Operator Theory:
Advances and Applications, Vol. 153, 123–141

Statistical Properties of Disordered Quantum Systems

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Abstract. We discuss two different approaches to the study of the long-time behavior of some disordered quantum anharmonic chains.

Mathematics Subject Classification (2000). Primary 82C10, 47A35. Secondary 82B44, 46L55.

Keywords. Quantum dynamics; Non equilibrium statistical mechanics; Ergodic theory; Disordered systems; Non commutative dynamical systems.

1. Introduction

Notwithstanding the rising interest in the ergodic properties of infinite systems, only a limited amount of results are available for non linear quantum evolutions. This is particularly true if one wants to gain a knowledge of invariant states that are not necessarily KMS states for the system. The latter problem is of interest since, if there exist invariant non-KMS states which are stable with respect to a wide class of perturbations, then their physical relevance could be comparable with the one of the KMS states. Indeed, it is shown in [6], that this is the case for an infinite chain with one defect, in which only one spring is subject to a small anharmonic perturbation. More precisely, starting from the model studied in [4], it was considered in proper units, the Hamiltonian

$$H(q,p)=\frac{1}{2}\sum_{i\in\mathbb{Z}}p_i^2+\frac{1}{2}\left(\frac{1}{M}-1\right)p_0^2+\frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2$$
$$+\sum_{i\in\mathbb{Z}}\frac{\kappa}{2}q_i^2+\frac{K}{2}q_0^2+V(q_0)\,, \tag{1.1}$$

where the non linear part V is small and regular in some appropriate sense.

It is well-known (see, e.g., [3]) that the corresponding linear system (i.e., when $V\equiv 0$) exhibits a multitude of quasi-free states invariant with respect to the free dynamics α_t^0, due to the integrable nature of the Hamiltonian. It is proven in [6] that, for such a class of perturbations $V \neq 0$, the infinite dynamics α_t associated to (1.1) is well defined – on an appropriate C^*-algebra $\mathfrak{M}$ and is described by a uniformly convergent series with respect to the time.

Recall that for a dynamical system $(\mathfrak{A}, \gamma_t, \varphi)$ with φ invariant for the dynamics, forward mixing (or equivalently strong clustering) means that

$$\lim_{t\to+\infty} \varphi(B^*\gamma_t(A)B) = \varphi(B^*B)\varphi(A) \tag{1.2}$$

for each $A, B \in \mathfrak{A}$.[1] Our definition of (forward) mixing slightly differs from the standard one

$$\lim_{t\to+\infty} \varphi(B\gamma_t(A)) = \varphi(A)\varphi(B)\,, \tag{1.3}$$

see, e.g., [9], see also [11] for very recent developments on ergodicity in quantum setting. We adopt the former as it is more suitable for applications to quantum mechanics (e.g., [8] where (1.2) is referred as the property of "return to equilibrium").

It is expected that (1.2) is stronger than (1.3). However we have

Proposition 1.1. *Property* (1.2) *implies* (1.3). *Conversely, if the dynamical system* $(\mathfrak{A}, \gamma_t)$ *is asymptotically abelian in norm, or if* Ω_φ *is cyclic for* $\pi_\varphi(\mathfrak{A})'$, *then* (1.2) *and* (1.3) *are equivalent,* $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ *being the GNS triplet of* φ.

Proof. It is immediate to verify that (1.2) and (1.3) are equivalent if $(\mathfrak{A}, \gamma_t)$ is asymptotically abelian in norm.

As Ω_φ is cyclic for $\pi_\varphi(\mathfrak{A})$, (1.2) implies that, if $\Psi \in \mathcal{H}_\varphi$, then $\langle \Psi, \pi_\varphi(\gamma_t(A))\Psi\rangle \longrightarrow \|\Psi\|^2\varphi(A)$ which gives by polarization

$$\begin{aligned}\varphi(B\gamma_t(A)) &\equiv \langle \pi_\varphi(B^*)\Omega_\varphi, \pi_\varphi(\gamma_t(A))\Omega_\varphi\rangle \\ &\longrightarrow \left(\frac{1}{4}\sum_{\omega^4=1} \omega\|(\pi_\varphi(B^*) + \omega I)\Omega_\varphi\|^2\right)\varphi(A) \equiv \varphi(A)\varphi(B)\,,\end{aligned}$$

that is (1.2) $\Rightarrow$ (1.3).

Let Ω_φ be cyclic for $\pi_\varphi(\mathfrak{A})'$. As Ω_φ is cyclic for $\pi_\varphi(\mathfrak{A})$, we get $\langle \Phi, U_t\Psi\rangle \longrightarrow \langle \Phi, \Omega_\varphi\rangle\langle\Omega_\varphi, \Psi\rangle$, for each $\Phi, \Psi \in \mathcal{H}_\varphi$, where U_t is the unitary implementation of γ_t in the GNS representation. By applying this with $\Phi = T^*\pi_\varphi(B)\Omega_\varphi$, $\Psi = \pi_\varphi(A)\Omega_\varphi$ for $T \in \pi_\varphi(\mathfrak{A})'$, we obtain $\langle \Omega_\varphi, \pi_\varphi(B^*)\pi_\varphi(\gamma_t(A))T\Omega_\varphi\rangle \longrightarrow \langle \Omega_\varphi, \pi_\varphi(B^*)T\Omega_\varphi\rangle\varphi(A)$. As Ω_φ is cyclic also for $\pi_\varphi(\mathfrak{A})'$, the last result is true for generic $\Psi \in \mathcal{H}_\varphi$ instead of $T\Omega_\varphi$. The assertion (1.3) $\Rightarrow$ (1.2) when Ω_φ is also cyclic for $\pi_\varphi(\mathfrak{A})'$ follows by computing the last limit for $\Psi = \pi_\varphi(B)\Omega_\varphi$. □

The results contained in [6] can be summarized as follows.

Theorem 1.2. *Suppose that the potential* V *in* (1.1) *is sufficiently small and regular. Then there exists a class of states on* $\mathfrak{M}$, *which is invariant and mixing w.r.t. the linear dynamics* α^0_t, *such that the following assertions hold true.*

[1] The backward mixing is defined analogously.

(i) *To any state ω in the class mentioned above, it corresponds a unique state ω_∞ invariant w.r.t. the non linear dynamics α_t, which is obtained as the following pointwise limit*

$$\omega_\infty(A) = \lim_{t\to\pm\infty} \omega(\alpha_t A)\,.$$

(ii) *The dynamical system $(\mathfrak{M}, \alpha_t, \omega_\infty)$ is mixing.*

In the situation described above, the perturbed dynamical system $(\mathfrak{M},\alpha_t,\omega_\infty)$ exhibits the same strongly ergodic properties as the unperturbed system $(\mathfrak{M},\alpha^0_t,\omega)$. Namely, there exists a non trivial class of states, non necessarily KMS, which are "stable" for suitable perturbations of the dynamics.

From the technical point of view, the results in [6] are based on a sort of L^1-asymptotic abelianess for the free dynamics, in accordance with the standard technique used in perturbation theory, see, e.g., [9] and the literature cited therein. Loosely speaking, it is shown that, calling $\{W(\lambda e_k)\}_{\lambda\in\mathbb{R}}$ the Weyl operators associated to the variables of the kth particle, it holds true for each fixed j,k in the chain,

$$\int_{\mathbb{R}^+} \| \, [W(e_j), \alpha^0_t W(e_k)] \| \, \mathrm{d}t < +\infty\,.$$

Of course, one can criticize [6] by saying that it is limited to a bounded perturbation of a linear infinite system, and that such a small and local perturbation cannot suffice to alter the statistical properties of an infinite system. In other words, the allowed perturbations are unreasonably restricted from the physical point of view. Accordingly, the stability of non-KMS states with respect to such perturbations is really too limited to imply something about the realistic stability of infinite states.

It is then natural to address the possibility to investigate a translation invariant perturbation, that is a locally small perturbation, but present at each site of the chain. Of course, in such a case the perturbation would be unbounded. To treat such a model with a perturbative approach in the spirit of [6], it would be necessary to have a sort of L^1 space-time asymptotic abelianess, that is

$$\sup_{k\in\mathbb{Z}} \Big[\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^+} \| \, [W(e_k), \alpha^0_t W(e_j)] \, \| \, \mathrm{d}t \Big] < +\infty\,. \tag{1.4}$$

Unfortunately, harmonic lattices, or harmonic lattices with finitely many defects, do not seem to enjoy such strong ergodic properties. Nevertheless, there exist translation invariant non linear dynamics for which (1.4) is satisfied. These are related to the attempt to make sense of the dynamics associated to the formal random Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \kappa_i q_i^2 + \frac{1}{2}\sum_{|i-j|=1} (q_i - q_j)^2 + V(q_i - q_j)\,, \tag{1.5}$$

where the κ_i are random variables on a common probability space satisfying certain reasonable properties.[2] The model described by the Hamiltonian (1.5) can be thought as an anharmonic chain with infinitely many random defects subjected to neighbor interactions. The construction of such a dynamics is based on the perturbation of a quasi-free dynamics given by Bogoliubov transformations acting on a phase-space of a CCR algebra, defined by averaging the commutators with respect to the random environment. One then considers quasi-free states ω determined by the truncated quenched state. In such a context, it is proven in [7], the analogous version of Theorem 1.2, see the next sections for further details.

A natural criticism to the above result is that to average and then quantize it is not the same as to quantize and then average. In other words, it could be physically more relevant to study the dynamics associated to the (formal) Hamiltonian (1.5) for each fixed realization of the spring constants $\{\kappa_i\}_{i\in\mathbb{Z}}$, and then average with respect to the randomness. These two alternatives are better illustrated in Section 2. Unfortunately, it is not clear how to define the infinite dynamics for each spring configuration, since (1.4) does not hold pointwise. In addition, the C^*-algebra in such a case would inevitably have a non trivial center, and this also contributes to spoil the ergodic properties (again see Section 2 for more details). On such issues, no rigorous result relative to long-time behavior seems to be available in literature.

In order to pursue the program related to the latter approach, some new preliminary results on the long-time behavior can be obtained-limited to finite systems and to linear dynamics. This is done in Section 4.

2. Disordered harmonic chain. Two alternative constructions

Let us describe more precisely how the model associated to the Hamiltonian (1.5) can be investigated.

We start by considering the disordered system obtained by putting $V = 0$ in (1.5). By disordered we mean that all the elastic constants $\{\kappa_i\}_{i\in\mathbb{Z}}$ appearing in (1.5), are random variables defined on a probability space (X, P). Namely, let us consider the infinite block matrix

$$A = \begin{pmatrix} 0 & I \\ \Delta - K & 0 \end{pmatrix} \tag{2.1}$$

where Δ is the discrete Laplacian given by $(\Delta v)_i = v_{i+1} + v_{i-1} - 2v_i$, and $K = K(\xi)$ is a random multiplication operator given, for generic $\xi \in X$, by $(K(\xi)v)_i := \kappa_i(\xi)v_i$. The block matrix $A(\xi)$ acts componentwise on the double sequence $\{(q_i, p_i)\}_{i\in\mathbb{Z}}$. It is the generator of the linear flow at fixed environment $\{\kappa_i(\xi)\}_{i\in\mathbb{Z}}$. We assume that all κ_i are identically equidistributed independent random variables with law p supported in some interval $[a, b] \subset \mathbb{R}^+$. In such a way, we

[2] We have chosen the unity of measure such that the mass of all particles is equal to one, and the Planck constant $\hbar = 1$.

get for the sample space, $X \equiv [a,b]^{\mathbb{Z}}$, and for the law, $P \equiv \prod_{i\in\mathbb{Z}} p$. To insure that the results in the present section apply to non trivial cases, it suffices to assume that

$$p(\,\mathrm{d}x) = f(x)\,\mathrm{d}x$$

with $f \in \mathcal{D}((a,b))$; see [7], Section 3.

Such a system can be seen both as a collection of Hamiltonian systems, one for each fixed environment (that is for almost all the realizations of the random variables), as well as a single Hamiltonian system on an appropriate (huge) phase space. Let us briefly discuss the second alternative. We start with a classical disordered one-dimensional chain described by the random Hamiltonian

$$H_\xi(q,p) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \kappa_i(\xi)q_i^2 + \frac{1}{2}\sum_{|i-j|=1}(q_i - q_j)^2$$

which takes into account the quadratic part of (1.5) and where, for simplicity, the system is a finite one. Consider the infinite-dimensional symplectic space $(\mathcal{V},\Omega)$ consisting of an appropriate linear subspace $\mathcal{V}$ of measurable functions on X with values in the phase-space $\mathbb{R}^{2n}$ equipped with the symplectic form Ω given by

$$\Omega((q,p),(Q,P)) := \int_X \sigma((q(\xi),p(\xi)),(Q(\xi),P(\xi)))\mu(\,\mathrm{d}\xi)\,. \tag{2.2}$$

Here, σ is the canonical symplectic form on the phase-space $\mathbb{R}^{2n}$, and μ is any measure on X in the measure-class determined by P.

The equations of motion associated to the functional Hamiltonian

$$\mathcal{H}(q,p) := \int_X H_\xi(q(\xi),p(\xi))\mu(\,\mathrm{d}\xi)\,, \tag{2.3}$$

read $(\dot q(\xi),\dot p(\xi)) = A(\xi)(q(\xi),p(\xi))$ and determine a Hamiltonian flow T_t^*. Such a flow determines uniquely, almost everywhere, the time evolution at fixed environment. In such a situation, *the above points of view are essentially equivalent.* Moreover, in the second case it is natural to quantize the symplectic space $(\mathcal{V},\Omega)$ by specifying the measure in Formulae (2.2), (2.3) as the probability P describing the randomness. In this way we have a genuine CCR-algebra: the commutation function between (linear combinations of) fields and conjugate momenta is a multiple of the identity operator determined by the symplectic form (2.2). Such an approach was firstly pursued in [7] in order to study the long-time behavior of anharmonic crystals in the setting of perturbation theory. The arising model exhibits good strongly ergodic properties which are stable for a wide class of perturbations.

This approach differs from the natural procedure in the alternative point of view: that is first quantize at fixed environment and then study the collection of the resulting systems. The algebra of observables in this case can be taken to be the "product" of the CCR-algebras corresponding to each fixed environment.[3]

[3]More precisely, the algebra of observables is modeled by a direct integral, see Section 4.

This is a different algebra than the previous one. In fact, it has a non trivial center and it is not a CCR.

To have a more concrete idea of the situation, consider the trivial case in which the system consists of a one-dimensional harmonic oscillator with two possible values of the spring. In this case, the largest possible $\mathcal{V}$ is isomorphic to $\mathbb{R}^4$. For such a choice, the CCR in our quantization scheme (i.e., the approach followed in [7]) is naturally represented on the Hilbert space $L^2(\mathbb{R}^4)$. The generated von Neumann algebra is $\mathcal{B}(L^2(\mathbb{R}^4))$, i.e., the algebra of all the bounded operators on $L^2(\mathbb{R}^4)$. The alternative quantization scheme yields the algebra $\mathcal{B}(L^2(\mathbb{R}^2)) \oplus \mathcal{B}(L^2(\mathbb{R}^2))$. The non trivial center consists of all the operators with two diagonal blocks proportional to the identity. Notice that there is no canonical way to compare these algebras, as well as the Hilbert spaces on which they canonically act. However, both algebras contain a common remarkable subalgebra, that is the algebra generated by the constant functions at time $t = 0$.

To insure strong ergodic properties in the case of the Hamiltonian (1.5), one must choose $\mathcal{V}$ rather small, this is precisely what is done in [7], where $\mathcal{V}$ consists of the smallest space containing the constant functions and invariant with respect to the linear evolution. Further, only quasi-free states are considered. Such states are defined on the Weyl operators as

$$\omega(W(v)) = e^{\mathbb{E}(-\frac{1}{2}B(v,v))}\,, \tag{2.4}$$

where B is a random two-point function which uniquely determines the quasi-free state, and $\mathbb{E}$ denotes the average w.r.t. the randomness, see Formulae (3.6), (3.7).

On the contrary, in the environment-by-environment scheme, it is natural to define the *quenched*, or eventually, the *annealed* states. For example, the quenched state ω_q and the annealed one ω_a at inverse temperature β are defined as

$$\omega_q(A) = \mathbb{E}\left(\frac{\mathrm{Tr}(e^{-\beta H}A)}{\mathrm{Tr}(e^{-\beta H})}\right)\,, \quad \omega_a(A) = \frac{\mathbb{E}(\mathrm{Tr}(e^{-\beta H}A))}{\mathbb{E}(\mathrm{Tr}(e^{-\beta H}))}\,,$$

when the r.h.s. are well defined. In the case at hand, the quenched state can be defined in the environment-by-environment scheme as

$$\omega_q(W(v)) = \mathbb{E}\left(e^{-\frac{1}{2}B(v,v)}\right)\,. \tag{2.5}$$

Notice that the quenched state (2.5) is not quasi-free. To compare the two states one must consider them on some common subalgebra, in our case the algebra generated by the constant functions. On such a subalgebra, the states described by (2.4) are the truncated functionals associated to the quenched states given in (2.5), see, e.g., [3].[4]

Let us see in more detail what can be said in the two approaches just outlined.

[4]In addition, their two point functions coincide even at different times, that is

$$\partial_\lambda\partial_\mu\omega(W(\lambda v)W(\mu T_t w))\big|_{\lambda=\mu=0} = \partial_\lambda\partial_\mu\omega_q(W(\lambda v)W(\mu T_t w))\big|_{\lambda=\mu=0}\,.$$

3. Disordered chains. The approach of [7]

In the present section, we report the main results of [7]. Consider the sequences $v = \{v_m^k\}$ where $k = 1, 2$ distinguishes the position from the momentum, and $m \in \mathbb{Z}$ is the index relative to the site. Define then the norm

$$\|v\| := \sum_{m\in\mathbb{Z}} e^{\epsilon|m|}(|v_m^1| + |v_m^2|)$$

where $\epsilon > 0$ is a fixed constant determined by the linear part of the Hamiltonian (1.5). Let $L^2 := \{v \mid \|v\| < \infty\}$. The space L^2 becomes in a natural way a phase-space when it is equipped with the complex conjugation C, and the commutation function

$$\theta(u, v) = \left\langle u, \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix} v \right\rangle , \tag{3.1}$$

see, e.g., [1, 2]. Further, for each $\xi \in X$, let $A(\xi)$ be the random operator (2.1) on the phase-space (L^2, C, θ). The associated one parameter group of Bogoliubov automorphisms is given by

$$T_t(\xi)v := e^{tA(\xi)^*} v\,. \tag{3.2}$$

The group $T_t(\xi)$ is easily computed ([5], Formula (1.2)) obtaining

$$T_t(\xi) = \begin{pmatrix} \cos(H(\xi)^{1/2}t) & -H(\xi)^{1/2}\sin(H(\xi)^{1/2}t) \\ H(\xi)^{-1/2}\sin(H(\xi)^{1/2}t) & \cos(H(\xi)^{1/2}t) \end{pmatrix} . \tag{3.3}$$

As already mentioned, we will restrict our discussion to a rather small C^*-algebra. We start by considering the constant functions $v(\xi) = v$, and the functions $T_t(\xi)v$ generated by the dynamics. We call the collection $\mathbf{v} := (v, t)$ of such functions *deterministic variables*.

The space $\mathbf{D}(L)$, spanned of all deterministic variables, is a phase-space in a natural way, if one defines as $\mathbf{C}$ the usual complex conjugation on functions, and the "commutation function" $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta}(\mathbf{u}, \mathbf{v}) := \int_X \theta(\mathbf{u}(\xi), \mathbf{v}(\xi)) P(\,\mathrm{d}\xi)\,. \tag{3.4}$$

If $\mathbf{v} = (v, \tau)$, the map (3.2) again defines a one parameter group of Bogoliubov automorphisms $\mathbf{T}_t$ on all of $\mathbf{D}(L)$ by

$$\mathbf{T}_t\mathbf{v}(\xi) := T_{t+\tau}(\xi)v\,. \tag{3.5}$$

To define a state, we consider fiberwise the two-point function

$$B_F(u, v) = \left\langle u(\xi), \begin{pmatrix} F(H(\xi)) & \frac{i}{2}I \\ -\frac{i}{2}I & H(\xi)F(H(\xi)) \end{pmatrix} v(\xi) \right\rangle , \tag{3.6}$$

where F is a suitable bounded Borel function on the common spectrum (see, e.g., [10]) of almost all $H(\xi)$.[5] The two-point function $\mathbf{B}_F$ on the phase-space

[5] One must assume $F(x) > 0$, and $xF(x)^2 \geq \frac{1}{4}$ in order to insure that the form be positive defined.

$(\mathbf{D}(L), \mathbf{C}, \boldsymbol{\theta})$ is given by

$$\mathbf{B}_F(\mathbf{u},\mathbf{v}) = \int_X B_F(\mathbf{u}(\xi),\mathbf{v}(\xi))P(\mathrm{d}\xi)\,. \tag{3.7}$$

The corresponding quasi-free state ω on the Weyl algebra associated to the real part of $(\mathbf{D}(L), \mathbf{C}, \boldsymbol{\theta})$ is obtained by

$$\omega(W(\mathbf{u})) = e^{-\frac{1}{2}\mathbf{B}_F(\mathbf{u},\mathbf{u})}\,.$$

Note that ω is invariant for the one parameter group of Bogoliubov automorphisms (3.5). The commutation rule (3.4) translates on the Weyl operators as

$$W(\mathbf{u})W(\mathbf{v}) = e^{-\frac{1}{2}\boldsymbol{\theta}(\mathbf{u},\mathbf{v})}W(\mathbf{u}+\mathbf{v})\,.$$

Let $\mathfrak{W}$ be the C^*-algebra generated by all the Weyl operators associated to the GNS representation of the quasi-free state ω which is kept fixed during the analysis.

While the C^*-algebra $\mathfrak{W}$ is large enough to accommodate the linear dynamics, it cannot be invariant under the non linear time evolution. On the other hand, $\mathfrak{W}''$ is most likely too large for our purposes, and the dynamics on it may very well have poor ergodic properties. This problem can be solved by enlarging $\mathfrak{W}$ enough to accommodate the non linear dynamics, but not as much as to spoil its ergodic properties.

Let M be the collection of all the triples $\mathfrak{m} = (m, \mathbf{h}, \mu)$, where $m \in \mathbb{N}$, $\mathbf{h} : \mathbb{R}^m \to \mathbf{D}(L)$ and μ is an appropriate measure on $\mathbb{R}^m$, see [7], Section 6 for more details. For each $\mathfrak{m} \in M$ define

$$W(\mathfrak{m}) = \int_{\mathbb{R}^m} W(\mathbf{h}(x))\mu(\mathrm{d}x)\,,$$

where the integrals are meant w.r.t the weak (or, equivalently strong) operator topology of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ being the Hilbert space of the GNS representation of the state under consideration. Next, we consider the linear set of operators

$$\mathcal{M} := \mathrm{span}\,\{W(\mathfrak{m}) \mid \mathfrak{m} \in M\}\,.$$

It is easy to check that $\mathcal{M}$ is a $*$-subalgebra of $\mathfrak{W}''$. Define $\mathfrak{M}$ as the C^*-algebra generated by $\mathcal{M}$. Notice that $\mathfrak{W} \subset \mathfrak{M} \subset \mathfrak{W}''$. Denoting again by ω and α_t^0 the natural extensions of the state and the linear dynamics to all of $\mathfrak{M}$, it is shown in [7] that the resulting dynamical system $(\mathfrak{M}, \alpha_t^0, \omega)$ is mixing.

In order to treat the non linear part V in (1.5), we consider perturbations associated to functions given by the (inverse) Fourier transform of "good" measures. Namely,

$$V(x) := \int_{\mathbb{R}^d} e^{i\lambda\cdot x}\nu(\mathrm{d}\lambda)$$

of a complex bounded Borel measure ν on $\mathbb{R}$ satisfying certain boundedness properties. The corresponding perturbation is obtained by an integral in the weak operator topology of $\mathcal{B}(\mathcal{H})$; see [7], Section 5 for further details. Let the dynamics

$\{\alpha_t^{\Lambda_N}\}_{\Lambda_N \subset \mathbb{Z}}$ restricted to an increasing sequence of finite boxes $\{\Lambda_N\}$ be defined by the Dyson expansion (see, e.g., [3, 9]).

The main results of [7] are summarized in the following

Theorem 3.1. *There exists a one parameter group of automorphisms* $\{\alpha_t\}_{t\in\mathbb{R}}$ *of* $\mathfrak{M}$ *such that, for each* $A \in \mathfrak{M}$*, the following pointwise norm-limit holds true*

$$\alpha_t A = \lim_{\Lambda_N \uparrow \mathbb{Z}} \alpha_t^{\Lambda_N} A\,.$$

In addition, the $\sigma(\mathfrak{M}^*, \mathfrak{M})$*-limit*

$$\lim_{t\to\pm\infty} \alpha_t^* \omega =: \omega_\infty$$

exists and defines an invariant state which is mixing for the dynamics α_t*.*

The proof of the above theorem is based on a uniform control in time of the power series defining the dynamics (see [6, 7]). In turns, this depends on the fact that the linear dynamics satisfies a kind of space-time asymptotic abelianess (1.4).

In conclusion, the model constructed in [7] and briefly outlined here, exhibits good strongly ergodic properties which are stable for a wide class of infinitely extended perturbations of the dynamics.

4. Disordered chains. The quenched approach

As already mentioned, there are serious technical problems to develop, in the case described in this section, a program similar to that of the previous one. To clearly illustrate the situation, we will limit ourselves to the study of finite linear systems. Although such results are limited, they are the first necessary step in order to understand ergodic properties exhibited by disordered systems. Further, they set the stage for the possible subsequent steps:

(1) take the thermodynamic limit,
(2) consider non linear perturbations.

In order to investigate the long-time behavior of the disordered harmonic chain, we start again with the Hamiltonian

$$H(q,p) := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + k_i q_i^2 + \frac{1}{2}\sum_{i=1}^{n-1}(q_{i+1} - q_i)^2$$

describing the finite system of particles allocated to the sites $1, 2, \ldots, n$ on the line, with free boundary conditions. The i-particle is subjected to the elastic force described by the elastic constant k_i, and a harmonic uniform nearest neighbor interaction. The infinitesimal generator of the Hamilton flow is given as usual by the truncation of the matrix (2.1).

In order to introduce the disorder, we fix a normalized L^1 function h on the hypercube $Q := [a, b]^n$ with $0 < a < b < +\infty$, and define on the hypercube

$$\mathrm{d}\mu_h(k) := h(k)\,\mathrm{d}^n k\,.$$

As already mentioned, to construct the appropriate C^*-algebra of observables, we consider the direct integral $\int_Q^{\oplus} \Gamma(\mathbb{C}^n)\, \mathrm{d}\mu_h(k)$ where $\Gamma(\mathbb{C}^n)$ is the Fock space for the n-dimensional CCR. Define $\mathfrak{A}$ as the C^*-algebra generated by

$$\left\{ \int_Q^{\oplus} W(u(k))\, \mathrm{d}\mu_h(k)\,, u \in L^2_{\mathbb{R}}\big(Q,\, \mathrm{d}\mu_h; \mathbb{C}^{2n}\big) \right\} ,$$

where $W(u(k))$ is the Weyl operator associated to $u(k) \in \mathbb{R}^{2n}$ acting on $\Gamma(\mathbb{C}^n)$. With an abuse of language, we call also

$$W(u) := \int_Q^{\oplus} W(u(k))\, \mathrm{d}\mu_h(k)$$

a *Weyl operator*.

It is easy to check that the Weyl operators satisfy the commutation relation

$$W(u)W(v) = \left(\int_Q^{\oplus} e^{-\frac{1}{2}\theta(u(k),v(k))} I_k \, \mathrm{d}\mu_h(k) \right) W(u+v) \tag{4.1}$$

where θ is given in (3.1), and I_k is the identity operator acting on $\Gamma(\mathbb{C}^n)$ relative to the fibre k.

On the C^*-algebra $\mathfrak{A}$, it is naturally acting the one parameter group of automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$, described on the Weyl operators by

$$\alpha_t W(u) = W(T_t u)$$

where T_t are made of Bogoliubov automorphisms acting fiberwise as

$$\big(T_t u\big)(k) := T_t(k) u(k)\,,$$

and $T_t(k)$ is given by (3.3).

Consider a two-point function B_F as in (3.6) based on the function F which we suppose to be continuous. For each environment k, consider the state φ_k given fiberwise by

$$\varphi_k(W(u(k))) := e^{-\frac{1}{2}B_F(u(k),u(k))}\,. \tag{4.2}$$

The state φ, defined on the Weyl operators by

$$\varphi(W(u)) := \int_Q \varphi_k(W(u(k))\, \mathrm{d}\mu_h(k)\,, \tag{4.3}$$

is uniquely extended on all of $\mathfrak{A}$. By construction, such a state is invariant w.r.t. the dynamics.

Let $B \in \mathfrak{A}$ such that $\varphi(B^*B) > 0$, be fixed. As usual, one can consider the state

$$\varphi_B := \frac{\varphi(B^* \,\cdot\, B)}{\varphi(B^*B)} \tag{4.4}$$

where φ is given in (4.3). Of course, in general φ_B will not be an invariant state, yet it is meaningful to ask if it has or not an asymptotic limit.

Theorem 4.1. *The pointwise limit*

$$\lim_{t\to+\infty}\varphi_B(\alpha_t(A)) =: \varphi_{B,\infty}(A)$$

exists and defines an invariant state on $\mathfrak{A}$.[6]

Proof. By compactness, the net $\{\varphi(B^*\alpha_t(\,\cdot\,)B)\}_{t\in\mathbb{R}}$ has weak*-cluster points. The assertion will follow by showing that there is only one cluster point. By a standard approximation argument, it suffices to investigate the three-point function $I(t) := \varphi(W(u)W(T_tv)W(w))$ where u, v, w are generic elements of $L^2\big(Q,\, \mathrm{d}\mu_h;\mathbb{R}^{2n}\big)$. By the commutation rule (4.1), we compute

$$I(t) = \int_Q e^{-\frac{1}{2}\big\{\theta(u,w)+\theta(u-w,T_tv)\big\}}e^{-\frac{1}{2}B_F(u+T_tv+w,u+T_tv+w)}\,\mathrm{d}\mu_h\,.$$

Notice that the first exponent is purely imaginary, and the second one is negative. Further, the one parameter group

$$T_t : L^2\big(Q,\, \mathrm{d}\mu_h;\mathbb{C}^{2n}\big)\mapsto L^2\big(Q,\, \mathrm{d}\mu_h;\mathbb{C}^{2n}\big)$$

is equibounded. Hence, for each fixed $\varepsilon>0$, we can choose smooth functions h_ε, u_ε, v_ε, w_ε, F_ε such that for every $t\in\mathbb{R}$, we have $|I(t)-I_\varepsilon(t)|<\varepsilon$. Here, I_ε is the corresponding function constructed as above by using the mentioned list of smooth functions. We obtain

$$\begin{aligned}|I(t)-I(t')| &< |I(t)-I_\varepsilon(t)|+|I_\varepsilon(t)-I_\varepsilon(t')|+|I_\varepsilon(t')-I(t')|\\ &< 2\varepsilon+|I_\varepsilon(t)-I_\varepsilon(t')|\,.\end{aligned}$$

Namely, the above 3ε-trick allows us to reduce ourselves to the case when all the involved functions are smooth.

We write

$$I(t) = \int_Q h(k)e^{-\frac{1}{2}\mathcal{E}(k)}e^{-\frac{1}{2}E(k,t)}\,\mathrm{d}^nk$$

where

$$\begin{aligned}\mathcal{E}(k) &:= \theta(u(k),w(k))+B_F(u(k)+w(k),u(k)+w(k))+B_F(v(k),v(k))\\ E(k,t) &:= \theta(u(k)-w(k),T_t(k)v(k))+2B_F(u(k)+w(k),T_t(k)v(k))\,.\end{aligned}$$

After some computations, we obtain

$$\begin{aligned}E(k,t) = \sum_{l=1}^n\Bigg(&\cos(\sqrt{\lambda_l(k)}\,t)\sum_{\sigma=1,2}\langle\xi^\sigma(k)+2\eta^\sigma(k)|e_l(k)\rangle\langle e_l(k)|v^\sigma(k)\rangle\\ &+\sin(\sqrt{\lambda_l(k)}\,t)\sum_{\sigma=1,2}\langle\hat\xi^\sigma(k)+2\hat\eta^\sigma(k)|e_l(k)\rangle\langle e_l(k)|v^\sigma(k)\rangle\Bigg)\end{aligned}$$

[6] The analogous limit at $-\infty$ exists as well, and defines another (in general different) invariant state on $\mathfrak{A}$.

where

$$\xi(k) := \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix} (u(k) - w(k)) ,$$
$$\hat{\xi}(k) := \begin{pmatrix} -iH(k)^{-1/2} & 0 \\ 0 & -iH(k)^{1/2} \end{pmatrix} (u(k) - w(k)) ,$$
$$\eta(k) := \begin{pmatrix} F(H(k)) & 0 \\ 0 & H(k)F(H(k)) \end{pmatrix} (u(k) + w(k)) ,$$
$$\hat{\eta}(k) := \begin{pmatrix} 0 & H(k)^{1/2}F(H(k)) \\ -H(k)^{1/2}F(H(k)) & 0 \end{pmatrix} (u(k) + w(k)) ,$$

$\{\lambda_l(k)\}_{l=1}^n$ and $\{e_l(k)\}_{l=1}^n$ being the finite sequences of the eigenvalues and the corresponding eigenvectors of the matrix $H(k)$.

Notice that we have for such a matrix,

$$H(k) = \begin{pmatrix} (k_1+2) & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & (k_2+2) & -1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & -1 & (k_l+2) & -1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & -1 & (k_{n-1}+2) & -1 \\ \cdot & \cdot & \cdot & \cdot & 0 & -1 & (k_n+2) \end{pmatrix} .$$

This is exactly the situation described in Appendix A. Namely, the eigenvalues of $H(k)$ are the n distinct roots of the orthogonal polynomial $P_n(k+2,x)$ corresponding to the parameters $(k_1+2,\dots,k_n+2)$. The functions

$$\omega(k) := (\sqrt{\lambda_1(k)},\dots,\sqrt{\lambda_n(k)}) \tag{4.5}$$

are, locally, smooth functions of $(k_1,\dots,k_n)$. Further, as the eigenvalues $(\lambda_1(k), \dots, \lambda_n(k))$ are strictly positive under our assumptions, there exists local inverses for $\omega(k)$ apart from a closed null-set $N\subset\mathbb{R}^n$, see Proposition A.3.

Let now $\{O_m\}_{m=1}^\infty$ be an open covering of the (open) set $\overset{\circ}{Q}\setminus N$ of full Lebesgue measure, together with a partition of unity $\{\psi_m\}_{m=1}^\infty$ subordinate to the mentioned covering, such that for each m, the function (4.5) admits the inverse K_m in O_m. We have

$$I(t) = \sum_{m=1}^\infty \int_{K_m(\operatorname{supp}(\psi_m))} \psi_m(K_m(\omega))h(K_m(\omega)) \times e^{-\frac{1}{2}\mathcal{E}(K_m(\omega))} e^{-\frac{1}{2}E(K_m(\omega),t)} \big|\det\partial_\omega K_m(\omega)\big| \,\mathrm{d}^n\omega$$

where the last series is summable, uniformly in $t\in\mathbb{R}$; see, e.g., [12], Theorem 3.12.

As $\mathcal{E}(k)$ and $E(k,t)$ are smooth functions,[7] we can apply Proposition B.1 after exchanging the sum with the limit. This leads to the assertion. □

Notice that, taking into account (B.1), the limit of the three-point function $\lim_{t\to+\infty}\varphi(W(u)W(T_t v)W(w))$, can be explicitly computed.

The above theorem differs substantially from the mixing property (1.2). In fact, mixing would imply $\varphi_{B,\infty}=\varphi$. It is not hard to show that this is generically impossible. This is a consequence of the abundance of invariant states and shows that the asymptotic state depends on the initial conditions. However, Theorem 4.1 tells us that the disordered chain exhibit a "good" long-time behavior, contrary to the usual chain (non disordered) where the dynamics is quasi-periodic.

Equally well, one can consider the "quenched" state φ^B associated to an element B as above. It is given in a natural way on the Weyl operators by

$$\varphi^B(W(u)) := \int_Q \frac{\varphi_k(B(k)^*W(u(k))B(k))}{\varphi_k(B(k)^*B(k))}\,\mathrm{d}\mu_h(k)\,, \tag{4.6}$$

where φ_k is given in (4.2), and $\{B(k)\}_{k\in Q}$ is the measurable decomposition of B which always exists; see, e.g., [14], Section IV.8.

Suppose that the measurable function $k\mapsto\varphi_k(B(k)^*B(k))$ is essentially bounded from below by a constant $\delta>0$. Define fiberwise the operator $\widetilde{B}$ as

$$\widetilde{B}(k) := \varphi_k(B(k)^*B(k))^{-1/2}B(k)\,.$$

Clearly, $\widetilde{B}$ belongs to $\mathfrak{A}$. In addition, $\varphi^B=\varphi_{\widetilde{B}}$, that is most of the states described in (4.6) are particular cases of those defined in (4.4).

Appendix A. Note on orthogonal polynomials

Consider a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers, together with the associated sequence of polynomials uniquely defined by

$$\begin{gathered} P_{-1}(x) := 0\,, \qquad P_0(x) := 1\,, \\ P_j(x) := (x-a_j)P_{j-1}(x) - P_{j-2}(x)\,. \end{gathered} \tag{A.1}$$

It is well known that the P_n are the orthogonal polynomials associated to some probability measure on the real line. Hence, the P_n have n distinct roots with nice separation properties, see, e.g., [13].

Consider, for each $n\in\mathbb{N}$, the finite sequence of parameters $a:=(a_1,\dots,a_n)$, together with the corresponding polynomial P_n. Let $\bar{P}^n_j$ be the polynomial constructed by the relations (A.1) using the parameters $(a_{n-j+1},\dots,a_n)$. To simplify

[7] This follows as, for all k nearby $\bar{k}$,

$$|e_l(k)\rangle\langle e_l(k)| = \frac{1}{2\pi i}\oint_\gamma (zI-H(k))^{-1}\,\mathrm{d}z$$

where γ is a sufficiently small fixed circle surrounding counterclockwise the eigenvalue $\lambda_l(\bar{k})$ corresponding to the one-dimensional projector $|e_l(\bar{k})\rangle\langle e_l(\bar{k})|$.

notations, we omit for the moment to make explicit the dependence of the polynomials on the parameters.

Lemma A.1. *The relation*

$$Q_j^n(x) := \partial_{a_j} P_n(x) = -\bar{P}_{n-j}^n(x) P_{j-1}(x)$$

holds true for each $n \in \mathbb{N}$, $(a_1, \dots, a_n) \in \mathbb{R}^n$ *and* $x \in \mathbb{R}$.

Proof. The proof is by induction over n. For $n = 1$ we have

$$\partial_{a_1} P_1(x) = \partial_{a_1}(x - a_1) = -1 = -\bar{P}_0^1(x) P_0(x) \,.$$

Let us suppose the lemma true for n, then

$$\begin{aligned} \partial_{a_{n+1}} P_{n+1}(x) &= -P_n(x) = -\bar{P}_0^{n+1}(x) P_n(x), \\ \partial_{a_n} P_{n+1}(x) &= (x - a_{n+1}) \partial_{a_n} P_n(x) = -(x - a_{n+1}) \bar{P}_0^n(x) P_{n-1}(x) \\ &= -\bar{P}_1^{n+1}(x) P_{n-1}(x) \,. \end{aligned}$$

For $j < n$ we obtain

$$\begin{aligned} \partial_{a_j} P_{n+1}(x) &= (x - a_{n+1}) \partial_{a_j} P_n(x) - \partial_{a_j} P_{n-1}(x) \\ &= -(x - a_{n+1}) \bar{P}_{n-j}^n(x) P_{j-1}(x) + \bar{P}_{n-j-1}^{n-1}(x) P_{j-1}(x) \\ &= -\bar{P}_{n+1-j}^{n+1}(x) P_{j-1}(x) \,, \end{aligned}$$

which concludes the proof. □

Note that Lemma A.1 implies that the Q_j^n are all polynomials of degree $n-1$ in x. We can thus set

$$Q_j^n(x) =: \sum_{k=0}^{n-1} \alpha_{j,k}^n x^k \tag{A.2}$$

where the coefficients depends on the parameters a. Define the $n \times n$ matrix A_n as that having the coefficients appearing in the r.h.s. of (A.2). Consider, for a finite sequence $\lambda := (\lambda_1, \dots, \lambda_n)$ of unknowns, the $n \times n$ matrix $\Lambda \equiv \Lambda(a, \lambda)$ whose entries are given by

$$[\Lambda(a, \lambda)]_{i,j} := \partial_{a_j} P_n(\lambda_i) \,.$$

Lemma A.2. *The following holds true.*

$$\det \Lambda = \det \big(A_n\big) \det \begin{pmatrix} 1 & \lambda_i & \dots & \lambda_i^{n-1} \end{pmatrix} \,. \tag{A.3}$$

Moreover, the polynomial $\det \Lambda$ *is not identically zero.*

Proof. Let $\mathbb{P}_n$ be the set of permutations of $\{0, \dots, n-1\}$. Then, by the multilinearity of the determinant, we get

$$\det \Lambda = \sum_{\sigma \in \mathbb{P}_n} \prod_{j=1}^n \alpha_{j,\sigma(j-1)}^n \det \left(\lambda_i^{\sigma(i-1)} \right)$$

and the first half follows by reordering the determinants in the r.h.s..

As we have for the Vandermonde determinant,

$$\det\begin{pmatrix}1 & \lambda_i & \dots & \lambda_i^{n-1}\end{pmatrix} = \prod_{\substack{i,j=1\\ i>j}}^{n} (\lambda_i - \lambda_j)\,,$$

we obtain that $\det\Lambda \equiv 0$ is equivalent to $\det A_n \equiv 0$. To rule out this last possibility, it suffices to show that it is different from zero for a specific choice of the parameters a. We compute it for $a_i = 0$ for each $i < n$, and $a_n = 1$. Let us call $\hat{P}, \hat{Q}$ and $\hat{\alpha}$ the polynomials and coefficients corresponding to parameters $a_i = 0$ for all i.

Clearly, $\bar{P}^i_j(x) = \hat{P}_j(x)$ for all $i < n$, while

$$\bar{P}^n_j(x) = (x-1)\hat{P}_{j-1}(x) - \hat{P}_{j-2}(x) = \hat{P}_j(x) - \hat{P}_{j-1}(x)\,.$$

Hence, for $j < n$,

$$Q^n_j(x) = -\hat{P}_{n-j}(x)\hat{P}_{j-1}(x) - \hat{P}_{n-j-1}(x)\hat{P}_{j-1}(x) = \hat{Q}^n_j(x) - \hat{Q}^{n-1}_j(x)\,.$$

Next, notice that $\hat{P}_j$ is even for j even and odd for j odd. Thus $\hat{Q}^n_j$ is even for n odd and vice versa. This means that the non zero coefficients $\alpha^n_{k,j}$ and $\alpha^{n-1}_{k,j}$ fall in different columns. In addition, for $j < n-1$,

$$\begin{aligned}\hat{Q}^n_j(x) &= -x\hat{P}_{n-j-1}(x)\hat{P}_{j-1}(x) + \hat{P}_{n-j-2}(x)P_{j-1}(x)\\ &= x\hat{Q}^{n-1}_j(x) - \hat{Q}^{n-2}_j(x)\,.\end{aligned}$$

Let us see what the above equations means in terms of the coefficients $\hat{\alpha}$. For $j < n-1$ we have

$$\begin{aligned}&\hat{\alpha}^n_{j,0} = -\hat{\alpha}^{n-2}_{j,0}\,,\\ &\hat{\alpha}^n_{j,k} = \hat{\alpha}^{n-1}_{j,k-1} - \hat{\alpha}^{n-2}_{j,k}\,, \quad k = 1,\dots,n-3\,,\\ &\hat{\alpha}^n_{j,n-2} = \hat{\alpha}^{n-1}_{j,n-3}\,,\\ &\hat{\alpha}^n_{j,n-1} = \hat{\alpha}^{n-1}_{j,n-2}\,.\end{aligned} \tag{A.4}$$

The above considerations imply that A_n has the following form

$$A_n = \begin{pmatrix} \hat{\alpha}^n_{1,0} - \hat{\alpha}^{n-1}_{1,0} & \dots & \hat{\alpha}^n_{1,n-2} - \hat{\alpha}^{n-1}_{1,n-2} & \hat{\alpha}^n_{1,n-1}\\ \vdots & \vdots\vdots\vdots & \vdots & \vdots\\ \hat{\alpha}^n_{n-1,0} - \hat{\alpha}^{n-1}_{n-1,0} & \dots & \hat{\alpha}^n_{n-1,n-2} - \hat{\alpha}^{n-1}_{n-1,n-2} & \hat{\alpha}^n_{n-1,n-1}\\ \hat{\alpha}^n_{n,0} & \dots & \hat{\alpha}^n_{n,n-2} & 1\end{pmatrix}.$$

We restrict ourselves to the even case, the other one being similar.

$$A_n = \begin{pmatrix} -\hat{\alpha}^{n-1}_{1,0} & \hat{\alpha}^n_{1,1} & -\hat{\alpha}^n_{1,2} & \dots & -\hat{\alpha}^{n-1}_{1,n-2} & \hat{\alpha}^n_{1,n-1}\\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\\ -\hat{\alpha}^{n-1}_{n-1,0} & \hat{\alpha}^n_{n-1,1} & -\hat{\alpha}^n_{n-1,2} & \dots & -\hat{\alpha}^{n-1}_{n-1,n-2} & \hat{\alpha}^n_{n-1,n-1}\\ 0 & \hat{\alpha}^n_{n,1} & 0 & \dots & 0 & 1\end{pmatrix}.$$

Now, we construct a new matrix by keeping the columns with $k = 2l$ and substituting the columns with $k = 2l+1$ with the sum of the same column $2l+1$

and the previous column $2l$. Clearly, the new matrix has the same determinant as the original one. Keeping in mind (A.4), and developing the determinant w.r.t. the last column (which has all zeroes but in the last entry), it yields

$$\begin{aligned}|\det A_n| &= \left|\det\begin{pmatrix}\hat\alpha^{n-1}_{1,0} & \hat\alpha^{n-2}_{1,1} & \hat\alpha^{n-1}_{1,2} & \dots & \hat\alpha^{n-2}_{1,n-3} & \hat\alpha^{n-1}_{1,n-2}\\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\\ \hat\alpha^{n-1}_{n-2,0} & \hat\alpha^{n-2}_{n-2,1} & \hat\alpha^{n-1}_{n-2,2} & \dots & \hat\alpha^{n-2}_{n-2,n-3} & \hat\alpha^{n-1}_{n-1,n-1}\\ \hat\alpha^{n-1}_{n-1,0} & 0 & \hat\alpha^{n-1}_{n-1,2} & \dots & 0 & 1\end{pmatrix}\right|\\ &= |\det A_{n-1}|\,.\end{aligned}$$

By the same arguments for the odd case, we conclude that $|\det A_n| = |\det A_{n-1}|$ for each n, which means by induction, $|\det A_n| = 1$ for each n as $\det A_2 = 1$. □

Consider the polynomial $P_n \equiv P_n(a,x)$ in the unknown x, constructed by (A.1) relatively to the parameters $a \equiv (a_1,\dots,a_n)$. A trivial application of the Implicit Function Theorem tells us that the n distinct roots

$$\lambda(a) := (\lambda_1(a),\dots,\lambda_n(a)) \tag{A.5}$$

of $P_n(a,x)$ depends smoothly on the parameters a.[8] In other words, defining

$$F(a,\lambda) := (P_n(a,\lambda_1),\dots,P_n(a,\lambda_n))\,,$$

the equation

$$F(a,\lambda) = 0$$

defines at least locally, the n distinct roots $\lambda(a)$ as smooth functions of the parameters a. It is crucial for our aims to ask for local invertibility of the function (A.5).

Proposition A.3. *Apart from a negligible set in the parameters a, the map $a \in \mathbb{R}^n \mapsto \lambda(a) \in \mathbb{R}^n$ defines a local diffeomorphism.*

Proof. For the cases under consideration, we have for the Jacobian matrix

$$\frac{\partial\lambda}{\partial a} = \left(\frac{\partial F}{\partial\lambda}(a,\lambda(a))\right)^{-1}\frac{\partial F}{\partial a}(a,\lambda(a))\,,$$

at least locally. Hence, it is sufficient to investigate the function $(\alpha,\lambda) \in \mathbb{R}^{2n} \mapsto \det\frac{\partial F}{\partial a}(a,\lambda) \equiv \det\Lambda(a,\lambda) \in \mathbb{R}$ under the conditions $\lambda_i \neq \lambda_j$, $i \neq j$, see Footnote 8. Taking into account Lemma A.2, the last determinant factorizes into two pieces, where the first one depends only on the parameters, and the second one never vanishes in our context. Further, it is also shown in Lemma A.2, that such a determinant is not identically zero. The negligible set we are looking for is precisely the locus of zeroes of the non trivial polynomial $\det A_n$ in (A.3), which is well known to be negligible w.r.t. the n-dimensional Lebesgue measure. □

[8]This easily follows from the fact that $P_n(a,x)$ has n distinct roots which means that $\partial_x P_n(a,x)\big|_{x=\lambda_i(a)} \neq 0\,,\ i = 1,\dots,n$.

Appendix B. A convergence property

Consider a real-valued C^1 function h supported into $(0,L)^n$, sequences $\{a_i\}_{i=1,\dots,n}$, $\{b_i\}_{i=1,\dots,n}$ of complex-valued C^1 functions on $\mathbb{R}^n$, and finally a holomorphic function $f:\mathbb{C}\mapsto\mathbb{C}$.

Proposition B.1. *Under the above conditions, we have*

$$\lim_{t\to+\infty}\int_{[0,L]^n}\mathrm{d}^n\omega\, h(\omega) f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin\omega_i t+b_i(\omega)\cos\omega_i t\big)\Big) \tag{B.1}$$
$$=\int_{[0,L]^n}\mathrm{d}^n\omega\, h(\omega)\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n}\mathrm{d}^n x\, f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin x_i+b_i(\omega)\cos x_i\big)\Big).$$

Proof. Taking into account the support of h, we can write

$$\Big|\int_{[0,L]^n}\mathrm{d}^n\omega\, h(\omega) f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin\omega_i t+b_i(\omega)\cos\omega_i t\big)\Big)$$
$$-\int_{[0,L]^n}\mathrm{d}^n\omega\, h(\omega)\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n}\mathrm{d}^n x\, f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin x_i+b_i(\omega)\cos x_i\big)\Big)\Big|$$
$$\le\sum_{j_1,\dots,j_n=0}^{\left[\frac{tL}{2\pi}\right]}\int_{Q_j}\mathrm{d}^n\omega\,\Big|h(\omega)-h\big(\frac{2\pi}{t}j\big)\Big|\,\Big|f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin\omega_i t+b_i(\omega)\cos\omega_i t\big)\Big)\Big|$$
$$+\sum_{j_1,\dots,j_n=0}^{\left[\frac{tL}{2\pi}\right]}\Big|h\big(\frac{2\pi}{t}j\big)\Big|\Big|\int_{Q_j}\mathrm{d}^n\omega\,\Big|f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin\omega_i t+b_i(\omega)\cos\omega_i t\big)\Big)$$
$$-f\Big(\sum_{i=1}^n\big(a_i\big(\frac{2\pi}{t}j\big)\sin\omega_i t+b_i\big(\frac{2\pi}{t}j\big)\cos\omega_i t\big)\Big)\Big|$$
$$+\Big|\sum_{j_1,\dots,j_n=0}^{\left[\frac{tL}{2\pi}\right]}\Big[\big(\frac{2\pi}{t}\big)^n h\big(\frac{2\pi}{t}j\big)\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n}\mathrm{d}^n x\, f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin x_i+b_i(\omega)\cos x_i\big)\Big)$$
$$-\int_{Q_j}\mathrm{d}^n\omega\, h(\omega)\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n}\mathrm{d}^n x\, f\Big(\sum_{i=1}^n\big(a_i(\omega)\sin x_i+b_i(\omega)\cos x_i\big)\Big)\Big]\Big|$$

where Q_j is the (small) hypercube of side $\frac{2\pi}{t}$ constructed starting from the point $\big(\frac{2\pi}{t}j_1,\dots,\frac{2\pi}{t}j_n\big)$. Here, we made the change of variables $(x_1,\dots,x_n)=(t\omega_1,\dots,t\omega_n)$ in the last piece.

We majorize the first two pieces by

$$2\pi\|\,\mathrm{d}h\|_\infty\|f\|_\infty\sqrt{n}L^n\frac{1}{t}$$
$$2\pi\|h\|_\infty\|f'\|_\infty\Big[\sum_{i=1}^n\big(\|\,\mathrm{d}a_i\|_\infty+\|\,\mathrm{d}b_i\|_\infty\big)\Big]\sqrt{n}L^n\frac{1}{t}$$

respectively, where $\| \ \|_\infty$ is the supremum norm on suitable compact sets. In order to estimate the last piece, we note that the first part is precisely the Riemann sums of the second part. Analogously, it is majorized by

$$2\pi\| \,\mathrm{d}G\|_\infty \sqrt{n} L^n \frac{1}{t}$$

where G is the smooth function given by

$$G(\omega) := \frac{h(\omega)}{(2\pi)^n} \int_{[0,2\pi]^n} \mathrm{d}^n x \, f\bigg(\sum_{i=1}^n \big(a_i(\omega) \sin x_i + b_i(\omega) \cos x_i \big) \bigg) .$$

Collecting together all pieces, we conclude that

$$\begin{aligned} &\bigg| \int_{[0,L]^n} \mathrm{d}^n\omega \, h(\omega) f\bigg(\sum_{i=1}^n \big(a_i(\omega) \sin \omega_i t + b_i(\omega) \cos \omega_i t \big) \bigg) \\ &- \int_{[0,L]^n} \mathrm{d}^n\omega \, h(\omega) \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \mathrm{d}^n x \, f\bigg(\sum_{i=1}^n \big(a_i(\omega) \sin x_i + b_i(\omega) \cos x_i \big) \bigg) \bigg| \\ &= \mathrm{O}\left(\frac{1}{t} \right) \end{aligned}$$

which leads to the assertion. □

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