INVARIANT MEASURES AND THEIR PROPERTIES. A FUNCTIONAL ANALYTIC POINT OF VIEW

CARLANGELO LIVERANI

Abstract. In this series of lectures I try to illustrate systematically what I call the "functional analytic approach" to the study of the statistical properties of Dynamical Systems. The ideas are presented via a series of examples of increasing complexity, hoping to give in this way a feeling of the breadth of the method.

Contents

Date: Rome, November 18, 2002.
I wish to thank the Scuola Normale Superiore, the "Center Ennio de Giorgi" and S.Marmi for giving me the opportunity to elaborate on the point of view here presented.

A foreword

This text grew out of a series of five lectures that I gave at the Center Ennio de Giorgi during the Research Trimester on Dynamical Systems (Pisa, February 1-April 30, 2002). I must say that, while writing, I added some more material that I felt naturally belonged to the logic of the argument but I did not have the time to cover in the actual lectures. As a consequence the present text corresponds more to a ten rather than a five lectures course. Nevertheless, I kept the original division since it is quite natural. In fact, it is rather tempting to expand the material even further and make this into a full blown graduate course. I am unsure if I will ever fall for such a temptation. In the mean time my task here is to present systematically an approach to the problem of statistical properties of Dynamical Systems and try to show how far it can be carried out. This reflects my opinion that the best way to test a point of view is to try to push it to its limits. Personally, I think that the approach presented here may be pushed much further. For example, extending it to non uniformly hyperbolic cases or more sophisticated objects such as dynamical ζ *functions. I hope that this presentation will motivate others to do so.*

1. **First Lecture** (the problem)

This series of lectures are dedicated to the study of the *statistical properties* of Dynamical Systems.

I must immediately emphasize that the material and the point of view presented here are a bit single minded and have no intention whatsoever to constitute a review of the field. In fact, if one wishes to get a brief general introduction to the field I warmly recommend the papers by Young [91, 92] and Viana [88]; for a recent overview on the statistical properties of Dynamical Systems see Baladi's book [4] and [34] for a detailed general introduction to the field of Dynamical Systems.

Before starting, however, I must at least mention that the present approach has a long history, beginning at least with the work of Sinai and then Ruelle, but twisted through the results of Lasota-Yorke and Keller, just to mention very few of the main actors.

1.1. **The general Problem.**

Consider a Dynamical Systems (X, T) where X is a measurable space and $T : X \to X$ a measurable map.

The term *statistical properties* of a dynamical system is a very loose expression but it roughly relates to the properties of the evolution of measures. It is an interesting fact of life that very complex Dynamical Systems become much simpler once one studies the evolution of measures rather than of points.

Given a probability measure μ on X, one can define $T_*\mu(A) := \mu(T^{-1}A)$ for each measurable set A.[1] Clearly $T_* : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the set of the probability measures on X.[2] Indeed, since $T^{-1}X = X$, if $\mu \in \mathcal{P}(X)$, then $T_*\mu(X) = \mu(X) = 1$.

[1] The operator T_* is often called *Transfer operator*, in analogy with the related object in Statistical Mechanics, or *Perron-Frobenius operator*, borrowing the terminology from the theory of positive matrices and operators, or also *Ruelle-Perron-Frobenius operator* in recognition of the rôle of D.Ruelle in emphasizing its importance and in the study of its properties.

[2] In these notes we will consider only probabilities measures. Nonetheless, it must be said that there exists a rich and very interesting theory of σ-finite invariant measures, [1].

When investigating the properties of the dynamical system $(\mathcal{P}(X), T_*)$, the first relevant question is the study of the fixed points, that is the invariant measures: $\mu(A) = \mu(T^{-1}A)$ for each measurable set A.[3]

Given an invariant measure μ, one can define the measurable dynamical system (X, T, μ). For such a dynamical system several natural questions concerning statistical properties arise. The first question is the identification of the invariant sets. One of the most interesting possibilities being when all the invariant sets[4] are trivial–either of zero or of full measure–such systems are called *ergodic*. This corresponds to a study of the properties of the invariant measure itself.

If one is interested in the behavior of nearby measures, a first possibility is to consider the set $\mathcal{A}(\mu) := \{\nu \in \mathcal{P}(X) \mid \nu \ll \mu\}$, that is the set of measures absolutely continuous with respect to μ. In several cases (certainly in all the ones discussed below) it happens that $T_*\mathcal{A}(\mu) \subset \mathcal{A}(\mu)$, in this case the map is called *nonsingular* with respect to μ. If μ is an attractor for $\mathcal{A}(\mu)$, then the measurable dynamical system (X, T, μ) is called *mixing*. This is a very interesting notion which has received a lot of attention due to its importance both in practical and theoretical contexts. For mixing systems a further obvious issue concerns the speed with which measures in $\mathcal{A}(\mu)$ are attracted to μ, the so called *speed of mixing*.

To understand more global properties of $(\mathcal{P}(X), T_*)$ one can try to gain a better knowledge of the behavior of a larger class of measures, for example one can study the asymptotic behavior of the measures that are absolutely continuous with respect to a given (not necessarily invariant) measure m (e.g. Lebesgue).

It is also possible to investigate stronger statistical properties of (X, T, μ) (e.g., Central Limit Theorems, K-property, Benoully property, etc.–see [34] for a more complete introductory discussion). Yet, here we will limit our investigations to the above concepts.

Of course the first question is: do invariant measures always exist?

The answer is, in general, negative. Consider indeed the following two trivial examples

(1) $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^2 + 2$. Clearly any invariant measure must be supported on $[2, \infty)$. On the other hand, if $x \geq 2$, $T^{-1}x \subset (-\infty, x - 1)$. Accordingly, for any bounded set A there exists $n \in \mathbb{N}$ such that $T^{-n}A \subset (-\infty, 2)$, hence $\mu(A) = \mu(T^{-n}A) = 0$.[5]

(2) $T : [0, 1] \to [0, 1]$

$$T(x) = \begin{cases} \dfrac{1}{2}x + \dfrac{1}{4} & \forall x \neq \dfrac{1}{2} \\ 0 & \text{for } x = \dfrac{1}{2} \end{cases}$$

Clearly, $\mu(\{1/2\}) = 0$ since the point has no preimage. Thus letting $A = [0, 1/2) \cup (1/2, 1]$, for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $T^n A \subset (1/2 - \varepsilon, 1/2 + \varepsilon)$, so $\mu((1/2 - \varepsilon, 1/2 + \varepsilon)) \geq \mu(T^n A) = \mu(A) = \mu([0, 1])$.

---

[3] Notice that if T is invertible and T^{-1} is measurable, then the above relation is equivalent to $\mu(A) = \mu(TA)$ for each measurable set A.

[4] That is, the measurable sets A such that $T^{-1}A \subset A$.

[5] Here no invariant measure at all does exist (a part from $\mu = 0$, which clearly always exists but we do not take into considerations since it yields no information whatsoever on the systems). If we wish to exclude only invariant probability measures, then the obvious example $Tx = x + 1$ suffices.

Hence, if the measure is Borel, by the regularity of the measure follows $\mu([0,1]) = \mu(\{1/2\}) = 0$.

In the two counterexamples above the obstructions derive, in the first case, from the non-compactness of the space and, in the second case, from the discontinuity of the map. Essentially these are the only possible obstructions as is illustrated in the next section.

1.2. **Existence (Kryloff–Bogoliouboff).**

Proposition 1.1 (Kryloff–Bogoliouboff [50])**.** *If X is a compact metric space and $T : X \to X$ is continuous, then there exists at least one invariant (Borel) measure.*

Proof. Consider any Borel probability measure ν and define the following sequence of measures $\{\nu_n\}_{n\in\mathbb{N}} := \{T_*^n \nu\}$. Next, define

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\nu_i.$$

Again $\mu_n(X) = 1$, so the sequence $\{\mu_i\}_{i=1}^{\infty}$ is contained in a weakly compact set (the unit ball) and therefore admits a weakly convergent subsequence $\{\mu_{n_i}\}_{i=1}^{\infty}$; let μ be the weak limit.[6] We claim that μ is T invariant. Since μ is a Borel measure it suffices to verify that for each $f \in \mathcal{C}^0(X)$ holds $\mu(f \circ T) = \mu(f)$. Let f be a continuous function, then by the weak convergence we have[7]

$$\begin{aligned}\mu(f\circ T) &= \lim_{j\to\infty}\frac{1}{n_j}\sum_{i=0}^{n_j-1}\nu_i(f\circ T) = \lim_{j\to\infty}\frac{1}{n_j}\sum_{i=0}^{n_j-1}\nu(f\circ T^{i+1})\\ &= \lim_{j\to\infty}\frac{1}{n_j}\left\{\sum_{i=0}^{n_j-1}\nu_i(f) + \nu(f\circ T^{n_j}) - \nu(f)\right\} = \mu(f).\end{aligned}$$

□

1.3. **Why is this not enough?**

The problem with the above result is not so much in the hypotheses of the Theorem (although discontinuous systems do have an important rôle in Dynamical Systems–just think of billiard systems or Poincarè maps of flows, for example) but rather in the lack of information about the invariant measure.

Indeed, in general there may be a lot of invariant measures and not all of them may be relevant for the study of a system. For example, if the system has a periodic orbit $\{T^i x_0\}_{i=1}^{n}$, $x_0 \in X$, then the Dirac measure that assigns to each point of the orbit the same weight is obviously invariant.

---

[6] This depends on the Riesz-Markov Representation Theorem that states that the space of Borel measures $\mathcal{M}(X)$ is exactly the dual of the Banach space $\mathcal{C}^0(X)$. Since the weak convergence of measures in this case corresponds exactly to the weak-* topology, the result follows from the Banach-Alaoglu theorem stating that the unit ball of the dual of a Banach space is compact in the weak-* topology.

[7] Note that it is essential that we can check invariance only on continuous functions: if we would have to check it with respect to all bounded measurable functions we would need that μ_n converges in a stronger sense (*strong convergence*) and this may not be true. Note as well that this is the only point where the continuity of T is used: to insure that $f \circ T$ is continuous and hence that $\mu_{n_j}(f \circ T) \to \mu(f \circ T)$.

A simple possibility is to look for measures that are *affiliated* to some reference measure. Instead of going into some abstract discussion about a precise technical meaning of the word *affiliated* let us be very concrete and see immediately a simple example in which the reference measure is Lebesgue.

1.4. **A simple example: smooth expanding maps.** Let T be a $\mathcal{C}^2(\mathbb{T}^d, \mathbb{T}^d)$ expanding map.[8] As already mentioned we would like to restrict the measures we are interested in to measures that are related to Lebesgue. An interesting way to do so is to consider the following semi–norm

$$\|\mu\| := \sup_{\substack{|\varphi|_\infty \leq 1 \\ \varphi \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}^d)}} \sum_{i=1}^d \mu(\partial_i \varphi_i) \tag{1.1}$$

and consider the set of measures $\mathcal{B} := \{\mu \in \mathcal{P}(\mathbb{T}^d) \mid \|\mu\| < \infty\}$.

The next little lemma will helps us to understand what type of measures we are considering.

Lemma 1.1. *If $\|\mu\| < \infty$ then μ is absolutely continuous with respect to the Lebesgue measure m and, in addition, the Radon-Nykodim derivative is a function of bounded variation.*

Proof. Let $\mu \in \mathcal{B}$. For each $\varepsilon \in (0, 1)$ let J_ε be the smoothing operator

$$J_\varepsilon \varphi(x) := \int_{\mathbb{T}^d} \varepsilon^{-d} j(\varepsilon^{-1}(x - y))\varphi(y)dy$$

where, $j \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^+)$, $\text{supp}(j) \subset [-1/2, 1/2]^d$ and $\int_{\mathbb{R}^d} j(y)dy = 1$. Moreover define, as usual,

$$J_\varepsilon^* \mu(\varphi) := \mu(J_\varepsilon \varphi).$$

Then $w\text{-}\lim_{\varepsilon \to 0} J_\varepsilon^* \mu = \mu$.[9] In addition, since

$$J_\varepsilon 1 = 1, \text{ and } \partial_i(J_\varepsilon \varphi) = -J_\varepsilon \partial_i \varphi,$$

it holds $J_\varepsilon^* \mu(1) = \mu(1) = 1$ (that is $J_\varepsilon^* : \mathcal{P}(\mathbb{T}^d) \to \mathcal{P}(\mathbb{T}^d)$) and

$$\begin{aligned} \|J_\varepsilon^* \mu\| &= \sup_{\substack{|\varphi|_\infty \leq 1 \\ \varphi \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}^d)}} \sum_{i=1}^d \mu(J_\varepsilon \partial_i \varphi_i) = \sup_{\substack{|\varphi|_\infty \leq 1 \\ \varphi \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}^d)}} \sum_{i=1}^d \mu(\partial_i (J_\varepsilon \varphi_i)) \\ &\leq \sup_{\substack{|J_\varepsilon \varphi|_\infty \leq 1 \\ \varphi \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}^d)}} \sum_{i=1}^d \mu(\partial_i (J_\varepsilon \varphi_i)) = \|\mu\| \end{aligned} \tag{1.2}$$

Moreover, $|J_\varepsilon^* \mu(\varphi)| \leq \varepsilon^{-d} |j|_\infty \, |\mu| \, |\varphi|_{L^1}$, hence $J_\varepsilon^* \mu$ is absolutely continuous with respect to Lebesgue. Let $h_\varepsilon \in L^\infty(\mathbb{T}^d, \mathbb{R})$ be the density of $J_\varepsilon^* \mu$. Clearly, $h_\varepsilon \in BV(\mathbb{T}^d, \mathbb{R})$, the space of functions of bounded variation, in addition they have uniformly bounded variation. Accordingly, $\{h_\varepsilon\}_{\varepsilon > 0}$ is a relatively compact sequence

[8] That is $\|DT\| \geq \lambda > 1$.

[9] By w-lim we mean the limit in the weak (or weak-$*$ for the functional analytic oriented) topology.

in $L^1(\mathbb{T}^d,\mathbb{R})$ ([23]). Consider any convergent subsequence $\{h_{\varepsilon_j}\}_{j\in\mathbb{N}}$, and let $h\in BV(\mathbb{T}^d,\mathbb{R})$ be its limit, then for each $\varphi\in\mathcal{C}^0(\mathbb{T}^d,\mathbb{R})$ holds

$$\mu(\varphi)=\lim_{j\to\infty}J^*_{\varepsilon_j}\mu(\varphi)=\lim_{j\to\infty}\int_{\mathbb{T}^d}h_{\varepsilon_j}\varphi=\int_{\mathbb{T}^d}h\varphi.$$

□

Remark 1.1. *Note that the closure of the space of measures with $\mathcal{C}^1$ density by the above norm yields the space of measures with densities in the Sobolev space $W_{1,1}$. It is an interesting exercise to check that the following holds unchanged for such a space as well.*[10]

The basic idea of the present approach is the realization that the operator T_* evolving the measures is a *regularizing operator*, if properly viewed. This is made precise by the following.

Lemma 1.2 (Lasota-Yorke type inequality)**.** *For each $\mu\in\mathcal{B}$ holds,*

$$\begin{aligned}&T_*\mu(1)=\mu(1)\\&\|T_*\mu\|\le\lambda^{-1}\|\mu\|+B\mu(1).\end{aligned}$$

Proof. We have already discussed the first inequality, for the second

$$\begin{aligned}\sum_i T_*\mu(\partial_i\varphi_i)&=\sum_i\mu((\partial_i\varphi_i)\circ T)\\&=\sum_i\mu(\partial_i((DT^{-1}\varphi)\circ T)_i)-\sum_{ij}\mu(\varphi_j\circ T\partial_i[\partial_jT_i^{-1}(Tx)]).\end{aligned}\tag{1.3}$$

The result follows since

$$\sup_i|(DT^{-1}\varphi)\circ T)_i|_\infty=\sup_i|DT^{-1}\varphi_i|_\infty\le\lambda^{-1}|\varphi|_\infty,$$

while $|\sum_{ij}\mu(\varphi_j\circ T\partial_i[\partial_jT_i^{-1}(Tx)]|_\infty\le|D^2T|_\infty d|\varphi|_\infty=:B|\varphi|_\infty$. □

Lemma 1.2 readily implies that all the Kryloff-Bogoliouboff accumulations points, starting from a measure absolutely continuous with respect to the Lebesgue measure m, have density in $BV(\mathbb{T}^d,\mathbb{R})$.[11]

Due to this state of affairs, one can study directly the evolution of the densities rather than the evolutions of the measures. To do so set $d\mu=hdx$ and define the operator $\mathcal{L}:\mathrm{BV}(\mathbb{T}^d,\mathbb{R})\to\mathrm{BV}(\mathbb{T}^d,\mathbb{R})$ by

$$\mathcal{L}h:=\frac{dT_*\mu}{dx}.$$

A direct computation yields the well known formula

$$\mathcal{L}f(x)=\sum_{y\in T^{-1}x}|\det(D_yT)|^{-1}f(y).\tag{1.4}$$

[10]In fact, changing a bit the norms (that is considering measures with densities in $\mathcal{C}^1$), one can prove–with a bit more work–that the limiting object of the Kryloff-Bogoliouboff procedure are measures with $\mathcal{C}^1$ densities, when starting by measures in such a class. This is a nice exercise as well.

[11]In fact, iterating the Lemma yields $\|T_*^n\mu\|\le\lambda^{-n}\|\mu\|+B(1-\lambda^{-1})^{-1}$. The result follows then by standard approximation arguments and the compactness of the unit ball of BV in L^1 (see, e.g. the proof of Lemma 1.3).

It is an helpful exercise to redo all the above arguments directly for the operator $\mathcal{L}$. The logic is exactly the same, only one must use the $\mathrm{BV}(\mathbb{T}^d, \mathbb{R})$ norm rather than the $\|\cdot\|$ norm (which, in fact, is exactly the same, once one considers densities rather than measures).

A natural question is if the procedure outlined above is able to provide all the invariant measures we may be interested in. To get a feeling of the situation just consider the next lemma.

Lemma 1.3. *For each invariant measurable set A of positive Lebesgue measure there exists an invariant measure $\mu \in \mathcal{B}$ supported in A.*

Proof. Let χ_A be the characteristic function of the set A, and consider the measure $m_A(\varphi) := m(\chi_A \varphi) m(A)^{-1}$. The idea is to apply the Kryloff-Bogoliouboff procedure to m_A, the problem being that m_A may not belong to $\mathcal{B}$ (it will not if χ_A is not of bounded variation). Yet, since BV is dense in L^1, for each $\varepsilon > 0$ we can consider $g_\varepsilon \in \mathrm{BV}$ such that $\int g_\varepsilon = 1$ and $|\chi_A - g_\varepsilon|_{L^1} \leq \varepsilon$. Let $m_\varepsilon(\varphi) := m(g_\varepsilon \varphi)$ and let μ_A and μ_ε be weak limits of some subsequence $\frac{1}{n_j}\sum_{i=0}^{n_j-1} T_*^i m_A$ and $\frac{1}{n_j}\sum_{i=0}^{n_j-1} T_*^i m_\varepsilon$, respectively. Clearly μ_A is supported in A. In addition, on the one hand

$$|\mu_A(\varphi) - \mu_\varepsilon(\varphi)| \leq \frac{1}{n_j}\sum_{i=0}^{n_j-1} m(|\chi_A - g_\varepsilon|\,|\varphi \circ T^i|) \leq |\varphi|_\infty \varepsilon,$$

and, on the other hand,

$$\|\mu_\varepsilon\| \leq \lim_{j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1} \lambda^{-i}\|\mu_\varepsilon\| + B(1-\lambda^{-1})^{-1} = B(1-\lambda^{-1})^{-1}.$$

Thus, by Lemma 1.1 it follows that there exists $h_\varepsilon \in \mathrm{BV}$, for which the variation satisfies $\bigvee h_\varepsilon \leq B(1-\lambda^{-1})^{-1}$, such that $\mu_\varepsilon(\varphi) = \int h_\varepsilon \varphi$. But then, by the compactness of the unit ball of BV in L^1, [23], it follows that there exists a subsequence ε_j such that h_{ε_j} converges in L^1 to a function of bounded variation, $\bigvee h \leq B(1-\lambda^{-1})^{-1}$. Accordingly,

$$\mu_A(\varphi) = \int h\varphi,$$

that is $\mu_A \in \mathcal{B}$. $\square$

The above lemma gives us quite a deep information on the structure of the invariant sets of positive Lebesgue measure. For example, in the one dimensional case ($d = 1$), since $h \in \mathrm{BV}$, its support must contain an interval, that is A must contain an interval. This eliminates the possibility of an invariant Cantor set of positive Lebesgue measure; this property is at times called *local ergodicity.*

The arguments in this section can be easily generalized. For example one can investigate cases in which the map is more or less regular.

For simplicity let us restrict our discussion to the one dimensional case ($d = 1$).

1.5. **More Regularity.** Suppose that $T \in \mathcal{C}^{n+1}(\mathbb{T}^1, \mathbb{T}^1)$, for $n \in \mathbb{N}$. One can then consider the norms, for $k \leq n$,

$$\|\mu\|_k := \sup_{\substack{\varphi \in \mathcal{C}^k \\ |\varphi|_{\mathcal{C}^0} \leq 1}} \mu(\varphi^{(k)}).$$

A trivial analogous of Lemma 1.1 shows that if $\sum_{j=0}^{n}\|\mu\|_j<\infty$ then μ is absolutely continuous with respect to Lebesgue with a density $n-1$ time differentiable and with the $n-1$ derivative being a bounded variation function.

It is an easy exercise to verify the formula

$$\frac{d^k}{dx^k}[(DT)^{-k}\varphi\circ T]=\varphi^{(k)}\circ T+\sum_{j=0}^{k-1}\alpha_j(T)\varphi^{(j)}\circ T,$$

where α_j depends only on the first $k-j$ derivatives of T. Using such a formula one can easily generalize the computation in the proof of Lemma 1.2 whereby obtaining, for each $k\leq n$,

$$\|T_*\mu\|_k\leq\lambda^{-n}\|\mu\|_k+B\sum_{j=0}^{k-1}\|\mu\|_j. \tag{1.5}$$

The above estimates give, for $k=1$, the same statement of Lemma 1.2. In particular, as we have already remarked,

$$\|T_*^m\mu\|_1\leq B(1-\lambda^{-1})^{-1}\quad\forall m\in\mathbb{N}.$$

Using (1.5) iteratively on k one obtains, by the same token, that there exist $C>0$ such that

$$\|T_*^m\mu\|_n\leq C\quad\forall m\in\mathbb{N}.$$

In analogy with what we have seen so far, the above inequality implies that there are invariant measures in $\mathcal{B}_n:=\{\mu\in\mathcal{P}(X)\mid\sum_{i=1}^{n}\|\mu\|_i<\infty\}$.

1.6. **Less Regularity.** There are two possibilities: maps that are only $\mathcal{C}^{1+\alpha}(\mathbb{T}^1,\mathbb{T}^1)$, $\alpha\in(0,1)$, or maps that can have discontinuities. We will concentrate on the second case being more interesting.[12]

Let us consider the one dimensional piecewise smooth expanding case. That is a map $T:[0,1]\to[0,1]$ such that there exists a partition $\mathcal{Z}$, in intervals, of $[0,1]$ such that, for each $Z\in\mathcal{Z}$, the map T restricted to $\mathring{Z}$ is $\mathcal{C}^2$. To further simplify matters we assume strong expansivity, that is $|DT|\geq\lambda>2$.[13] In this case we can use the same semi-norm used for $\mathcal{C}^2$ maps, (1.1), that is

$$\|\mu\|:=\sup_{\substack{\varphi\in\mathcal{C}^1\\|\varphi|_\infty\leq 1}}\mu(\varphi').$$

Remark 1.2. *Note that, due to the discontinuity of the map, the original Kryloff-Bogoliouboff argument 1.1 does not apply to the present situation.*

To overcome such a problem we will implement the same argument of Proposition 1.1 but without using the weak compactness of the measures, instead we will use the compactness of the unit ball of the function of bounded variation in L^1, [23].

Indeed, let us start by noticing the following.

[12] For the reader interested in the first case, consider the semi–norms:

$$\|\mu\|_\alpha:=\sup_{\substack{\varphi\in\mathcal{C}^1\\|\varphi|_{\mathcal{C}^{1-\alpha}}\leq 1}}\mu(\varphi').$$

[13] In the general case in which one has only $|DT|\geq\lambda>1$, once must consider the powers of the map T in order to gain enough expansion.

Lemma 1.4. *For piecewise smooth maps $T_*\mathcal{B} \subset \mathcal{A}(m)$, the set of absolutely continuous measures with respect to Lebesgue.*

Proof. If $\|\mu\| < \infty$, then Lemma 1.1 implies that μ is absolutely continuous with respect to Lebesgue, it is then trivial to see that $T_*\mu \in \mathcal{A}(m)$. □

Accordingly, if $\|\mu\| < \infty$, then, for each $\varphi \in \mathcal{C}^1$,

$$T_*\mu(\varphi') = \sum_{Z\subset\mathcal{Z}} T_*\mu(\varphi'\chi_Z) = \sum_{Z\subset\mathcal{Z}} T_*\mu(\varphi'\chi_{\mathring{Z}}),$$

since, by Lemma 1.4, $T_*\mu$ gives zero weight to points.

For each $Z \in \mathcal{Z}$, define ϕ_Z to be linear and such that $\phi_Z = \varphi$ on ∂Z, then define $\psi_Z = \varphi - \phi_Z$, on Z, and extend ψ_Z to all $[0,1]$ by setting it equal to zero outside Z. We obtain in this way a continuous function. Moreover, for each $x \in \mathring{Z}$,

$$|\phi_Z'(x)| \leq \frac{2|\varphi|_\infty}{|Z|}.$$

Thus,

$$\begin{aligned}
T_*\mu(\varphi') &= \sum_{Z\subset\mathcal{Z}} \mu(\psi_Z' \circ T) + \mu(\phi_Z' \circ T\chi_{\mathring{Z}}) \\
&\leq \sum_{Z\subset\mathcal{Z}} \mu((\psi_Z \circ T(DT)^{-1})') + B\mu(1)|\varphi|_\infty \\
&= \mu\big(\big(\sum_{Z\subset\mathcal{Z}} \psi_Z \circ T(DT)^{-1}\big)'\big) + B\mu(1)|\varphi|_\infty.
\end{aligned}$$

We are left with the problem that the function $\bar\psi := \sum_{Z\subset\mathcal{Z}} \psi_Z \circ T(DT)^{-1}$ it is not $\mathcal{C}^1$, in fact its derivative has a finite number of points of discontinuity. Nevertheless, for each ε we can find a continuous function g_ε, $|g_\varepsilon|_\infty \leq |\bar\psi'|_\infty$, such that it differs from $\bar\psi'$ only on a set of Lebesgue measure ε. Then, if we define $\psi_\varepsilon(x) := \bar\psi(0) + \int_0^x g_\varepsilon(z)dz$, holds $|\bar\psi - \psi_\varepsilon|_\infty \leq 2|\bar\psi'|_\infty\varepsilon$. Hence,

$$\mu(\bar\psi') \leq \mu(\psi_\varepsilon') + 2|\bar\psi'|_\infty\|\mu\|\varepsilon \leq \|\mu\|(2|\varphi|_\infty + 4|\bar\psi'|_\infty\varepsilon).$$

Since ε is arbitrary, this implies

$$T_*\mu(\varphi') \leq (2\lambda^{-1}\|\mu\| + B)|\varphi|_\infty.$$

We have thus, also in this case, the Lasota-Yorke inequality[14]

$$\|T_*\mu\| \leq 2\lambda^{-1}\|\mu\| + B.$$

We can thus redo the Kryloff–Bogoliouboff argument: let $\mu_0 \in \mathcal{B}$ be a probability measure, then let $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} T_*^i\mu_0$. By iterating the Lasota–Yorke inequality follows

$$\|\mu_n\| \leq \frac{1}{n}\sum_{i=0}^{n-1}(2\lambda^{-1})I\|\mu_0\| + B(1-2\lambda^{-1})^{-1} \leq \frac{\|\mu_0\|}{n(1-2\lambda^{-1})} + \frac{B}{1-2\lambda^{-1}}.$$

By Lemma 1.1 we know that the μ_n are absolutely continuous with respect to Lebesgue and that their density h_n are functions of bounded variation with uniformly bounded variation norm. The already mentioned compactness of the unit BV ball in L^1 implies that there exists a subsequence h_{n_j} such that it converges in L^1 to a function h of bounded variation norm bounded by $B(1-2\lambda^{-1})^{-1}$. Let

[14]In fact, this is the original Lasota-Yorke inequality,[62].

μ be the measure with density h, then $\mu \in \mathcal{B}$ and it is trivial to check that it is an invariant measure.

Remark 1.3. *Note that we have a stronger convergence than in the usual Kryloff–Bogoliouboff argument (μ_{n_j} converges on all L^∞ functions), this is an aspect of the stronger control that we must have on the system in order to control the effect of the discontinuities.*

2. **Second Lecture** (THREE LESS TRIVIAL EXAMPLES)

In the previous lecture we have considered only probability measures, this is a very good point of view for a probabilist but not so exciting for an analyst: the set of probability measures is not a vector space. It is natural to ask if the previous arguments can be turned into a more functional analytic setting. This will be done while treating the next examples. We will see in the following lectures that such a functional analytic point of view will prove far reaching.

2.1. **Coupled map lattices.**

Let $\tilde{T}$ be a smooth expanding map of the circle. Define $T : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ as the product of such maps. Next, let $F_\varepsilon : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ be a diffeomorphism close to the identity. More precisely, assume that $(DF_\varepsilon)^{-1} = \mathrm{Id} + \varepsilon A$ and that

$$|A_{ij}| \le Ce^{-a|i-j|} \text{ and } |\partial_k A_{ij}| \le Ce^{-a|i-k|-a|j-k|}.$$

We finally consider that map $T_\varepsilon := F_\varepsilon \circ T$.[15] Next, we want to consider an appropriate Banach space on which to analyze the dynamics. In analogy with the previous lecture, let us consider the set of signed measures $\mathcal{M}(\mathbb{T}^{\mathbb{Z}})$ and define on it the two norms[16]

$$|\mu| := \sup_{\substack{\varphi \in \mathcal{C}^0(\mathbb{T}^{\mathbb{Z}}, \mathbb{R}) \\ |\varphi|_\infty \le 1}} \mu(\varphi)$$

$$\|\mu\|_1 := \sup_{\substack{\varphi \in \mathcal{C}^1(\mathbb{T}^{\mathbb{Z}}, \mathbb{R}^{\mathbb{Z}}) \\ \sum_{i \in \mathbb{Z}} |\varphi_i|_\infty \le 1}} \sum_{i \in \mathbb{Z}} \mu(\partial_i \varphi_i)$$

The first is the usual norm in $\mathcal{M}(\mathbb{T}^{\mathbb{Z}})$, the second is a generalization of the bounded variation norm to the infinite dimensional setting. We will work in the space $\mathcal{B} := \{\mu \in \mathcal{M}(\mathbb{T}^{\mathbb{Z}}) \mid \|\mu\|_1 < \infty\}$ which is a Banach space once it is equipped with the norm $\|\cdot\| := \|\cdot\|_1 + |\cdot|$.

Let us gain some understanding of what $\mathcal{B}$ looks like. Given $\mu \in \mathcal{M}(\mathbb{T}^{\mathbb{Z}})$ and $\Lambda \subset \mathbb{Z}$ we can define the marginal $\mu_\Lambda \in \mathcal{M}(\mathbb{T}^\Lambda)$ by the relation

$$\mu(\varphi) =: \mu_\Lambda(\varphi) \quad \forall \varphi \in \mathcal{C}^0(\mathbb{T}^\Lambda, \mathbb{R}).$$

Now, if Λ is a fine set, it follows that $\|\mu_\Lambda\| < \infty$. We know already (Lemma 1.1) that this means that μ_Λ is absolutely continuous with respect to Lebesgue with density of bounded variation. Accordingly, all the measures in $\mathcal{B}$ have finite dimensional

[15] The meaning of the above inequalities is that the maps are weakly coupled and the coupling is quite local, that is a motion of one coordinate is not influenced from distant coordinates, at least for a short time.

[16] Actually, $\|\cdot\|_1$ is only a semi-norm, indeed $\|\otimes_{\mathbb{Z}} m\|_1 = 0$.

marginals which are absolutely continuous with respect to Lebesgue.[17] This is clearly a natural generalization of what we have seen in the finite dimensional case.

To check that the Lasota-York inequality holds just remember equation (1.3) and notice that[18]

$$|(DT_\varepsilon)^{-1}| \leq \lambda^{-1}(1+\varepsilon(1-e^{-a})^{-1})$$

and

$$\begin{aligned}\sum_{ij} |\varphi_j \partial_i[(D\tilde{T})^{-1}(\mathrm{Id}+\varepsilon A)_{ij}]|_\infty &\leq \sum_{ij} |\varphi_j|_\infty [B\delta_{ij} + \varepsilon(B+\lambda^{-1})C]e^{-a|i-j|}\\ &\leq \sum_i |\varphi_i|_\infty [B + \varepsilon C(B+\lambda^{-1})(1-e^{-a})^{-1}]\\ &=: B_1 \sum_i |\varphi_i|_\infty.\end{aligned}$$

Accordingly, for ε small enough there exists $\theta \in (\lambda^{-1}, 1)$ such that

$$\|T_{\varepsilon *}\mu\| \leq \theta\|\mu\| + B_1|\mu|.$$

To conclude we have a last problem: in infinite dimensions the unit ball of $\mathcal{B}$ is no longer compact in L^1. This is a serious problem as far as the study of the statistical properties is concerned (see the third lecture), one that it is not clear how to deal with up to now. Yet, limited to the properties of the invariant measures, such a strong property it is not needed, in fact it suffices that the unit $\mathcal{B}$ ball is sequentially closed in the weak topology. Clearly, in such a case, all the Kryloff–Bogoliouboff accumulation point will have $\|\cdot\|$ norm bounded by $B_1(1-\theta)^{-1}+1$, since the iteration of the Lasota-Yorke inequality yields, for probability measures,

$$\|T_{\varepsilon *}^n\mu\| \leq \theta^n\|\mu\| + B_1|(1-\theta)^{-1}\mu| \leq B_1(1-\theta)^{-1}+1.$$

This implies the existence of invariant measures with finite dimensional marginals that are absolutely continuous with respect to Lebesgue and with BV density.

Lemma 2.1. *The $\mathcal{B}$ unit ball is sequentially closed in the weak topology.*

Proof. Let the sequence $\{\mu_n\} \subset \mathcal{B}$ converge weakly to μ and suppose that $\|\mu_n\| \leq 1$. Then, for each function $\varphi \in \mathcal{C}^1(\mathbb{T}^\mathbb{Z}, \mathbb{R}^\mathbb{Z})$ such that $\sum_i |\varphi_i|_\infty \leq 1$, holds

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \mu(\partial_i\varphi_i) &= \lim_{l\to\infty}\sum_{i=-l}^{l} \mu(\partial_i\varphi_i) = \lim_{l\to\infty}\lim_{n\to\infty}\sum_{i\in\Lambda}\mu_n(\partial_i\varphi_i)\\ &\leq \lim_{l\to\infty}\lim_{n\to\infty}\|\mu_n\|\sum_{i=-l}^{l}|\varphi_i|_\infty \leq 1.\end{aligned}$$

Hence $\|\mu\| \leq 1$. □

For a generalization of the above results to the discontinuous case see [45]. Instead consult [35, 5] for much stronger results in the smooth (analytic) case (in the latter work a different approach, based on Statistical Mechanical methods–cluster expansion– and pioneered in [14, 12, 13], is used).

---

[17] More is true: the bounded variation norm of the density of the Λ marginal is bounded by $|\Lambda|\,\|\mu\|_1$, thus there is a precise control on how the measure becomes singular with respect to Lebesgue as Λ increases.

[18] The operator norm is taken with respect to the ℓ^1 vector space norm $|v|_1 := \sum_{i\in\mathbb{Z}}|v_i|$.

2.2. **Partially hyperbolic systems.** Let us consider a partially hyperbolic system (X, T) where X is a Riemannian compact manifold and $T \in \mathcal{C}^2(X, X)$.

This means that the tangent space is naturally split into two space $E_0(x) \oplus E_u(x) = \mathcal{T}_x X$ such that[19]

$$\|D_x T|_{E_u(x)}\| \geq \lambda > 1$$
$$\|D_x T|_{E_0(x)}\| \leq 1.$$

This implies the existence of an unstable foliation $\{W^u(x)\}$, [69, 70]. For the following it is important to know more precisely the properties of such a foliation. This subject has been widely investigated, I refer to [63] and [34] for an introduction to the field and to [76, 26] for more complete results. The easiest way to do so is to describe the local picture near an arbitrary point z.

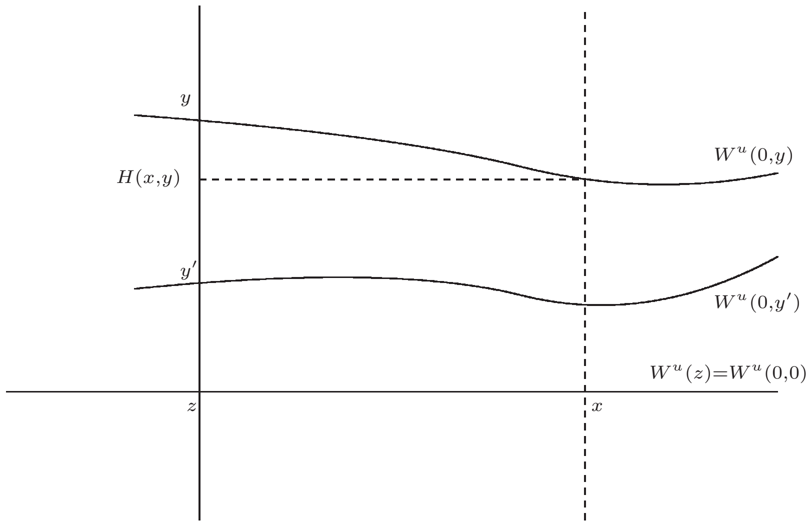

FIGURE 2.1. Unstable foliation

The foliation can be described (locally) by a function H (see 2.1) with the following properties

(1) $H(0, 0) = 0$; $H(\xi, 0) = \xi$.
(2) $H(\cdot, y)$ is the graph of $W^u(0, y)$.
(3) $H(\cdot, y)$ id $\mathcal{C}^2$ for each y
(4) $H(x, \cdot)$ is $\mathcal{C}^\alpha$ for some $\alpha > 0$ (this is nothing else than the Holonomy between the affine spaces $\{(0, y)\}_{y\in\mathbb{R}^{d_0}}$ and $\{(x, y)\}_{y\in\mathbb{R}^{d_0}}$).
(5) $J_x H(\cdot, y) \in \mathcal{C}^1$.

Due to the above properties it is possible to straighten locally the foliation via the change of coordinates

$$\Psi(\xi, \eta) = (\xi, H(\xi, \eta)). \tag{2.1}$$

Note that such a change of coordinate is only Hölder but it is absolutely continuous and

$$J\Psi = J_\eta H$$

[19]This it is not the general definition, yet the following argument holds in greater generality.

Let us consider the set of continuous vector fields in the unstable direction $\mathcal{V} := \{v \in \mathcal{C}^0(X, \mathcal{T}X) \mid v(x) \in E^u(x); \textit{smooth when restricted to unstable manifolds}\}$. The first task is to define a *divergence* for such vector fields

Lemma 2.2. *There exists a functional* u-div : $\mathcal{V} \to \mathcal{C}^0(X)$ *such that, for each* $h \in \mathcal{C}^1$*, holds*

$$\int_X v(h) = \int_X h \, \text{u-div}\, v. \tag{2.2}$$

Proof. Note that (2.2) always defines a functional but, in general, we have only $\text{u-div}\, v \in (\mathcal{C}^1)^*$, our task is thus to prove the extra regularity. If $v \in E_u(x)$ then $v = (w, \partial_\xi H w)$. Accordingly the vector fields in $\mathcal{V}$ can be described as $(w, \partial_\xi H w)$ with $w \in \mathcal{C}^1$. Thus

$$v(h) = \sum_i w_i \partial_{x_i} h + (\partial_\xi H w)_i \partial_{y_i} h.$$

Hence, for each $h \in \mathcal{C}^1$, holds

$$\begin{aligned}
\int v(h) &= \int \left\{ \sum_i w_i \partial_{x_i} h + (\partial_\xi H w)_i \partial_{y_i} h \right\} J\Psi d\xi d\eta \\
&= \int \sum_i w_i \partial_{\xi_i} (h \circ \Psi) J\Psi d\xi d\eta \\
&= \int d\eta \int \sum_i \partial_{\xi_i} (w_i J\Psi) h \circ \Psi d\xi \\
&= \int h \left\{ \sum_i \partial_{\xi_i} (w_i J\Psi)(J\Psi)^{-1} \right\} \circ \Psi^{-1}.
\end{aligned}$$

□

We then consider the set $\mathcal{M}(X)$ of signed measures on X and we define on it the following norms[20]

$$\begin{aligned}
|\mu| &:= \sup_{\substack{\varphi \in \mathcal{C}^0 \\ |\varphi|_\infty \leq 1}} \mu(\varphi) \\
\|\mu\|_1 &:= \sup_{\substack{v \in \mathcal{V} \\ |v|_\infty \leq 1}} \mu(\text{u-div}\, v).
\end{aligned} \tag{2.3}$$

Not surprisingly we will restrict ourselves to $\mathcal{B} := \{\mu \in \mathcal{M}(X) \mid \|\mu\|_1 < \infty\}$. The next Lemma will give an idea of which type of measures we are talking about, but to state it properly some notation is needed. Let us consider a neighborhood U that can be covered by a coordinate chart of the type previously described, so that in the new coordinates the unstable foliation consists of hyper-planes. Clearly in U the unstable foliation gives rise to a natural, measurable, partition, let us call it $\mathcal{F}_u$.

Lemma 2.3. $\|\mu\|_1 < \infty$ *implies that, for each* $\varphi \in \mathcal{C}^0(X, \mathbb{R})$ *supported in* U*,*[21]

$$\mathbb{E}_\mu(\varphi \mid \mathcal{F}_u) = \mathbb{E}_m(h\varphi \mid \mathcal{F}_u)$$

[20]Again $\|\cdot\|_1$ is a semi-norm, indeed $\|m\|_1 = 0$.

[21]In the following I will use the probabilistic notation $\mathbb{E}_\mu(f)$ for $\mu(f)$, so that I can more naturally work with the conditional expectation $\mathbb{E}(\cdot \mid \mathcal{F}_u)$.

where $h|_{\mathcal{F}_u}$ is BV m–a.e..

Proof. Let v be supported in a ball $B \subset U$ and ϕ be $\mathcal{F}_u$ measurable. Then $\phi|_\gamma =$ const for all $\gamma \in \mathcal{F}_u$ and $v(\phi) = 0$. Then

$$\mathbb{E}_\mu(\text{u-div}(\phi v)) = \mathbb{E}_\mu(\phi\, \text{u-div}\, v) = \mathbb{E}_\mu(\phi \mathbb{E}(\text{u-div}\, v \mid \mathcal{F}_u)).$$

Taking the sup on such ϕ, $|\phi|_\infty \leq 1$, we have

$$\mathbb{E}_\mu(|\mathbb{E}(\text{u-div}\, v \mid \mathcal{F}_u)|) \leq \|\mu\|_1 |v|_\infty$$

Hence there exists $A \in L^1(X, \mu)$ such that

$$\mathbb{E}_\mu(\text{u-div}\, v \mid \mathcal{F}_u)(x) \leq A(x)\|\mu\|_1 |v|_\infty$$

and the lemma follows. □

The above lemma characterizes the measures locally, but this suffice since it is always possible to reduce all the considerations to small neighborhood by using a smooth partition of unity.

As an interesting example of measures in $\mathcal{B}$ consider

$$\mu(\varphi) := \int_{W^u} h\varphi$$

where W^u is a regular piece of unstable manifold and $h \in \mathcal{C}^1(W^u, \mathbb{R})$.

Let us see how the dynamics acts on such norms.

$$T_*\mu(\varphi) = \mu(\varphi \circ T) \leq |\mu||\varphi|_\infty$$

thus $|T_*\mu| \leq |\mu|$.

Moreover if $\frac{d\mu}{dm} = h \in \mathcal{C}^1(X, \mathbb{R})$, then $\frac{dT_*^n\mu}{dm} = \mathcal{L}^n h$, where the operator $\mathcal{L}$ is defined by $\mathcal{L}^n h = |\det DT^n|^{-1} h \circ T^{-n} := g_n h \circ T^{-n} \in \mathcal{C}^{(1)}(X, \mathbb{R})$, thus

$$T_*^n\mu(\text{u-div}\, v) = \int v(\mathcal{L}^n h) = \int \frac{v(g_n)}{g_n}\mathcal{L}^n h + \int \bar{v}_n(h)$$

where

$$\bar{v}_n = (DT^{-n}v) \circ T^n.$$

Clearly $\bar{v}_n \in \mathcal{V}$ and $|\bar{v}_n|_\infty \leq \lambda^{-n}|v|_\infty$. Accordingly,

$$\|T_*^n\mu\|_1 \leq \lambda^{-1}\|\mu\|_1 + B|\mu|. \tag{2.4}$$

This, as we have already seen, implies that there exists invariant measures in $\mathcal{B}$, these are commonly called SRB (Sinai-Ruelle-Bowen) measures.[22]

2.3. **Non-uniform expansion.** The next logical step is to investigate situations in which some non-uniform hyperbolicity takes place. In general this is a very hard problem not yet well understood. Although many results exists ([91] for an overview) no general theory seems to be available as yet. Here, I will discuss the simplest possible example: a non-uniformly expanding map.

[22] The closeness of the unit ball of $\mathcal{B}$ with respect to the weak topology is left as an exercise to the reader.

To simplify our discussion even further let us consider a very concrete family of maps.[23] Let $0 < \gamma < 1$ and define the map $T : [0, 1] \to [0, 1]$

$$T(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \forall x \in [0, 1/2) \\ 2x - 1 & \forall x \in [1/2, 1] \end{cases} \tag{2.5}$$

This kind of maps are called *intermittent* and were addressed by Prellberg and Slawny in [75], they have a relationship with a statistical model introduced by Fisher [24] and then studied by Gallavotti [25]. In the papers [27], [90], the dynamical behavior of these maps was taken as a model for the intermittency of turbulent flows [74]. The existence of absolutely continuous invariant measures for such maps was first proved in [86] and their statistical properties have been widely investigated in more recent years [61, 94, 32, 95, 51, 83]. Here we will discuss only the existence of invariant measures by adopting our present approach.

Let us consider the semi norm

$$\|\mu\|_\alpha := \sup_{\substack{\varphi(0)=0=\varphi(1)\\ |\varphi|_{\mathcal{C}^0}+H^0_\alpha(\varphi)\leq 1}} \mu(\varphi')$$

$$H^0_\alpha(\varphi) := \sup_{x\in[0,1]} \frac{|\varphi(x)|}{x^\alpha}$$

where we chose $\alpha > \gamma$.

Lemma 2.4. *The space $\mathcal{B}_\alpha := \{\mu \in \mathcal{M}(\mathbb{T}) \mid \|\mu\|_\alpha < \infty\}$ consists of measures that are absolutely continuous with respect to Lebesgue and with density h_μ such that, setting $\bar h_\mu(x) := x^\alpha h_\mu(x)$, $\bar h_\mu \in L^\infty$. In addition*

$$|\bar h_\mu|_{L^\infty} \leq 2(\alpha + 2)\|\mu\|_\alpha + (\alpha + 1)|\mu|.$$

Proof. Let $\mu \in \mathcal{B}_\alpha$, then for each $\varphi \in L^1$, $\int_{[0,1]} x^\alpha \varphi(x) = 0$, holds

$$\int_{[0,1]} x^\alpha \varphi(x)\mu(dx) = \int_{[0,1]} \frac{d}{dx}\left(\int_0^x \xi^\alpha \varphi(\xi)d\xi\right)\mu(dx).$$

Now, let $\phi(x) := \int_0^x \xi^\alpha \varphi(\xi)d\xi$, it is easy to see that $\phi(0) = \phi(1) = 0$, $|\phi|_{\mathcal{C}^0} \leq |\varphi|_{L^1}$, $H_\alpha(\phi)(0) \leq |\varphi|_{L^1}$. Accordingly,

$$\bar\mu(\varphi) := \int_{[0,1]} x^\alpha \varphi(x)\mu(dx) \leq 2\|\mu\|_\alpha|\varphi|_{L^1}.$$

Thus, for each $\varphi \in L^1$, holds, setting $\bar\varphi := \int x^\alpha \varphi(x)$,

$$\bar\mu(\varphi) = \bar\mu(\varphi - (\alpha + 1)\bar\varphi) + (\alpha + 1)\bar\varphi\bar\mu(1) \leq (2(\alpha + 2)\|\mu\|_\alpha + (\alpha + 1)|\mu|)|\varphi|_{L^1}.$$

This implies that $\bar\mu$ is absolutely continuous with respect to Lebesgue and that its density $\bar h_\mu \in L^\infty([0, 1], m)$, with $|\bar h_\mu|_\infty \leq 2(\alpha + 2)\|\mu\|_\alpha + (\alpha + 1)|\mu|$. □

To continue we need to show that the operator enjoys some regularization property, nevertheless we cannot hope in a Lasota-Yorke type inequality as the ones already seen. Indeed, such an inequality would imply an exponentially fast convergence to the invariant measure (see the next lecture) while it is known for some time that such maps exhibits only a polinomially fast convergence to equilibrium

[23] One could consider more general examples (e.g. any piecewise smooth expanding maps with a finite number of neutral periodic orbits of the type $T^q x = x + x^{1+\gamma}$, $\gamma \in (0, 1)$) but it would make little difference in the following, apart from making the exposition less readable.

[52, 66]. Nevertheless, it is possible to obtain a weaker Lasota-Yorke type inequality which suffices for our purposes.

Lemma 2.5. *For each $\alpha > \beta > \gamma$ there exist $C > 0$ such that*
$$\|T_*^n \mu\|_\alpha \leq \sigma_{\alpha\beta}(n)\|\mu\|_\beta + B_{\alpha\beta}(n)|\mu|, \quad \forall n \in \mathbb{N},$$
where $\sigma_{\alpha\beta}(n) := Cn^{-\frac{\alpha-\beta}{\gamma}}$ and $B_{\alpha,\beta}(n) := Cn^{\frac{(1-\alpha)(\alpha-\beta)}{\gamma(\alpha-\gamma)}}$.

Before proving such a non-uniform version of the Lasota-Yorke inequality, let us see how it solves our problems. For each $\epsilon > 0$ sufficiently small let $\alpha_k := \gamma + \epsilon\gamma(1 + \frac{1}{k}) < 1$, set $\delta_k := \alpha_{k+1} - \alpha_k$, $n_k = e^{a\epsilon^{-1}k^2}$, $m_k := \sum_{j=1}^k n_j$, $\sigma_k := \sigma_{\alpha_k,\alpha_{k+1}}(n_k)$ and $B_k := B_{\alpha_k,\alpha_{k+1}}(n_k)$.[24] Then for $\|\mu\|_\gamma < \infty$ holds

$$\begin{aligned}
\|T_*^{m_k}\mu\|_{\gamma+2\epsilon} \leq \|T_*^{m_k}\mu\|_{\alpha_1} &\leq \prod_{j=1}^k \sigma_j \|\mu\|_{\alpha_k} + \sum_{l=1}^{k-1} B_l \prod_{j=1}^l \sigma_j |\mu| \\
&\leq \prod_{j=1}^k \sigma_j \|\mu\|_\gamma + \sum_{l=1}^{k-1} B_l \prod_{j=1}^l \sigma_j |\mu|.
\end{aligned} \tag{2.6}$$

Now $\sigma_k \leq Ce^{-a} =: \sigma_* < 1$, provided a has been chosen large enough. Thus $\prod_{j=1}^k \sigma_j \leq \sigma_*^k$ and
$$\|T_*^{m_k}\mu\|_{\gamma+2\epsilon} \leq \sigma_*^k \|\mu\|_\gamma + C\sum_j^{k-1} \sigma_*^j e^{\frac{1-\gamma}{\epsilon\gamma}a}|\mu| \leq \sigma_*^k\|\mu\|_\gamma + B_\varepsilon|\mu|.$$
From the above inequality follows, in the usual way, that each invariant measure constructed with the Kryloff-Bogoliouboff argument, starting from an absolutely continuous measure, will yield an invariant measure with $\|\cdot\|_\alpha$ norm finite, for each $\alpha > \gamma$.

To prove Lemma 2.5 we need a good control on the distortion of the map.

Let $\mathcal{Z}_n$ be the dynamical partition of level n (that is, $\mathcal{Z}_n$ is a maximal partition such that T^n is one-one on its elements). To have an idea of how it looks like let $a_0 = 1$, $a_1 := 1/2$ and $\{a_{n+1}\} := T^{-1}a_n \cap [0, 1/2]$. Then $[0, a_n] \in \mathcal{Z}_n$.

Lemma 2.6. *There exists a constant $C > 0$ such that*
$$\frac{1}{Cn^{\frac{1}{\gamma}}} \leq a_n \leq \frac{C}{n^{\frac{1}{\gamma}}}\,; \quad \frac{2^\gamma}{C^{\gamma+1}n^{\frac{1}{\gamma}+1}} \leq a_{n-1} - a_n \leq \frac{C^{\gamma+1}2^\gamma}{n^{\frac{1}{\gamma}+1}}\,.$$
Proof. Clearly, $a_{n+1} \leq a_n$. Hence
$$a_{n+1} = a_n - 2^\gamma a_{n+1}^{\gamma+1} \geq a_n - 2^\gamma a_n^{\gamma+1} \geq \frac{1}{Cn^{\frac{1}{\gamma}}}\left[1 - \frac{2^\gamma}{c^\gamma n}\right] \geq \frac{1}{C(n+1)^{\frac{1}{\gamma}}},$$
provided C is large enough.

On the other hand, for C large, $a < Cn^{-\frac{1}{\gamma}}$ for each a_n such that $a_n - 2^\gamma a_n^{\gamma+1} < 2^{-\gamma}a_n$. For larger n we have
$$a_{n+1} = a_n - 2^\gamma(a_n - 2^\gamma a_{n+1}^{\gamma+1})^{\gamma+1} \leq a_n - 2^\gamma 2^{-\gamma(\gamma+1)}a_n^{\gamma+1} \leq \frac{C}{(n+1)^{\frac{1}{\gamma}}},$$
provided, again, C is large enough. Finally, the last inequality follows from the already established ones and $a_{n-1} - a_n = 2^\gamma a_n^{\gamma+1}$. $\square$

[24] All this choices are largely arbitrary, many others would do as well.

Our basic distortion result is the following.[25]

Lemma 2.7. *There exists $C > 0$ such that for each $n \in \mathbb{N}$, $Z \in \mathcal{Z}_n$ and $x, y \in Z$ holds true*

$$e^{-\frac{C}{n-m+1}} \leq \frac{D_x T^m}{D_y T^m} \leq e^{\frac{C}{n-m+1}} \qquad \forall m \leq n.$$

Proof. For $m \leq n$, holds

$$\begin{aligned}\prod_{j=0}^{m-1} \frac{D_{T^j x} T}{D_{T^j y} T} &\leq \prod_{j=0}^{m-1} e^{|\ln |D_{T^j x} T| - \ln |D_{T^j y} T|} \leq \prod_{j=0}^{m-1} e^{(a_{n-j-1} - a_{n-j}) a_{n-j}^{\gamma-1}} \\ &\leq e^{C \sum_{j=0}^{m-1} (n-j)^{-1-\frac{1}{\gamma}} (n-j)^{1-\frac{1}{\gamma}}} \leq e^{C \sum_{j=n-m+1}^{n} j^{-2}} \leq e^{\frac{C}{n-m+1}}.\end{aligned}$$

This conclude the proof, the other bound being completely similar. □

Proof of Lemma 2.5. For each $\varphi \in \mathcal{C}^1$, holds

$$\begin{aligned}T_*^n \mu(\varphi') = \mu(\varphi' \circ T^n) &= \mu([(DT^n)^{-1} \varphi \circ T^n]') + \mu((D^2 T^n)(DT^n)^{-2} \varphi \circ T^n) \\ &=: \mu(\phi_n') + \mu(\psi_n).\end{aligned} \tag{2.7}$$

Note that ϕ_n is $\mathcal{C}^1$ on the interior of each $Z \in \mathcal{Z}_n$. In addition, $\phi_n \in \mathcal{C}^0$ since for each $a \in \partial Z \in \mathcal{Z}_n$ holds $T^n a \in \{0, 1\}$ hence $\phi_n(a) = 0$. Nevertheless, it may not belong to the class of test function sine it is not necessarily $\mathcal{C}^1$. On the other hand we have already seen in the subsection devoted to piecewise smooth maps how to deal with such a problem once it is known that the function is uniformly $\mathcal{C}^1$ outside the boundaries of the partition, so we will not replay the same argument. Thus, we need a careful estimate of the norm of two functions ϕ_n and ψ_n.
Estimate of the norm of $\phi_n := (DT^n)^{-1} \varphi \circ T^n$. It is convenient to partition the orbit of x according to the time spent in a neighborhood of the fixed point. Define $U_0 = (0, 1/2)$ and let $U = (0, a_{n_*})$ be a fixed neighborhood of the fixed point. Define $n_0(x) := \inf\{k \in \mathbb{N} \mid T^k x \notin U_0\}$. Next, let $\kappa(x) = \chi_{U_0}(x)(1 - \chi_{U_0}(Tx)) + \chi_U(Tx)(1 - \chi_U(x))$, clearly $\kappa(x)$ is equal one if and only if the point x enters U or exits U_0 in one time step.[26] We can then define $n_{i+1}(x) := \inf\{k \geq n_i(x) \mid \kappa(T^k x) = 1\}$. Clearly, the stretches of trajectories between n_{2i} and n_{2i+1} belong to the complement of U, while the pieces of trajectories between n_{2i-1} and n_{2i} belong to U_0, but starting from inside U.

Let us start analyzing a piece of trajectory in U. If $z \in U$, then $T^j z \in U$ for each $j < n_0(z)$, by definition. In fact, $z \in [a_{n_0(z)}, a_{n_0(z)+1}]$. The first interesting fact follows from Lemmata 2.6, 2.7: for each $j \leq n_0(z)$ holds true

$$D_z T^j \geq C^{-1} \frac{a_{n_0(z)-j} - a_{n_0(z)+1-j}}{a_{n_0(z)} - a_{n_0(z)+1}} \geq C^{-3} \frac{(n_0(z)+1)^{\frac{1}{\gamma}+1}}{(n_0(z) - j + 1)^{\frac{1}{\gamma}+1}}. \tag{2.8}$$

On the other hand, outside U the map enjoys a minimal amount $\sigma > 1$ of expansion at each step.

[25] Here an in the following C will be used to designate fixed, possibly different, constants.
[26] By χ_A we mean the characteristic function of the set A.

Clearly, the worst possible case is if all the trajectory belongs to U_0, in such a case $n_0(x) \geq n$ and

$$\begin{aligned}|\phi_n(x)| &\leq C^3 \left(\frac{n_0(x)-n+1}{n_0(x)+1}\right)^{1+\frac{1}{\gamma}} |\varphi(T^n(x))| \\ &\leq C^3 \left(\frac{n_0(x)-n+1}{n_0(x)+1}\right)^{1+\frac{1}{\gamma}} (n_0(x)-n)^{-\frac{\alpha}{\gamma}} H^0_\alpha(\varphi) \\ &\leq C^3 n_0(x)^{-1-\frac{1}{\gamma}} (n_0(x)-n+1)^{1+\frac{1-\alpha}{\gamma}} H^0_\alpha(\varphi).\end{aligned}$$

Maximizing on $n_0(x)$ yields

$$|\phi_n|_\infty \leq C(\alpha) n^{-\frac{\alpha}{\gamma}} H^0_\alpha(\varphi). \tag{2.9}$$

The last task is to compute the β-Hölder constant of ϕ_n at zero. Let $y < x$, then

$$|\phi_n(x)| \leq |(D_x T^n)^{-1}| \, |\varphi(T^n x)| \leq H^0_\alpha(\varphi) |(D_x T^n)^{-1}| \, |T^n x|^\alpha.$$

Again the worst possibility turns out to be the case in which all the trajectory lies in U_0. In such a case, $n < n_0(x)$. Thus

$$|\phi_n(x)| \leq H^0_\alpha(\varphi) \left(\frac{n_0(T^n x+1)}{n_0(T^n x)+n+1}\right)^{1+\frac{1}{\gamma}} n_0(T^n x)^{-\frac{\alpha}{\gamma}} (n_0(T^n x)+n)^{\frac{\beta}{\gamma}} x^\beta$$

Taking the maximum on the possible values of $n_0(T^n x)$ yields

$$H_\alpha(\phi_n)(0) \leq C_8(\alpha-\beta)^{\frac{\alpha-\beta}{\gamma}} n^{-\frac{\alpha-\beta}{\gamma}} H^0_\alpha(\varphi). \tag{2.10}$$

Estimate of the norm of $\psi_n := (D^2 T^n)(DT^n)^{-2} \varphi \circ T^n$. We first use the formula

$$(D^2_x T^n)(D_x T^n)^{-2} = \sum_{i=0}^{n-1} (D_{x_i} T^{n-i})^{-1} \frac{D^2_{x_i} T}{D_{x_i} T} \tag{2.11}$$

where $x_i := T^i x$. According to (2.8), if $z \in U$, then

$$D_z T^{n_0(z)} \geq C^{-3} n_0(z)^{\frac{1}{\gamma}+1} \geq C^{-3} n_*^{\frac{1}{\gamma}+1} \geq 2$$

provided we have chosen n_* large enough. Thus in a stretch of trajectory inside U_0 we get a fixed total expansion, on the other hand outside U we have some fixed amount $\sigma > 1$ of expansion at each iteration. It thus makes sense to consider a stretch of trajectory in U_0 as a single step. Of course, this is useful only if uniform bounds on the distortion hold. By formulae (2.11), (2.8) follows, for each $z \in U$ and $m \leq n_0(z)$,

$$\begin{aligned}\sum_{i=0}^{m-1} (D_{T^i z} T^{m-i})^{-1} \frac{D^2_{T^i z} T}{D_{T^i z} T} &\leq C^4 \sum_{i=1}^{m} \frac{(n_0(z)-m+1)^{1+\frac{1}{\gamma}}}{(n_0(z)-i)^{1+\frac{1}{\gamma}}} (n_0(z)-i+1)^{-1-\frac{1}{\gamma}} \\ &= C^4 (n(z)-m+1)^{1+\frac{1}{\gamma}} \sum_{i=n_0(z)-m}^{n(z)} i^{-2} \\ &\leq C(n(z)-m+1)^{\frac{1}{\gamma}} \frac{m}{n_0(z)}.\end{aligned}$$

Using the above formula one can perform the sum over the pieces of trajectories that belong to U_0. Accordingly, one obtains that the sum is bounded by a fixed constant unless $T^n x \in U_0$. In such a case one has to analyze separately the last terms of the sum, the one that correspond to the last run in U. Let k be the last entrance time in U, then calling $z = T^k x$ holds $n_0(T^n x) = n_0(z) - n + k > 0$ and

$$\sum_{i=k}^{n-1} (D_{x_i} T^{n-i-1})^{-1} \frac{D^2_{x_i} T}{D_{x_i} T} \le C \sum_{i=k}^{n-1} \left(\frac{n_0(z) - n + k}{n_0(z) - n + i}\right)^{1+\frac{1}{\gamma}} \left(\frac{1}{n_0(z) - n + i}\right)^{\frac{\gamma-1}{\gamma}}$$
$$\le C_1 (n_0(z) - n + k)^{\frac{1}{\gamma}} \frac{n-k}{n_0(z)} \le C_1 n_0(T^n x)^{\frac{1}{\gamma}} \frac{1}{1 + \frac{n_0(T^n x)}{n-k}}.$$

Accordingly, if $T^n x \notin U$, we have that $\psi_n(x) \le C_2 |\varphi|_\infty$. On the other hand, if $T^n x \in U$, then

$$\begin{aligned} |\psi_n(x)| &\le C_1 |\varphi(T^n x)| n_0(T^n x)^{\frac{1}{\gamma}} \frac{1}{1 + \frac{n_0(T^n x)}{n-k}} \\ &\le C_1 H^0_\alpha(\varphi) n_0(T^n x)^{\frac{1-\alpha}{\gamma}} \frac{n-k}{n - k + n_0(T^n x)} \\ &\le C_1 H^0_\alpha(\varphi) n_0(T^n x)^{\frac{1-\alpha}{\gamma}}. \end{aligned} \tag{2.12}$$

Let us consider two cases: first $n_0(T^n x) \le n^{\bar\delta}$. In such a situation we have

$$|\psi_n(x)| \le C_1 H^0_\alpha(\varphi) n^{\frac{(1-\alpha)\bar\delta}{\gamma}}. \tag{2.13}$$

On the other hand, if $n_0(T^n x) > n^{\bar\delta}$, then let $\Omega_{k,n,m}$ be the set of points x such that $T^{k-1} x \in [1/2, 1] =: I_1$ and $T^k x \in [a_{m+n-k+1}, a_{m+n-k}] =: \Delta_{m+n-k}$. Clearly, for $x \in \Omega_{k,n,m}$ holds $n_0(T^n x) = m$ and the last entrance time in U is exactly k. Let $\Omega := \cup_{m=n^{\bar\delta}}^\infty \cup_{k=0}^n \Omega_{k,n,m}$, then ψ_n is uniformly bonded by (2.13) on the complement of Ω. Finally, let us define $\bar\psi_n(x) := x^{-\beta} \psi_n(x)$. With such notations and using all the above estimates we can write

$$\begin{aligned} \mu(\psi_n) &\le \mu(\chi_\Omega \psi_n) + C_3 n^{\frac{(1-\alpha)\bar\delta}{\gamma}} |\varphi|_{\mathcal{C}^\alpha} |\mu| \\ &\le 3\{\|\mu\|_\beta + |\mu|\} |\chi_\Omega \bar\psi_n|_{L^1} + C_3 n^{\frac{(1-\alpha)\bar\delta}{\gamma}} H^0_\alpha(\varphi) |\mu|, \end{aligned}$$

where we have used Lemma 2.4. We are then left with the estimate of the L^1 norm of $\chi_\Omega \bar\psi_n$.

Clearly $\Omega_{n-k+m} = T^{-k+1}(T^{-1}\Delta_{m+n-k} \cap I_1) =: \bar\Delta_{m+n-k}$ and will thus consist of an interval J_Z into each element $Z \in \mathcal{Z}^{k-1}$. Since $T^{k-1} J_Z \subset I_1$ our distortion estimates imply that

$$\frac{|I_1|}{|\bar\Delta_{m+n-k}|} \le C \frac{|Z|}{|J_Z|},$$

that is $|J_Z| \le C |\bar\Delta_{m+n-k}|\, |Z|$. Let us understand a bit better how the elements of $\mathcal{Z}^{k-1}$ are distributed. For each Δ_ℓ, $T^\ell \Delta_\ell = I_1$ by construction. This means that, if $\ell \le k$

$$\begin{aligned} \sum_{\substack{Z \in \mathcal{Z}^{k-1} \\ Z \subset \Delta_\ell}} |J_Z| &\le C \sum_{\substack{Z \in \mathcal{Z}^{k-1} \\ Z \subset \Delta_\ell}} |\bar\Delta_{m+n-k}| |Z| \le C^2 \sum_{\substack{Z' \in \mathcal{Z}^{k-1-\ell} \\ Z' \subset I_1}} |\Delta_{m+n-k}| |Z'| |\Delta_\ell| \\ &\le C^2 |\Delta_{m+n-k}|\, |\Delta_\ell| \end{aligned}$$

where we have used again our distortion estimates.

Hence,

$$\begin{aligned}\int_\Omega |\bar\psi_n| &= \sum_{m=n^{\bar\delta}}^\infty \sum_{k=0}^n \sum_{\ell=1}^k \int_{\Omega_{m,n,k}\cap\Delta_\ell} |\bar\psi_n| \\ &\le \sum_{m=n^{\bar\delta}}^\infty \sum_{k=0}^n \sum_{\ell=1}^k C_5 H^0_\alpha(\varphi) m^{\frac{1-\alpha}{\gamma}} \frac{n-k}{n-k+m} \ell^{\frac{\beta}{\gamma}} (m+n-k)^{-\frac{1}{\gamma}-1} \ell^{-\frac{1}{\gamma}-1} \\ &\le C_6 H^0_\alpha(\varphi) \sum_{m=n^{\bar\delta}}^\infty \sum_{k=0}^n m^{\frac{1-\alpha}{\gamma}} k (m+k)^{-\frac{1}{\gamma}-2} \\ &\le C_7 H^0_\alpha(\varphi) \sum_{m=n^{\bar\delta}}^\infty m^{-\frac{\alpha}{\gamma}} \le C_8 H^0_\alpha(\varphi) n^{(1-\frac{\alpha}{\gamma})\bar\delta}\end{aligned}$$

Conclusion. By the above results we can estimate the terms in (2.7) as

$$|T^{*n}\mu(\varphi')| \le \{C_9 n^{-\frac{\alpha-\beta}{\gamma}} + C_{10} n^{(1-\frac{\alpha}{\gamma})\bar\delta}\}\|\mu\|_\beta + C_{11} n^{\frac{1-\alpha}{\gamma}\bar\delta}|\mu|.$$

We finally choose $\bar\delta := \frac{\alpha-\beta}{\alpha-\gamma}$ and the lemma follows. □

3. Third Lecture (BEYOND EXISTENCE: STATISTICAL PROPERTIES)

In the previous lectures we have successfully investigated the invariant measures of a variety of systems, still there are plenty of reasons to be unhappy about the above considerations. A most obvious question is: what about uniqueness?

Clearly in the generality discussed so far one cannot hope that uniqueness always holds. For example, consider a partially hyperbolic system consisting of an expanding map times identity. Clearly any measure obtained by the unique absolutely continuous invariant measure for the expanding system times any other measure, on the space on which acts the identity, is an invariant measure.[27]

Nevertheless, in many cases it is possible to take the previous analysis much further. Let us start by analyzing the simplest case: the smooth expanding maps.

3.1. The spectral picture.

The idea is to use the dynamical knowledge gained so far and transform it into informations on the spectrum of the operator T_* on the Banach space $\mathcal{B} := \{$complex valued measures μ such that $\|\mu\| < \infty\}$.[28]

First of all Lemma 1.2 implies that the spectral radius is bounded by one and the existence of invariant measures shows that $1 \in \sigma(T_*)$, that is the spectral radius is exactly one.

More can be said thanks to the following abstract result.

Theorem 3.1 (Hennion-Neussbaum argument [28])**.** *Consider two Banach spaces $\mathcal{B} \subset \mathcal{B}_w$, $\|\cdot\| \ge \|\cdot\|_w$, and an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ such that, for some $M > \theta > 0$, $A, B, C > 0$, and for each $n \in \mathbb{N}$, holds true*

$$\|\mathcal{L}^n f\|_w \le C M^n \|f\|_w; \quad \|\mathcal{L}^n f\| \le A\theta^n \|f\| + B M^n \|f\|_w$$

[27] It is also easy to make counterexamples by using expanding discontinuous maps, it is an helpful exercise to try.

[28] Since here we want to discuss spectral theory it is convenient to consider measures and function with complex values. Such an extension is totally standard, thus we will not comment further on it.

Then the spectral radius of $\mathcal{L}$ is bounded by M. If, in addition, $\mathcal{L}$ is compact as an operator from $\mathcal{B}$ to $\mathcal{B}_w$, then $\mathcal{L}$ is quasi compact and its essential spectral radius[29] *is bounded by θ.*

Proof. The first assertion is trivial, for the second start by noticing that Nussbaum's formula [67] asserts that if r_n is the inf of the r such that $\{\mathcal{L}^n f\}_{\|f\|\le 1}$ can be covered by a finite number of balls of radius r, then the essential spectral radius of $\mathcal{L}$ is given by $\lim_{n\to\infty} \sqrt[n]{r_n}$. Let $B_1 := \{f \in \mathcal{B} \mid \|f\| \le 1\}$. By hypotheses $\mathcal{L}B_1$ is relatively compact in $\mathcal{B}_w$. Thus, for each $\varepsilon > 0$ there are $f_1, \dots, f_{N_\varepsilon} \in \mathcal{L}B_1$ such that $\mathcal{L}B_1 \subseteq \bigcup_{i=1}^{N_\epsilon} U_\epsilon(f_i)$, where $U_\epsilon(f_i) = \{f \in \mathcal{B} \mid \|f - f_i\|_w < \epsilon\}$. For $f \in \mathcal{L}B_1 \cap U_\epsilon(f_i)$, holds

$$\|\mathcal{L}^{n-1}(f - f_i)\| \le A\theta^{n-1}\|f - f_i\| + \frac{B}{M}^{n-1}\|f - f_i\|_w \le A(\theta + BM)\theta^{n-1} + BM^n\epsilon\ .$$

Choosing ϵ sufficiently small we can conclude that for each $n \in \mathbb{N}$ the set $\mathcal{L}^n(B_1)$ can be covered by a finite number of $\|\cdot\|$–balls of radius const. θ^n centered at the points $\{\mathcal{L}^{n-1} f_i\}_{i=1}^{N_\epsilon}$. □

In the case at hand the above Theorem can be immediately applied with $M = 1$ and $\theta = \lambda^{-1}$ thanks to Lemma 1.2 while the compactness follows readily from the compactness of the unit BV ball in L^1. We have thus the spectral picture sketched in Figure 3.2.

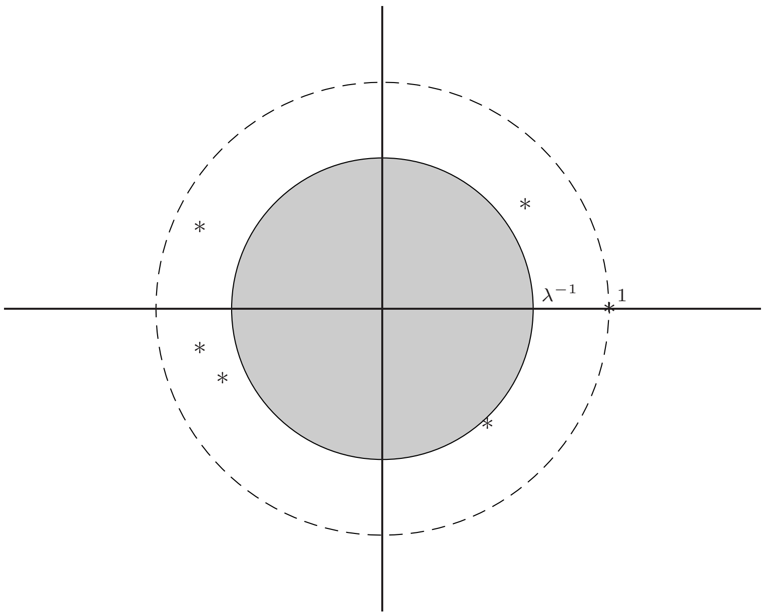

FIGURE 3.2. The spectrum of the transfer operator

Let Π_1 be the projector on the eigenvalue one, then $\Pi_1 T_* = T_*\Pi_1$. Again by 1.2 follows that $\|(T_*\Pi_1)^n\| = \|T_*^n\Pi_1\| \le D(1 - \lambda^{-1})^{-1}$. This implies that $T_*\Pi_1$

[29]By *essential spectrum* I mean the complement of the point spectrum with finite multiplicity.

cannot contain a Jordan block, otherwise the norm of the powers would grow at least linearly. Accordingly, $T_*\Pi_1 = \Pi_1 T_* = \Pi_1$ and thus Π_1 can be written as:[30]

$$\Pi_1\mu = \sum_{i=1}^{\ell} \Psi_i(\mu)\mu_i$$

where $\Psi_i \in \mathcal{B}^*$, $\Psi(T_*\mu) = \Psi_i(\mu)$, $T_*\mu_i = \mu_i$ and ℓ is the dimension of the eigenspace associated to the eigenvalue 1. By Lemma 1.2 follows

$$\|\Pi_1\mu\| = \|\Pi_1 T_*^n \mu\| \le \lambda^{-n}\|\Pi_1\|\,\|\mu\| + B(1-\lambda^{-1})^{-1}\|\Pi_1\|\,|\mu|$$

thus $\|\Pi_1\mu\| \le (1-\lambda^{-1})^{-1}B\|\Pi_1\|\,|\mu|$. That is, Π_1, and hence the Ψ_i, is continuous over the measures. For each each $h \in BV$ let $\bar\Psi_i(h) := \Psi_i(\mu_h)$, where $d\mu_h = hdm$. Then $|\bar\Psi_i(h)| = |\Psi_i(\mu_h)| \le C|\mu_h| \le C|h|_{L^1}$. Since BV is dense in L^1, $\bar\Psi$ has a unique continuous extension as a functional on L^1. Thus, $\bar\Psi_i$ belongs to the dual of L^1 and can then be identified with an L^∞ function $\bar\psi_i$: $\bar\Psi_i(\mu_h) = \int \bar\psi_i h = \mu_h(\bar\psi_i)$. Then $|\bar\psi_i|_\infty \le C$ and $\bar\psi_i \circ T = \bar\psi_i$.[31]

Thus, since $\int \bar\psi_i h_j = \delta_{ij}$, the space of invariant functions is at least ℓ dimensional. On the other hand, the level set of an invariant function is an invariant set and to any invariant set can be associated at least an invariant measure, see Lemma 1.3, it follows that there are exactly ℓ invariant sets of positive Lebesgue measure.

The above considerations imply immediately the dichotomy below.

Lemma 3.1. *If the systems is mixing then it mixes exponentially fast on $\mathcal{B}$.*

This means, in particular, that there exists $\sigma > 0$ such that for each $\mu \in \mathcal{B}$, $\mu(1) = 0$, and $\varphi \in L^\infty$, calling h the density of μ, holds true

$$\left|\int h\varphi \circ T^n\right| \le \|\mu\|\,|\varphi|_{L^\infty} e^{-\sigma n}.$$

In fact, it is possible to show that all the systems under considerations are mixing.[32] Yet, for simplicity we will prove this fact only in the case $d = 1$.

First of all, we remarked after Lemma 1.3 that each invariant set must contain an interval. Since the image of an interval must be eventually all the space–due to the expansivity of the map–it follows that there can be only one invariant set: $\mathbb{T}^1$. This implies that $\ell = 1$ and that the eigenvector is an ergodic measure.[33]

Note that the above argument can be applied verbatim to any power T^q of the map. Hence all the powers of T are ergodic.

The mixing follows analogously since all the arguments carried out for Π_1 hold unchanged for Π_α, $\alpha \in \sigma(T_*)$, $|\alpha| = 1$. The only difference is that the corresponding functions $\bar\psi_i$ satisfy $\bar\psi_i \circ T = \alpha\bar\psi_i$. Conversely, if there exists $\psi \in L^\infty(\mathbb{T}^1, m)$ such

---

[30] Of course, this decomposition it is not unique, indeed one can choose any basis of the eigenspace.

[31] An alternative approach is to notice that the spectral picture implies that Π_1 is the limit, in $\mathcal{B}$, of $\frac{1}{n}\sum_{i=0}^{n-1} T_*^i$. This implies immediately that Π_1 is weakly continuous on $\mathcal{B}$. Since $\mathcal{B}$ is weakly dense in $\mathcal{M}(\mathbb{T}^d)$ it follows that Π_1, and hence also the Ψ_i, can be extended to a weakly continuous operator (functional) on $\mathcal{M}(\mathbb{T}^d)$. One can then define $\bar\psi_i(x) = \Psi_i(\delta_x)$. Note that, in this way, one gets the extra information that the $\bar\psi_i$ must be continuous. Of course, such an argument, contrary to the one in the main text, would not hold if T it is not continuous.

[32] This is a consequence of the smoothness of the map and the connectedness of $\mathbb{T}^d$, it could be false if the map is not continuous or the space is not connected.

[33] Or, if one prefers, that the ergodic decomposition associated to the Lebesgue measure is trivial.

that $\psi \circ T = \alpha\psi$, then $\alpha \in \sigma(T_*)$. Indeed, for each $\mu \in \mathcal{B}$ holds $(\alpha - T_*)\mu(\psi) = 0$, that is Range$(\alpha - T_\alpha) \neq \mathcal{B}$. But then, since for each $n \in \mathbb{N}$ we have $\bar{\psi}_i^n \in L^\infty$ and

$$\bar{\psi}_i^n \circ T = (\bar{\psi}_i \circ T)^n = \alpha^n \bar{\psi}_i^n,$$

$\alpha^n \in \sigma(T_*)$ for all $n \in \mathbb{N}$. Since $\sigma(T_*) \cap \{z \in \mathbb{C} \mid |z| = 1\}$ consists at most of finitely many points (by the quasi-compactness of T_*) it follows that there must exist $q \in \mathbb{N}$ such that $\alpha^q = 1$. This implies $\bar{\psi}_i \circ T^q = \bar{\psi}_i$, hence all the $\bar{\psi}_i$ must be constant by the ergodicity of T^q. Accordingly the equation $\bar{\psi}_i \circ T = \alpha\bar{\psi}_i$ cannot be satisfied. This shows that there cannot be eigenvalue on the unit circle beside one. The quasi-compactness implies then the existence of a spectral gap which yields, as announced, exponential mixing on $\mathcal{B}$. The mixing follows then by a standard approximation argument.

3.2. **Perturbations.** The previous results start to be rather satisfactory, yet to be really satisfied one would like to be able to answer to questions like

- Can we compute the invariant measure with a preassigned precision?
- Can we compute the rate of decay of correlation with a preassigned precision?
- Is the system stable under random perturbation? (If not, we may be talking about objects that are not observable in reality)
- do nearby systems behave similarly?

The answer to the above questions can be obtained via perturbation theorems. Few such results are available (e.g., see [47], [89] for a review and [6] for some more recent results), here we will follow mainly the theory developed in [46] adapted to the special cases at hand.

For simplicity let us work directly with the densities. Then $\mathcal{L}$ is the transfer operator for the densities, see equation 1.4. We will start by considering an abstract family of operators $\mathcal{L}_\varepsilon$ satisfying the following properties.

Condition 3.1. *Consider a family of operators $\mathcal{L}_\varepsilon$ with the following properties*

(1) A uniform Lasota-Yorke inequality:

$$\|\mathcal{L}_\varepsilon^n h\|_{BV} \leq A\lambda^{-n}\|h\|_{BV} + B|h|_{L^1}, \quad |\mathcal{L}_\varepsilon^n h|_{L^1} \leq C|h|_{L^1} \;;$$

(2) $\int \mathcal{L}h(x)dx = \int h(x)dx$;

(3) For $L : BV \to BV$ define the norm

$$|||L||| := \sup_{\|h\|_{BV} \leq 1} |Lf|_{L^1},$$

that is the norm of L as an operator from $BV \to L^1$. Then we require that there exists $D > 0$ such that

$$|||\mathcal{L} - \mathcal{L}_\varepsilon||| \leq D\varepsilon.$$

Condition 3.1-(3) specifies in which sense the family $\mathcal{L}_\varepsilon$ can be considered an approximation of the unperturbed operator $\mathcal{L}$. Notice that the condition is rather weak, in particular the distance between $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ as operators on BV can be always larger than 1. Such a notion of closeness is completely inadequate to apply standard perturbation theory, to get some perturbations results it is then necessary to drastically restrict the type of perturbations allowed, this is done by Conditions

3.1-(1,2) which state that all the approximating operators enjoys properties very similar to the limiting one.[34]

Let us see immediately why the above setting is relevant to the issues under consideration.

Example 3.1. *The $\mathcal{L}_\varepsilon$ are Perron-Frobenius (Transfer) operators of maps T_ε which are $\mathcal{C}^1$–close to T, that is $d_{\mathcal{C}^1}(T_\varepsilon, T) = \varepsilon$ and such that $d_{\mathcal{C}^2}(T_\varepsilon, T) \leq M$, for some fixed $M > 0$. In this case the uniform Lasota-Yorke inequality is trivial. On the other hand, for all $\varphi \in \mathcal{C}^1$ holds*

$$\int (\mathcal{L}_\varepsilon f - \mathcal{L} f)\varphi = \int f(\varphi \circ T_\varepsilon - \varphi \circ T).$$

Now let $\Phi(x) := (D_x T)^{-1} \int_{Tx}^{T_\varepsilon x} \varphi(z)dz$, since

$$\Phi'(x) = -(D_x T)^{-1} D_x^2 T \Phi(x) + D_x T_\varepsilon (D_x T)^{-1} \varphi(T_\varepsilon x) - \varphi(Tx)$$

follows

$$\int (\mathcal{L}_\varepsilon f - \mathcal{L} f)\varphi = \int f\Phi' + \int f(x)[(D_x T)^{-1} D_x^2 T \Phi(x) + (1 - D_x T_\varepsilon (D_x T)^{-1})\varphi(T_\varepsilon x)].$$

Given that $|\Phi|_\infty \leq \lambda^{-1}\varepsilon|\varphi|_\infty$ and $|1 - D_x T_\varepsilon (D_x T)^{-1}|_\infty \leq \lambda^{-1}\varepsilon$, we have

$$\int (\mathcal{L}_\varepsilon f - \mathcal{L} f)\varphi \leq \|f\|_{BV}\lambda^{-1}|\varphi|_\infty\varepsilon + |f|_{L^1}\lambda^{-1}(B+1)\varepsilon|\varphi|_\infty \leq D\|f\|_{BV}\varepsilon|\varphi|_\infty.$$

By Lebesgue dominate convergence theorem we obtain the above inequality for each $\varphi \in L^\infty$, and taking the sup on such φ yields the wanted inequality.

$$|\mathcal{L}_\varepsilon f - \mathcal{L} f|_{L^1} \leq D\|f\|_{BV}\varepsilon.$$

We have thus seen that all the requirements in Condition 3.1 are satisfied. See [41] *for a more general setting including piecewise smooth maps.*

Let us mention a couple of other interesting examples that, although not exhaustive, give a good idea of the applicability of the present setting. The explicit verification of Condition 3.1 in these cases is left to the reader (or referred to the references).

Example 3.2. *$\mathcal{L}_\varepsilon$ is the transition operator of a stochastically perturbed map T. For example, we can consider an operator J_ε of the type introduced in Lemma 1.1 and define $\mathcal{L}_\varepsilon := J_\varepsilon \mathcal{L}$. This corresponds to moving a point with the deterministic map T and then distributing it in a small ε neighborhood according to a probability distribution determined by the kernel j. In such a case ε is the "size" of the perturbation, see* [41, 8, 6, 9] *for more details.*

Example 3.3. *$\mathcal{L}_\varepsilon$ is the transition operator for the Ulam-type discretization of T with grid size ε. This means that one chooses a partition $\mathcal{Z}$ of intervals of size smaller than ε, then defines the conditional expectation*

$$P_\varepsilon f(x) := \sum_{Z \in \mathcal{Z}} \chi_Z(x) \frac{1}{|Z|} \int_Z f$$

[34] Actually only Condition 3.1-(1) is needed in the following. Condition 3.1-(2) simply implies that the eigenvalue one is common to all the operators. If 3.1-(2) is not assumed, then the operator $\mathcal{L}_\varepsilon$ will always have one eigenvalue close to one, but the spectral radius could vary slightly, see [60] for such a situation.

and sets $\mathcal{L}_\varepsilon := P_\varepsilon \mathcal{L}$. *The interest of such type of perturbations (discretizations) of a dynamical system lies in the fact that the range of* $\mathcal{L}_\varepsilon$ *is finite dimensional. Thus the operator* $\mathcal{L}_\varepsilon$ *is nothing else than a matrix. Accordingly its eigenvalues and eigenvectors can be explicitly computed and, if they are close to the ones of the unperturbed system, this provides a possible tool for investigating the spectrum of* $\mathcal{L}$ *itself. In fact, this strategy goes back to Ulam* [87]. *For more details on this example see* [53, 41, 7, 16, 9] *and for related work also* [33, 64, 36, 48].

Comforted by the fact that we are talking about problems of practical interest, let us go back to our more abstract setting and see what can be done. To state the result consider, for each operator L, the set

$$V_{\delta,r}(L) := \{z \in \mathbb{C} \mid |z| \le r \text{ or } \operatorname{dist}(z, \sigma(L)) \le \delta\}.$$

Since the complement of $V_{\delta,r}(L)$ belongs to the resolvent of L it follows that

$$H_{\delta,r}(L) := \sup\left\{\|(z-L)^{-1}\|_{\mathrm{BV}} \mid z \in \mathbb{C} \setminus V_{\delta,r}(L)\right\} < \infty.$$

By $R(z)$ and $R_\varepsilon(z)$ we will mean respectively $(z-\mathcal{L})^{-1}$ and $(z-\mathcal{L}_\varepsilon)^{-1}$.

Theorem 3.2 ([46])**.** *Consider a family of operators* $\mathcal{L}_\varepsilon : BV \to BV$ *satisfying Conditions 3.1. Let* $H_{\delta,r} := H_{\delta,r}(\mathcal{L})$; $V_{\delta,r} := V_{\delta,r}(\mathcal{L})$, $r > \lambda^{-1}$, $\delta > 0$, *then, if* $\varepsilon \le \varepsilon_1(\mathcal{L}, r, \delta)$, $\sigma(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$. *In addition, if* $\varepsilon \le \varepsilon_0(\mathcal{L}, r, \delta)$, *there exists* $a > 0$ *such that, for each* $z \notin V_{\delta,r}$, *holds true*

$$|||R(z) - R_\varepsilon(z)||| \le C\varepsilon^a.$$

Proof.[35] To start with we collect some trivial, but very useful algebraic identities.

For each operator $L : \mathrm{BV} \to \mathrm{BV}$ and $n \in \mathbb{Z}$ holds

$$\frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}L)^i(z-L) + (z^{-1}L)^n = \mathrm{Id} \tag{3.1}$$

$$R(z)(z-\mathcal{L}_\varepsilon) + \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L}) + R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L}) = \mathrm{Id} \tag{3.2}$$

$$(z-\mathcal{L}_\varepsilon)\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right] = \mathrm{Id} - (z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z) \tag{3.3}$$

$$\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right](z-\mathcal{L}_\varepsilon) = \mathrm{Id} - (z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L}), \tag{3.4}$$

where we have set $G_{n,\varepsilon} := \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L}_\varepsilon)^i$.

Let us start applying the above formulae. For each $h \in \mathrm{BV}$ and $z \notin V_{r,\delta}$ holds

$$\begin{aligned}\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\|_{\mathrm{BV}} &\le (r\lambda)^{-n}A\|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\|_{\mathrm{BV}} + \frac{B}{r^n}|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h|_{L^1}\\ &\le [(r\lambda)^{-n}A2C_1 + Br^{-n}D\varepsilon]H_{r,\delta}\|h\|_{\mathrm{BV}} < \|h\|_{\mathrm{BV}}\end{aligned}$$

Thus $\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)\|_{\mathrm{BV}} < 1$ and the operator on the right hand side of (3.3) can be inverted by the usual Neumann series. Accordingly, $(z-\mathcal{L}_\varepsilon)$ has a well defined right inverse. Analogously,

$$\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{\mathrm{BV}} \le (r\lambda)^{-n}A\|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{\mathrm{BV}} + Br^{-n}|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h|_{L^1}.$$

[35] This proof is simpler than the one in [46], yet it gives worst bounds, although sufficient for the present purposes.

This time to continue we need some informations on the L^1 norm of the resolvent. Let $g \in \mathrm{BV}$, then equation (3.1) yields

$$\begin{aligned}|R(z)g|_{L^1} &\leq \frac{1}{r}\sum_{i=0}^{n-1}|(z^{-1}\mathcal{L})^i g|_{L^1} + \|R(z)(z^{-1}\mathcal{L})^n g\|_{\mathrm{BV}}\\ &\leq \frac{1}{r^n(1-r)}|g|_{L^1} + H_{\delta,r}A(r\lambda)^{-n}\|g\|_{\mathrm{BV}} + H_{\delta,r}Br^{-n}|g|_{L^1}\\ &\leq r^{-n}(H_{\delta,r}B + (1-r)^{-1})|g|_{L^1} + H_{\delta,r}A(r\lambda)^{-n}\|g\|_{\mathrm{BV}}\end{aligned}$$

Substituting, we have

$$\begin{aligned}\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\|_{\mathrm{BV}} &\leq \{(r\lambda)^{-n}AH_{\delta,r}2C_1[1+Br^{-n}]\\ &+ Br^{-2n}[H_{\delta,r}B + (1-r)^{-1}]D\varepsilon\}\|h\|_{\mathrm{BV}} < 1,\end{aligned}$$

again, provided ε is small enough and choosing n appropriately. Hence the operator on the right hand side of (3.4) can be inverted, thereby providing a left inverse for $(z - \mathcal{L}_\varepsilon)$. This implies that z does not belong to the spectrum of $\mathcal{L}_\varepsilon$.

To investigate the second statement note that (3.2) implies

$$R(z) - R_\varepsilon(z) = \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z) - R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z).$$

Accordingly, for each $\varphi \in \mathrm{BV}$ holds

$$|R(z)\varphi - R_\varepsilon(z)\varphi|_{L^1} \leq \{r^{-n}(1-r)^{-1}\varepsilon + H_{\delta,r}(\lambda r)^{-n}2AC_1 + H_{\delta,r}B\varepsilon\}\|R_\varepsilon(z)\varphi\|_{\mathrm{BV}}.$$

□

Remark 3.1. *It takes little work to use the above Hölder continuity of the spectral data to answer the questions posed at the beginning of the section (limited to the case of smooth piecewise expanding maps). Just remember the examples.*

3.3. **Differentiability of SRB-measures.** The previous sections established the continuity of the spectral data (and of the invariant measure in particular) with respect to various type of perturbations. It provides also an estimate of the modulus of continuity, for example for the invariant measure the modulus of continuity must be, at least, $x \ln x$. Nevertheless, in certain cases, the above estimate it is not optimal since it does not take into account extra smoothness properties of the map. Let us see how this works in our usual simple example: let us consider a $\mathcal{C}^3(\mathbb{T}^1, \mathbb{T}^1)$ expanding map. According to our discussion in section 1.5 we can consider three norms for the density of the measures we are interested in:

$$\|h\|_0 := |h|_{L^1}; \quad \|h\|_1 := |h|_{W_{1,1}} = |h|_{L^1} + |h'|_{L^1}; \quad \|h\|_2 := |h|_{W_{1,2}} = |h|_{W_{1,1}} + |h''|_{L^1},$$

and the three corresponding Sobolev spaces $\mathcal{B}_0 = L^1(\mathbb{T}^1, \mathbb{R})$, $\mathcal{B}_1 = W_{1,1}(\mathbb{T}^1, \mathbb{R})$ and $\mathcal{B}_2 = W_{1,2}(\mathbb{T}^1, \mathbb{R})$.[36] It is easy to get the Lasota-Yorke inequality for these spaces and the compactness of the unit ball of $\mathcal{B}_1$ in $\mathcal{B}_0$ and of $\mathcal{B}_2$ in $\mathcal{B}_1$ are well known. The application of the arguments developed so far implies that $\mathcal{L}$ is ergodic and has a spectral gap both in $\mathcal{B}_1$ and in $\mathcal{B}_2$. For the perturbed family of operators let us consider, for example, a smooth random perturbation. Then we have $\mathcal{L}_\varepsilon h_\varepsilon = h_\varepsilon$ for the perturbed density. Let us define the quantities $\tilde{\mathcal{L}}_\varepsilon := \mathcal{L}_\varepsilon - \mathcal{L}$; $\tilde{h}_\varepsilon := h_\varepsilon - h$.

[36] We choose to work with Sobolev spaces rather than with spaces of bounded variations for a precise reason, see the discussion of equation (3.6).

By the general theory there exists $M > 0$ such that $\|h\|_2 + \|h_\varepsilon\|_2 \le M$. In addition, $\mathcal{L}_\varepsilon h_\varepsilon = h_\varepsilon$ implies

$$(\mathrm{Id} - \mathcal{L}_\varepsilon)\tilde{h}_\varepsilon = \tilde{\mathcal{L}}_\varepsilon h.$$

Since $\int \tilde{\mathcal{L}}_\varepsilon h = 0$ and the spectral gap implies that the spectral radius of $\mathcal{L}_\varepsilon$ on $\mathbb{V}_i^0 := \{h \in \mathcal{B}_i \mid \int h = 0\}$, $i \in \{1, 2\}$, is strictly less than one, we can write

$$\tilde{h}_\varepsilon = (\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1}\tilde{\mathcal{L}}_\varepsilon h. \tag{3.5}$$

By the same arguments described in section 3.2 (see section 4.2 for the general setting) it follows that $(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1}$ are uniformly bounded, by some constant C_0, as an operator on $\mathcal{B}_1$ while $|||\tilde{\mathcal{L}}_\varepsilon|||_{\mathcal{B}_2 \to \mathcal{B}_1} \le C\varepsilon$, hence

$$\|\tilde{h}_\varepsilon\|_1 \le C_0 \|\tilde{\mathcal{L}}_\varepsilon h\|_1 \le C_0 C\varepsilon \|h\|_2,$$

that is h_ε is Lipsichtz as a function of ε in $\|\cdot\|_1$ norm. To get differentiability it is necessary a bit more: the existence of an operator $\hat{\mathcal{L}} : \mathcal{B}_2 \to \mathcal{B}_1$ such that

$$\lim_{\varepsilon \to 0} ||\varepsilon^{-1}\tilde{\mathcal{L}}_\varepsilon g - \hat{\mathcal{L}} g||_1 = 0 \quad \text{for all } g \in \mathcal{B}_2. \tag{3.6}$$

Let us assume (3.6) for the time being, we will come back to it at the end of the section. Then, since on $\mathbb{V}_1^0$ (remember that by the spectral gap there exists $\nu \in (0, 1)$ such that $\|\mathcal{L}_\varepsilon^n\|_1 \le C\nu^n$)

$$(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1} = \sum_{k=0}^{\infty} \mathcal{L}_\varepsilon^k$$

and

$$\begin{aligned}
|||\mathcal{L}_\varepsilon^k - \mathcal{L}^k|||_{\mathcal{B}_1 \to \mathcal{B}_0} &\le \sum_{j=0}^{k-1} |||\mathcal{L}_\varepsilon^j (\mathcal{L}_\varepsilon - \mathcal{L})\mathcal{L}^{k-1-j}|||_{\mathcal{B}_1 \to \mathcal{B}_0} \\
&\le C \sum_{j=0}^{k-1} |||(\mathcal{L}_\varepsilon - \mathcal{L})\mathcal{L}^{k-1-j}|||_{\mathcal{B}_1 \to \mathcal{B}_0} \le C\varepsilon \sum_{j=0}^{k-1} \|\mathcal{L}^{k-1-j}\|_1 \le C\varepsilon k,
\end{aligned}$$

it follows

$$\begin{aligned}
|||(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1} - (\mathrm{Id} - \mathcal{L})^{-1}|||_{\mathbb{V}_1^0 \to \mathbb{V}_0^0} &\le \sum_{k=0}^{L-1} |||\mathcal{L}_\varepsilon^k - \mathcal{L}^k|||_{\mathcal{B}_1 \to \mathcal{B}_0} \\
&\quad + \sum_{k=L}^{\infty} \{\|\mathcal{L}^n|_{\mathbb{V}_1^0}\|_1 + \|\mathcal{L}_\varepsilon^n|_{\mathbb{V}_1^0}\|_1\} \\
&\le C\{L\varepsilon + \nu^L\} \le C\varepsilon \ln \varepsilon^{-1}.
\end{aligned} \tag{3.7}$$

Accordingly, setting $\hat{h} = (\mathrm{Id} - \mathcal{L})^{-1}\hat{\mathcal{L}} h$, holds

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \|\varepsilon^{-1}\tilde{h}_\varepsilon - \hat{h}\|_0 = \lim_{\varepsilon \to 0} \|\varepsilon^{-1}(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1}\tilde{\mathcal{L}}_\varepsilon h - (\mathrm{Id} - \mathcal{L})^{-1}\hat{\mathcal{L}} h\|_0 \\
&\le \lim_{\varepsilon \to 0} \|(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1}[\varepsilon^{-1}\tilde{\mathcal{L}}_\varepsilon h - \hat{\mathcal{L}} h]\|_1 + \|(\mathrm{Id} - \mathcal{L}_\varepsilon)^{-1}\hat{\mathcal{L}} h - (\mathrm{Id} - \mathcal{L})^{-1}\hat{\mathcal{L}} h\|_0 \\
&\le C_0 \lim_{\varepsilon \to 0} \|\varepsilon^{-1}\tilde{\mathcal{L}}_\varepsilon h - \hat{\mathcal{L}} h\|_1 = 0
\end{aligned}$$

which is the announced differentiability of the invariant density. To be precise we have seen that $h_\varepsilon \in \mathcal{C}^1(\mathbb{R}, \mathcal{B}_0)$, as a function of ε.

Equation (3.6). Let us assume, as before, that the random perturbation is given by $\mathcal{L}_\varepsilon := J_\varepsilon \mathcal{L}$, where J_ε is defined in Lemma 1.1. Let $\gamma := \int j(\xi)|\xi| d\xi$ and define $\hat{J}f := \gamma f'$. Clearly $\hat{J}$ is a bounded operator from $\mathcal{B}_2$ to $\mathcal{B}_1$.

Lemma 3.2. *For each $f \in \mathcal{B}_2$ holds*

$$\lim_{\varepsilon\to 0} \|\varepsilon^{-1}(J_\varepsilon - Id)f - \hat{J}f\|_1 = 0.$$

Proof. Let us start with the L^1 norm. For each $f \in \mathcal{C}^2$ holds

$$\begin{aligned}
\varepsilon^{-1}(J_\varepsilon - \mathrm{Id})f - \hat{J}f &= -\varepsilon^{-1}f(x) + \varepsilon^{-1}\int_{\mathbb{T}^1} j_\varepsilon(x-y)f(y)dy - \gamma f'(x) \\
&= \int_{\mathbb{T}^1} \varepsilon^{-1} j_\varepsilon(x-y)\int_{[x,y]} f'(z)dzdy - \gamma f'(x) \\
&= \int_{\mathbb{T}^1} \varepsilon^{-1} j_\varepsilon(\xi)\int_{[0,\xi]} f'(x-\eta)d\eta d\xi - \gamma f'(x) \\
&= \int_{\mathbb{T}^1} d\xi \varepsilon^{-1} j_\varepsilon(\xi)\int_{[0,\xi]} \{f'(x-\eta) - f'(x)\}d\eta
\end{aligned}$$

Differentiating

$$\frac{d}{dx}[\varepsilon^{-1}(J_\varepsilon - \mathrm{Id})f - \hat{J}f] = \int_{\mathbb{T}^1} d\xi \varepsilon^{-1} j_\varepsilon(\xi)\int_{[0,\xi]} \{f''(x-\eta) - f''(x)\}d\eta$$

Now, since $f \in \mathcal{C}^2$, both quantities inside the integrals converge everywhere to zero when $\varepsilon \to 0$, and by the Lebesgue dominated convergence theorem the Lemma follows for $\mathcal{C}^2$ functions. Since the above representations easily imply that $\varepsilon^{-1}(J_\varepsilon - \mathrm{Id})$ are uniformly bounded operators from $\mathcal{B}_2$ to $\mathcal{B}_1$, the result for all $W_{1,2}$ follows by density.[37] □

We have thus the announced equation (3.6) setting $\hat{\mathcal{L}} := \hat{J}\mathcal{L}$.

4. **Fourth Lecture** (The uniformly hyperbolic case)

In the last lecture we have carried out our program till some of its most extreme consequences for the case of smooth expanding maps, the next natural question is: In which generality can this program be carried out?

A part from obvious possibilities to investigate generalizations to higher dimension and piecewise smooth maps (see[4] for informations on such possibilities) the first natural candidate are hyperbolic systems. The problem here is clearly the presence of a stable direction. A moment thought shows that it is not clear at all how T_* can be seen as a regularizing operator when a stable direction is present. To understand this better let us consider the simplest possible example.

4.1. **Another *simple* example: an attracting fixed point.**

Consider a map $T \in \mathcal{C}^2(U, U)$, $U \subset \mathbb{R}^d$ compact and convex. Suppose that the maps contracts:

$$\|D_xT\| \leq \lambda^{-1}; \quad \lambda > 1.$$

[37] Here finally is the difference between working with Sobolev spaces and spaces with derivatives of bounded variation. All the rest would be exactly the same but this last fact would not be true and we could not conclude the argument. I leave this as a topic for the reader to meditate.

Clearly, such a map has only one fixed point x_* and each smooth measure, when iterated, converges weakly to δ_{x_*}. So the regularity properties seem to deteriorate rather than to improve.

Yet, nowhere is written that we must restrict our attention to the space of measures, that is $\mathcal{M}(U) = \mathcal{C}^0(U,\mathbb{R})^*$. Let us consider $\mathcal{C}^\alpha(U,\mathbb{R})^*$, $\alpha > 0$, instead. More precisely, let us fix some $\delta > 0$ and consider the norms

$$|\varphi|_{\mathcal{C}^\alpha} := |\varphi|_\infty + H_\alpha(\varphi)\ ; \quad H_\alpha(\varphi) := \sup_{|x-y|\le\delta} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha},$$
$$\|\mu\|_{s,\alpha} := \sup_{|\varphi|_{\mathcal{C}^\alpha}\le 1} \mu(\varphi).$$

Then, since $|\varphi\circ T|_\infty \le |\varphi|_\infty$ and $H_\alpha(\varphi\circ T) \le \lambda^{-\alpha} H_\alpha(\varphi)$, it holds that

$$\|T_*\mu\|_\alpha \le \|\mu\|_\alpha. \tag{4.1}$$

Moreover setting, for each $\varepsilon > 0$,

$$\mathbb{A}_\varepsilon\varphi(x) = \frac{1}{m(B_\varepsilon(x))}\int_{B_\varepsilon(x)} \varphi dm$$

we have, for $\alpha \in (0,1)$,

$$|\varphi - \mathbb{A}_\varepsilon\varphi|_\infty \le \varepsilon^\alpha H_\alpha(\varphi)$$
$$H_\alpha(\varphi - \mathbb{A}_\varepsilon\varphi) \le 2H_\alpha(\varphi)$$
$$H_1(\mathbb{A}_\varepsilon\varphi) \le \varepsilon^{\alpha-1} H_\alpha(\varphi)$$

from which the Lasota-Yorke inequality follows. Indeed, for each $\varphi \in \mathcal{C}^\alpha$, $|\varphi|_{\mathcal{C}^\alpha} \le 1$ and $\sigma \in (\lambda^{-\alpha}, 1)$, holds

$$\begin{aligned} T_*^n\mu(\varphi) &= \mu(\varphi\circ T^n) = \mu((\varphi - \mathbb{A}_\varepsilon\varphi)\circ T^n) + \mu((\mathbb{A}_\varepsilon\varphi)\circ T^n) \\ &\le \sup_{|\phi|_{\mathcal{C}^\alpha}\le(\varepsilon^\alpha + 2\lambda^{-n\alpha})} \mu(\phi) + \varepsilon^{\alpha-1}\|\mu\|_{s,1} \le A\sigma^n\|\mu\|_{s,\alpha} + B\|\mu\|_{s,1}, \end{aligned}$$

provided we have chosen A, B large and ε small enough. While (4.1), for $\alpha = 1$, yields $\|T_*\mu\|_{s,1} \le \|\mu\|_{s,1}$.

Moreover, since the unit ball of $\mathcal{C}^1$ is compact in $\mathcal{C}^\alpha$, it follows that the unit ball of $(\mathcal{C}^\alpha)^*$ is compact in $(\mathcal{C}^1)^*$.[38] We have thus all the ingredients to apply the strategy outlined in the previous lecture (as it will be further remarked in the next section).

In particular, calling $\mathcal{B} := \overline{\mathcal{M}(U)}^{\|\cdot\|_{s,\alpha}}$ and $\mathcal{B}_w := \overline{\mathcal{M}(U)}^{\|\cdot\|_{s,1}}$ we have that $T_* : \mathcal{B} \to \mathcal{B}$ is a quasi-compact operator and its spectral radius is one. This means that $T_* = \sum_{i=1}^{\ell} T_*\Pi_i + R$ where $\sigma(T_*\Pi_i) = \{0, \alpha_i\}$, $|\alpha_i| = 1$, and $\|R^n\| \le C\theta^n$ for some $C \ge 0$ and $\theta < 1$. But then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T_*^i = \Pi_0,$$

where $\sigma(T_*\Pi_0) = \{0,1\}$. Moreover, since $\|T_*^n\| \le 1$ implies no Jordan blocks, it holds true $T_*\Pi_0 = \Pi_0 T_* = \Pi_0$.

[38] To see this one can, for example, apply Theorem 4.210 of [37] to the trivial embedding.

Accordingly, for each $\varphi \in \mathcal{C}^\alpha$ and each measure $\mu \in \mathcal{M}(U)$

$$\Pi_0\mu(\varphi) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu(\varphi \circ T^i) \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |\mu|\,|\varphi|_\infty \leq |\mu|\,|\varphi|_\infty.$$

This means that $\Pi_0\mathcal{M}(U) \subset \mathcal{M}(U)$. But the range of Π_0 is finite dimensional, hence its range must be contained in $\mathcal{M}(U)$, since $\mathcal{M}(U)$ is dense in $\mathcal{B}$ by construction. In other words, all the $\mu \in \mathcal{B}$ such that $T_*\mu = \mu$ must be measures, and since there is only one invariant measure it follows that δ_{x_*} it is not only the unique invariant measure, but also the unique invariant distribution.

Remark 4.1. *Note that the above arguments says nothing of interest on the regularity of the invariant measure (only that it is a measure, which it is obvious). Yet, it says that one can consider invariant distributions rather than invariant measures and still be able to characterize them. This is very important if one is interested in the the spectral picture, since eigenvectors with eigenvalue less then one are often only distributions.*[39]

4.2. **The general framework.** Before proceeding further it is helpful to remark that most of what we have done so far can be seen as special cases of a rather general scheme. Such a scheme can be summarized by the following ingredients.

(1) Two Banach spaces $\mathcal{B}_i$ (let $\|\cdot\|$ be the norm of $\mathcal{B}_1$ and $|\cdot|$ the norm of $\mathcal{B}_2$).
(2) A domination between the norms: $\|\cdot\| \geq |\cdot|$.
(3) An operator $\mathcal{L} : \mathcal{B}_i \to \mathcal{B}_i$. ($\mathcal{L}$ is the Transfer operator)
(4) A regularization property:

$$\|\mathcal{L}^n f\| \leq A\lambda^{-n}\|f\| + B|f|, \quad |\mathcal{L}^n f| \leq C|f|$$

with $\lambda > 1$. (Lasota-Yorke type inequality)
(5) Compactness: $\mathcal{L}\{f \in \mathcal{B}_1 \mid \|f\| = 1\}$ is pre-compact in $\mathcal{B}_2$.
(6) Invariant functional: exists $\ell \in \mathcal{B}_2^*$ such that $\mathcal{L}^*\ell = \ell$.

Remark 4.2. *Note that the last point corresponds, in the previous examples, to* $1 \circ T = 1$. *This is connected to our choice to restrict the analysis to measures related to Lebesgue. If one wants to choose a conformal measure as reference measure then the above setting still applies, provided it is slightly generalized. This generalization is important (for example it is very natural to investigate the measure of maximal entropy), yet it exceeds our scopes, see* [52, 4] *for examples.*

The consequences of the above setting are briefly summarized as follows.

- (4) $\Longrightarrow \sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \mid |z| \leq 1\}$.
- (6) $\Longrightarrow 1 \in \sigma(\mathcal{L})$.
- (4,5) $\Longrightarrow \sigma_{ess}(\mathcal{L}) \subset \left\{z \in \mathbb{C} \;\middle|\; |z| \leq \lambda^{-1}\right\}$ (Hennion-Neussbaum argument)

If 1 is a simple eigenvalues and no other eigenvalues of modulus one are present then the projection Π_1 on the associated eigenvector is given by

$$\Pi_1 f = h\ell(f)$$

where $\mathcal{L}h = h$.

[39]As we have already remarked such eigenvalues determine the speed of decay of the correlations and are often called *resonances* [80].

Next, to consider the problems related to stability and computability, define the norm of $\mathcal{L}$ seen as an operator from $\mathcal{B}_1$ to $\mathcal{B}_2$, that is

$$|||\mathcal{L}||| := \sup_{\|f\|\leq 1} |\mathcal{L}f|.$$

Let $\mathcal{L}_i$ be two operators that satisfy (1-6), and

$$|||\mathcal{L}_1 - \mathcal{L}_2||| \leq \varepsilon.$$

For any operator $\mathcal{L}$, let us consider the set

$$V_{\delta,r}(\mathcal{L}) := \{z \in \mathbb{C} \mid |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(\mathcal{L})) \leq \delta\}.$$

Since the complement of $V_{\delta,r}(\mathcal{L})$ belongs to the resolvent of $\mathcal{L}$ it follows that

$$H_{\delta,r}(\mathcal{L}) := \sup\left\{\|(z-\mathcal{L})^{-1}\| \mid z \in \mathbb{C}\setminus V_{\delta,r}(\mathcal{L})\right\} < \infty.$$

Theorem 4.1 ([46]). *Consider two operators $\mathcal{L}_i : \mathcal{B}_1 \to \mathcal{B}_1$ satisfying (1-6). Let $H^i_{\delta,r} := H_{\delta,r}(\mathcal{L}_i)$; $V^i_{\delta,r} := V_{\delta,r}(\mathcal{L}_i)$, then there exists $a, b \in \mathbb{R}^+$ such that, if $\varepsilon \leq \varepsilon_1(\mathcal{L}_1, r, \delta)$,*

$$\|(z-\mathcal{L}_2)^{-1} f\| \leq a\|f\| + b|f|.$$

In addition, setting $\eta := \frac{\ln r/\alpha}{\ln \alpha^{-1}}$, if $\varepsilon \leq \varepsilon_0(\mathcal{L}_1, r, \delta)$, for each $z \in \mathbb{C}\setminus V^1_{\delta,r}$ it holds true

$$|||(z-\mathcal{L}_1)^{-1} - (z-\mathcal{L}_2)^{-1}||| \leq \varepsilon^\eta \left[a\|(z-\mathcal{L}_1)^{-1}\| + b\|(z-\mathcal{L}_1)^{-1}\|^2\right].$$

This allows to obtain very strong spectral stability results as we have already seen.

4.3. **Uniformly hyperbolic systems.** In this section we will consider Anosov diffeomorphisms $T : \mathtt{M} \to \mathtt{M}$ where T is of class $\mathcal{C}^3$ and $\mathtt{M}$ is a smooth compact Riemannian manifold. The idea is to extend much of the results of the previous lecture to this setting. In fact, since the details starts to be a bit more technical this section (and the next lecture) will be less detailed. I will only try to make clear which type of results are possible and refer to the original papers for the complete technical details.

Let us remind that by Anosov we mean (as usual) that there exists a direct sum decomposition of the tangent bundle $\mathcal{T}\mathtt{M}$ into continuous sub-bundles E^s and E^u, that is $\mathcal{T}_x\mathtt{M} = E^s_x \oplus E^u_x$, and constants $A \geq 1$ and $0 < \lambda_s < 1 < \lambda_u$ such that[40]

$$\begin{aligned} &(d_xT)(E^s_x) = E^s_{Tx}, \quad \|(d_xT^n)_{|E^s_x}\| \leq A\lambda_s^n, \\ &(d_xT)(E^u_x) = E^u_{Tx}, \quad \|(d_xT^{-n})_{|E^u_x}\| \leq A\lambda_u^{-n}, \end{aligned} \tag{4.2}$$

for all $x \in \mathtt{M}$ and $n \geq 0$. Let $d_{u/s} = \dim(E^{u/s})$. It is well known that for such maps there exist stable and unstable foliations $(W^s(x))_{x\in\mathtt{M}}$ and $(W^u(x))_{x\in\mathtt{M}}$, [34, 63, 29]. Each single $W^{s/u}(x)$ is an immersed $\mathcal{C}^3$ sub-manifold of $\mathtt{M}$, and $\mathcal{T}_yW^{s/u}(x) = E^{s/u}_y$ for any $y \in W^{s/u}(x)$. The dependence of $E^{s/u}_x$ and $W^{s/u}(x)$ on x, however, is only Hölder in general–see [76, 26] for exact results. We will denote by τ the optimal common Hölder-exponent for both distributions. This exponent depends in a well understood way on the various contraction and expansion coefficients of the map, and there are a number of cases where the foliations are indeed $\mathcal{C}^{1+\alpha}$ for some $\alpha > 0$ (for example when $\dim(\mathtt{M}) = 2$).

[40] By dT we denote the differential of T, clearly $d_xT : \mathcal{T}_x\mathtt{M} \to \mathcal{T}_{Tx}\mathtt{M}$. Similarly, if $f : \mathtt{M} \to \mathbb{R}$ is differentiable, then $d_xf : \mathcal{T}_x\mathtt{M} \to \mathbb{R}$.

It is immediate to verify that, if μ is absolutely continuous with respect to the Riemannian volume m, then $T_*\mu$ is absolutely continuous with respect to m. Given this fact, it is possible to define the evolution of the corresponding densities:

$$\mathcal{L}\frac{d\mu}{dm} := \frac{d(T_*\mu)}{dm}.$$

The above defined operator $\mathcal{L}$ is usually called the Perron-Frobenius or the Ruelle-Perron-Frobenius or the Transfer operator.

A direct computation shows that $\mathcal{L}$ has the following representation:

$$\mathcal{L}f = f\circ T^{-1}\cdot\frac{d(T_*m)}{dm} =: f\circ T^{-1}\cdot g \quad \forall f\in L^1(\mathtt{M},m) \tag{4.3}$$

where $g\in\mathcal{C}^2(\mathtt{M})$.[41]

To treat this case we will try to combine the approaches that were successful in the expanding case and the one that worked in the contracting case. We start by defining a suitable set of test functions to control the stable direction. For points $x,y\in\mathtt{M}$ with $y\in W^s(x)$ we define $d^s(x,y)$ as the distance between x and y within the Riemannian manifold $W^s(x)$ (which inherits its Riemannian structure from $\mathtt{M}$). Fix some $\delta>0$. For $0<\beta\leq 1$ and bounded measurable $\varphi:\mathtt{M}\to\mathbb{R}$, we define[42]

$$H^s_\beta(\varphi) := \sup_{d^s(x,y)\leq\delta}\frac{|\varphi(x)-\varphi(y)|}{d^s(x,y)^\beta}\,, \tag{4.4}$$

which means that the supremum is taken over all pairs of points x and y such that $y\in W^s(x)$. Clearly, H^s_β is a semi norm, and we use it to define

$$\mathcal{D}_\beta := \{\varphi:\mathtt{M}\to\mathbb{R}\;:\;\varphi \text{ measurable}, |\varphi|_\infty\leq 1, H^s_\beta(\varphi)\leq 1\}. \tag{4.5}$$

In order to control the unstable direction we provide a set $\mathcal{V}$ of measurable test vector fields $v:\mathtt{M}\to\mathcal{T}\mathtt{M}$ adapted to the unstable foliation in the sense that $v(x)\in E^u_x$ for all $x\in\mathtt{M}$.

Given the fact that $x\mapsto E^u_x$ is, in general, only τ-Hölder for some $\tau<1$ we cannot ask the vector fields to be globally more regular than that. By a slight abuse of notation we define[43]

$$H^s_\beta(v) := \sup_{d(x,y)^s\leq\delta}\frac{\|v(x)-v(y)\|}{d^s(x,y)^\beta}\;. \tag{4.6}$$

Then we will consider the vector fields

$$\mathcal{V}_\beta := \{v\in\mathcal{V}\;:\;|v|_\infty\leq 1;\; H^s_\beta(v)\leq 1\}\;. \tag{4.7}$$

From now on we will always assume

$$0\;<\;\beta\;<\;\gamma\;\leq 1\;.$$

[41] If $\mathtt{M}=\mathbb{T}^d$ then $g=|\det(DT)|^{-1}$.

[42] The δ in the definition is fixed once and for all, yet it must satisfy various smallness requirements that depend only on $(\mathtt{M},T)$.

[43] To compute the difference between two tangent vectors at different (close) points we parallel transport one of them to the tangent space of the other along the geodesic. We will not mention this explicitly, since it is completely trivial on $\mathtt{M}=\mathbb{T}^d$ and it is a routine operation on general Riemannian manifolds, see e.g. [18, Section 2.3].

Using the above defined classes of test functions and test vector fields we now define the norms that will describe our Banach spaces of generalized functions. For $f \in \mathcal{C}^1(\mathtt{M}, \mathbb{R})$ let

$$
\begin{aligned}
\|f\|_s &:= \sup_{\varphi \in \mathcal{D}_\beta} \int_{\mathtt{M}} f\varphi \, dm \\
\|f\|_u &:= \sup_{v \in \mathcal{V}_\beta} \int_{\mathtt{M}} df(v) dm \\
\|f\| &:= \|f\|_u + b\|f\|_s \\
\|f\|_w &:= \sup_{\varphi \in \mathcal{D}_\gamma} \int_{\mathtt{M}} f\varphi \, dm.
\end{aligned}
\tag{4.8}
$$

where $\int_{\mathtt{M}} df(v)dm$ is short hand for $\int_{\mathtt{M}} d_x f(v(x))m(dx)$. The constant $b \geq 1$ must be chosen sufficiently large. Except for $\|\cdot\|_u$, which is only a semi norm, all these expressions define norms on $\mathcal{C}^1(\mathtt{M}, \mathbb{R})$ and $\|f\|_w \leq \|f\|_s \leq b^{-1}\|f\|$. Note that the above norms are inhomogeniously anisotropic because the stable and unstable directions are treated differently and may change from point to point.

Definition 4.1. *$\mathcal{B}(\mathtt{M})$ and $\mathcal{B}_w(\mathtt{M})$ denote the completions of $\mathcal{C}^1(\mathtt{M}, \mathbb{R})$ w.r.t. the norms $\|\cdot\|$ and $\|\cdot\|_w$, respectively.*

Each $f \in \mathcal{C}^1(\mathtt{M}, \mathbb{R})$ naturally gives rise to a bounded linear functional on $\mathcal{C}^1(\mathtt{M}, \mathbb{R})$ by virtue of

$$
\langle f, \varphi \rangle := \int_{\mathtt{M}} f\varphi \, dm \; .
$$

Obviously, $\|f\|^*_{\mathcal{C}^1} \leq \|f\|_w \leq \|f\| \leq \|f\|_{\mathcal{C}^1}$. Therefore there exist canonical continuous embedding (not necessarily one-to-one)

$$
\mathcal{C}^1(\mathtt{M}, \mathbb{R}) \to \mathcal{B}(\mathtt{M}) \to \mathcal{B}_w(\mathtt{M}) \to \mathcal{C}^1(\mathtt{M}, \mathbb{R})^* \; .
$$

In fact, each $f \in \mathcal{B}_w$ defines a bounded linear functional on $\mathcal{C}^1(\mathtt{M}, \mathbb{R})$ by $\langle f, \varphi \rangle := \lim_{n\to\infty} \langle f_n, \varphi \rangle$ where $f_n \in \mathcal{C}^1(\mathtt{M}, \mathbb{R})$ and $\lim_{n\to\infty} \|f - f_n\|_w = 0$. In the same way one can embed $\mathcal{B}(\mathtt{M})$ into $\mathcal{B}_w(\mathtt{M})$.

We now state, without proof, the two basic fact that hold true in (and justify the) above setting.

Lemma 4.1 ([11])**.** *Suppose $\beta < \min\{\tau, 1\}$ and $\gamma \in (\beta, 1]$. Then $\mathcal{L}$ extends naturally to a bounded linear operator on both $\mathcal{B}_w(\mathtt{M})$ and $\mathcal{B}(\mathtt{M})$. In addition, for each $\sigma > \max\{\lambda_u^{-1}, \lambda_s^\beta\}$, we can choose constants b and δ in (4.4) - (4.8) for which there exists $B > 0$ such that, for each $f \in \mathcal{B}(\mathtt{M})$, we have*

$$
\|\mathcal{L}^n f\|_w \leq A\,\|f\|_w \qquad \text{and} \qquad \|\mathcal{L}^n f\| \leq 3A^2\sigma^n\|f\| + B\|f\|_w \qquad \text{for } n = 1, 2, \ldots
$$

where A is the constant from (4.2).

Proposition 4.1 ([11])**.** *If $\gamma \cdot \min\{\tau, 1\} > \beta$, then the ball $\mathcal{B}_1 := \{f \in \mathcal{B}(\mathtt{M}) : \; \|f\| \leq 1\}$ is relatively compact in $\mathcal{B}_w(\mathtt{M})$.*

The next Theorem follows by the usual Hennion-Neussbaum argument.

Theorem 4.2. *Suppose that $\gamma \cdot \min\{\tau, 1\} > \beta$. Then, for each $\sigma > \max\{\lambda_u^{-1}, \lambda_s^\beta\}$, the operator $\mathcal{L} : \mathcal{B}(\mathtt{M}) \to \mathcal{B}(\mathtt{M})$ has essential spectral radius bounded by σ and is thus quasi compact.*

An immediate consequence of Theorem 4.2 is that for each constant $r \in (\sigma, 1)$ the portion

$$\mathrm{sp}_r(\mathcal{L}) := \mathrm{sp}(\mathcal{L}) \cap \{z \in \mathbb{C} : |z| > r\}$$

of the spectrum consists of finitely many eigenvalues $\lambda_1, \dots, \lambda_p$ of finite multiplicity. See Figure 4.3 for a depiction of the above facts.

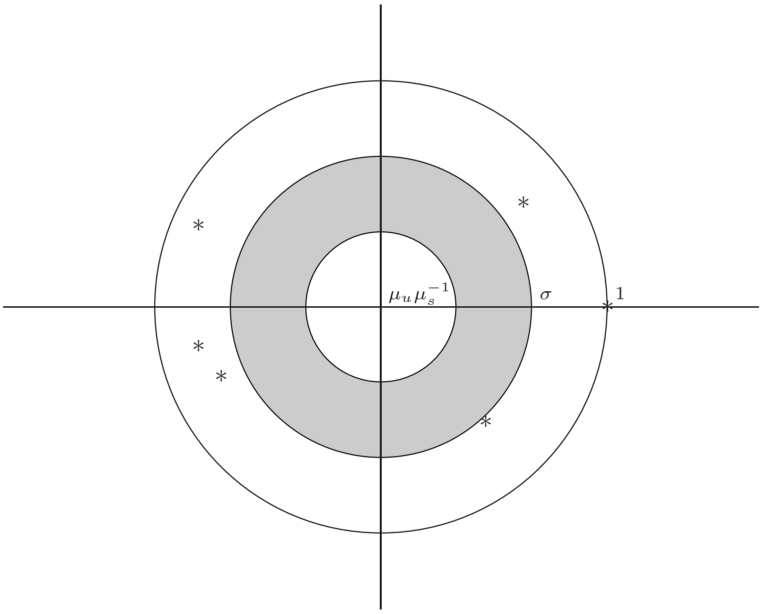

FIGURE 4.3. Region containing the spectrum of the transfer operator

We have thus gained a spectral picture for Anosov maps completely analogous to the one obtained for expanding maps. by arguments similar to the one already carried out in section 4.1 and in section 3.1 it is easy to prove that all the eigenspaces associated to the eigenvalues of modulus one must be contained in the spaces of measures $\mathcal{M}(\mathtt{M})$. Moreover, such measures have conditional expectations with respect to the unstable foliations that are absolutely continuous with respect to the Lebesgue measure, that is they are SRB measures (this can be proven in analogy with Lemma 2.3, or see [11] for details).

Since it is well known that, for the systems at hand, the SRB measure, μ_{SRB} is unique [3], and the same holds for all the powers of the map, one can easily prove that the Dynamical Systems $(\mathtt{M}, T, \mu_{SRB})$ is mixing. The already established spectral gap implies then that the system mixes exponentially fast on $\mathcal{C}^1$ observable.

It is then natural to investigate the stability of such a picture. At the moment only partial results are available [11]. Just to give a flavor of the situation let us quote the following result.

Let $J_\varepsilon : \mathcal{C}^0 \to \mathcal{C}^0$ be the operator defined by

$$J_\varepsilon f(x) := \int_{\mathbb{T}^2} \varepsilon^{-2} j(\varepsilon^{-1}\|x-y\|) f(y) dy$$

where $j \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^+)$, $\mathrm{supp}(j) \subset [0,1]$, $\int j(r)dr = 1$. The first interesting fact is that J_ε can be extended to an operator on $\mathcal{B}$ which enjoys nice properties.

Lemma 4.2. *Assume that* $\mathtt{M} = \mathbb{T}^2$. *The unstable foliation is* $\mathcal{C}^\tau$, $\tau - 1 \geq \alpha \geq \gamma > \beta > 0$. *Then any smooth averaging operator can be extended in a unique way to a continuous operator on* $\mathcal{B}(\mathtt{M})$ *and one can choose* δ *(small) and* b *(large) in such a way that there exists* $K > 0$ *with*

$$\|J_\varepsilon\|_w \leq K \qquad \text{and} \qquad \|J_\varepsilon f\| \leq 3\|f\| + K\|f\|_w$$

for all sufficiently small $\varepsilon > 0$ *and all* $f \in \mathcal{B}(\mathtt{M})$.

We can then define the operator $\mathcal{L}_\varepsilon := J_\varepsilon \mathcal{L}$. As already noticed $\mathcal{L}_\varepsilon$ is the transfer operator of a random process consisting in moving a point by the map T followed by a random jump determined by the transition kernel J_ε. In other words at each step of time the point is displaced at random in a small neighborhood of its current position, thus we have a small random perturbation of the deterministic system. The goal is clearly to apply the perturbation theory described in the previous lecture. This is possible thanks to the next two results.

Lemma 4.3 ([11])**.** *Assume that* $\mathtt{M} = \mathbb{T}^2$. *There is a constant* $B' > 0$ *such that, for sufficiently small* $\varepsilon > 0$ *and each* $n \in \mathbb{N}$

$$\|\mathcal{L}_\varepsilon^n\|_w \leq B' \quad \text{and} \quad \|\mathcal{L}_\varepsilon^n f\| \leq 9A^2\sigma^n\|f\| + B'\|f\|_w$$

for all $f \in \mathcal{B}(\mathtt{M})$.

Lemma 4.4 ([11])**.** *Assume that* $\mathtt{M} = \mathbb{T}^2$. *There exists a constant* $K > 0$ *such that*

$$|||J_\varepsilon - Id||| \leq K\varepsilon^{\gamma-\beta}.$$

Accordingly $|||\mathcal{L}_\varepsilon - \mathcal{L}||| \leq \|\mathcal{L}\|K\varepsilon^{\gamma-\beta}$ which suffice to apply Theorem 4.1 and obtain the wanted spectral stability.[44]

5. **Fifth Lecture** (Geodesic flows)

In this last lecture we will discuss flows. Since some aspect are a bit technical–although very similar to what we have discussed already–I will mostly restrict to the ideas referring to the original article [58] for the nasty details.

We will consider a $\mathcal{C}^4$, $d+1$ dimensional, connected compact Riemannian manifold M with everywhere negative curvature and the associated geodesic flow[45] on the unitary tangent bundle $\mathtt{M}$. The flow $T_t : \mathtt{M} \to \mathtt{M}$ so defined satisfies the following conditions.

Lemma 5.1 ([49])**.** *At each point* $x \in \mathtt{M}$ *there exists a splitting of the tangent space* $\mathcal{T}_x\mathtt{M} = E^s(x) \oplus E^0(x) \oplus E^u(x)$. *The splitting is invariant with respect to* T_t, E^0 *is one dimensional and coincides with the flow direction, in addition there exists* $\mu > 0$ *such that*

$$\|dT_t v\| \leq e^{-\mu t}\|v\| \quad \text{for each } v \in E^s \text{ and } t \geq 0$$
$$\|dT_t v\| \geq e^{\mu t}\|v\| \quad \text{for each } v \in E^u \text{ and } t \leq 0.$$

[44]Note that the above estimates imply only Hölder continuity in ε even for the SRB measure. This is a consequence of the choice of the spaces. The situation could be improved in the two dimensional case, since the foliation is, in this case, $\mathcal{C}^{1+\alpha}$ regular but for the higher dimensional case where the foliation can be only Hölder continuous it is not clear how to improve the above approach. Nevertheless, it is known that the SRB measure is differentiable in quite general situations [81, 82, 22], so it is clear that more work is needed to understand the real potentiality of the present point of view.

[45]That is $T_0 = \mathrm{Id}$ and $T_{t+s} = T_t \circ T_s$ for each $t, s \in \mathbb{R}$.

That is, the flow is Anosov.[46]

Remark 5.1. *We will restrict our discussion geodesic flows but all is said holds more generally for* contact flows.[47]

5.1. **A bit of history.** For geodesic flows (and contact flows as well) the Riemannian volume (contact volume) is invariant, hence the issue of finding "nice" invariant measure is trivial.[48] On the contrary the investigation of the statistical properties of the flow are far from trivial. Here is a bit of history of the subject.

Although the interest in geodesic flows is very old, for convenience we can start our history with Hopf who in 1939 proved the ergodicity of geodesic flows on surfaces of *constant* negative curvature, [30, 31]. His argument, to this days called *the Hopf argument*, proved a very far reaching tool to investigate ergodicity in hyperbolic systems (see [59] for a modern application of such ideas). The mixing of such systems was proved by Sinai [85] in 1961.

The next step was undertaken by Anosov in 1963, [2]. He proved ergodicity of geodesic flows in negative curvature. Thanks to his work it was possible to extend the Hopf strategy to the a large class of systems (not by chance now called *Anosov*) even when the stable and unstable foliation are not $\mathcal{C}^1$. This was an impressive technical advance. After about ten years Ornstein and Weiss [68] established that this more general class of systems was mixing (in fact Bernoulli).

The study of the rate of mixing starts with the work of Collet, Epstein, Gallavotti in 1984, [17] soon followed by other similar results [65, 77, 71]. In these papers it is proven the exponential decay of correlations for geodesic flows in *constant* negative curvature (in two and three dimensions). Unfortunately, the technique used there are based on representation theory and therefore use in an essential way the group structure of the constant curvature case.

The possibility to obtain a more dynamical proof of the rate of decay of correlation came about thanks to the monumental work of Sinai, Ruelle and Bowen that developed the so called *thermodynamic formalism* for Dynamical Systems (see [34] for a review of such theory). This approach was extend to flows in a sequence of works [84, 79, 73].

Building on such foundations Chernov in 1998 was finally able to produce a dynamical approach to the study of decay of correlation for flows, [15]. He managed to prove sub-exponential decay of correlations for geodesic flows on negative curvature surfaces. Shortly after Dolgopyat, in a series of papers [19, 20, 21], proved exponential decay of correlations for Anosov flows in negative curvature with $\mathcal{C}^1$ foliations (e.g.: negative curvature surfaces and 1/4-pinched manifolds).

At this point the reader may guess that all axion A flows enjoy exponential decay of correlations for smooth observable, this turns out to be false. Indeed, Ruelle in 1983 produced an example of a suspension over a shift with piecewise constant ceiling which it is mixing but exhibits arbitrarily slow correlation decay, [79] (see also [72]). Such an example is not Anosov, yet it is hyperbolic (Axiom A). To this

[46] More in general, one can allow a constant $A > 0$ in front of the exponential term, but this can be handled exactly as in the following.

[47] By *contact flow* we mean that there exists a $\mathcal{C}^{(2)}$ one form α on $\mathtt{M}$, such that there exists a $\mathcal{C}^{(2)}$ one form α on $\mathtt{M}$, such that $\alpha \wedge (d\alpha)^d$ is nowhere zero, which is left invariant by T_t (that is $\alpha(dT_t v) = \alpha(v)$ for each $t \in \mathbb{R}$ and tangent vector $v \in \mathcal{T}\mathtt{M}$).

[48] Yet, there exists other very important invariant measures that one may be interest in investigating such as the measure of maximal entropy.

days it is not known if all Anosov flows mixes exponentially fast, yet we are going to see that this is the case for contact Anosov flows.

5.2. **Looking at a generator.** We can now start to describe the spaces on which we will consider the operators T_t and $\mathcal{L}_t$.[49] First of all let us fix $\delta > 0$ sufficiently small and define

$$H_{s,\beta}(\varphi) := \sup_{d_s(x,y)\leq\delta} \frac{|\varphi(x)-\varphi(y)|}{d_s(x,y)^\beta}; \quad |\varphi|_{s,\beta} := |\varphi|_\infty + H_{s,\beta}(\varphi). \tag{5.1}$$

Definition 5.1. *In the following by the Banach space $\mathcal{C}^\alpha_s(\mathtt{M},\mathbb{C})$ we will mean the closure of $\mathcal{C}^1(\mathtt{M},\mathbb{C})$ with respect to the norm $|\cdot|_{s,\alpha}$. Similar definitions hold with respect to the metric d_u and the Riemannian metric d (giving the space of Hölder function $\mathcal{C}^\alpha$).*

Let us also define the unit ball $\mathcal{D}_\beta := \{\varphi \in \mathcal{C}^\beta_s(\mathtt{M},\mathbb{C}) \mid |\varphi|_{s,\beta} \leq 1\}$. For a given $\beta < 1$, and $f \in \mathcal{C}^1(\mathtt{M},\mathbb{C})$, let

$$\begin{aligned} &\|f\|_w := \sup_{\varphi\in\mathcal{D}_1} \int_\mathtt{M} \varphi f \\ &\|f\| := \|f\|_s + \|f\|_u\ ; \quad \|f\|_s := \sup_{\varphi\in\mathcal{D}_\beta} \int_\mathtt{M}\varphi f\ ; \quad \|f\|_u := H_{u,\beta}(f). \end{aligned} \tag{5.2}$$

Let $\mathcal{B}(\mathtt{M},\mathbb{C})$ and $\mathcal{B}_w(\mathtt{M},\mathbb{C})$ be the completion of $\mathcal{C}^1(\mathtt{M},\mathbb{C})$ with respect to the norms $\|\cdot\|$ and $\|\cdot\|_w$ respectively. Note that such spaces are separable by construction and are all contained in $(\mathcal{C}^\beta)^*$, the dual of the β-Hölder functions.

It is well known that the strong stable and unstable foliations and the Jacobian of the holonomies associated to the stable and unstable foliations for an Anosov flow are τ-Hölder. From now on we will assume

$$0 < \beta < \tau^2. \tag{5.3}$$

Again it is possible to prove some regularization properties of the family $\mathcal{L}_t$.

Lemma 5.2 (Lasota-Yorke type inequality). *$\{\mathcal{L}_t\}_{t\geq 0}$ is a strongly continuous semi group on $\mathcal{B}$ and $\mathcal{B}_w$. In addition,*

$$\begin{aligned} &\|\mathcal{L}_t f\|_w \leq \|f\|_w\ ; \quad \|\mathcal{L}_t f\| \leq \|f\| \\ &\|\mathcal{L}_t f\| \leq 3e^{-\lambda\beta t}\|f\| + B\|f\|_w. \end{aligned}$$

Hence there exists a generator $X : D(X) \subset \mathcal{B} \to \mathcal{B}$ and the following formula for the resolvent holds true

$$(z-X)^{-1} = R(z) = \int_0^\infty e^{-zt}\mathcal{L}_t dt\ ; \quad \forall z \in \mathbb{C}, \Re(z) > 0.$$

Unfortunately the unit ball of $\mathcal{B}$ it is not compact in $\mathcal{B}_w$. This is due to the obvious fact that our norms do not induce any control on the flow direction. Nevertheless, the above formula it is very inspiring since the operator $R(z)$ can be written as an integral along the flow direction, it is thus not inconceivable that the range of $R(z)$ be regular also in the flow direction, whereby yielding some compactness property.

[49]Clearly $\mathcal{L}_t$ is defined by $\int_\mathtt{M} \varphi\circ T_t h dm = \int_\mathtt{M} \varphi \mathcal{L}_t h dm$.

Lemma 5.3. *Let $z = a + ib$, $a > 0$, then*

$$\|R(z)\|_w \leq a^{-1}\ ; \quad \|R(z)\| \leq a^{-1}$$
$$\|R(z)^n f\| \leq \frac{3}{(a+\lambda\beta)^n}\|f\| + a^{-n}B\|f\|_w.$$

In addition, $R(z) : \mathcal{B}(\mathtt{M}, \mathbb{C}) \to \mathcal{B}_w(\mathtt{M}, \mathbb{C})$ is compact.

We can then investigate the spectrum of X by studying the resolvent.

Note that $\zeta \in \rho(R(z))$ is equivalent to $z - \zeta^{-1} \in \rho(X)$. Thus the bound on the spectral radius of $R(z)$ implies that $\sigma(X)$ lies in the left half plane (see Figure 5.4). This information is of little interest (it followed already from the fact that $\mathcal{L}_t$ is bounded in the future). On the other hand, Lemma 5.3 implies, via the usual arguments (see Theorem 3.1) the quasi compactness of $R(z)$. This means that one can get some information on the spectrum of X on the left of the imaginary axis (see Figure 5.5). In fact, by varying z, it follows that there is a strip on the left of the imaginary axis that contains only point spectrum of finite multiplicity without accumulation points (see Figure 5.6).

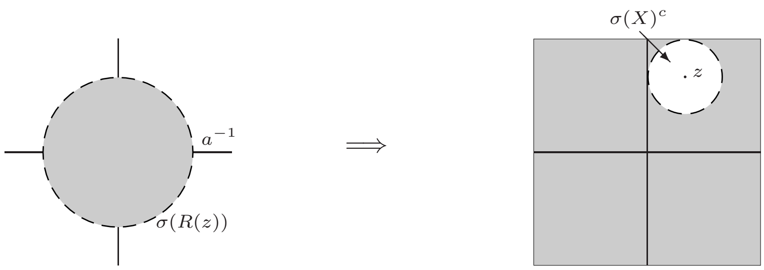

FIGURE 5.4. Relation between the spectral radius of $R(z)$ and $\sigma(X)$.

FIGURE 5.5. Relation between the quasi compactness of $R(z)$ and $\sigma(X)$

By arguments similar to the one seen for expanding maps the ergodicity of the flow implies that 0 is a simple eigenvalue, while the mixing implies $\sigma(X) \cap \{ib\}_{b\in\mathbb{R}} = \{0\}$.

Remark 5.2. *Up to now the results hold for each Anosov flow and they correspond to the to the existence of a strip on which the Laplace transform of the correlation functions (or the dynamical zeta function) is meromorphyc.*

Yet, we cannot conclude much about decay of correlations from the above results. The obstacle is that, although the imaginary axis is free of eigenvalues (apart from

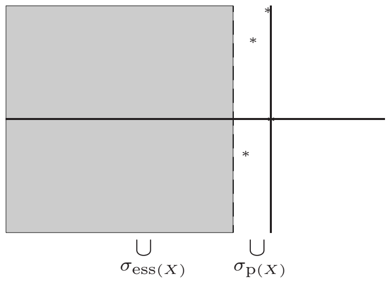

FIGURE 5.6. The spectrum of the generator X.

zero), nothing prevent the eigenvalues to accumulate to the imaginary axis very far from the real axis. To exclude such a phenomenon it is necessary to have an estimate of the norm of the resolvent for large values of $\Im(z)$. This turns out to be the hardest part of the argument and it is based on the following result.

Lemma 5.4 (Dolgopyat inequality)**.** *There exist $c, b_*, \nu > 0$ such that, for all $a \in [1,2]$, $|b| \geq b_*$, $z = a + ib$, and $n \geq c \ln |b|$,*

$$\|R(z)^n\| \leq (a+\nu)^{-n}.$$

Hence,

$$r_{\text{sp}}(R(z)) \leq (a+\nu)^{-1}.$$

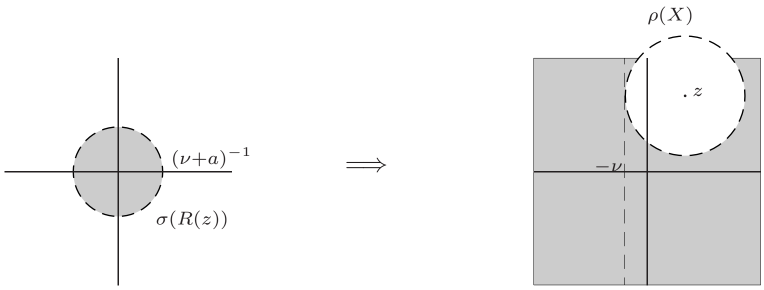

FIGURE 5.7. Ruling out eigenvalues close to the imaginary axis

Remark 5.3. *In fact, Dolgopyat result holds for a different operator on a different space, yet the cancellation mechanism relevant to the proof is similar. In his case he uses $\mathcal{C}^{(1)}$ foliations. Here the contact structure suffices to yield the result regardless the regularity of the foliation.*

By Lemma 5.4 one obtains that there exists a strip on the left of the imaginary axis that, far away from the real axis, does not contain any eigenvalues (see Figure 5.7). Putting this information together with what we know already we obtain a spectral gap for the generator.

To conclude a little more work is needed. The problem is that $\mathcal{L}_t$ grows exponentially in the past and so no obvious relation between the spectrum of X and the spectrum of $\mathcal{L}_t$ does exist. We conclude thanks to some form of the inverse Laplace transform.

Lemma 5.5. *If $f \in D(X^2)$, then*

$$\mathcal{L}_t f = \frac{1}{2\pi} \lim_{w\to\infty} \int_{-w}^{w} e^{at+ibt} R(a+ib) f \, db.$$

All the above facts easily imply the main result.

Theorem 5.1. *For a $\mathcal{C}^4$ geodesic flow T_t on a manifold M of everywhere negative curvature, the operators $\mathcal{L}_t$ form a strongly continuous group on $\mathcal{B}(\mathcal{M}, \mathbb{C})$. In addition, there exists $\sigma, C_1 > 0$ such that, for each $f \in \mathcal{C}^1$, $\int f = 0$, the following holds*

$$\|\mathcal{L}_t f\| \le C_1 e^{-\sigma t} |f|_{\mathcal{C}^\alpha}.$$

Note that the above theorem implies the exponential decay of correlation for all Hölder observable, yet this does not imply that the operators $\mathcal{L}_t$ are quasi-compact, and it is possible that they are not. This may explain the rather indirect route needed to attain the above result.

5.3. **Few rough ideas about the proofs.** The proof of the previous results follows lines very similar to what we have done in the previous lectures. In particular the easy inequalities

$$H_{u,1}(R(z)f) \le \frac{1}{a+\lambda} H_{u,1}(f); \quad H_{s,\beta}(R(z)^*\varphi) \le \frac{1}{a+\lambda\beta} H_{s,\beta}(\varphi)$$

are the basis to prove Lemmata 5.2 and 5.3. From Lemma 5.3 follows

$$\begin{aligned}
\|R(z)^{3n} f\| &\le \frac{3}{(a+\lambda\beta)^n} \|R(z)^{2n} f\| + a^{-n} B \|R(z)^{2n} f\|_w \\
&\le 3(a+\lambda\beta)^n a^{2n} \|f\| + a^{-n} B \|R(z)^{2n} f\|_w \\
&\le \frac{1}{2(a+\nu)^{3n}} \|f\| + a^{-n} B \|R(z)^{2n} f\|_w.
\end{aligned}$$

To prove Lemma 5.4, we have thus to estimate the weak norm of $R(z)^n f$ in terms of the strong norm of f. By using a and idea that, by now, should be familiar we write

$$\int R(z)^{2n} f \varphi \sim \int f R(z)^{*n} \mathbb{A}_\delta^u R(z)^{*n} \varphi.$$

Where $\mathbb{A}_\delta^u$ is the average of a piece of strong unstable manifold of size δ.

$$\mathbb{A}_\delta^u R(z)^{*n} \varphi(x) = \frac{1}{(n-1)!} \int_{\cup_{t=0}^\infty T_t W_\delta^u(x)} t^{n-1} e^{-zt} \varphi(\xi) J_u T_{-t}(\xi).$$

Note that, for large t, the image of the unstable manifold becomes extremely long and invades all the space.

The idea is then to sum together different pieces of such a long manifold. Different pieces can be compared along the stable holonomy yielding a cancellation due to the non-joint integrability of stable and unstable manifold. This comes about since the image of the strong unstable manifold by the stable holonomy is tilted in the flow direction (see Figure 5.8). Accordingly, the oscillating factor e^{ibt}, which it is constant along strong unstable manifolds, gets mapped into something that oscillate along a fixed strong unstable manifold whereby producing the wanted cancellations (see Figure 5.8 where the strips symbolize the different phases of the oscillating factor).

The non-joint integrability of the foliation is expressed quantitatively by the contact form.

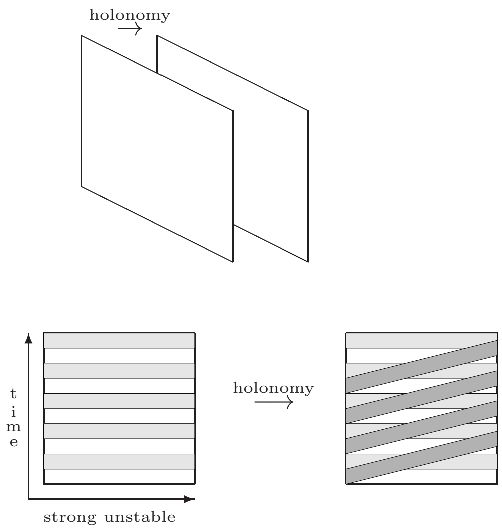

FIGURE 5.8. The Chernov-Dolgopyat cancellation mechanism

To be more precise consider a point $x \in \mathtt{M}$ and a small neighborhood $B_\delta(x)$. And a coordinate system (u,t,s) such that $\{(u,0,0)\} = W^u(x)$, $(0,t,0) = T_t x$ and $\{(0,0,s)\} = W^s(x)$. Let $y \in B_\delta(x) \cap W^s(x)$ and $y' \in B_\delta(x) \cap W^u(x)$. Moreover let $z' = W^u(y) \cap W^{sc}(y')$ and $z = W^s(y') \cap W^{uc}(y)$. By construction z and z' are on the same flow orbit. Thus there exists $\Delta(y,y')$ such that $T_{\Delta(y,y')} z = z'$. The function $\Delta(y,y')$ is called *temporal distance*, see Figure 5.9 for a pictorial definition.

The relation between the temporal distance and the contact form is made precise by the following Lemma.

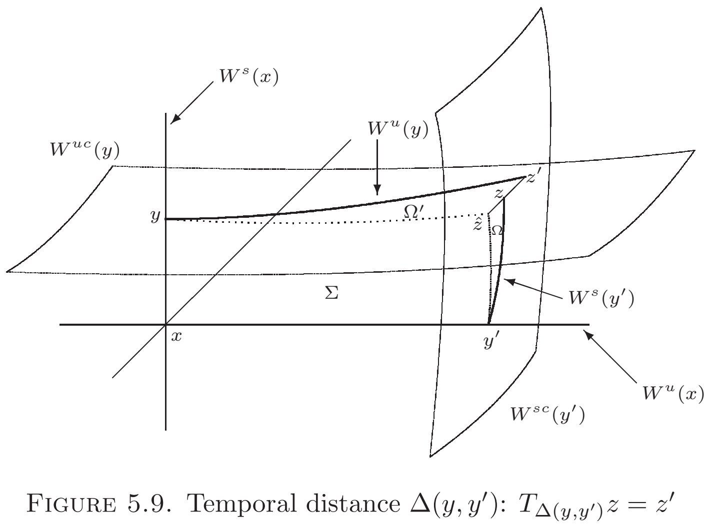

FIGURE 5.9. Temporal distance $\Delta(y,y')$: $T_{\Delta(y,y')} z = z'$

Lemma 5.6. *Let $v \in E^u(x)$, $w \in E^s(x)$ be such that $exp_x(v) = y'$ and $exp_x(w) = y$,[50] then*

$$\Delta(y, y') = d\alpha(v, w) + \mathcal{O}(|v|^\tau |w|^2 + |w|^\tau |v|^2).$$

The above formula allows to make quantitatively precise the cancellations depicted in Figure 5.8.

References

[1] J.Aaronson, *An introduction to infinite ergodic theory,* American Mathematical Society, Mathematical Surveys and Monographs, **50** (1997).

[2] D.V.Anosov, *Geodetic flows on closed Riemannian manifolds with negative curvature,* Proc.Inst.Steklov **90**, 1–235 (1967).

[3] D.V.Anosov, Ya.G.Sinai *Certain smooth ergodic systems,* Russ. Math. Surv. **22**, 103 – 167 (1982).

[4] V.Baladi, *Positive transfer operators an decay of correlations*, World scientific, Singapore (2000).

[5] V.Baladi, H.H.Rugh, *Floquet spectrum of weakly coupled map lattices*, Comm. Math. Phys. **220** (2001), no. 3, 561–582.

[6] V.Baladi, L.-S.Young, *On the spectra of randomly perturbed expanding maps*, Comm. Math. Phys., **156**:2 (1993), 355-385; **166**:1 (1994), 219–220.

[7] M.L.Blank, *Stochastic properties of deterministic Dynamical Systems,* Sov. Sci. Rev. C. maniath./Phys. **6** (1987), 243-271.

[8] M.L.Blank, *Small perturbations of chaotic Dynamical Systems,* Usp. maniat. Nauk **44** (1989), 3-28.

[9] M.L.Blank, G.Keller, *Stochastic stability versus localization in chaotic dynamical systems,* Nonlinearity **10** (1997), 81-107.

[10] M.L.Blank, G. Keller, *Random perturbations of chaotic Dynamical Systems: Stability of the spectrum,* Preprint (1998).

[11] M.Blank, G.Keller, C.Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*,Nonlinearity, 15, n.6, 1905-1973 (2002).

[12] J.Bricmont, A.Kupianen, *Coupled analytic maps*, Nonlinearity, **8**, (1995), 379–396.

[13] J.Bricmont, A.Kupianen, *High temperature expansions and dynamical systems*, Communication Math.Phys., **178**, (1996), 703–732.

[14] L.A.Bunimovich, Ya.G.Sinai, *Spacetime chaos in coupled map lattices*, Nonlinearity 1 (1988), no. 4, 491–516.

[15] N.Chernov, *Markov approximations and decay of correlations for Anosov flows*, Annals of Mathematics, **147** (1998), 269–324.

[16] C.Chiu, Q.Du, T.Y.Li, *Error estimates of the Markov finite approximation to the Frobenius-Perron operator,* Nonlinear Analysis, Theory, Methods & Appl. **19** (1992), 291-308.

[17] P.Collet, H. Epstein, G. Gallavotti, *Perturbations of geodesic flows on surfaces of constant negative curvature*, Communications in Mathematical Physics, **95** (1984), 61–112.

[18] M.P. Do Carmo, *Riemannian Geometry,* Birkhäuser, Boston (1992).

[19] D.Dolgopyat, *Decay of correlations in Anosov flows*, Annals of mathematics, **147** (1998), 357-390.

[20] D.Dolgopyat, *Prevalence of rapid mixing in hyperbolic flows*, Ergodic Theory and Dynamical Systems, **18**, 1097-1114 (1998).

[21] D.Dolgopyat, *Prevalence of rapid mixing-II: topological prevalence*, Ergodic Theory and Dynamical Systems, **20** (2000), no. 4, 1045–1059.

[22] D.Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems*, preprint.

[23] L.E.Evans, R.F.Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, New York, London, Tokyo (1992).

[24] B.U.Felderhof, M.E.Fisher, Ann. Phys. (N.Y.), **58** (1970).

[50] Here the exponential function is meant with respect to the restriction of the metric to $W^u(x)$ and $W^s(x)$, respectively.

[25] G.Gallavotti, *Funzioni zeta e insiemi basilari*, Accad. Lincei Rend. Sc. fis. mat. e nat., **61** (1976), 309–317.

[26] B.Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergod. Th.& Dynam. Sys., **14** (1994), 645-666.

[27] P.Gaspard, X.J.Wang, *Sporadicity: between periodic and chaotic dynamical behaviors*, Proc. Math. Acad. Sci. USA, **85** (1988), 4591.

[28] H.Hennion, *Sur un théorème spectral et son application aux noyaux Lipchitziens*, Proceedings of the American Mathematical Society, **118** (1993), 627–634.

[29] M.Hirsch, C.Pugh, M.Shub, *Invariant Manifolds*, Lecture Notes in Math. **583** (1977).

[30] E.Hopf, *Statistik der Geodätischen Linien in Mannigfaltigkeiten Negativer Krümmung*, Ber. Verh. Sächs. Akad. Weiss., Leipzig **91**, 261–304, (1939).

[31] E.Hopf, *Statistik der Lösungen Geodätischen Probleme vom unstabilen Typus. II*, Mathematische Annalen **117**, 590–608 (1940).

[32] H.Hu, *Decay of correlations for piecewise smooth maps with indifferent fixed point*, to appear in Ergodic Theory and Dynamical Systems.

[33] F.Y.Hunt, W.Miller, *On the approximation of invariant measures,* J. Stat. Phys. **66** (1992), 535-548.

[34] A.Katok, B.Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, **54**, G.-C.Rota editor, Cambridge University Press, Cambridge (1995).

[35] T.Fischer, H.H.Rugh, *Transfer operators for coupled analytic maps*, Ergodic Theory Dynam. Systems **20** (2000), no. 1, 109–143.

[36] G.Froyland, *Computer-assisted bounds for the rate of decay of correlations,* Commun. Math. Phys. **189** (1997), 237-257.

[37] T.Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York (1966).

[38] G.Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan,* C. R. Acad. Sci. Paris, Série A, **289**, (1979), 625-627 (Kurzfassung der Dissertation).

[39] G.Keller, *Un théorème de la limite centrale pour une classe de transformations monotones par morceaux,* C. R. Acad. Sci. Paris, Série A, **291**, (1980), 155-158.

[40] G.Keller, *On the rate of convergence to equilibrium in one-dimensional systems,* Commun. Math. Phys., **96**, (1984), 181-193.

[41] G.Keller, *Stochastic stability in some chaotic Dynamical Systems,* Monatshefte Math. **94** (1982), 313-333.

[42] G.Keller, *Coupled map lattice via transfer operators on functions of bounded variation. Stochastic and spatial structures of Dynamical Systems (Amsterdam, 1995)*, 71–80, Konink. Nederl. Akad. Wetensch. Verh. Afd. Natuurk. Eerste Reeks, **45**, North-Holland, Amsterdam, 1996.

[43] G.Keller, *G. Mixing for finite systems of coupled tent maps*, Tr. Mat. Inst. Steklova 216 (1997), Din. Sist. i Smezhnye Vopr., 320–326.

[44] G.Keller, *Equilibrium states in ergodic theory*, London Mathematical Soceity Students Texts **42**, Camridge University Press, Cambrindge (1998).

[45] G.Keller, M.Künzle, *Transfer operators for coupled map lattices.* Ergodic Theory Dynamical Systems **12** (1992), no. 2, 297–318.

[46] G.Keller, C.Liverani, *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, (4) **XXVIII** (1999), 141-152.

[47] Yu.Kifer, *Random Perturbations of Dynamical Systems*, Progress in Probability and Statistics 16, Birkhäuser, Boston (1988).

[48] M.Keane, R.Murray, L.S.Young, *Computing invariant measures for expanding circle maps,* Nonlinearity **11** (1998), 27-46.

[49] W.P.A.Klingenberg, *Riemannian Geometry*, second edition, Walter de Gruyter, Berlin, New York (1995).

[50] N.Kriloff, N.Bogoliouboff, *La théorie générale de la mesure dans son application à l'étude des systéme dynamiques de la mécanique non linéaire*, Annals of mathematics, **38** no.1, (1937), 65-113.

[51] S.Isola, *Isola, Stefano Renewal sequences and intermittency*, J. Statist. Phys. **97** (1999), no. 1-2, 263–280.

[52] A.Lambert, S.Siboni, S.Vaienti, *Statistical properties of a non-uniformly hyperbolic map of the interval*, Journal of Statistical Physics **71**, n. 5/6 (1993) 1305–1330.

[53] T.Y.Li, *Finite approximations for the Frobenius-Perron operator: A solution to Ulam's conjecture,* J. Approx. Th. **17** (1976), 177-186.

[54] C.Liverani, *Decay of Correlations*, Annals of Mathematics, **142**, pp. 239-301, (1995).

[55] C.Liverani, *Decay of Correlations in Piecewise Expanding maps*, Journal of Statistical Physics, **78**, 3/4, pp. 1111-1129, (1995).

[56] C.Liverani, *Rigorous numerical investigation of the statistical properties of piecewise expanding maps–A feasibility study*, Nonlinearity, 14, n. 3, pp. 463–490, (2001).

[57] C.Liverani, *Flows, Random Perturbations and Rate of Mixing*, Ergodic Theory and Dynamical Systems, **18** (1998), 1421–1446.

[58] C.Liverani, *On Contact Anosov Flows*, preprint.

[59] C.Liverani, M.Wojtkowski, *Ergodicity in Hamiltonian Systems*, Dynamics Reported, **4**, pp. 130-202, (1995).

[60] C.Liverani, V. Maume-Deschamps, *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set*, to appear on Annales Ist.H. Poincarè.

[61] C.Liverani, B. Saussol and S.Vaienti, *A Probabilistic Approach to Intermittency*, Ergodic Theory and Dynamical Systems, **19**, (1999), 671–685.

[62] A.Lasota, J.A.Yorke, *On the existence of invariant measures for piecewise monotonic transformations,* Transact.Amer.Math.Soc., **186**, (1973), 481-488.

[63] R.Mañe, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin Heidelberg (1987).

[64] W.Miller, *Stability and approximation of invariant measures for a class of nonexpanding transformations,* Nonlinear Analysis, Theory, Methods & Appl. **23** (1994), 1013-1025.

[65] C.Moore, *Exponential decay of correlation coefficients for geodesic flows*, in Group Representation Ergodic Theory, Operator Algebra and Mathematical Physics, Springer, Berlin, (1987).

[66] M.Mori, *On the intermittency of a piecewise linear map (Takahashi model)*, Tokyo Journal of Mathematics **16**, n. 2 (1993), 411–428.

[67] R.D.Nussbaum, *The radius of the essential spectrum*, Duke Math. J., **37** (1970), 473-478.

[68] D.S.Ornstein, B.Weiss, *Geodesic flows are Bernoullian*, Israel J. Math., **14** (1973), 184–198.

[69] Y.Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Mathematics of the USSR, Isvestia, **10**, no. 6,(1976), 1261–1305.

[70] Y.Pesin, *Characteristic exponents and smooth ergodic theory*, Russian mathematical surveys, **32**, no.4 (1977), 1195–1228.

[71] M.Pollicott, *Exponential mixing for geodesic flow on hyperbolic three manifold*, Journal of Statistical Physics, **67** (1992), 667-673.

[72] M.Pollicott, *A complex Ruelle-Perron-Frobenius theorem and two counterexamples*, Ergodic Theory and Dynamical Systems, **4** (1984), 135-146.

[73] M.Pollicott, *On the rate of mixing of Axiom A flows*, Inventiones Mathematicae, **85** (1985), 413–426.

[74] Y.Pomeau, P.Manneville, *Intermittent transition to turbulence in dissipative Dynamical Systems*, Communication in Mathematical Physics, **74** (1980), 189–197.

[75] T.Prellberg, J.Slawny, *Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions*, Journal of Statistical Physics, **66**, n. 1/2 (1991), 503–514.

[76] C.Pugh, M.Shub, A.Wilkinson, *Hölder Foliations*, Duke Math. J., **86** (1997), 517-546.

[77] M.Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory and Dynamical Systems, **7** (1987), 267–288.

[78] D.Ruelle, *Thermodynamics formalism*, Addison-Wesley, New York, 1978.

[79] D.Ruelle, *Flots qui ne mélange pas exponentialment*, C.R.Acad. Sc. Paris, **296** (1983), 191–193.

[80] D.Ruelle, *Ruelle, David Locating resonances for Axiom A Dynamical Systems*, J. Statist. Phys. **44** (1986), no. 3-4, 281–292.

[81] D.Ruelle, *Differentiation of SRB states*, Comm. Math. Phys. **187** (1997), no. 1, 227–241.

[82] D.Ruelle, *Differentiation of SRB states: Correction and complements*, preprint.

[83] O.Sarig, *Subexponential decay of correlations*, to appear in Invent. Math.

[84] Ya.G.Sinai, *Gibbs measures in ergodic theory*, Russian Math. Surveys **27**, 21-69 (1972).

[85] Ya.G.Sinai, *Geodesic flows on compact surfaces of negative curvature*, Soviet. Math. Dokl. **2** (1961), 106-109.

[86] M.Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Israel Journal of Mathematics, **37** n. 4 (1980), 303–314.

[87] S.M.Ulam, *Problems in Modern Mathematics*, New York, Interscience (1960).

[88] M.Viana, *Dynamics: a probabilistic and geometric perspective*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 557–578.

[89] M.Viana, *Stochastic dynamics of deterministic systems*, Lecture Notes, XXI Braz. math. Colloq., IMPA, Rio de Janeiro (1997).

[90] X.J.Wang, *Statistical physics of temporal intermittency*, Phys. Rev., **A40** (1989), 6647.

[91] L.-S.Young, *Developments in chaotic dynamics*, Notices Amer. Math. Soc. 45 (1998), no. 10, 1318–1328.

[92] L.-S.Young, *What are SRB measures and which dynamical systems have them*, To appear in JSP volume dedicated to Ruelle and Sinai.

[93] L.-S.Young, *Statistical Properties of Dynamical Systems with Some Hyperbolicity,* Annals of Mathematics (2), **147**, (1998), no. 3, 585–650.

[94] L.-S.Young, *Recurrence times and rates of mixing,* Israel J. of Math., **110**, (1999), 153–188.

[95] M.Yuri, *Decay of correlations for certain multi-dimensional maps*, Nonlinearity **9**, n. 6 (1996), 1439–1461

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy

E-mail address: `liverani@mat.uniroma2.it`