xi)}}{\frac{1}{2}\frac{\partial^2 H_\epsilon}{\partial p^2}(q, \xi)}$$

for some $\xi \in [p_*, p_* + \delta p]$. By the explicit form of the Hamiltonian (3.4), it follows

$$\frac{\delta_E}{4} \leq \delta p(q) \leq 4\delta_E$$

provided q is large enough. Accordingly a trajectory of the flow that enters in the neighborhood at a distance δ_E from the stable manifold will exit the neighborhood at a distance proportional to δ_E of the unstable one.

Next, consider that if the trajectory gets closer that δ to the hyperbolic fixed point (in the blowed up coordinates) and further away than δ from its stable manifold, then it will take a time $\mathcal{O}(\ln \delta)$ to get to a distance of order one. Accordingly the trajectories discussed above will spend at most a time $\mathcal{O}(\epsilon^{-1})$ in a neighborhood of the hyperbolic fixed points. Thus, provided that L has been chosen large enough, the above scenario will hold also for a trajectory of the map T due to (3.7), (3.8). Notice that if $\delta_E < Le^{-C\epsilon^{-1}}$, then the upper bound will still hold while the existence of the separatrices will anyhow constrain the motion from entering the internal region. □

3.4. **A cone filed.** Let us fix $T \in \mathcal{T}_{\varepsilon,c}$. Let $S_\varepsilon := \{(x, y) \in \mathbb{T}^2 \mid |x| \leq \varepsilon^{-\frac{1}{4}}\}$, $M_\varepsilon := \{(x, y) \in \mathbb{T}^2 \mid \cos x \leq (1 + 2\varepsilon)^{-1}\}$ and Ω be the all region outside the separatrices of the map. Obviously, $\overline{\Omega}$ is an invariant compact set. Clearly M_ε is more or less the strip $|x| \leq 2\epsilon$, while the fixed points are roughly at $|x| = \sqrt{6}\epsilon$, thus well outside M_ε (see figure 1). Note that the stable and unstable manifolds of the hyperbolic fixed points divide S_ε into five separate open regions: the region belonging to the elliptic island, two thin sectors TS on the left and right of the hyperbolic fixed points and two fat sectors FS below and above the island (see figure 1). We define a cone field $\mathcal{C}$ on Ω as follows: $\mathcal{C}(x) := \mathcal{C}_+$ for all $x \notin S_\varepsilon \cap FS$; if $x \in S_\varepsilon \cap FS$, then let $n \in \mathbb{N}$ the smallest integer such that $T^{-n}x \notin S_\varepsilon$ (clearly such an n exists finite if $x \in S_\varepsilon \cap FS$). Define $v_n := D_{T^{-n}x}DT^n(1,0)$ and for each $v \in \mathbb{R}^2$ let α_n, β_n be such that $v = \alpha_n v_n + \beta_n(0,1)$, define then $\mathcal{C}(x) := \{v \in \mathbb{R}^2 \mid \alpha_n\beta_n \geq 0\}$.[13]

Lemma 3.3. *The cone field $\mathcal{C}$ is eventually strictly invariant on Ω.*

Proof. Notice that

$$DT_\varepsilon(x,y)(1,u) = \begin{pmatrix} 1+h'_\varepsilon(x) & 1 \\ h'_\varepsilon(x) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ u \end{pmatrix} = (1 + h'_\varepsilon(x) + u)(1, F(x,y,u)).$$

where

$$F(x,y,u) := \frac{h'_\varepsilon(x) + u}{1 + h'_\varepsilon(x) + u}.$$

The vector $(0,1)$ is always rotated clockwise by an uniform amount by the Jacobian of the map T_ε, hence also the Jacobian of T does the same. Thus, all is needed is to check the lower edge of the cone. First recall that for each $(x,y) \notin M_\varepsilon$, $D_{(x,y)}T_\varepsilon\mathcal{C}_+ \subset \text{int}(\mathcal{C}_+) \cup \{0\}$, thus the cone field is strictly invariant outside $S_\varepsilon \cap FS$. Second notice that, by definition, the lover edge is exactly invariant as long as the trajectory lies in $S_\varepsilon \cap FS$, accordingly all we need to check is that, upon exiting such a region, the lover edge belongs to the interior of $\mathcal{C}_+$. To show this we will follows the lower edge along trajectories increasingly closer to the separatrix using first the map T_ε, then the flow Ψ_ϵ and finally the dynamics on the separatrix itself as efficient approximations of the behavior of trajectories of the map T.

If $y \geq L\sqrt{\varepsilon}$, for L large enough, the trajectory can spend only one time step in M_ε and, if $T(x,y) \in M_\varepsilon$, then $D_{(x,y)}T_\varepsilon^2\mathcal{C}_+ \subset \mathcal{C}_+$. The same result holds trivially for the map T. Accordingly, the cone field is invariant for T, as long as the trajectory does not enter in a $\sqrt{\varepsilon}$ neighborhood of $\{x = 0\}$. Next, let us look more carefully at what happens in such a neighborhood.

First of all notice that if $|x| \geq \varepsilon^{\frac{1}{4}}$, then

$$F(x,y,0) \geq h_\varepsilon(x) \geq \frac{\varepsilon^{\frac{1}{2}}}{4}.$$

In addition, there exists $c > 0$ such that, if $c\varepsilon^{\frac{1}{2}} \leq |x| \leq \varepsilon^{\frac{1}{4}}$ and $u \in [0, \frac{\varepsilon^{\frac{1}{2}}}{4}]$, then

$$F(x,y,u) \geq u.$$

This implies that $\mathcal{C}(x) \subset \{(1,u) \in \mathbb{R}^2 \mid u \geq \frac{\varepsilon^{\frac{1}{2}}}{4}\}$ until $|x| \geq c\varepsilon^{\frac{1}{2}}$. Moreover, if $|x| \leq c\varepsilon^{\frac{1}{2}}$ and $u \in [0, \frac{\varepsilon^{\frac{1}{2}}}{4}]$, then

$$F(x,y,u) \geq u - C\varepsilon$$

for some $C > 0$. Finally, if $|y| \geq M\varepsilon$ and $|x| \leq c\varepsilon^{\frac{1}{2}}$, then the trajectory exists from the $c\varepsilon^{\frac{1}{2}}$ neighborhood of zero in a time at most $2M^{-1}\varepsilon^{-\frac{1}{2}}$, provided M is chosen large enough. Accordingly, the second component of the normalized image of the vector $(1, \frac{\varepsilon^{\frac{1}{2}}}{4})$, when the point exits the dangerous region, is larger than $\frac{\varepsilon^{\frac{1}{2}}}{4} - C2M^{-1}\varepsilon^{\frac{1}{2}} \geq \frac{\sqrt{\varepsilon}}{8}$ provided M has been chosen large enough.

By Lemma 3.2 this shows that the cone field $\mathcal{C}$ is eventually strictly invariant for T unless the trajectory enters the region $R_\varepsilon := \{(x,y) \in \mathbb{R}^2 \mid |x| \leq c\varepsilon^{\frac{1}{2}};\ |y| \leq M\varepsilon\}$. This last possibility requires a finer analysis.

[13] That is, $\mathcal{C}(x)$ is the sector between the vector v_n and the vector $(0,1)$.

For each point $\xi \in R_\varepsilon$ let $\overline{V}(\xi) = (1, V(\xi))$, be the flow direction and let N be the time at which the point exits R_ε. The image vector will be $d\Psi_\epsilon^N \overline{V} = \lambda(1, V)$, $\lambda > 0$. Obviously, $d\Psi_\epsilon^N \overline{V} \in \mathcal{C}_+$ uniformly, see figure 1. Accordingly, Lemma 3.2 implies that if the trajectory enters in $S_\varepsilon \cap FS$ at a distance larger than $\mathcal{O}(e^{-c_0\epsilon^{-1}})$, then $n = \mathcal{O}(\epsilon^{-1})$ and $DT^N\overline{V} \in \mathcal{C}_+$, provided c_0 is chosen small enough. Since, upon entering R_ε, $V(\xi) < 0$ while $V(T^n\xi) > 0$, see figure 1, it follows $v_n > 0$, that is $DT^N\mathcal{C}_+ \subset \mathcal{C}_+$ (that is, the lover edge of the cone is always above the flow direction).

This easy analysis holds for all the maps in a $e^{-c\varepsilon^{\frac{1}{2}}}$ neighborhood of T_ϵ. Unfortunately, it fails for trajectories that border the elliptic island at a distance smaller than $e^{-c_0\epsilon^{-1}}$. For a simple analysis of these last trajectories it is necessary to assume the existence of a separatrix (i.e. $T \in \mathcal{T}_{\varepsilon,c}$). In such a case it is possible to compare their behavior with the behavior over the separatrix. In order to achieve this some preliminary considerations are in order.

We start by noticing that, for each $u \geq 0$

$$\frac{\partial F(x,y,u)}{\partial x} = \frac{(1+\varepsilon)\sin x}{(1+h_\varepsilon'(x)+u)^2}. \tag{3.9}$$

Thus, for $x \leq -\varepsilon$ we have $\frac{\partial F(x,y,u)}{\partial x} \leq -\frac{|x|}{2}$ and the same holds for the analogous quantity of the map T. Now let $(1, v_*)$ be the unstable direction of the hyperbolic fixed point on the left, and let (x_*, y_*) be its coordinates. It is easy to show that $|x_* + \sqrt{6\varepsilon}| \leq C\varepsilon$. If $x \leq x_*$ and $D_{(x,y)}T(1, v_*) =: \lambda(1, u)$, then the above facts imply that $u \geq v_*$. In other words $\mathcal{C}(x) \subset \{(u, v) \in \mathbb{R}^2 \mid \frac{v}{u} \geq v_*\}$ for each (x, y) above the stable manifold and with $x \leq x_*$. Next, let $(x, \gamma(x))$ be the equation of the upper separatrix. Note that $\gamma'(x_*) = v_*$.

Sub-lemma 3.4. *For each ε small enough, the separatrix is convex, more precisely $\gamma'' \leq -\frac{1}{2\sqrt{3}}$.*

Proof. Since the stable and unstable manifolds are continuous in the $\mathcal{C}^r$ topologies [9], it suffices to prove the lemma for the flow. This is best done in the blow up coordinates; recall that in such coordinates the separatrix reads $(q, p_*(q))$.[14] Using again the stability of the invariant manifolds it suffices to prove the result for the Hamiltonian, see (3.4),

$$H_0(q,p) = \frac{1}{2}p^2 + V(q).$$

For such an Hamiltonian the energy level of the separatrix is $H_0 = \frac{3}{2}$ and the separatrix p_*^0 reads, for all $|q| \leq \sqrt{6}$,

$$p_*^0(q) = \sqrt{3 - q^2 + \frac{1}{12}q^4} = \sqrt{3}(1 - \frac{1}{6}q^2),$$

Thus $(p_*^0)'' = -\frac{1}{\sqrt{3}}$ and, as already mentioned, by stability analogous estimates follow for $p_*'' = \gamma''$ and, finally, for the second derivative of the separatrix of the map T. □

In addition, since $T(x,y) = (x+y+h_\varepsilon(x)+\delta\alpha_\varepsilon(x,y), y+h_\varepsilon(x)+\delta\beta_\varepsilon(x,y))$, with δ exponentially small, for each two points (x, y), (x, y'), $y' \geq y$, setting $(x_1, y_1) :=$

[14] Note that, $\gamma'' = p_*''$.

$T(x, y)$ and $(x'_1, y'_1) := T(x, y')$, one has

$$\begin{aligned}y'_1 - y_1 &= x + y' + h_\varepsilon(x) + \delta\alpha_\varepsilon(x, y') - x - y - h_\varepsilon(x) - \delta\alpha_\varepsilon(x, y)\\ &= (1 - \mathcal{O}(\delta))(y' - y) \geq \frac{y' - y}{2} > 0,\end{aligned}$$

that is, higher points are pushed more on the right (the map is close to a twist). Accordingly, we have for (x, y), $x \geq x_*$ and $y > \gamma(x)$, and setting $(x_1, \gamma(x_1)) := T(x, \gamma(x))$, $(x'_1, y'_1) := T(x, y)$,

$$\begin{aligned}\lambda(x)(1, u) := D_{(x,y)}T(1, \gamma'(x)) =& D_{(x,\gamma(x))}T(1, \gamma'(x)) + \mathcal{O}(\delta|y - \gamma(x)|)\\ =& \tilde{\lambda}(x)(1, \gamma'_T(x_1)) + \mathcal{O}(\delta|y - \gamma(x)|)).\end{aligned}$$

The above inequality shows that the evolution of the tangent vectors is sharply controlled by the evolution on the separatrix. We are now ready to exploit the convexity of the latter. Since $x'_1 - x_1 = y'_1 - \gamma(x_1) \geq \frac{y-\gamma(x)}{2}$, it follows

$$\begin{aligned}u &= \gamma'_T(x_1) + \mathcal{O}(\delta|y - \gamma(x)|))\\ &> \gamma'_T(x'_1) + \frac{1}{2\sqrt{3}}|y - \gamma(x)| + \mathcal{O}(\delta|y - \gamma(x)|))\\ &\geq \gamma'_T(x'_1) + \frac{1}{4\sqrt{3}}|y - \gamma(x)| > \gamma'_T(x'_1).\end{aligned} \tag{3.10}$$

In other words, calling (x^*, y^*) the coordinates of the hyperbolic fixed point on the right, $\mathcal{C}((x, y)) \subset \{(u, v) \in \mathbb{R}^2 \mid \frac{v}{u} \geq \gamma'(x)\}$, for all $x \leq x^*$.

The same argument as in equation (3.10) shows that if (x, y) and $(x_{-1}, y_{-1}) := T^{-1}(x, y)$ are such that $x \geq x^*$ and $x_{-1} < x^*$, $y_{-1} > \gamma(x_{-1})$ then there exists $c_* > 0$ such that $\mathcal{C}((x, y)) \subset \{(u, v) \in \mathbb{R}^2 \mid \frac{v}{u} \geq v^* + c_*(y - y^*)\}$, where $v^* := \gamma'(x^*)$. Clearly $(1, v^*) \in \mathbb{R}^2$ is the stable direction of the hyperbolic fixed point (x^*, y^*). Then equation (3.9) implies that, for $(1, v) \in \mathcal{C}$, setting $D_{(x,y)}T^n(1, v) =: \lambda_n(1, v_n)$ and $D_{(x^*,y^*)}T^n(1, v) =: \tilde{\lambda}_n(1, \tilde{v}_n)$, $v_n > \tilde{v}_n$. By usual distortion arguments, we can essentially consider the evolution linear until the distance from the fixed point is of order $\sqrt{\varepsilon}$, this will take a time of about $n \sim \ln(y - y^*)\epsilon^{-1}$, at the same time, under the action of $D_{(x^*,y^*)}T$, the stable component of $(1, v)$ will shrink by a factor $(y - y^*)^{-1}\epsilon$ and the unstable component will expand by the same factor. This clearly means that $\tilde{v}_n > 0$. Hence the cone field will be strictly invariant upon exiting the region $S_\varepsilon \cap FS$. □

3.5. **Positive entropy.** We have thus proved that for each map $T \in \mathcal{T}_{\varepsilon,c}$ there exists a measurable cone field $\mathcal{C}$ which is eventually strictly invariant on the invariant set Ω (the region outside the separatrices). It follows from [22] that the Lyapunov exponents are positive in Ω. Since the cone field is continuous (actually constant) on the open set $U := \Omega \backslash S_\varepsilon$ it follows by [12, 8] that U belongs to one ergodic component. Since Ω is equal almost everywhere to the union of the images of U it follows that Ω consists of only one ergodic component. In addition, (Ω, T) is mixing, [8].

Accordingly, the entropy $h(T)$ of T is given by

$$h(T) := \int_\Omega \lambda^+(x)dx + \mathcal{O}(\varepsilon^{\frac{3}{2}}) = \lambda^+_\Omega + \mathcal{O}(\varepsilon^{\frac{3}{2}}).$$

Moreover, calling v_u the unstable direction, $\|v_u\| = 1$,

$$\lambda_\Omega^+ = \frac{1}{m(\Omega)} \int_\Omega \ln \|DTv_u\|.$$

Next, notice that, outside a $\sqrt{\varepsilon}$ neighborhood of zero, $v_u \in \mathcal{C}_+$. On the other hand $\|DT_0 - DT\| = \mathcal{O}(\varepsilon)$, it is then easy to verify that, calling v_u^0 the unit unstable vector of T_0, outside a $\varepsilon^{\frac{1}{4}}$ neighborhood of zero it holds true $v_u - v_u^0 = \mathcal{O}(\varepsilon)$. We can then compute

$$\lambda_\Omega^+ = \int_{\mathbb{T}^2} \ln \|DT_0 v_u^0\| + \mathcal{O}(\varepsilon^{\frac{1}{2}}) = h(T_0) + \mathcal{O}(\varepsilon^{\frac{1}{2}}).$$

Accordingly one can construct a one parameter family of maps (exponentially close to the initial one) for which the metric entropy is continuous at $\varepsilon = 0$. Similar arguments can be used to show continuity at $\varepsilon \neq 0$ for ε sufficiently small.

Remark 3.5. *Notice that if we choose the special map constructed in Lemma 3.1, then also the time averages for L^∞ functions, with respect to T_ε or T, differ (in L^1) of an exponentially small amount for an exponentially long time.*

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