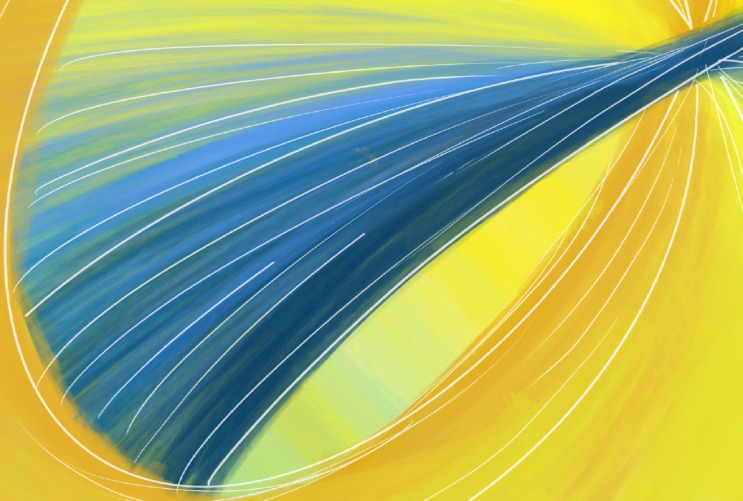
Transfer Operators in Hyperbolic Dynamics
An introduction



Mark F. Demers
Niloofar Kiamari
Carlangelo Liverani

impa

33º Colóquio Brasileiro de Matemática

Transfer Operators in Hyperbolic Dynamics
An introduction

Transfer Operators in Hyperbolic Dynamics - an introduction
Primeira impressão, julho de 2021

Publicado no Brasil / Published in Brazil.

ISBN 978-65-89124-26-9
MSC (2020) Primary: 37C30, Secondary: 37D20, 37C05, 37E35, 47B38, 37A25

Coordenação Geral Carolina Araujo

Produção Books in Bytes **Capa** Izabella Freitas & Jack Salvador

Realização da Editora do IMPA
IMPA
Estrada Dona Castorina, 110
Jardim Botânico
22460-320 Rio de Janeiro RJ

www.impa.br
editora@impa.br

Preface

This text is a result of the notes written for several Schools. It started with a series of lectures, *Probability and uniformly hyperbolic systems*, given by Carlangelo Liverani in Coimbra in 2008 and the lectures delivered by Mark Demers and Carlangelo Liverani at the *International Conference on Statistical Properties of Non-equilibrium Dynamical Systems*, SUSTC, Shenzhen, July 27 – August 2, 2016. It was then modified and used for the lectures *An introduction to the statistical properties of hyperbolic dynamical systems*, delivered by Carlangelo Liverani at the TMU-ICTP School, Tehran, May 5 – 10, 2018. It has finally reached its present state for the lectures by Carlangelo Liverani at the 33º Colóquio Brasileiro de Matemática.

Our aim is not to make a review of the field, but rather to introduce the reader to some basic modern techniques used to study the statistical properties of *chaotic* systems. Here by *chaotic* we mean uniformly hyperbolic systems. That is, systems that display a strong uniform sensitivity with respect to initial conditions. We will stress in particular the so-called *functional approach*, but we will also provide a simple introduction to the use of *standard pairs* and *Hilbert metrics*, and discuss some of the relations among these tools.

The goal is to provide the reader with a quick introduction to the literature. On the one hand we describe in detail the main techniques when applied to the simplest cases, providing full proofs for the essential general facts of the theory. On the other hand we try to flesh out the fundamental ideas necessary to understand the current literature, while avoiding the most technical details.

This note is a partial update with respect to the small review Liverani (2003).

For a much more in depth and technical discussion of transfer operators see Baladi (2000, 2018).

Our main focus, the functional approach, has its origin in the study of the Koopman operator Koopman (1931) (acting on L^2) starting, at least, with the von Neumann mean ergodic theorem von Neumann (1932) and further developed by the Russian school, see Cornfeld, Fomin, and Sinai (1982). An important development of this point of view occurred with the study of the transfer operator in symbolic dynamics by Sinai (1968, 1972b), Ruelle (1976a, 1978) and Bowen (1970, 2008).

Next, the functional approach developed further thanks to the work of Lasota and Yorke (1973), Ruelle (1976b), Keller (1979), Hofbauer and Keller (1982) and, more recently, Fried (1986), Rugh (1992, 1996), and Kitaev (1999), just to mention a few. This has eventually led to the current theory, which has assumed its present form starting with Blank, Keller, and Liverani (2002).

The basic idea is to study directly the spectrum of the Ruelle transfer operator without coding the system (even though the theory can be applied also to the transfer operator of a system after inducing). In order to do so, it is necessary to consider the action of the transfer operator on an appropriate Banach (or Hilbert) space or, more generally, in an appropriate topology. The non trivial part of the theory rests in the identification of the appropriate topological spaces which, to be effective, must reflect the geometric features of the dynamics.

In this note we will discuss only uniformly hyperbolic systems, yet the techniques presented here are relevant also in the non uniformly hyperbolic case, although they must be supplemented with essential new ideas such as Young towers, started by Young (1998); coupling, as introduced by Dolgopyat (2000) and Young (1999); and Operator Renewal Theory, whose development is due to Sarig (1999). In fact, it may be interesting to combine different techniques in order to develop a more effective theory: examples of attempts in this direction are De Simoi and Liverani (2016) and Maume-Deschamps (2001).

Another of our goals is to explain which properties the above mentioned Banach spaces must enjoy and to provide a guide for how to construct and adapt them to the peculiarities of the systems at hand. Also, we will briefly discuss the idea of coupling in an especially simple case, but we will not provide any details regarding Young towers or Operator Renewal Theory. More generally, we will not discuss non-uniform hyperbolicity nor general partial hyperbolicity (for the latter we refer to the book Bonatti, Díaz, and Viana (2005)).

The plan of the exposition is as follows: we start, in Chapter 1, discussing the simplest possible case, smooth expanding maps of the circle. This allows us to il-

lustrate, in the simplest possible setting, the power of the functional approach and the type of results that can be obtained once such machinery is in place. In particular, we will show how important properties of the system such as exponential decay of correlations, the Central Limit Theorem, Large deviation results, stability and linear response easily follow from the spectral properties of the transfer operator.

In Chapter 2, we will discuss the case of attractors, where the need to consider spaces of distributions first becomes apparent.

In Chapter 3 we develop the theory for the case of toral automorphisms. This may seem a bit silly as toral automorphisms can be studied directly using Fourier series. Yet, this will allow us to illustrate, in the most elementary manner, the main ideas of the theory, including anisotropic Banach spaces and coupling.

Then, in Chapter 4, we collect all the ideas previously illustrated and extend them to study general uniformly hyperbolic maps. This gives a precise taste of what the full theory looks like for uniformly hyperbolic systems.

Next, we discuss non-singular flows. By non-singular we mean that the vector field generating the flow has no zeros. This implies that a Lyapunov exponent (the one in the flow direction) is necessarily zero. Hence, this is one of the simplest possible partially hyperbolic systems. The other simple classes of partially hyperbolic systems are skew-products and group extensions. Some of these can be treated with similar techniques, but we will not discuss them explicitly in this note.

We will restrict ourselves to the case of *contact* flows. Although much of the present theory can be applied, with few changes, to more general hyperbolic flows, the contact flow case is the simplest example and hence well suited to an introductory discussion.

There are three main steps in adapting the analysis of the discrete time transfer operator for hyperbolic maps to the semi-group of continuous time transfer operators for hyperbolic flows:

1. Adapt Banach spaces used for hyperbolic maps to the setting of hyperbolic flows: the presence of the neutral flow direction makes this a nontrivial change.

2. Contrary to the discrete-time case, we do not prove the quasi-compactness of the transfer operator for the time-one map of the flow, but rather for the generator of the semi-group of transfer operators for the flow; this involves the use of the resolvent to ‘integrate out’ the neutral direction.

3. The use of the contact form to estimate an oscillatory integral and derive a spectral gap for the generator of the semi-group and an estimate for the resolvent close to the imaginary axis (the Dolgopyat-type estimate).

It then follows from some general considerations that a spectral gap for the generator of the semi-group implies exponential decay of correlations for the flow. This approach is detailed in Chapter 5.

At last Chapter 6 discusses the extension of these ideas to hyperbolic billiards. Note that hyperbolic billiards have serious discontinuities, hence albeit the overall strategy is the same as in the smooth case, there are crucial technical problems to overcome, problems that delayed the extension of the theory to this type of system for almost 20 years.

The notes also include several appendices. These are aimed at providing the reader with some basic knowledge that, while necessary to fully understand the main text, is not necessarily common knowledge.

Appendix A contains some very basic facts concerning functional analysis. These are normally covered in a graduate functional analysis course, but, just in case the reader was distracted, here we provide the minimum necessary to understand the main text.

Appendix B is devoted to a full exposition of the Hennion–Neussbaum theory. Such a theory underlies much of the current approach, yet it is impossible to find a full exposition of such results that has as prerequisite only the content of a standard first functional analysis course. We think that it is better to have full control of the main instruments used in the field, hence we attempt to fill this expository gap.

Appendix C presents a simplified version of the perturbative theory developed in Keller and Liverani (1999) and Gouëzel and Liverani (2006). Although not necessary to understand the main text, this theory is by now a standard tool to study the dependence of the statistical properties of a system on a parameter or external influences. Hence, it is natural to add it for completeness.

Appendix D contains the basics of projective cones and Hilbert metrics. Part of this material can also be found in other books (e.g., Viana (1997)) but we add it for completeness. Also we emphasize the connection with the Banach space approach, which is not common knowledge.

Appendix E contains hints to the solutions of the problems in the text. We strongly recommend that the readers look at this appendix only as a last resort and only after some hard thinking in order to find a solution.

To conclude we would like to thank all the people that provided us with helpful suggestions related to this text. They are too many to name but, at least, we must mention Viviane Baladi, Oliver Butterley, Jacopo de Simoi, Dmitry Dolgopyat and Sébastien Gouëzel.

Contents

1 Expanding maps

We start by discussing *smooth expanding maps*. By a smooth expanding map we mean a map $f \in C^r(\mathbb{T},\mathbb{T})$,[1] $r \geqslant 2$, such that $\inf_x |f'(x)| \geqslant \lambda_\star > 1$. Clearly $(f,\mathbb{T})$ is a topological, actually differentiable, dynamical system. Our first goal is to view it as a measurable dynamical system, hence we need to select an invariant probability measure.

1.1 Invariant measures

Deterministic systems often have a lot of invariant measures. In particular, to any periodic orbit is associated an invariant measure (the average along the orbit). Given such plentiful possibilities, we need a criteria to select relevant invariant measures. A common choice is to consider measures that can be obtained by pushing forward a measure absolutely continuous with respect to Lebesgue.

More precisely, let $d\mu = h(x)dx$, $h \in L^1(\mathbb{T},\mathrm{Leb})$ and define,[2] for all $\varphi \in$

[1] By $\mathbb{T}$ we mean the one dimensional torus $\mathbb{R}/\mathbb{Q}$. While C^r, as usual, denotes the set of functions r times differentiable with continuous derivatives.

[2] Obviously, Leb stands for the Lebesgue measure.

$\mathcal{C}^0(\mathbb{T}, \mathbb{R})$, the average

$$\mu(\varphi) = \int_{\mathbb{T}} \varphi(x)\mu(dx)$$

and the push-forward

$$f_*\mu(\varphi) = \mu(\varphi \circ f).$$

Note that if μ is a probability measure (i.e., $\mu(1) = 1$ and $h \geqslant 0$ implies $\mu(h) \geqslant 0$), then $f_*\mu$ is also a probability measure. Then

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \mu \right\}_{n \in \mathbb{N}}$$

is a weakly compact[3] set, hence it has accumulation points. On can easily check that such accumulation points are invariant measures for f, that is fixed points for f_* (this is, essentially, the proof of the Krylov–Bogoliubov Theorem). It is then natural to study the fixed points of f_*.

To this end we need to understand a bit better the action of f_*.

For example, if μ is a delta function supported on a point $\bar{x}$, that is $\mu(\varphi) = \varphi(\bar{x}) = \delta_{\bar{x}}(\varphi)$, then $f_*\mu(\varphi) = \mu(\varphi \circ f) = \varphi(f(\bar{x})) = \delta_{f(\bar{x})}(\varphi)$, so the action of f_* on atomic measures reproduces the dynamics of the map f on points. However, for measures absolutely continuous with respect to Lebesgue, the situation is different. Assume, that $\{p_1, \ldots, p_m\}$ is an open partition of $\mathbb{T}$ (that is, each p_i is an open interval, $\cup_{i=1}^m \bar{p}_i = \mathbb{T}$ and $p_i \cap p_j \neq \emptyset$ implies $i = j$) and that $f : p_i \to \mathbb{T} \setminus \{0\}$ is one-to-one and onto. Then we can set $\phi_{p_i} = f|_{p_i}^{-1} : \mathbb{T} \setminus \{0\} \to p_i$. Then

$$f_*\mu(\varphi) = \mu(\varphi \circ f) = \int_{\mathbb{T}} \varphi \circ f(x) \cdot h(x)dx = \sum_{i=1}^m \int_{p_i} \varphi \circ \phi_{p_i}^{-1}(x) \cdot h(x)dx$$

$$= \sum_{i=1}^m \int_{\mathbb{T}} \varphi(y) \cdot \frac{h(\phi_{p_i}(y))}{\left|f'(\phi_{p_i}(y))\right|} dy = \int_{\mathbb{T}} \varphi(x) \sum_{y \in f^{-1}(x)} \frac{h(y)}{|f'(y)|} dx.$$

In other words, $\frac{d(f_*\mu)}{d\,\mathrm{Leb}} = \mathcal{L}h$ where

$$\mathcal{L}h(x) = \sum_{f(y)=x} \frac{h(y)}{|f'(y)|}. \tag{1.1.1}$$

[3]Recall that in the weak topology, $\mu_n \to \mu$ if and only if $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$ for all $\varphi \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$.

The operator $\mathcal{L}$ is called the *(Ruelle) transfer operator*. Of course, to properly define such an operator we must specify on which space it acts. Since

$$\int |\mathcal{L}h(x)|dx \leqslant \int \mathcal{L}|h|(x)dx = \int 1 \circ f(x)|h(x)|dx = \int |h(x)|dx,$$

it follows that $\mathcal{L}$ is well defined as an operator from $L^1(\mathbb{T}, \text{Leb})$ to itself; moreover it is a contraction on $L^1(\mathbb{T}, \text{Leb})$. In addition, if $d\mu_* = h_* dx$ is an invariant measure, then

$$h_* dx = d\mu_* = df_*\mu_* = \mathcal{L}h_* dx,$$

that is $\mathcal{L}h_* = h_*$. Conversely, if $\mathcal{L}h_* = h_*$, then

$$d\mu_* = h_* dx = \mathcal{L}h_* dx = df_*\mu_*$$

so that $d\mu_* = h_* dx$ is an invariant measure.

We have thus reduced the problem of studying the invariant measures absolutely continuous with respect to Lebesgue to the problem of studying the operator $\mathcal{L}$, more precisely the eigenspace associated to the eigenvalue one.
We want thus to investigate the spectral theory of the operator $\mathcal{L}$. Unfortunately, the spectrum of $\mathcal{L}$ on L^1 turns out to be the full unit disk, a not very useful fact, e.g. see Keller (1984) or Collet and Isola (1991).

1.2 Lasota–Yorke like inequalities and physical measures

As before, let $f \in \mathcal{C}^r(\mathbb{T}, \mathbb{T})$, $r \geqslant 2$, such that $\inf_x |f'(x)| \geqslant \lambda_\star > 1$. Following Lasota–Yorke, we look then at the action of $\mathcal{L}$ on $W^{1,1}$: [4]

$$\frac{d}{dx}\mathcal{L}h = \mathcal{L}\left(\frac{h}{f'}\right) - \mathcal{L}\left(h\frac{f''}{(f')^2}\right). \tag{1.2.1}$$

The above implies the so-called *Lasota–Yorke inequalities*

$$\begin{aligned} &\|\mathcal{L}h\|_{L^1} \leqslant \|h\|_{L^1} \\ &\|(\mathcal{L}h)'\|_{L^1} \leqslant \lambda_\star^{-1}\|h'\|_{L^1} + D\|h\|_{L^1}, \end{aligned} \tag{1.2.2}$$

[4] Recall that $g \in W^{1,1}$ if $g \in L^1$ and $g' \in L^1$. Note that the formula follows by differentiating Equation (1.1.1), using the chain rule and the formula for the derivative of the inverse function.

where $D = \left\| \frac{f''}{(f')^2} \right\|_\infty$. Such inequalities imply that $\mathcal{L}$ is well defined as an operator from $W^{1,1}$ to itself. In addition, when acting on $W^{1,1}$ it is a quasi-compact operator, see Theorem 1.1 for the exact statement. That is, the spectrum $\sigma_{W^{1,1}}(\mathcal{L}) \subset \{z \in \mathcal{C} \ : \ |z| \leqslant 1\}$ while the essential spectrum is strictly smaller: ess-$\sigma_{W^{1,1}}(\mathcal{L}) \subset \{z \in \mathcal{C} \ : \ |z| \leqslant \lambda_\star^{-1}\}$.[5] To illustrate the above facts, let us consider first the special case in which the distortion $D = \|\frac{f''}{(f')^2}\|_{L^\infty}$ is small, more precisely $\lambda_\star^{-1} + D < 1$.

Note that, if $\mathrm{Leb}(h) = 0$, then also $\mathrm{Leb}(\mathcal{L}h) = 0$, hence the space $\mathbb{V} = \{h \in L^1 \ : \ \mathrm{Leb}(h) = 0\}$ is invariant under $\mathcal{L}$. Also, if $h \in \mathbb{V}$, then, since $W^{1,1} \subset \mathcal{C}^0$, by the mean value theorem for integrals there must exist x_* such that $h(x_*) = 0$, thus

$$\|h\|_{L^1} = \int_{\mathbb{T}} |h(x)| = \int_{\mathbb{T}} \Big| \int_{x_*}^{x} h'(y) \Big| \leqslant \|h'\|_{L^1}.$$

Next, let us define the norm $\|h\|_{W^{1,1}} = \|h'\|_{L^1} + a\|h\|_{L^1}$ for some $a > 0$ to be chosen shortly.[6] Accordingly, for $h \in \mathbb{V}$, inequality (1.2.2) implies

$$\begin{aligned} \|\mathcal{L}h\|_{W^{1,1}} &\leqslant \lambda_\star^{-1}\|h'\|_{L^1} + (D+a)\|h\|_{L^1} \leqslant (\lambda_\star^{-1} + D + a)\|h'\|_{L^1} \\ &\leqslant (\lambda_\star^{-1} + D + a)\|h\|_{W^{1,1}}. \end{aligned} \tag{1.2.3}$$

We can then choose a such that $\nu := \lambda_\star^{-1} + D + a < 1$, which implies that $\mathcal{L}$ is a strict contraction on $\mathbb{V}$, that is $\sigma_{W^{1,1}}(\mathcal{L}|_{\mathbb{V}}) \subset \{z \in \mathbb{C} \ : \ |z| \leqslant \nu\}$. Note that the dual operator $\mathcal{L}^*$ satisfies $\mathcal{L}^* \,\mathrm{Leb} = \mathrm{Leb}$, hence $1 \in \sigma(\mathcal{L}^*)$ and then $1 \in \sigma(\mathcal{L})$. Thus we have that there exists $h_* \in L^1$ such that $\mathcal{L}h = h_* \,\mathrm{Leb}(h) + Qh$, where $\|Q\|_{W^{1,1}} \leqslant \nu$ and $\mathrm{Leb}\, Q = Qh_* = 0$. Hence, (1.2.3) implies that, for each $h \in W^{1,1}$,

$$\left\| \mathcal{L}^n h - h_* \int h \right\|_{W^{1,1}} = \left\| \mathcal{L}^n \left(h - h_* \int h \right) \right\|_{W^{1,1}} \leqslant \nu^n \left\| h - h_* \int h \right\|_{W^{1,1}}$$

We have just proven that $d\mu_* = h_*(x)dx$ is the only invariant measure of f absolutely continuous with respect to Lebesgue.[7]

As already mentioned, the above spectral decomposition, and hence the uniqueness of the invariant measure absolutely continuous with respect to Lebesgue,

[5] See Appendix B.1 for a definition of the essential spectrum.

[6] Note that all such norms are equivalent, so the choice of a special value of a is only a matter of convenience.

[7] To complete the argument, use that $W^{1,1}$ is dense in L^1.

holds in much greater generality, in particular for each $f \in \mathcal{C}^2$ such that $|f'| \geqslant \lambda_\star > 1$, due to the following abstract theorem, see Appendix B for a full proof, requiring only a basic knowledge of functional analysis, of the following result, there corresponding to Theorem B.14,[8] and of a more general statement as well, Theorem B.15.

Theorem 1.1 (Hennion (1993))**.** *Let $\mathcal{B} \subset \mathcal{B}_w$ be two Banach spaces, $\|\cdot\|$ and $\|\cdot\|_w$ being the respective norms, satisfying $\|\cdot\|_w \leqslant \|\cdot\|$. In addition, let $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ be a linear operator such that there exists $M, C, \theta > 0$ and $n_0 \in \mathbb{N}$ such that $\mathcal{L}^{n_0} : \mathcal{B} \to \mathcal{B}_w$ is a compact operator and for each $n \in \mathbb{N}$ and $v \in \mathcal{B}$,*

$$\|\mathcal{L}^n v\|_w \leqslant CM^n \|v\|_w$$
$$\|\mathcal{L}^n v\| \leqslant C\theta^n \|v\| + CM^n \|v\|_w,$$

then the spectral radius of $\mathcal{L}$ is bounded by M and its essential spectral radius is bounded by θ.

Remark 1.2. *In the following we will mostly use the above Theorem when $M = 1$. Also, the compactness of the operator (for each $n_0 \in \mathbb{N}$) will often follow by checking that the unit ball in $\mathcal{B}$, $\{v \in \mathcal{B} \ : \ \|v\| \leqslant 1\}$, is relatively compact in $\mathcal{B}_w$. Finally, if one can prove that there exist eigenvalues outside the essential spectrum (as we have done before), then Theorem 1.1 implies that the operator is* quasi compact *(that is, the maximal part of the spectrum consists of a point spectrum).*

Let us see you Theorem 1.1 can be used to study the statistical properties of expanding maps.

Proposition 1.3. *For each $f \in \mathcal{C}^r(\mathbb{T}, \mathbb{T})$, $r \geqslant 2$, with $\inf_x |f'(x)| \geqslant \lambda_\star > 1$, there exists $h_* \in W^{1,1}$, $h_* > 0$, such that, for all $\alpha > 0$ there exists $\nu_\alpha \in (\lambda_\star^{-1}, 1)$ such that, for all $h \in \mathcal{C}^\alpha$ and $\varphi \in \mathcal{C}^0$, we have*

$$\left| \int_{\mathbb{T}} \varphi \circ f^n h - \int_{\mathbb{T}} \varphi h_* \int_{\mathbb{T}} h \right| \leqslant \|\varphi\|_{\mathcal{C}^0} \|h\|_{\mathcal{C}^\alpha} \nu_\alpha^n.$$

Proof. By Equation (1.2.2) and Theorem 1.1 we know that $\sigma_{W^{1,1}}\{\mathcal{L}\}$ has only finitely many eigenvalues of finite multiplicity on the circle $\{|z| = 1\}$ and that there exists $\nu \in (\lambda_\star^{-1}, 1)$ such that the rest of the spectrum is contained in the disk $\{|z| < \nu\}$.

[8] In fact, Theorem B.14 is a bit more general than Theorem 1.1.

It follows that there exists a finite set $\Theta \subset [0, 2\pi)$ such that we can write[9]

$$\mathcal{L} = \sum_{\theta \in \Theta} e^{i\theta} \Pi_\theta + Q$$

where Π_θ are finite rank operators such that $\Pi_\theta \Pi_{\theta'} = \delta_{\theta,\theta'} \Pi_\theta$, $\Pi_\theta Q = Q \Pi_\theta = 0$ and the spectral radius of Q is smaller than ν. Moreover, since $1 \in \mathcal{L}^*$, we have $\{0\} \in \Theta$. It follows that, for all $\theta \in \Theta$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-ik\theta} \mathcal{L}^k = \sum_{\theta' \in \Theta} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{ik(\theta'-\theta)} \Pi_\theta + e^{-ik\theta} Q^k = \Pi_\theta. \tag{1.2.4}$$

Let $h_* = \Pi_0 1$, then $\mathcal{L} h_* = \mathcal{L} \Pi_0 1 = \Pi_0 1 = h_*$. Note that, by the above equation $h_* \geqslant 0$. Since $h_* \in W^{1,1} \subset \mathcal{C}^0$, we have that if there exists $\bar{x} \in \mathbb{T}$ such that $h_*(x) = 0$, then

$$0 = h_*(\bar{x}) = (\mathcal{L}^n h_*)(\bar{x}) = \sum_{y \in f^{-n}(\bar{x})} \frac{1}{|(f^n)'(y)|} h_*(y).$$

Thus $h_*(y) = 0$ for all $y \in f^{-n}(\bar{x})$, $n \in \mathbb{N}$. But since the map is expanding, for each interval I there exists n such that $f^n(I) = \mathbb{T}$, hence the preimages of $\bar{x}$ are dense and since h_* is continuous this would imply $h_* \equiv 0$, which is not possible. It follows that $h_* > 0$.

On the other hand, if $\Pi_\theta h = e^{i\theta} h$, then by Equation (1.2.4) we have

$$|h| \leqslant \Pi_0 |h|$$

$$\int_{\mathbb{T}} \Pi_0 |h| - |h| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\mathbb{T}} \mathcal{L}^k |h| - \int_{\mathbb{T}} |h| = 0,$$

from which it follows that $\mathcal{L}|h| = \Pi_0 |h| = |h|$. But then we can choose β such that $h_* - \beta |h| \geqslant 0$ and there exists $\bar{x}$ such that $h_*(\bar{x}) - \beta |h|(\bar{x}) = 0$. Then, by the same argument used above, it must be $h_* = \beta |h|$, which means that Π_0 is a rank one projector. Accordingly, it must be that $h = e^{i\varphi} h_*$ for some $\varphi \in \mathcal{C}^0((0,1), \mathbb{R})$. This implies

$$e^{i\theta + i\varphi} h_* = e^{i\theta} h = \mathcal{L} h = \mathcal{L} e^{i\varphi} h_*$$

[9] Remark that there cannot be Jordan blocks with eigenvector of modulus one, since this would imply that $\|\mathcal{L}^n\|$ grows polynomially, contrary to Equation (1.2.2).

that is

$$\mathcal{L}\left[h_* - e^{i(\varphi-\theta-\varphi\circ f)}h_*\right] = 0.$$

Integrating and taking the real part we have

$$\int_{\mathbb{T}} h_*[1-\cos(\varphi-\theta-\varphi\circ f)] = 0$$

which is possible only if $\varphi-\theta-\varphi\circ f = 0$. But multiplying by h_* and integrating again we have

$$\theta = \int_{\mathbb{T}} \theta h_* = \int_{\mathbb{T}} (\varphi-\theta-\varphi\circ f)h_* = 0.$$

which shows that 1 is the only peripheral eigenvalue and it is simple. This proves that for all $h \in W^{1,1}$ and $\varphi \in \mathcal{C}^0$, we have

$$\left|\int_{\mathbb{T}} \varphi\circ f^n h - \int_{\mathbb{T}} \varphi h_* \int_{\mathbb{T}} h\right| \leqslant \|\varphi\|_{\mathcal{C}^0}\|h\|_{W^{1,1}} e^{-\nu n}.$$

The Proposition then follows by a standard approximation argument.

Problem 1.4. *Complete the proof of the Proposition.*

As an alternative, you can see Baladi (2000) for a more exhaustive discussion. □

Remark 1.5. *The proof of the above theorem shows that ν_α is either the second largest eigenvalue or it is arbitrarily close to $\lambda_\star^{-1}$. It is then natural to ask the question if there exist maps that have discrete eigenvalues larger than $\lambda_\star^{-1}$, beside* 1. *The answer is affirmative, see Keller and Rugh (2004) for details.*

Problem 1.6. *Derive further* (1.2.1) *to obtain a Lasota–Yorke inequality with respect to the norms $W^{p,1}$, $W^{p-1,1}$, $p \leqslant r-1$. Show then that the essential spectral radius of $\mathcal{L}$ when acting on $W^{p,1}$ is bounded by $\lambda_\star^{-p}$.*

The previous problem shows that our game can be played with many norms. This is an important fact since, on the one hand, different norms provide different types of convergence and, on the other hand, certain norms are better suited to capture particular features of the problems. To get a better idea of the possibilities, solve the next problem.

Problem 1.7. *For a $\mathcal{C}^r$ expanding map, obtain a Lasota–Yorke inequality with respect to the norms $\mathcal{C}^p$, $\mathcal{C}^{p-1}$, $1 \leqslant p \leqslant r-1$. Show then that the essential spectral radius of $\mathcal{L}$ when acting on $\mathcal{C}^p$ is bounded by $\lambda_\star^{-p}$.*

An interesting consequence of the above analysis is that smooth expanding maps admit a unique *physical measure*. A measure μ is a physical measure if there exists a measurable set A (called the *basin of attraction*) of positive Lebesgue measure such that, for all $\varphi \in \mathcal{C}^0$ and $x \in A$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^n(x) = \mu(\varphi).$$

Problem 1.8. *Show that if there exists $h_* \in L^1$, $h_* > 0$, such that for all $h \in L^1$ we have $\lim_{n\to\infty} \mathcal{L}^n h = h_* \int h$,[10] then $d\mu_* = h_*(x)dx$ is the unique physical measure of the system and the basin of attraction is the whole space, except for a zero Lebesgue measure set.*

The above problem shows that, for the uniqueness of the physical measure, the speed of convergence is immaterial. Yet, if one has estimates on the speed of convergence (as in our case), then it is possible to obtain a much more useful bound. To see this, for $\varphi \in \mathcal{C}^1(\mathbb{T}^1, \mathbb{C})$, let us set $\hat\varphi = \varphi - \mu_*(\varphi)$ and compute

$$\begin{aligned}
&\left\| \sum_{k=0}^{n-1} \varphi \circ f^k(x) - n\mu_*(\varphi) \right\|^2_{L^2(\mu_*)} = \sum_{k,j=0}^{n-1} \int \overline{\hat\varphi} \circ f^k(x) \cdot \hat\varphi \circ f^j(x) \cdot h_*(x)dx \\
&= \sum_{k=0}^{n-1} \int |\hat\varphi(x)|^2 \cdot h_*(x)dx + 2\sum_{k>j}^{n-1} \sum_{j=0}^{n-2} \int \overline{\hat\varphi} \circ f^{k-j}(x) \cdot \hat\varphi(x) \cdot h_*(x)dx \\
&= n\|\hat\varphi\|_{L^2(\mu)} + 2\sum_{l=1}^{n-1} (n-l) \int \overline{\hat\varphi} \circ f^l(x) \cdot \hat\varphi(x) \cdot h_*(x)dx \\
&= n\left[\|\hat\varphi\|_{L^2(\mu)} + 2\sum_{l=1}^{\infty} \int \overline{\hat\varphi} \circ f^l(x) \cdot \hat\varphi(x) \cdot h_*(x)dx \right] \\
&\quad - 2\sum_{l=n}^{\infty} n \int \overline{\hat\varphi}(x) \cdot \mathcal{L}^l(\hat\varphi \cdot h_*)(x)dx - 2\sum_{l=1}^{n-1} l \int \overline{\hat\varphi}(x) \cdot \mathcal{L}^l(\hat\varphi \cdot h_*)(x)dx.
\end{aligned} \tag{1.2.5}$$

[10] The limit is meant in the L^1 topology.

Note that[11]

$$|\mathcal{L}^l(\hat{\varphi}\cdot h_*)(x)| \leqslant h_*(x)\left|\int(\hat{\varphi}\cdot h_*)(x)dx\right| + \|Q^l(\varphi h_*)\|_{W^{1,1}}$$
$$= \|Q^l(\varphi h_*)\|_{W^{1,1}} \leqslant C_\#\|\varphi\|_{\mathcal{C}^1}\nu^l$$

for some $\nu < 1$. Thus the quantity in the last line of (1.2.5) is uniformly bounded in n and the quantity in the square bracket is well defined.
Accordingly,

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ f^k(x) - \mu_*(\varphi)\right\|^2_{L^2(\mu)} \leqslant C_\#\frac{\|\varphi\|_{\mathcal{C}^1}}{n}. \tag{1.2.6}$$

The above is a refinement, in the special case of expanding maps, of Von Neumann's Mean Ergodic Theorem. Indeed, Von Neumann's Theorem, together with the ergodicity of μ_*, implies that the left hand side of the equation (1.2.6) tends to zero but without any information on the speed of convergence. Since $h_* > 0$, it also provides an alternative solution to Problem 1.8. In addition it can be used to prove the almost sure convergence of the ergodic averages.[12] The latter follows also from the Birkhoff Ergodic Theorem since $h_* > 0$. Summarizing: the ergodic average converges Lebesgue almost everywhere to the average with respect to the unique invariant measure absolutely continuous with respect to Lebesgue. A natural question is: what is the exact speed of convergence?

1.3 Standard Pairs

Let us revisit what we have learned about smooth expanding maps of the circle using a different technique: *standard pairs*.

This tool is less powerful than the spectral decomposition of the transfer operator, but much more flexible; it is then instrumental in the study of less trivial systems. We present it in a very simplified manner and such a simplification is possible only because we treat very simple systems: smooth expanding maps. Once we fix some $a > 0$, a standard pair is a couple $\ell = (I, \rho)$ where $I = [\alpha, \beta] \subset \mathbb{T}$

[11] Here, and the following, we will use $C_\#, c_\#$ to mean a generic constant, depending only on the choice of f, which value can change from one occurrence to the next.

[12] Use the usual trick to study the sum in blocks of size 2^k.

and $\rho \in \mathcal{C}^1(I, \mathbb{R}_{\geqslant 0})$ such that $\rho \in \mathcal{C}_a(I)$, where

$$\mathcal{C}_a(I) := \left\{ \rho \in \mathcal{C}^0(I) \ : \ \rho \geqslant 0, \ \frac{\rho(x)}{\rho(y)} \leqslant e^{a|x-y|}, \forall x, y \in I \right\},$$

and $\int_I \rho = 1$. We fix some $\delta < 1/2$ and denote by $\mathfrak{L}_a$ the set of all possible such objects satisfying $\delta \leqslant |I| \leqslant 2\delta$.

To a standard pair $\ell = (I, \rho)$ is uniquely associated the probability measure.

$$\mu_\ell(\varphi) = \int_I \varphi(x)\rho(x)dx.$$

Remark 1.9. *For further use we call $\ell = (I, \rho)$, where $|I| \leqslant \delta$ and/or $\rho \in \mathcal{C}_b(I)$ for some $b > a$, a* prestandard pair.

Remark 1.10. *In this particular case we could have considered only the case $I = \mathbb{T}$, but this would not have illustrated the flexibility of the method nor prepared us for future developments.*

Lemma 1.11. *There exists $a_0 > 0$ such that, for all $a \geqslant a_0$ and $\ell \in \mathfrak{L}_a$ there exists $N \in \mathbb{N}$ and $\{\ell_i\}_{i=1}^N \subset \mathfrak{L}_a$ such that*

$$f_* \mu_\ell = \sum_{i=1}^N p_i \mu_{\ell_i},$$

where $\sum_i p_i = 1$.

Proof. Note that, if we choose 2δ small enough, then f is invertible on each interval I of length smaller than I. Hence, calling ϕ the inverse of $f|_I$.

$$f_* \mu_\ell(\varphi) = \int_{f(I)} \rho \circ \phi \cdot |\phi'| \cdot \varphi$$

Note that by hypothesis $f(I)$ is longer than $\lambda_\star$ times I. If it is longer than 2δ, then we can divide it into subintervals of length between δ and 2δ. Let $\{I_i\}$ denote the collection of such a partition of $f(I)$. Also, letting $p_i = \int_{I_i} \rho \circ \phi \cdot |\phi'|$ and $\rho_i = p_i^{-1} \rho \circ \phi \cdot |\phi'|$, we have

$$f_* \mu_\ell(\varphi) = \sum_i p_i \int_{I_i} \rho_i \cdot \varphi.$$

Note that $\int_{I_i} \rho_i = 1$ by construction and that $\rho_i \in \mathcal{C}_a(I_i)$ follows by the same computations done in Section 1.2. □

We have thus seen that convex combinations of standard pairs, which we will write $\{p_i, \ell_i\}$ (called *standard families*), are invariant under the dynamics. This is a different way to restrict the action of the transfer operator to a suitable class of measures. In fact, it is not so different from the previous one as (finite) standard families yield measures absolutely continuous with respect to Lebesgue and with densities that are piecewise $\mathcal{C}^1$.

1.3.1 Coupling

Given two measures μ, ν on two probability spaces X, Y, respectively, a *coupling* of μ, ν is a probability measure α on $X \times Y$ such that, for all $f \in \mathcal{C}^0(X, \mathbb{R})$ and $g \in \mathcal{C}^0(Y, \mathbb{R})$ we have

$$\int_{X\times Y} f(x)\alpha(dx, dy) = \mu(f)$$
$$\int_{X\times Y} g(y)\alpha(dx, dy) = \nu(g).$$

That is, the marginals of α are exactly μ and ν.

Problem 1.12. *Let X be a compact Polish*[13] *space, let d be the distance and consider the Borel σ-algebra on X. For each pair of probability measures μ, ν let $\mathcal{G}(\mu, \nu)$ be the set of couplings of μ and ν.*

1. *Show that $\mathcal{G}(\mu, \nu) \neq \emptyset$.*

2. *Show that*
$$\boldsymbol{d}(\mu, \nu) = \inf_{\alpha\in\mathcal{G}(\mu,\nu)} \int_{X^2} d(x, y)\alpha(dx, dy)$$
is a distance (called the Kantorovich distance in the space of measures).

3. *Show that the topology induced by $\boldsymbol{d}$ on the set of probability measures is the weak topology.*

4. *Discuss the cases $X = [0, 1]$, $d(x, y) = |x-y|$ and $X = [0, 1]$, $d_0(x, y) = 0$ iff $x = y$ and $d_0(x, y) = 1$ otherwise.*

See the end of Section 2.1 for a generalization of the distance $\boldsymbol{d}$.

[13] That is a complete, separable, metric space.

Decay of correlations via coupling

In this section we provide an alternative approach to exponential mixing for expanding maps based on the above mentioned ideas. Let us consider any two standard pairs $\ell, \widetilde{\ell} \in \mathfrak{L}_a$. First of all note that there exists $n_0 \in \mathbb{N}$ such that, for each I of length δ, $f^{n_0} I = \mathbb{T}^1$. We then consider the standard pairs $\{\ell_i = (I_i, \rho_i)\}$ and $\{\widetilde{\ell}_i = (\widetilde{I}_i, \widetilde{\rho}_i)\}$ into which, according to Lemma 1.11, we can decompose the measures $f_*^{n_0}\mu_\ell$ and $f_*^{n_0}\mu_{\widetilde{\ell}}$, respectively. Let $\pi_i, \widetilde{\pi}_i$ be the corresponding weights in the convex combination of the standard families.
Choose any of the intervals $\{I_i\}$, say I_0. There are, at most, three intervals $\widetilde{I}_i$, let us call them $\widetilde{I}_0, \widetilde{I}_1, \widetilde{I}_2$, whose union covers I_0. Then there must exist an interval $\widetilde{I}_i$, say $\widetilde{I}_0$, such that $J = \widetilde{I}_0 \cap I_0$ is an interval of length at least $\delta/3$. Let J be the central third of $\widetilde{J}$. We can write $I_0 = J_1 \cup J \cup J_2$ and $\widetilde{I}_0 = \widetilde{J}_1 \cup J \cup \widetilde{J}_2$. Note that the subintervals $J, J_i . \widetilde{J}_i$ are, by construction, of size at least $\delta/9$.

Next, we define $z_J = \int_J \rho_0$, $\rho_J = z_J^{-1}\rho_0$; $\widetilde{z}_J = \int_J \widetilde{\rho}_0$, $\widetilde{\rho}_J = \widetilde{z}_J^{-1}\widetilde{\rho}_0$; $z_{J_i} = \int_{J_i} \rho_0$, $\rho_{J_i} = z_{J_i}^{-1}\rho_0$; $z_{\widetilde{J}_i} = \int_{\widetilde{J}_i} \widetilde{\rho}_0$, $\widetilde{\rho}_{\widetilde{J}_i} = z_{\widetilde{J}_i}^{-1}\widetilde{\rho}_0$.

Note that (J, ρ_J), $(J, \widetilde{\rho}_J)$, (J_i, ρ_{J_i}) and $(\widetilde{J}_i, \widetilde{\rho}_{\widetilde{J}_i})$ are all prestandard pairs. Obviously, they will appear in the convex combination defining the measures $f_*^{n_0}\mu_\ell$ and $f_*^{n_0}\mu_{\widetilde{\ell}}$ with the weights $p_J = \pi_0 z_J$, $\widetilde{p}_J = \widetilde{\pi}_0 \widetilde{z}_J$, $p_i = \pi_0 z_{J_i}$ and $\widetilde{p}_i = \widetilde{\pi}_0 z_{\widetilde{J}_i}$ respectively.

For simplicity we rename our collection of intervals so that they become $\{J, I_i\}$ and $\{J, \widetilde{I}_i\}$ and, together with the corresponding densities that we rename ρ_J, ρ_i and $\widetilde{\rho}_J, \widetilde{\rho}_i$, form standard and prestandard pairs. Similarly, we rename the weights to read p_J, p_i and $\widetilde{p}_J, \widetilde{p}_i$. This allows us to write

$$f_*^n \mu_\ell(\varphi) = p_J \int \rho_J \varphi + \sum_i p_i \int_{I_i} \rho_i \varphi$$
$$f_*^n \mu_{\widetilde{\ell}}(\varphi) = \widetilde{p}_J \int \widetilde{\rho}_J \varphi + \sum_i \widetilde{p}_i \int_{\widetilde{I}_i} \widetilde{\rho}_i \varphi.$$

Note that there exists a fixed constant $c_0 > 0$ such that $\min\{p_J, \widetilde{p}_J\} \geqslant 4c_0$. In addition, by definition $\inf\{\rho_J, \widetilde{\rho}_J\} \geqslant e^{-a2\delta} \geqslant 1/2$, provided δ has been chosen

small enough. Accordingly, setting $\bar\rho = \frac{1}{|J|}$, we can write

$$\begin{aligned} f_*^n \mu_\ell(\varphi) =& c_0 \int_J \bar\rho\varphi + (p_J - 2c_0)\int_J \rho_J\varphi + c_0\int_J (2\rho_J - \bar\rho)\varphi \\ &+ \sum_i p_i \int_{I_i} \rho_i\varphi =: c_0\nu(\varphi) + (1-c_0)\nu_R(\varphi) \\ f_*^n \mu_{\tilde\ell}(\varphi) =& c_0 \int_J \bar\rho\varphi + (\tilde p_J - 2c_0)\int_J \tilde\rho_J\varphi + c_0\int_J (2\tilde\rho_J - \bar\rho)\varphi \\ &+ \sum_i \tilde p_i \int_{\tilde I_i} \tilde\rho_i\varphi =: c_0\nu(\varphi) + (1-c_0)\tilde\nu_R(\varphi), \end{aligned} \tag{1.3.1}$$

where $\nu, \nu_R, \tilde\nu_R$ are probability measures.[14]
We can then consider the coupling

$$\alpha_1(g) = c_0 \int \bar\rho(x) g(x,x) + (1-c_0)\nu_R \times \tilde\nu_R(g).$$

Problem 1.13. *For each $n \in \mathbb{N}$, calling $\bar\rho_n$ the density of $f_*^n\nu$, show that*

$$(f_* \times f_*)^n \alpha_1(g) = c_0 \int \bar\rho_n(x) g(x,x) + (1-c_0) f_*^n\nu_R \times f_*^n\tilde\nu_R(g).$$

Problem 1.14. *Show that there exists $n_1 \in \mathbb{N}$, such that both $f_*^{n_1}\nu_R$ and $f_*^{n_1}\tilde\nu_R$ admit a decomposition into standard families $\{p_i^1, \ell_i^1\}$ and $\{\tilde p_i^1, \tilde\ell_i^1\}$, respectively.*

The above Problem implies that, at time $\bar n = n_0 + n_1$, we can take any two standard pairs ℓ_i^1 and $\tilde\ell_j^1$, apply the same arguments used to derive Equation (1.3.1), and obtain

$$f_*^{n_0}\mu_{\ell_i^1} = c_0\nu_{i,j} + (1-c_0)\tilde\nu_R^{1,i,j} \ , \quad f_*^{n_0}\mu_{\tilde\ell_i^1} = c_0\nu_{i,j} + (1-c_0)\tilde\nu_R^{1,i,j} \ .$$

We can thus write $f_*^{n_0}\nu_R = \sum_{i,j} p_i^1 \tilde p_j^1 f_*^{n_0}\mu_{\ell_i^1}$ and $f_*^{n_0}\tilde\nu_R = \sum_{i,j} p_i^1 \tilde p_j^1 f_*^{n_0}\mu_{\tilde\ell_j^1}$ and, letting $\bar\rho_{i,j}$ be the density of the measure $\nu_{i,j}$, consider the coupling

$$\alpha_2(g) = \sum_{ij} p_i^1 \tilde p_j^1 c_0 \int_{J_{i,j}} \bar\rho_{i,j}(x) g(x,x) + (1-c_0)\nu_R^{1,i,j} \times \tilde\nu_R^{1,i,j}(g).$$

[14] Note that $f_*^n\mu_\ell(1) - c_0\nu(1) = 1 - c_0$, hence we have to renormalize by $1-c_0$ in order to have a probability measure.

Collecting the above considerations and recalling Problem 1.13, it follows that there exists a probability density $\bar{\rho}^1$ such that the measures $f_*^{2\bar{n}}\mu_\ell$ and $f_*^{2\bar{n}}\mu_{\tilde{\ell}}$ admit a coupling of the form

$$f_*^{n_1} \times f_*^{n_1}\alpha(g) = [c_0 + (1-c_0)c_0] \int \bar{\rho}^1(x) g(x,x) + (1-c_0)^2 f_*^{n_1} \nu_R^{1,i,j} \times f_*^{n_1} \tilde{\nu}_R^{1,i,j}(g).$$

But the above implies, using the discrete distance d_0 of Problem 1.12,

$$\boldsymbol{d}(f_*^{2\bar{n}}\mu_\ell, f_*^{2\bar{n}}\mu_{\tilde{\ell}}) \leqslant (1-c_0)^2.$$

By induction it then follows

$$\boldsymbol{d}(f_*^{k\bar{n}}\mu_\ell, f_*^{k\bar{n}}\mu_{\tilde{\ell}}) \leqslant (1-c_0)^k. \tag{1.3.2}$$

Remark 1.15. *Note that if $\mu = \sum_{i=1}^N p_i \mu_{\ell_i}$ is the measure associated to the standard family $\{p_i, \ell_i\}_{i=1}^N$, then it is absolutely continuous with respect to Lebesgue with density ρ given by*[15]

$$\rho(x) = \sum_i p_i \rho_i(x) \mathbb{1}_{I_i}(x) \leqslant \sum_i p_i e^a \leqslant e^a.$$

The above allows us to prove the following fact:

Theorem 1.16. *For each pair of measures μ, ν associated to standard families, and all observables $\varphi \in L^\infty$ we have,*

$$|f_*^n \mu(\varphi) - f_*^n \nu(\varphi)| \leqslant C e^{-cn} \|\varphi\|_{L^\infty}.$$

Proof. Let G be any coupling of $f_*^n\mu$ and $f_*^n\nu$, and let $g(x,y) = \varphi(x) - \varphi(y)$. Then

$$|f_*^n \mu(\varphi) - f_*^n \nu(\varphi)| = |G(g)| \leqslant G(d \cdot |g|) \leqslant \|g\|_\infty G(d) \leqslant 2\|\varphi\|_\infty G(d)$$

and the claim follows by Equation (1.3.2), taking the infimum over the couplings and setting $c = \bar{n}^{-1} \ln(1-c_0)^{-1}$. □

Thus, if we have a measure μ determined by a standard family it follows that $\{f_*^n\mu\}$ is a sequence of measures determined by standard families, hence it must be a Cauchy sequence (just apply the previous remark to μ and $f_*^n\mu$). If follows

[15] By $\mathbb{1}_J$ we mean the characteristic function of the set J.

that there exists a unique measure ν, with density $\rho \in L^\infty$ such that, for each standard family measure μ and measurable set A,

$$\lim_{n\to\infty} \mu(f^{-n}A) = \nu(A) = \nu(f^{-1}A).$$

Moreover it is easy to see that we can approximate any measure μ that is absolutely continuous with respect to Lebesgue by a sequence of measures $\{\mu_k\}$ that arise as standard families. It thus follows that the dynamical system $(\mathbb{T}, f, \nu)$ is mixing, i.e., for each measure μ absolutely with respect to Lebesgue (hence with respect to ν) we have

$$\lim_{n\to\infty} \mu(f^{-n}A) = \nu(A).$$

In particular, ν is the unique absolutely continuous invariant measure.

Remark 1.17. *Theorem 1.16 is equivalent to the proof of the existence of a spectral gap for the operator $\mathcal{L}$ established in Section 1.2. However, note that the standard pair method does not provide any further information on the spectrum. This is both its weakness and its strength.*

In the next section we discuss a further strategy to obtain similar results. Again, such a strategy only allows us to establish the equivalent of a spectral gap, yet it provides a sharper estimate on the size of the gap.

1.4 Projective cones and Hilbert metric

Projective metrics are widely used in geometry, not to mention the importance of their generalizations (e.g. Kobayashi metrics) for the study of complex manifolds, Isaev and Krantz (2000b). It may seem surprising that they play a major role. also in the study of statistical properties of dynamical systems,[16] e.g. see Dubois (2009), Ferrero and Schmitt (1988), Liverani (1995a,b), Liverani and Maume-Deschamps (2003), Rugh (2010), and Saussol (2000).

A quick introduction to the Hilbert metric can be found in Appendix D.

Problem 1.18. *Prove that for each $\sigma \in (\lambda_\star^{-1}, 1)$ and $a \geqslant D(\sigma - \lambda_\star^{-1})^{-1}$, setting*

$$\mathcal{C}_a = \left\{ h \in \mathcal{C}^1(\mathbb{T}, \mathbb{R}_{\geqslant 0}) \ : \ \frac{h(x)}{h(y)} \leqslant e^{a|x-y|} \right\},$$

[16] Actually, it is not too surprising since projective metrics provide a proof of the Perron–Frobenius theorem for matrices with positive elements, see Appendix D.3, and the transfer operator is a positive operator (that is, it maps positive functions in positive functions).

it holds true that

$$\mathcal{L}(\mathcal{C}_a) \subset \mathcal{C}_{\sigma a}.$$

Problem 1.19. *Prove that the diameter of $\mathcal{C}_{\sigma a}$ in the Hilbert metric of $\mathcal{C}_a$ is finite.*

Remark 1.20. *Thanks to the above two problems we could conclude by proving Theorem 1.21 using the cone $\mathcal{C}_a$. Yet, this gives a not-so-good estimate of the spectral gap. It is thus interesting to see how a more refined cone can be used to yield a better estimate.*

Consider a dynamical partition $\mathcal{P}_m$.[17] Let us define the convex cone

$$\mathcal{C}_{a,m} = \left\{ h \in C^{(0)}(\mathbb{T}) \;\middle|\; \mathbb{E}(h \mid \mathcal{F}_m) \geqslant 0;\ |h'|_1 \leqslant a \int_{\mathbb{T}} h \right\}. \tag{1.4.1}$$

Where $\mathcal{F}_m$ is the σ-algebra generated by the partition $\mathcal{P}_m$ and $\mathbb{E}(\cdot \mid \mathcal{F}_m)$ is the conditional expectation with respect to the Lebesgue measure. The first relevant fact consists in the following computation[18]

$$\begin{aligned} |(\mathcal{L}h)'|_1 &\leqslant \lambda_\star^{-1}|h'|_1 + B|h|_1 \leqslant \lambda_\star^{-1}|h'|_1 + B\int_{\mathbb{T}} h + B|h - \mathbb{E}(h \mid \mathcal{F}_m)|_1 \\ &\leqslant (\lambda_\star^{-1} + B\lambda_\star^{-m})|h'|_1 + B\int_{\mathbb{T}} h\,, \end{aligned}$$

for all $h \in \mathcal{C}_{a,m}$ since each $I \in \mathcal{P}_m$ satisfies $|I| \leqslant \lambda_\star^{-m}$.

The above means that if $\nu \in (\lambda_\star^{-1}, 1)$, $\lambda_\star^{-m} \leqslant \frac{\nu - \lambda_\star^{-1}}{B}$ and $a = 2B(1-\nu)^{-1}$, then $\mathcal{L}h$ satisfies the second condition defining the cone. What about the first condition?

$$\mathcal{L}^m h(x) \geqslant \sum_{y \in h^{-m}x} |D_y f^m|^{-1} \left\{ \mathbb{E}(h \mid \mathcal{F}_m)(y) - \int_{I(y)} |h'(\xi)| d\xi \right\}$$

where $I(y)$ is the element of $\mathcal{P}_m$ which contains the point y. To continue it is necessary to apply a standard type of argument in hyperbolic theory: a distortion

[17] By *dynamical partition* we mean a partition obtained in the following way. Let $\mathcal{P}_0$ be any partition such that f restricted to each element is one-to-one and the image of each of its elements is the whole circle. Then $\mathcal{P}_m = \bigvee_{i=0}^{m} f^{-i}\mathcal{P}_0$. It should be remarked that any sufficiently fine partition would do, see Liverani (1995b), and even a smooth partition of unity could be used. The special choice here is determined only for didactic reasons.

[18] Remember inequality (1.2.2).

estimate.

$$1 = \int_{\mathbb{T}} dx = \int_{I(y)} |D_\xi f^m| d\xi \geqslant \int_{I(y)} |D_y f^m| e^{-\sum_{i=1}^m |\ln |D_{f^i\xi} f| - \ln |D_{f^i y} f||}$$
$$\geqslant |I(y)||D_y f^m| e^{-(1-\lambda_\star^{-1})^{-1} B_0}$$

Let $D := e^{-(1-\lambda_\star^{-1})^{-1} B_0}$, then the above equations yield

$$\mathcal{L}^m h(x) \geqslant \sum_{I \in \mathcal{P}_m} \left\{ D \int_I h - \lambda_\star^{-m} \int_I |h'| \right\} \geqslant (D - \lambda_\star^{-m} a) \int_{\mathbb{T}} h \geqslant \frac{D}{2} \int_{\mathbb{T}} h \tag{1.4.2}$$

provided we choose m so that $D - 2\lambda_\star^{-m} a > 0$.

This means that, by choosing m such that $\lambda_\star^{-m} \leqslant \frac{D}{4B}(1-\nu)$, it holds that $\mathcal{L}^m \mathcal{C}_{a,m} \subset \mathcal{C}_{\sigma a, m}$ with $\sigma = \frac{1+\nu}{2} < 1$. In addition, it is easy to compute that[19]

$$\Delta := \operatorname{diam}(\mathcal{L}^m \mathcal{C}_{a,m}) \leqslant 2 \ln \left[2 \frac{1 + B + 2(1+D)\frac{\nu - \lambda_\star^{-1}}{1-\nu}}{D} \right] := 2 \ln \delta < \infty. \tag{1.4.3}$$

The estimate (1.4.3) can be used together with Theorem D.2 and Lemma D.4 to prove:

Theorem 1.21. *If $f : \mathbb{T} \to \mathbb{T}$ is twice differentiable and $|Df| \geqslant \lambda_\star > 1$, then there exists a unique invariant measure μ_*, absolutely continuous with respect to Lebesgue; moreover $h_* := \frac{d\mu_*}{dm} \in W^{1,1}$. The dynamical system $(\mathbb{T}, f, \mu_*)$ is mixing. In addition, there exists $\Lambda \in (0,1)$ such that for all measures μ absolutely continuous with respect to m such that $h := \frac{d\mu}{dm} \in W^{1,1}$, it holds that*

$$|\mu(\varphi \circ f^n) - \mu_*(v f \circ f^n)| \leqslant C_\# \Lambda^n \|h - h_*\|_{W^{1,1}} \|\varphi\|_{L^1}.$$

In addition,

$$\Lambda \leqslant \left[\tanh \frac{\Delta}{4} \right]^{\frac{1}{m}} = \left[\frac{\delta + 1}{\delta - 1} \right]^{\frac{1}{m}}.$$

[19] To see this, compute the distance of a generic element $h \in \mathcal{C}_{a,m}$ from 1. This is done by looking for λ, μ such that $\lambda \preceq f \preceq \mu$. This immediately yields $\lambda \leqslant \min\{\inf \mathbb{E}(h \mid \mathcal{F});\ \int_{\mathbb{T}} h - \frac{1}{a}|h'|_1\}$ and $\mu \geqslant \max\{\sup \mathbb{E}(h \mid \mathcal{F});\ \int_{\mathbb{T}} h + \frac{1}{a}|h'|_1\}$. Now if $h \in \mathcal{L}^m \mathcal{C}_{a,m}$, according to the above discussion it follows that $\lambda \leqslant \min\{\frac{D}{2};\ 1-\sigma\} \int_{\mathbb{T}} h := \alpha$ and $\mu \geqslant \max\{1 + a\lambda_\star^{-m} + B + D\lambda_\star^{-m} a;\ 1+\sigma\} \int_{\mathbb{T}} h := \beta$. Thus the distance between h and 1 is given by $\ln \frac{\beta}{\alpha}$. The diameter is then obviously less than twice such a distance.

Note that the bound for the contraction rate Λ it is now rather explicit. We do not insist on its actual value since the above bound is still too simplistic to be optimal. The goal here was only to emphasize the possibility to obtain explicit bounds.[20]

1.5 The Central Limit Theorem

Let $\varphi \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$ and set $\hat{\varphi} := \varphi - \mu_*(\varphi)$, then, by Equation (1.2.6), we know that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\varphi} \circ f^k(x) = 0 \quad \text{Leb}-\text{a.e.}.$$

Moreover, (1.2.6) suggests that $\frac{1}{n}\sum_{k=0}^{n-1} \hat{\varphi} \circ f^k(x)$ is of size $\mathcal{O}(n^{-\frac{1}{2}})$. It is then tempting to define

$$\Psi_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \hat{\varphi} \circ f^k.$$

The natural question is whether Ψ_n has a limit as $n \to \infty$. The answer depends on the meaning that we give to the word "limit". In fact, the answer may be positive only if we consider Ψ_n as a random variable with distribution $F_n(t) := \mu(\{x \ : \Psi_n(x) \leqslant t\})$.[21] Let us call $\mathbb{P}$ the associated probability. The goal of this section is to prove the following theorem.

Theorem 1.22. *Suppose that there does not exist $g \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$ such that $\hat{\varphi} = g - g \circ f$ (i.e., $\hat{\varphi}$ is not a continuous coboundary). Then there exists $\sigma, C > 0$ such that, calling $\mathbb{P}_{\mathcal{G}_\sigma}$ the probability distribution of a Gaussian random variable of zero average and variance σ, we have, for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, $|b-a| \leqslant 1$,*

$$|\mathbb{P}(\Psi_n \in [a,b]) - \mathbb{P}_{\mathcal{G}_\sigma}([a,b])| \leqslant C \left(\frac{e^{-\frac{a^2}{2\sigma^2}}}{n^{\frac{3}{10}}} + \frac{|b-a|}{n^{\frac{1}{2}}} \right)$$

[20] For the reader interested in sharp bounds see, e.g., Baladi and Young (1993, 1994), Keane, Murray, and Young (1998), Liverani (2001), Galatolo and Nisoli (2014), Galatolo, Nisoli, and Saussol (2015), Jenkinson, Pollicott, and Vytnova (2018).

[21] Recall that the initial measure μ has the form $d\mu = h dx$. Here, for simplicity, we assume $h \in \mathcal{C}^1$.

Remark 1.23. *Note that, if there exists $g \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$ such that $\hat{\varphi} = g - g \circ f$ (i.e., $\hat{\varphi}$* is a continuous coboundary*), then $\sigma = 0$ (see Lemma 1.28) and*

$$\Psi_n := \frac{1}{\sqrt{n}}(g - g \circ f^n).$$

Thus it converges uniformly to zero. Hence, the necessity of the assumption.

Remark 1.24. *Note that if $\hat{\varphi} = g - g \circ f$, then, if x belongs to a periodic orbit of period p, we have*

$$\sum_{k=0}^{p-1} \hat{\varphi} \circ f^k(x) = \sum_{k=0}^{p-1} g \circ f^k - \sum_{k=0}^{p-1} g \circ f^{k+1} = 0.$$

Hence, the assumption of Theorem 1.22 is checkable: it suffices to find a periodic orbit on which such a sum is not zero to verify the hypotheses of Theorem 1.22.

Remark 1.25. *Theorem 1.22 means that, if the precision of the instrument that performs the measure is compatible with the statistics, then the typical fluctuations in the measurements are of order $\frac{1}{\sqrt{n}}$ and Gaussian. This is well known by experimentalists who routinely assume that the result of a measurement is distributed according to a Gaussian.*[22]

Remark 1.26. *Note that Theorem 1.22 is sensitive to the size of the interval only if $|b - a| \geqslant C_{\#} n^{-\frac{1}{5}}$, to have a better resolution more work is needed. Also, if $\max\{|a|, |b|\} \geqslant C_{\#}\sqrt{\ln n}$ then $\mathbb{P}_{\mathcal{G}_\sigma}([a, b])$ is smaller than the error term, hence we do not obtain much information. If one wants to have a better knowledge on the tail of the distribution, then one has to study the* Large deviations*. These can in fact be studied by similar techniques, see Section 1.6.*

Be aware that the above result is far from optimal, it is intended only to give an idea of the results and techniques available. Sharper results can be obtained with more work (e.g. see Kasun and Liverani (2021) and references therein for more precise results).

The rest of the section is devoted to the proof of Theorem 1.22. The proof consists of several steps. We start by recalling the relation between the distribution

[22] Note however that our proof holds in a very special case that has little to do with a real experimental setting. To prove the analogous statement for a realistic experiment is a completely different ball game.

function F_n and the probability. The following Lemma holds in higher generality, see Varadhan (2001), but for the reader's convenience we provide a simple proof in our special case.

Lemma 1.27. *For each continuous function g holds*[23]

$$\mathbb{E}(g) := \mu(g(\Psi_n)) = \int_{\mathbb{R}} g(t) dF_n(t) \tag{1.5.1}$$

where the integral is a Riemann–Stieltjes integral.

Proof. We consider first the case $g \in \mathcal{C}^1_0$, then

$$\begin{aligned} \int_{\mathbb{R}} g dF_n &= -\int_{\mathbb{R}} F_n(t) g'(t) dt \\ &= -\int_{\mathbb{R}} dt \int_{\mathbb{T}^1} dx\, h(x) \mathbb{1}_{\{z\,:\,\Psi_n(z) \leqslant t\}}(x) g'(t). \end{aligned} \tag{1.5.2}$$

Applying Fubini yields

$$\begin{aligned} \int_{\mathbb{R}} g dF_n &= -\int_{\mathbb{T}^1} dx \int_{\mathbb{R}} dt\, h(x) \mathbb{1}_{\{z\,:\,\Psi_n(z) \leqslant t\}}(x) g'(t) \\ &= -\int_{\mathbb{T}^1} dx\, h(x) \int_{\Psi_n(x)}^{\infty} g'(t) dt = \int_{\mathbb{T}^1} dx\, h(x) g(\Psi_n(x)). \end{aligned}$$

The results for $g \in \mathcal{C}^0_0$ follows by density. To conclude note that (1.2.6) and Chebyshev's inequality imply

$$\begin{aligned} \mu(\{x \,:\, \Psi_n(x) \geqslant t\}) &\leqslant \int_{\mathbb{T}^1} dx\, h(x) \mathbb{1}_{\{z\,:\,\Psi_n(z) \geqslant t\}}(x) |\Psi_n(x)|^2 t^{-2} \\ &\leqslant \int_{\mathbb{T}^1} dx\, h(x) |\Psi_n(x)|^2 t^{-2} \leqslant t^{-2} C_{\#} \|\varphi\|_{\mathcal{C}^1}. \end{aligned}$$

Thus, if g and $\widetilde{g}_t$ differ only outside the set $\{|s| \geqslant t\}$, $\widetilde{g}_t \in \mathcal{C}^0_0$ and $\|\widetilde{g}_t\|_{\mathcal{C}^0} \leqslant \|g\|_{\mathcal{C}^0}$, by (1.5.2) we have

$$\left| \mu(g(\Psi_n)) - \int_{\mathbb{R}} \widetilde{g}_t(s) dF_n(s) \right| = |\mu(g(\Psi_n)) - \mu(\widetilde{g}_t(\Psi_n))| \leqslant \|g\|_{\mathcal{C}^0} t^{-2} C_{\#} \|\varphi\|_{\mathcal{C}^1}$$

and the Lemma follows by taking the limit for $t \to \infty$. □

[23] By $\mathbb{E}$ we mean the expectation with respect to the probability $\mathbb{P}$. So it is just a different notation (more probabilistic) for the expectation with respect to the measure $d\mu = h d\,\mathrm{Leb}$.

It is thus clear that if we can control the distribution F_n, we have a very sharp understanding of the probability to have small deviations (of order $\sqrt{n}$) from the limit.

This can be achieved in various ways. In the following, we choose to compute the *characteristic function*

$$\varphi_n(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} dF_n(t)$$

of the distribution F_n since this provides the strongest results, but see Liverani (1996) for a softer approach or De Simoi and Liverani (2015) and Dolgopyat (2005) for a more general approach. The characteristic function determines the distribution via the formula

$$F_n(b) - F_n(a) = \lim_{\Lambda\to\infty} \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{e^{-ia\lambda} - e^{-ib\lambda}}{i\lambda} \varphi_n(\lambda) d\lambda, \tag{1.5.3}$$

as can be seen in any basic book of probability theory, e.g. Varadhan (2001, 2007). In the case when there exists a density, that is an L^1 function f_n such that $F_n(b) - F_n(a) = \int_a^b f_n(t)dt$, then the formula above becomes simply

$$f_n(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\lambda} \varphi_n(\lambda) d\lambda, \tag{1.5.4}$$

and follows trivially from the inversion of the Fourier transform. Our next step is to find a convenient expression for φ_n. We follow a clever idea due to Nagaev (1957) and Guivarc'h and Hardy (1988). Recalling (1.5.1), we can write

$$\begin{aligned} \varphi_n(\lambda) &= \int_{\mathbb{T}^1} e^{i\lambda\Psi_n(x)} h(x) dx \\ &= \int_{\mathbb{T}^1} e^{i\frac{\lambda}{\sqrt{n}}\sum_{k=0}^{n-2}\hat{\varphi}\circ f^k} \circ f(x) \cdot e^{i\frac{\lambda}{\sqrt{n}}\varphi(x)} h(x) dx \\ &= \int_{\mathbb{T}^1} e^{i\frac{\lambda}{\sqrt{n}}\sum_{k=0}^{n-2}\hat{\varphi}\circ f^k(x)} \cdot \mathcal{L}\left(e^{i\frac{\lambda}{\sqrt{n}}\varphi} h\right)(x) dx. \end{aligned} \tag{1.5.5}$$

It is then natural to define, for each $\nu \in \mathbb{R}$, the operator

$$\mathcal{L}_\nu h(x) = \left[\mathcal{L}\left(e^{i\nu\varphi} h\right)\right](x). \tag{1.5.6}$$

Using such an operator we can rewrite (1.5.5) as

$$\begin{aligned}\varphi_n(\lambda) &= \int_{\mathbb{T}^1} e^{i\frac{\lambda}{\sqrt{n}}\sum_{k=0}^{n-2}\hat{\varphi}\circ f^k(x)}\cdot \mathcal{L}_{\frac{\lambda}{\sqrt{n}}}(h)(x)dx \\ &= \int_{\mathbb{T}^1} \mathcal{L}^n_{\frac{\lambda}{\sqrt{n}}}(h)(x)dx,\end{aligned} \tag{1.5.7}$$

where the last line is obtained by iterating the previous arguments.

To conclude we must understand the growth of $\mathcal{L}^n_{\frac{\lambda}{\sqrt{n}}}$. That is, we want to understand the spectrum of the operators $\mathcal{L}_\nu$ for moderately large ν. Since for $\nu = 0$ we know the spectrum, we can apply standard perturbation theory.

Lemma 1.28. *There exists $\nu_0, C_0 > 0$ and $\xi \in (0,1)$ such that, for all $\nu \in [0,\nu_0]$, we can write $\mathcal{L}_\nu = \lambda_\nu \Pi_\nu + Q_\nu$ where all the quantities are analytic in ν and* [24]

$$\Pi_\nu(\varphi) = h_\nu \ell_\nu(\varphi)\,, \forall \varphi \in W^{1,1} \quad \text{with } h_\nu \in W^{1,1}, \ell_\nu \in (W^{1,1})^*; \quad \ell_\nu(h_\nu) = 1$$

$$|\lambda_\nu - 1 - \frac{1}{2}\sigma^2\nu^2| \leqslant C_0\nu^3$$

$$\left\| \Pi_\nu - \Pi_0 - \nu\sum_{k=0}^{\infty}\mathcal{L}^k(\mathbb{1}-\Pi)\mathcal{L}_0'\Pi + \nu\sum_{k=0}^{\infty}\Pi\mathcal{L}_0'(\mathbb{1}-\Pi)\mathcal{L}^k(\mathbb{1}-\Pi)\right\|_{W^{1,1}} \leqslant C_0\nu^2$$

$$\sigma^2 = \int_{\mathbb{T}}\hat{\varphi}(x)^2 h_*(x)dx + 2\sum_{k=1}^{\infty}\int_{\mathbb{T}}\hat{\varphi}\circ f^k(x)\cdot\hat{\varphi}(x)\cdot h_*(x)dx$$

$$\|Q_\nu^n\|_{W^{1,1}} \leqslant C_0\xi^n,$$

where $\mathbb{1}$ is the identity operator and we have used $'$ for the derivative with respect to ν and set $\mathcal{L} = \mathcal{L}_0$, $\Pi = \Pi_0$.

In addition, $\sigma = 0$ iff there exists $g \in \mathcal{C}^0(\mathbb{T},\mathbb{R})$ such that $\hat{\varphi} = g - g\circ f$ (i.e., $\hat{\varphi}$ is a continuous coboundary*).*

Proof. The spectral decomposition $\mathcal{L}_\nu = \lambda_\nu\Pi_\nu + Q_\nu$, its analyticity and the bound on Q_ν follow by standard perturbation theory (see Appendix A.4 or, e.g., Kato (1995) if you want the general theory). Moreover, $\Pi_\nu^2 = \Pi_\nu$, $\mathcal{L}_\nu\Pi_\nu = \Pi_\nu\mathcal{L}_\nu = \lambda_\nu\Pi_\nu$ and $\Pi_\nu Q_\nu = Q_\nu\Pi_\nu = 0$. Recall that $\lambda_0 = 1$ and $\Pi_0 = h_* \otimes \text{Leb}$.

Next, we must Taylor expand in ν the various objects. First of all note that, since the projector $\Pi_0 = h_* \otimes \text{Leb}$ is a rank one operator, so is the projector Π_ν.

[24] By $(W^{1,1})^*$ we mean the dual space of $W^{1,1}$.

Hence, there exists a unique $h_\nu \in W^{1,1}$ with $\int_{\mathbb{T}} h_\nu(x)dx = 1$, in the range of Π_ν. Next, choose $\ell_\nu \in (W^{1,1})^*$ to have the same kernel as Π_ν and normalize it so that $\ell_\nu(h_\nu) = 1$; it follows that $\Pi_\nu(\varphi) = h_\nu \ell_\nu(\varphi)$. Moreover,

$$\mathcal{L}'_\nu \Pi_\nu + \mathcal{L}_\nu \Pi'_\nu = \lambda'_\nu \Pi_\nu + \lambda_\nu \Pi'_\nu.$$

Multiplying by Π_ν from the left, yields

$$\lambda'_\nu \Pi_\nu = \Pi_\nu \mathcal{L}'_\nu \Pi_\nu = \ell_\nu(\mathcal{L}'_\nu h_\nu)\Pi_\nu \tag{1.5.8}$$

which, since $\mathcal{L}'_\nu h = \mathcal{L}_\nu(i\hat{\varphi}h)$, gives

$$\lambda'_\nu = i\lambda_\nu \ell_\nu(\hat{\varphi}h_\nu)$$

and, in particular, $\lambda'_0 = 0$.

Next, setting $\widehat{\mathcal{L}}_\nu = \lambda_\nu^{-1}\mathcal{L}_\nu$, we have

$$(\mathbb{1} - \lambda_\nu^{-1} Q_\nu)(\mathbb{1} - \Pi_\nu)\Pi'_\nu = (\mathbb{1} - \widehat{\mathcal{L}}_\nu)\Pi'_\nu = \lambda_\nu^{-1}\left[\mathcal{L}'_\nu \Pi_\nu - \lambda'_\nu \Pi_\nu\right] = \\ = \lambda_\nu^{-1}(\mathbb{1} - \Pi_\nu)\mathcal{L}'_\nu \Pi_\nu$$

which implies

$$(\mathbb{1} - \Pi_\nu)\Pi'_\nu = \lambda_\nu^{-1} \sum_{k=0}^{\infty} \lambda_\nu^{-k} Q_\nu^k (\mathbb{1} - \Pi_\nu)\mathcal{L}'_\nu \Pi_\nu = \lambda_\nu^{-1} \sum_{k=0}^{\infty} \widehat{\mathcal{L}}_\nu^k (\mathbb{1} - \Pi_\nu)\mathcal{L}'_\nu \Pi_\nu. \tag{1.5.9}$$

Note that the above estimates imply that there exists $\nu_0 > 0$ such that the series is convergent for all $\nu \leqslant \nu_0$. Analogously, from $\Pi_\nu \mathcal{L}_\nu = \lambda_\nu \Pi_\nu$ we obtain

$$\Pi'_\nu(\mathbb{1} - \Pi_\nu) = \lambda_\nu^{-1} \sum_{k=0}^{\infty} \Pi_\nu \mathcal{L}'_\nu (\mathbb{1} - \Pi_\nu)\widehat{\mathcal{L}}_\nu^k (\mathbb{1} - \Pi_\nu). \tag{1.5.10}$$

Noticing that $\Pi'_\nu \Pi_\nu + \Pi_\nu \Pi'_\nu = \Pi'_\nu$, that is

$$\Pi'_\nu \Pi_\nu = (\mathbb{1} - \Pi_\nu)\Pi'_\nu,$$

implies $\Pi_\nu \Pi'_\nu \Pi_\nu = 0$ and $(\mathbb{1} - \Pi_\nu)\Pi'_\nu(\mathbb{1} - \Pi_\nu) = 0$. We can then write

$$\begin{aligned} \Pi'_\nu &= \Pi_\nu \Pi'_\nu \Pi_\nu + (\mathbb{1} - \Pi_\nu)\Pi'_\nu \Pi_\nu + \Pi_\nu \Pi'_\nu (\mathbb{1} - \Pi_\nu) + (\mathbb{1} - \Pi_\nu)\Pi'_\nu(\mathbb{1} - \Pi_\nu) \\ &= (\mathbb{1} - \Pi_\nu)\Pi'_\nu \Pi_\nu + \Pi_\nu \Pi'_\nu (\mathbb{1} - \Pi_\nu) \\ &= \lambda_\nu^{-1} \sum_{k=0}^{\infty} \widehat{\mathcal{L}}_\nu^k (\mathbb{1} - \Pi_\nu)\mathcal{L}'_\nu \Pi_\nu + \lambda_\nu^{-1} \sum_{k=0}^{\infty} \Pi_\nu \mathcal{L}'_\nu (\mathbb{1} - \Pi_\nu)\widehat{\mathcal{L}}_\nu^k (\mathbb{1} - \Pi_\nu). \end{aligned} \tag{1.5.11}$$

Finally, differentiating (1.5.8), we have

$$\lambda_\nu'' \Pi_\nu + \lambda_\nu' \Pi_\nu' = \Pi_\nu' \mathcal{L}_\nu' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu'' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu' \Pi_\nu'$$

which, multiplying both from left and right by Π_ν yields

$$\begin{aligned}\lambda_\nu'' \Pi_\nu &= \Pi_\nu \Pi_\nu' \mathcal{L}_\nu' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu'' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu' \Pi_\nu' \Pi_\nu \\ &= \Pi_\nu \Pi_\nu' (\mathbb{1} - \Pi_\nu) \mathcal{L}_\nu' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu'' \Pi_\nu + \Pi_\nu \mathcal{L}_\nu' (\mathbb{1} - \Pi_\nu) \Pi_\nu' \Pi_\nu.\end{aligned}$$

hence,

$$\lambda_\nu'' = \ell_\nu \big(\Pi_\nu' (\mathbb{1} - \Pi_\nu) \mathcal{L}_\nu' h_\nu + \mathcal{L}_\nu'' h_\nu + \mathcal{L}_\nu' (\mathbb{1} - \Pi_\nu) \Pi_\nu' h_\nu \big). \tag{1.5.12}$$

From the above and equations (1.5.9), (1.5.10) it follows

$$\lambda_0'' = -\int_{\mathbb{T}} \hat{\varphi}(x)^2 h_*(x) dx - 2 \sum_{k=1}^{\infty} \int_{\mathbb{T}} \hat{\varphi} \circ f^k(x) \hat{\varphi}(x) h_*(x) dx.$$

Note that (1.2.5) implies that $-\sigma^2 = \lambda_0'' < 0$, thus σ is well defined. We are left with the task of investigating the case $\sigma = 0$. Equation (1.2.5) implies that if $\sigma = 0$, then $\left\| \sum_{k=0}^{n-1} \hat{\varphi} \circ f^k(x) \right\|_{L^2(\mu)}$ is uniformly bounded in n. Accordingly it admits weakly convergent subsequences in L^2. Let $g \in L^2$ be an accumulation point, then for each $h \in W^{1,1}$ we have

$$\begin{aligned}\int g \circ f \cdot h \cdot h_* &= \lim_{j \to \infty} \int \sum_{k=1}^{n_j} \hat{\varphi} \circ f^k \cdot h \cdot h_* \\ &= -\int \hat{\varphi} \cdot h \cdot h_* + \lim_{j \to \infty} \int \sum_{k=0}^{n_j - 1} \hat{\varphi} \circ f^k \cdot h \cdot h_* + \int \hat{\varphi} \mathcal{L}^{n_j} (h \cdot h_*) \\ &= -\int \hat{\varphi} \cdot h \cdot h_* + \int g \cdot h \cdot h_*.\end{aligned}$$

Since $W^{1,1}$ is dense in L^2 it follows

$$\hat{\varphi} h_* = g h_* - g \circ f h_*,$$

where, without loss of generality, we can assume $\int g h_* = 0$.

It remains to prove that $g \in \mathcal{C}^0$. This follows from Livšic theory, see Livšic (1971a, 1972b), but let us provide a simple direct argument: Applying $\mathcal{L}$ to the last equation yields

$$\mathcal{L} \hat{\varphi} h_* = -(\mathbb{1} - \mathcal{L}) g h_*.$$

Since the above equation can be restricted to the space of zero average functions and $h_* > 0$ we can write

$$g = -\frac{1}{h_*}(\mathbb{1} - \mathcal{L})^{-1}\mathcal{L}\hat{\varphi}h_*,$$

and since $\hat{\varphi}h_* \in W^{1,1}$ we have $g \in W^{1,1}$. The claim follows recalling that $W^{1,1} \subset \mathcal{C}^0$. □

Using Lemma 1.28 and Equation (1.5.7) we can obtain the following result.

Theorem 1.29 (Central Limit Theorem)**.** *Suppose $\hat{\varphi}$ is not a continuous coboundary, then for each continuous function g we have*

$$\lim_{n\to\infty} \mathbb{E}(g(\Psi_n)) = \mathbb{E}_{\mathcal{G}_\sigma}(g),$$

where $\mathbb{E}_{\mathcal{G}_\sigma}$ is the expectation for a Gaussian random variable of zero average and variance σ.

Proof. For $|\lambda| \leqslant \nu_0\sqrt{n}$ we can use Lemma 1.28 and equation (1.5.7) to write

$$\varphi_n(\lambda) = e^{-\frac{\sigma^2\lambda^2}{2}+\mathcal{O}(1/\sqrt{n})} + \mathcal{O}(\xi^n). \tag{1.5.13}$$

Hence $\lim_{n\to\infty}\varphi_n(\lambda) = e^{-\frac{\sigma^2\lambda^2}{2}}$ which is the characteristic function of a Gaussian random variable of zero average and variance σ. The result follows since convergence of the characteristic functions implies weak convergence of the measures, see Varadhan (2001). □

The above result shows that our renormalized Birkhoff averages converge to a Gaussian random variable, yet in practice it is not very useful since it does not provide any information for the difference between Ψ_n and a Gaussian random variable when n is large, but finite. In the following we address this subtler problem.

It turns out that to have sharper results on the limiting distribution we need to control φ_n for larger λ. This is the meaning of the next Lemma.

Lemma 1.30. *For each $\nu \neq 0$ we have that the essential spectrum of $\mathcal{L}_\nu$ acting on $W^{1,1}$ is contained in $\{z \in \mathcal{C} : |z| \leqslant \lambda_\star^{-1}\}$ and $\sigma_{W^{1,1}}(\mathcal{L}_\nu) \subset \{z \in \mathcal{C} : |z| < 1\}$ provided $\hat{\varphi}$ is not a continuous coboundary.*

Proof. Since, for each $h \in \mathcal{C}^1$,

$$\|\mathcal{L}_\nu h\|_{L^1} \leqslant \|\mathcal{L}|h|\|_{L^1} \leqslant \|h\|_{L^1}$$
$$\frac{d}{dx}\mathcal{L}_\nu h = \mathcal{L}_\nu\left(\frac{h}{f'}\right) - \mathcal{L}_\nu\left(\frac{f''h}{(f')^2}\right) + i\nu\mathcal{L}(\hat{\varphi}'h)$$

we have the Lasota–Yorke inequality for the operator $\mathcal{L}_\nu$. Then Theorem 1.1 implies the inclusion $\sigma_{W^{1,1}}(\mathcal{L}_\nu) \subset \{z \in \mathcal{C} \,:\, |z| \leqslant 1\}$ and that the essential spectral radius is bounded by $\lambda_\star^{-1}$. Accordingly the spectral radius can equal one only if there exists $\theta \in \mathbb{R}$ and $h \in W^{1,1}$ such that $\mathcal{L}_\nu h = e^{i\theta}h$. But then $|h| \leqslant \mathcal{L}|h|$ which, integrating yields

$$0 \leqslant \int \mathcal{L}|h|(x) - |h|(x)dx = 0\,,$$

so that $\mathcal{L}|h| = |h|$. Since the eigenvalue one is simple for $\mathcal{L}$, it must be that $h(x) = e^{i\alpha_\nu(x)}h_*(x)$. As both h_ν and $h_* > 0$ are continuous, it follows that α_ν can be assumed to be a continuous function without loss of generality. In addition,

$$\mathcal{L}h_*(x) = h_*(x) = e^{-i\theta - i\alpha_\nu(x)}\mathcal{L}_\nu h(x) = \mathcal{L}\left(e^{-i\theta - i\alpha_\nu \circ f + i\alpha_\nu + i\nu\hat{\varphi}}h_*\right).$$

Taking the real part and integrating yields

$$0 = \int_{\mathbb{T}} \left[1 - \cos\left(\theta - \alpha_\nu \circ f(x) + \alpha_\nu(x) + \nu\hat{\varphi}(x)\right)\right] h_*(x)dx$$

which implies that there exists a function $N : \mathbb{T} \to \mathbb{Z}$ such that

$$\theta - \alpha_\nu \circ f(x) + \alpha_\nu(x) + \nu\hat{\varphi}(x) = 2N(x)\pi$$

Lebesgue almost surely. Hence N must be constant and, taking the average with respect to μ_*, it follows $2N\pi - \theta = 0$. Thus, dividing by ν, we see that $\hat{\varphi}$ is a continuous coboundary. □

Let $L \geqslant \nu_0 > 0$. By Lemma 1.30 we have that the spectral radius of $\mathcal{L}_{\frac{\lambda}{\sqrt{n}}}$, for $|\lambda| \in [\nu_0\sqrt{n}, L\sqrt{n}]$ is smaller than some $\gamma_L \in (0,1)$.[25] Thus, for $|\lambda| \in [\nu_0\sqrt{n}, L\sqrt{n}]$ we have that there exists $C_L > 0$ such that

$$|\varphi_n(\lambda)| \leqslant C_L\gamma_L^n. \tag{1.5.14}$$

[25] Indeed, the spectral radius is either smaller or equal than $\lambda_\star^{-1}$ or it is determined by the point spectrum, and hence varies continuously by standard perturbation theory.

While it is possible to obtain similar estimates for even larger λ, they are out of the scope of this note (see De Simoi and Liverani (2018, Appendix B) for details).

Unfortunately, our estimates do not allow us to use (1.5.3) to compute the distribution F_n. This problem can be bypassed in various ways, here we present what is probably the simplest solution: we smooth the density.

To this end let $\boldsymbol{Z}$ be a bounded, independent, zero average random variable so that $|\boldsymbol{Z}| \leqslant 1$ with smooth density $\psi \in \mathcal{C}^\infty$. We can then consider the random variable $\overline{\Psi}_{n,\varepsilon} = \Psi_n + \varepsilon \boldsymbol{Z}$ for some $\varepsilon > 0$. The random variable $\overline{\Psi}_{n,\varepsilon}$ admits a density, which we denote by $\mathcal{N}_{n,\varepsilon}$. In fact, denoting by $\widehat{\psi}$ the Fourier transform of ψ and using (1.5.4), we have

$$\begin{aligned}
\mathcal{N}_{n,\varepsilon}(y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\mathbb{E}(e^{i\lambda\overline{\Psi}_n})d\lambda \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda y}\mu(e^{i\lambda\Psi_n})\widehat{\psi}(\varepsilon\lambda)d\lambda \\
&= \frac{1}{2\pi}\int_{-\nu_0\sqrt{n}}^{\nu_0\sqrt{n}} e^{-i\lambda y}\left[e^{-\frac{\sigma^2\lambda^2}{2}+\mathcal{O}(1/\sqrt{n})}+\mathcal{O}(\xi^n)\right]\widehat{\psi}(\varepsilon\lambda)d\lambda \\
&\quad + \mathcal{O}(C_L\gamma_L^n) + \frac{1}{2\pi}\int_{|\lambda|\geqslant L\sqrt{n}} e^{-i\lambda y}\mu(e^{i\lambda\Psi_n})\widehat{\psi}(\varepsilon\lambda)d\lambda.
\end{aligned}$$

To conclude, recall that for all $p \in \mathbb{N}$, $|\widehat{\psi}(\nu)| \leqslant C_p\|\psi\|_{\mathcal{C}^{p+2}}|\nu|^{-p}$ for some $C_p > 0$. As an example let us choose $p = 5$. Thus, there exists $n_L \in \mathbb{N}$ such that, for all $n \geqslant n_L$,

$$\mathcal{N}_{n,\varepsilon}(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}} + \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{\varepsilon^5 L^4 n^2}\right).$$

In addition, note that

$$\mathbb{P}(\overline{\Psi}_{n,\varepsilon} \in [a+\varepsilon, b-\varepsilon]) \leqslant \mathbb{P}(\Psi_n \in [a,b]) \leqslant \mathbb{P}(\overline{\Psi}_{n,\varepsilon} \in [a-\varepsilon, b+\varepsilon]).$$

Hence, calling $\mathbb{P}_{\mathcal{G}_\sigma}$ the probability distribution of a Gaussian random variable of zero average and variance σ, we have

$$\begin{aligned}
\mathbb{P}(\Psi_n \in [a,b]) &\leqslant \int_{a-\varepsilon}^{b+\varepsilon} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}dy + |b-a|\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{\varepsilon^5 L^4 n^2}\right) \\
&\leqslant \mathbb{P}_{\mathcal{G}_\sigma}([a,b]) + \mathcal{O}\left(\varepsilon e^{-\frac{a^2}{2\sigma^2}}\right) + |b-a|\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{\varepsilon^5 L^4 n^2}\right)
\end{aligned}$$

Arguing similarly for the lower bond and choosing, for example, $\varepsilon = n^{-\frac{3}{10}}$ and $L = 1$ we have, for some $C > 0$,

$$|\mathbb{P}(\Psi_n \in [a,b]) - \mathbb{P}_{\mathcal{G}_\sigma}([a,b])| \leqslant C\left(\frac{e^{-\frac{a^2}{2\sigma^2}}}{n^{\frac{3}{10}}} + \frac{|b-a|}{n^{\frac{1}{2}}}\right),$$

which concludes the proof of Theorem 1.22.

Remark 1.31. *Note that, if we are not interested in the rate of convergence, then the information that we obtained on the spectral properties of $\mathcal{L}_v$ suffices to prove the Local Limit Theorem.*[26]

1.6 Large deviations

As discussed in Remark 1.26, Theorem 1.22 does not provide very good estimates for large deviations, e.g. deviations of the ergodic average from the expectation larger than $n^{-\alpha}$ for $\alpha < \frac{1}{2}$. In this section we provide the essentials on how to estimate such events.

Given $\varphi \in \mathcal{C}^1$, $n \in \mathbb{N}$ and $a \in \mathbb{R}_+$ let

$$\begin{aligned} A_{a,n}(\varphi) &:= \left\{ x \in \mathbb{T}^1 \ : \ \left| \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) - \mu_*(\varphi) \right| \geqslant a \right\} \\ \mathring{A}_{a,n}(\varphi) &:= \left\{ x \in \mathbb{T}^1 \ : \ \left| \frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) - \mu_*(\varphi) \right| > a \right\} \end{aligned} \tag{1.6.1}$$

By Problem 1.8 we have

$$\lim_{n\to\infty} \mathbb{P}(A_{a,n}(\varphi)) := \lim_{n\to\infty} \mu(A_{a,n}(\varphi)) = 0.$$

Our goal, in this section, is to compute more precisely the asymptotic of the probability $\mathbb{P}(A_{a,n}(\varphi))$.

Again, note that we can write $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) - \mu_*(\varphi) = \frac{1}{n}\sum_{k=0}^{n-1} \hat{\varphi} \circ f^k(x)$ where $\hat{\varphi} := \varphi - \mu_*(\varphi)$. Thus we can reduce the question to the study of

[26] One must use the usual trick to prove the Theorem first for functions with compactly supported Fourier transform and then extend the result by density.

zero average functions. Our goal is to prove the following Theorem. To state it we need to define the *rate function*:

$$\mathbb{I}(a) = - \sup_{\{\nu \in \mathcal{M}_f \,:\, \nu(\hat{\varphi}) \geq a\}} h_\nu(f) - \nu(\ln|f'|),$$

where $\mathcal{M}_f$ is the set of invariant probability measures invariant with respect to f, h_ν the Kolmogorov–Sinai entropy (see Katok and Hasselblatt (1995) for a description of the Kolmogorov–Sinai entropy and its properties), and where the sup takes the value $-\infty$ if the set $\mathcal{M}_f(a) := \{\nu \in \mathcal{M}_f \,:\, \nu(\hat{\varphi}) \geqslant a\}$ is empty.

Theorem 1.32 (Large Deviations)**.** *For each $a, \epsilon \geqslant 0$ the exists $n_0 \in \mathbb{N}$ and constants $c_{a,\epsilon}, C_a > 0$ such that, for all $n \geqslant n_0$,*

$$c_{a,\epsilon} e^{-(\mathbb{I}(a)+\epsilon)n} \leqslant \mathbb{P}(\mathring{A}_{a,n}(\varphi)) \leqslant \mathbb{P}(A_{a,n}(\varphi)) \leqslant C_a e^{-\mathbb{I}(a)n}.$$

The proof of the above Theorem is the content of the next three sections: in Section 1.6.1 we discuss the upper bound in terms of seemingly different rate functions $\widetilde{I}$ and $\mathbb{J}$, see Equation (1.6.14). In Section 1.6.2 we discuss the lower bound in terms of the rate function $\mathbb{J}$, and, finally, in Section 1.6.3 we show that $\mathbb{J} = \mathbb{I}$, hence concluding the theorem.

1.6.1 Large deviations. Upper bound

Note that it suffices to study the set

$$A^+_{a,n}(\varphi) := \left\{ x \in \mathbb{T}^1 \,:\, \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) - \mu_*(\varphi + a) \geqslant 0 \right\}.$$

since $A_{a,n}(\varphi) = A^+_{a,n}(\varphi) \cap A^+_{a,n}(-\varphi)$.
On the other hand, setting $\hat{\varphi} := \varphi - \mu_*(\varphi)$, for each $\lambda \geqslant 0$ we have

$$\mu(A^+_{a,n}(\varphi)) = \mu(\{x \,:\, e^{\lambda \sum_{k=0}^{n-1} (\hat{\varphi} \circ f^k(x) - a)} \geqslant 1\}) \leqslant e^{-n\lambda a} \mu(e^{\lambda \sum_{k=0}^{n-1} \hat{\varphi} \circ f^k})$$

Accordingly, arguing exactly as in Equations (1.5.5), (1.5.6) and (1.5.7) (and recalling that $d\mu = h\, d\,\mathrm{Leb}$),

$$\mu(A^+_{a,n}(\varphi)) \leqslant e^{-n\lambda a}\, \mathrm{Leb}(\mathcal{L}^n_\lambda h) \tag{1.6.2}$$

where we have defined the operator $\mathcal{L}_\lambda g := \mathcal{L}(e^{\lambda \hat{\varphi}} g)$, $\mathcal{L}$ being the Transfer operator of the map f.

Lemma 1.33. *For each $\lambda \in \mathbb{R}$ the operator $\mathcal{L}_\lambda$, acting on $\mathcal{C}^1$, has a simple maximal eigenvalue, i.e. is of Perron–Frobenius type. Accordingly, $\mathcal{L}_\lambda = \alpha_\lambda \Pi_\lambda + Q_\lambda$, where $\alpha_\lambda > 0$, Π_λ is rank one, $\Pi_\lambda^2 = \Pi_\lambda$, $\Pi_\lambda Q_\lambda = Q_\lambda \Pi_\lambda = 0$ and $\|Q_\lambda^n\| \leqslant C_\lambda \beta_\lambda^n$ for some $C_\lambda > 0$ and $\beta_\lambda < \alpha_\lambda$. Also, $\alpha_\lambda, \Pi_\lambda$ and Q_λ are analytic in λ.*

Proof. Consider the cone $\mathcal{C}_a = \{h \in \mathcal{C}^1 \ : \ h \geqslant 0, \ \frac{|h'(x)|}{h(x)} \leqslant a\}$. Computing as in the derivation of Equation (1.1.1), we see that, given $\sigma \in (\lambda^{-1}, 1)$, for each $\lambda \in \mathbb{R}$ there exists $a_\lambda > 0$ such that, for all $a \geqslant a_\lambda$, $\mathcal{L}_\lambda \mathcal{C}_a \subset \mathcal{C}_{\sigma a}$. The fact that $\mathcal{L}_\lambda$ is Perron–Frobenius type follows then by the analogue of Problem 1.19, Theorem D.2 and Lemma D.4 where the norm can be chosen to be $\|h\|_a = a\|h\|_{\mathcal{C}^0} + \|h'\|_{\mathcal{C}^0}$, which is equivalent to the $\mathcal{C}^1$ norm and, finally, Theorem D.8. The spectral decomposition follows from Lemma A.24 and the analyticity can be argued as in Problem A.29. □

Hence, there exists $c \in \mathbb{R}$ such that

$$\mu(A_{a,n}^+(\varphi)) \leqslant e^{-n(\lambda a - \ln \alpha_\lambda) + c}.$$

Since λ has been chosen arbitrarily we have obtained

$$\mu(A_{a,n}^+(\varphi)) \leqslant e^{-n\widetilde{I}(a) + c} \tag{1.6.3}$$

where $\widetilde{I}(a) := \sup_{\lambda \in \mathbb{R}^+} \{\lambda a - \ln \alpha_\lambda\}$. The problem is then reduced to studying the function $\widetilde{I}(a)$ which is a version of what it is commonly called the *rate function*. Note that $\widetilde{I}$ is not necessarily finite. Indeed, if $a > \|\hat{\varphi}\|_\infty$, then clearly $\mu(A_{a,n}^+(\varphi)) = 0$.

To better understand the rate function it is helpful to make a little digression into convex analysis.

Recall that a function $g : \mathbb{R}^d \to \mathbb{R}^d$ is convex if for each $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$ we have $g(ty + (1-t)x) \leqslant tg(y) + (1-t)g(x)$ (if the inequality is everywhere strict, then the function is *strictly convex*).

Problem 1.34. *Show that if $g \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, then g is convex iff $\frac{\partial^2 g}{\partial x^2}$ is a positive matrix.*[27] *Give a condition for strict convexity.*

Problem 1.35. *If a function $g : D \subset \mathbb{R}^d \to \mathbb{R}$, D convex,*[28] *is convex and bounded, then it is continuous.*

[27] A matrix $A \in GL(\mathbb{R}, d)$ is called *positive* if $A^T = A$ and $\langle v, Av \rangle \geqslant 0$ for each $v \in \mathbb{R}^d$.

[28] A set D is convex if, for all $x, y \in D$ and $t \in [0, 1]$, holds true $ty + (1-t)x \in D$.

Given a function $g : \mathbb{R}^d \to \mathbb{R}$ let us define its *Legendre transform* as

$$g^*(x) = \sup_{y \in \mathbb{R}^d} \{\langle x, y\rangle - g(y)\} . \tag{1.6.4}$$

Remark that g^* can take the value $+\infty$.

Problem 1.36. *Prove that g^* is convex.*

Problem 1.37. *Prove that $g^{**} \leqslant g$.*

Problem 1.38. *Prove that if $g \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ is strictly convex, then the function $h(y) := \frac{\partial g}{\partial y}(y)$ is invertible and g^* is strictly convex. Moreover, calling ℓ the inverse function of h, we have*

$$g^*(x) = \langle x, \ell(x)\rangle - g \circ \ell(x).$$

Problem 1.39. *Show that if $g \in \mathcal{C}^2$ is strictly convex, then $g^{**} = g$.*

Problem 1.40. *Show that, for each $x, y \in \mathbb{R}^d$, $\langle x, y\rangle \leqslant g^*(x) + g(y)$, (Young inequality).*

From the above discussion it follows that the rate function is defined very similarly to the Legendre transform of the logarithm of the maximal eigenvalue, which is commonly called the *pressure of $\hat{\varphi}$*.

In fact, setting $\mathbb{J}(a) = \max_{\lambda \in \mathbb{R}}(\lambda a - \ln \alpha_\lambda)$ we will see that, for $a \geqslant 0$, $\mathbb{J}(a) = \widetilde{I}(a)$. Unfortunately, to see that the rate function is exactly a Legendre transform takes some work. Let us start by studying the function α_λ.

Lemma 1.41. *There exists $h_\lambda \in \mathcal{C}^1$ and $\ell_\lambda \in (\mathcal{C}^1)^*$ such that $\Pi_\lambda(g) = h_\lambda \ell_\lambda(g)$, $\ell_\lambda(h_\lambda) = 1$, $\ell_\lambda(h_\lambda') = 0$. In addition, ℓ_λ is a measure and $\mu_\lambda(\cdot) := \ell_\lambda(h_\lambda \cdot)$ is an invariant probability measure. Moreover everything is analytic in λ.*

Proof. By Lemma 1.33 we know that Π_λ is rank one and analytic, hence $\widetilde{h}_\lambda = \Pi_\lambda 1 \in \mathcal{C}^1$ is analytic as well and $\Pi_\lambda(h) = \widetilde{h}_\lambda \widetilde{\ell}_\lambda(h)$ for some $\widetilde{\ell}_\lambda \in (\mathcal{C}^1)^*$, which must be analytic in λ as well. Also, since Π_λ is a projector, it must be $\widetilde{\ell}_\lambda(\widetilde{h}_\lambda) = 1$. Next, note that, by Lemma 1.33 again

$$|\widetilde{\ell}_\lambda(h)| = \lim_{n\to\infty} \frac{\left|\int_{\mathbb{T}} \widetilde{h}_\lambda \alpha_\lambda^{-n} \mathcal{L}_\lambda^n h\right|}{\int_{\mathbb{T}} \widetilde{h}_\lambda^2} \leqslant \lim_{n\to\infty} \frac{\left|\int_{\mathbb{T}} h_\lambda \alpha_\lambda^{-n} \mathcal{L}_\lambda^n 1\right|}{\int_{\mathbb{T}} \widetilde{h}_\lambda^2} \|h\|_{\mathcal{C}^0} = |\widetilde{\ell}_\lambda(1)| \|h\|_{\mathcal{C}^0}.$$

Hence, $\widetilde{\ell}_\lambda \in (\mathcal{C}^0)^*$. That is, it is a measure. Also, if $h \geqslant 0$, then

$$h_\lambda \ell_\lambda(h) = \lim_{n\to\infty} \alpha_\lambda^{-n} \mathcal{L}_\lambda^n h \geqslant 0,$$

hence it is a positive measure. Obviously, for all $h \in \mathcal{C}^1$,

$$\begin{aligned} \mathcal{L}_\lambda \widetilde{h}_\lambda &= \alpha_\lambda \widetilde{h}_\lambda, \\ \widetilde{\ell}_\lambda(\mathcal{L}_\lambda h) &= \alpha_\lambda \widetilde{\ell}_\lambda(h), \end{aligned} \tag{1.6.5}$$

and $\alpha_0 = 1$, $\widetilde{h}_0 = h_*$ and $\widetilde{\ell}_0 = \text{Leb}$. Notice that $\widetilde{h}_\lambda$ and $\widetilde{\ell}_\lambda$ are not uniquely defined: for any analytic function β_λ, with $\beta_0 = 0$, the eigenvectors $h_\lambda = e^{\beta_\lambda}\widetilde{h}_\lambda$ and $\ell_\lambda = e^{-\beta_\lambda}\widetilde{\ell}_\lambda$ are such that $\Pi_\lambda = h_\lambda \otimes \ell_\lambda$ and satisfy all the other properties as well. Thus $\ell_\lambda((h_\lambda)') = \widetilde{\ell}_\lambda(\widetilde{h}'_\lambda) + \beta'_\lambda$. Choosing $\beta_\lambda = -\int_0^\lambda \widetilde{\ell}_t(\widetilde{h}'_t)dt$ we obtain the wanted property $\ell_\lambda((h_\lambda)') = 0$. To conclude, note that

$$\mu_\lambda(h \circ f) = \alpha_\lambda^{-1}\ell_\lambda(\mathcal{L}_\lambda(h \circ f h_\lambda)) = \alpha_\lambda^{-1}\ell_\lambda(h\mathcal{L}_\lambda(h_\lambda)) = \ell_\lambda(hh_\lambda) = \mu_\lambda(h),$$

and $\mu_\lambda(1) = \ell_\lambda(h_\lambda) = 1$. □

Lemma 1.42. *The functions α_λ and $\ln \alpha_\lambda$ are convex. Moreover,*

$$\left|\frac{d}{d\lambda} \ln \alpha_\lambda\right| \leqslant |\hat{\varphi}|_\infty.$$

Proof. Note that

$$\frac{d^2}{d\lambda^2} \ln \alpha_\lambda = \frac{\alpha''_\lambda \alpha_\lambda - (\alpha'_\lambda)^2}{\alpha_\lambda^2}, \tag{1.6.6}$$

thus the convexity of $\ln \alpha_\lambda$ implies the convexity of α_λ.

In view of the above fact we can differentiate (1.6.5) obtaining

$$\mathcal{L}'_\lambda h_\lambda + \mathcal{L}_\lambda h'_\lambda = \alpha'_\lambda h_\lambda + \alpha_\lambda h'_\lambda. \tag{1.6.7}$$

Applying ℓ_λ yields

$$\frac{d\alpha_\lambda}{d\lambda} = \alpha_\lambda \ell_\lambda(\hat{\varphi} h_\lambda)) = \alpha_\lambda \mu_\lambda(\hat{\varphi}). \tag{1.6.8}$$

Thus $\alpha'_0 = 0$. Note that, as claimed,

$$\left|\frac{d}{d\lambda} \ln \alpha_\lambda\right| \leqslant |\mu_\lambda(\hat{\varphi})| \leqslant |\hat{\varphi}|_\infty.$$

Differentiating again yields

$$\frac{d^2\alpha_\lambda}{d\lambda^2} = \alpha_\lambda \mu_\lambda(\hat{\varphi})^2 + \alpha_\lambda \ell'_\lambda(\hat{\varphi} h_\lambda) + \alpha_\lambda \ell_\lambda(\hat{\varphi} h'_\lambda). \tag{1.6.9}$$

On the other hand, from (1.6.7) we have

$$(\mathbb{1}\alpha_\lambda - \mathcal{L}_\lambda) h'_\lambda = \mathcal{L}_\lambda(\varphi_\lambda h_\lambda),$$

where $\varphi_\lambda = \hat{\varphi} - \mu_\lambda(\hat{\varphi})$. Since, by construction, $\Pi_\lambda h'_\lambda = \Pi_\lambda(\varphi_\lambda h_\lambda) = 0$, the above equation can be studied in the space $\mathbb{V}_\lambda = (\mathbb{1} - \Pi_\lambda)\mathcal{C}^1$ in which $\mathbb{1}\alpha_\lambda - \mathcal{L}_\lambda$ is invertible.

Setting $\hat{\mathcal{L}}_\lambda := \alpha_\lambda^{-1}\mathcal{L}_\lambda$, we have

$$h'_\lambda = (\mathbb{1} - \hat{\mathcal{L}}_\lambda)^{-1} \hat{\mathcal{L}}_\lambda(\varphi_\lambda h_\lambda). \tag{1.6.10}$$

Using similar considerations on the equation $\ell_\lambda(\mathcal{L}_\lambda g) = \alpha_\lambda \ell_\lambda(g)$, we obtain

$$\begin{aligned} \alpha''_\lambda &= \alpha_\lambda \mu_\lambda(\hat{\varphi})^2 + \alpha_\lambda \ell_\lambda(\varphi_\lambda (\mathbb{1} - \hat{\mathcal{L}}_\lambda)^{-1}(\mathbb{1} + \hat{\mathcal{L}}_\lambda)(\varphi_\lambda h_\lambda)) \\ &= \alpha_\lambda \mu_\lambda(\hat{\varphi})^2 + \alpha_\lambda \sum_{n=1}^{\infty} \ell_\lambda(\varphi_\lambda \hat{\mathcal{L}}_\lambda^n (\mathbb{1} + \hat{\mathcal{L}}_\lambda)(\varphi_\lambda h_\lambda)) \\ &= \frac{(\alpha'_\lambda)^2}{\alpha_\lambda} + \left[\mu_\lambda(\varphi_\lambda^2) + 2 \sum_{n=1}^{\infty} \ell_\lambda(\varphi_\lambda \hat{\mathcal{L}}_\lambda^n(\varphi_\lambda h_\lambda)) \right] \alpha_\lambda. \end{aligned} \tag{1.6.11}$$

Finally, notice that

$$\ell_\lambda(\varphi_\lambda \hat{\mathcal{L}}_\lambda^n(\varphi_\lambda h_\lambda)) = \ell_\lambda(\hat{\mathcal{L}}_\lambda^n(\varphi_\lambda \circ f^n \varphi_\lambda h_\lambda)) = \mu_\lambda(\varphi_\lambda \circ f^n \varphi_\lambda)$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \mu_\lambda \left(\left[\sum_{k=0}^{n-1} \varphi_\lambda \circ f^k \right]^2 \right) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k,j=0}^{n-1} \mu_\lambda(\varphi_\lambda \circ f^k \varphi_\lambda \circ f^j) \\ &= \mu_\lambda(\varphi_\lambda^2) + \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n-1} (n-k) \mu_\lambda(\varphi_\lambda \circ f^k \varphi_\lambda) \\ &= \mu_\lambda(\varphi_\lambda^2) + 2 \sum_{k=1}^{\infty} \mu_\lambda(\varphi_\lambda \circ f^k \varphi_\lambda). \end{aligned} \tag{1.6.12}$$

The above two facts and equations (1.6.6), (1.6.11) yield

$$\frac{d^2}{d\lambda^2}\ln\alpha_\lambda = \lim_{n\to\infty}\frac{1}{n}\mu_\lambda\left(\left[\sum_{k=0}^{n-1}\varphi_\lambda\circ f^k\right]^2\right)\geqslant 0. \tag{1.6.13}$$

□

Note that equation (1.6.8) implies $\alpha_0' = 0$, hence $\alpha_\lambda' \geqslant 0$ for $\lambda \geqslant 0$. Since the maximum of $\lambda a - \ln\alpha_\lambda$ is taken either at $\alpha_\lambda a = \alpha_\lambda'$ or at infinity (if $a > \sup_{\lambda>0}\frac{\alpha_\lambda'}{\alpha_\lambda}$), it follows that

$$\widetilde{I}(a) = \sup_{\lambda\geqslant 0}(\lambda a - \ln\alpha_\lambda) = \sup_{\lambda}(\lambda a - \ln\alpha_\lambda) =: \mathbb{J}(a) \tag{1.6.14}$$

as announced. In fact, more can be said.

Lemma 1.43. *Either the rate function $\mathbb{J}$ is strictly convex, or there exists $\beta \in \mathbb{R}, \phi \in \mathcal{C}^0$ such that $\varphi - \beta = \phi - \phi\circ f$.*

Proof. By Problem 1.38 it suffices to prove that $\ln\alpha_\lambda$ is strictly convex. On the other hand equations (1.6.6) and (1.6.13) imply that if the second derivative of $\ln\alpha_\lambda$ is zero for some λ, then, recalling Lemma 1.33,

$$\begin{aligned}\mu_\lambda\left(\left[\sum_{k=0}^{n-1}\varphi_\lambda\circ f^k\right]^2\right) &= n\left[\mu_\lambda(\hat\varphi^2) + 2\sum_{k=1}^{n-1}\frac{n-k}{n}\mu_\lambda(\varphi_\lambda\circ f^k\,\varphi_\lambda)\right]\\ &= -2n\sum_{k=n}^{\infty}\ell_\lambda(\varphi_\lambda\hat{\mathcal{L}}_\lambda^k(\varphi_\lambda h_\lambda)) - 2\sum_{k=1}^{n-1}k\ell_\lambda(\varphi_\lambda\hat{\mathcal{L}}_\lambda^k(\varphi_\lambda h_\lambda)) - \alpha_\lambda\mu_\lambda(\hat\varphi)^2\\ &\leqslant C(\lambda)\left[n\beta_\lambda^n + \sum_{k=0}^{\infty}k\beta_\lambda^k\right]\end{aligned}$$

Accordingly, the sequence $\sum_{k=0}^{n-1}\varphi_\lambda\circ f^k$ is bounded in $L^2(\mathbb{T}^1,\mu_\lambda)$ and hence weakly compact. Let $\sum_{k=0}^{n_j-1}\varphi_\lambda\circ f^k$ be a weakly convergent subsequence.[29] That is, there exists $\phi_\lambda \in L^2$ such that for each $\varphi \in L^2$ holds

$$\lim_{j\to\infty}\mu_\lambda(\varphi\sum_{k=0}^{n_j-1}\varphi_\lambda\circ f^k) = \mu_\lambda(\varphi\phi_\lambda).$$

[29] Such a subsequence always exists, e.g. see Lieb and Loss (2001).

It follows that, for each $g \in \mathcal{C}^1$,

$$\begin{aligned}\mu_\lambda(g[\varphi_\lambda - \phi_\lambda + \phi_\lambda \circ f]) &= \mu_\lambda(g\varphi_\lambda) + \lim_{j\to\infty} \sum_{k=0}^{n_j-1} \mu_\lambda(g\varphi_\lambda \circ f^{k+1} - g\varphi_\lambda \circ f^k) \\ &= \lim_{j\to\infty} \mu_\lambda(g\varphi_\lambda \circ f^{n_j}) = \lim_{j\to\infty} \ell_\lambda(\varphi_\lambda \hat{\mathcal{L}}_\lambda^{n_j}(gh_\lambda)) \\ &= \mu_\lambda(g)\mu_\lambda(\varphi_\lambda) = 0.\end{aligned}$$

Thus, since $\mathcal{C}^1$ is dense in L^2, it follows that

$$\varphi_\lambda = \phi_\lambda - \phi_\lambda \circ f\,, \qquad \mu_\lambda - \text{a.s.} \tag{1.6.15}$$

A function with the above property is called a *coboundary*, in this case an L^2 coboundary since we know only that $\phi_\lambda \in L^2(\mathbb{T}, \mu_\lambda)$. In fact, this it is not enough to conclude the Lemma: we need to show, at least, that $\phi_\lambda \in \mathcal{C}^0$.

First of all notice that, since for each $\beta \in \mathbb{R}$ we have $\varphi_\lambda = \phi_\lambda + \beta - (\phi_\lambda + \beta) \circ T$, we can assume without loss of generality that $\mu_\lambda(\phi_\lambda) = 0$. But then

$$\hat{\mathcal{L}}_\lambda(\varphi_\lambda\, h_\lambda) = \hat{\mathcal{L}}_\lambda(\phi_\lambda\, h_\lambda) - \phi_\lambda\, h_\lambda = -(\mathbb{1} - \hat{\mathcal{L}}_\lambda)\phi_\lambda\, h_\lambda.$$

Hence

$$\phi_\lambda = h_\lambda^{-1}(\mathbb{1} - \hat{\mathcal{L}}_\lambda)^{-1}\hat{\mathcal{L}}_\lambda(\varphi_\lambda\, h_\lambda) \in W^{1,1} \subset \mathcal{C}^0.$$

□

Remark 1.44. *The above result is quite sharp. Indeed, it shows that if $\mathbb{J}$ is not strictly convex, then for each invariant measure ν one has $\nu(\varphi) = \beta$. So it suffices to find two invariant measures for which the average of φ differs (for example the average on two periodic orbits) to infer that $\mathbb{J}$ is strictly convex.*

Note that Equations (1.6.3) and (1.6.14) imply the upper bound

$$\mu(A_{a,n}^+(\varphi)) \leqslant C_\# e^{-n\mathbb{J}(a)}. \tag{1.6.16}$$

Problem 1.45. *Set $\sigma := \alpha''(0)$. Show that, for a small, $\mathbb{J}(a) = \frac{a^2}{2\sigma} + \mathcal{O}(a^3)$. Show that if $a > \sup_\lambda \mu_\lambda(\hat{\varphi})$, then $\mathbb{J}(a) = +\infty$. In particular, this implies that $\mathbb{J}(a) = +\infty$ if $a > \|\varphi\|_\infty$.*

The above discussion allows us to conclude

$$\mu(A_{a,n}^+(\varphi)) \leqslant \mu(\mathcal{L}_{\lambda\_}^n h) \leqslant C e^{-\frac{a^2}{2\sigma^2}n + \mathcal{O}(a^3 n)}.$$

Since similar arguments hold for the set $A^+_{a,n}(-\varphi)$, it follows that we have an exponentially small probability to observe a deviation from the average. Moreover, the expected size of a deviation is of order $n^{-\frac{1}{2}}$. To see if this is really the case we need a lower bound.

1.6.2 Large deviations. Lower bound

If $\mathbb{J}(a) = \infty$ then, by Equations (1.6.14) and (1.6.3), we have $\mu(A^+_{a,n}) = 0$, hence we can restrict ourselves to the case $\mathbb{J}(a) < \infty$. Note that, by Problem 1.45, this implies that it must be $a \leqslant \sup_\lambda \mu_\lambda(\hat{\varphi})$. In the case $a = \sup_\lambda \mu_\lambda(\hat{\varphi})$ we content ourselves with the trivial bound $\mu(\mathring{A}_{a,n}) \geqslant 0$. We can thus consider only the case $a < \sup_\lambda \mu_\lambda(\hat{\varphi})$.

Note that the derivative of $\lambda a - \ln \alpha_\lambda$, by Equation (1.6.8), is $a - \mu_\lambda(\hat{\varphi})$. Thus the maximum of $\lambda a - \ln \alpha_\lambda$ takes place for $\bar{\lambda}$ such that $a = \mu_{\bar{\lambda}}(\hat{\varphi})$.

For each $\delta \in (0, \frac{1}{4}[\sup_\lambda \mu_\lambda(\hat{\varphi}) - a])$ let $\bar{\lambda}_\delta$ be such that $a + \frac{3}{2}\delta = \mu_{\bar{\lambda}_\delta}(\hat{\varphi})$ and let $I_\delta = (a + \delta, a + 2\delta)$.

Recall that $S_n = \sum_{k=0}^{n-1} \hat{\varphi} \circ f^k$, then $\mu_\lambda(S_n) = n\mu_\lambda(\hat{\varphi})$ and, by (1.6.12),

$$\mu_\lambda\left(\left[\sum_{k=0}^{n-1} \hat{\varphi} \circ f^k - n\mu_\lambda(\hat{\varphi})\right]^2\right) \leqslant C_\lambda n,$$

where C_λ depends continuously by λ.

Next, we set $A_{n,I_\delta} = \{x \in \mathbb{T}^1 \,:\, \frac{1}{n} S_n(x) \in I_\delta\}$. Note that

$$A_{n,I_\delta} \subset \mathring{A}_{a,n}(\varphi). \tag{1.6.17}$$

Recalling the definition $\varphi_\lambda = \hat{\varphi} - \mu_\lambda(\hat{\varphi})$, we have

$$\begin{aligned}\mu_{\bar{\lambda}_\delta}(A^c_{n,I_\delta}) &\leqslant \mu_{\bar{\lambda}_\delta}\left(\left\{\left|\sum_{k=0}^{n-1} \varphi_{\bar{\lambda}_\delta} \circ f^k\right| \geqslant \delta n\right\}\right)\\ &\leqslant \delta^{-2} n^{-2} \mu_{\bar{\lambda}_\delta}\left(\left|\sum_{k=0}^{n-1} \varphi_{\bar{\lambda}_\delta} \circ f^k\right|^2\right) \leqslant C_a \delta^{-2} n^{-1}.\end{aligned}$$

It follows that there exists $n_{a,\delta} \in \mathbb{N}$ such that, for all $n \geqslant n_{a,\delta}$, $\mu_{\bar{\lambda}_\delta}(A_{n,I_\delta}) \geqslant \frac{1}{2}$.

Since, by Lemma 1.41, ℓ_λ is a measure, we can then write, for all $m \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{2} \leqslant \mu_{\bar\lambda_\delta}(A_{n,I}) &= \ell_{\bar\lambda_\delta}(\mathbb{1}_{A_{n,I}} h_{\bar\lambda_\delta}) \\ &\leqslant C_\# e^{-(n+m)\ln\alpha_{\bar\lambda_\delta}} \ell_{\bar\lambda_\delta}\left(\mathcal{L}^{n+m}_{\bar\lambda_\delta}(\mathbb{1}_{A_{n,I}})\right).\end{aligned} \tag{1.6.18}$$

To conclude we must analyse a bit the characteristic function of $A_{n,I}$. First of all, notice that if $|f^k x - f^k y| \leqslant \varepsilon$ for each $k \leqslant n$, then $|f^k x - f^k y| \leqslant \lambda_\star^{-n+k}\varepsilon$ for all $k \leqslant n$. Accordingly, for each $z \in [x, y]$

$$\begin{aligned}|D_x f^n - D_z f^n| &\leqslant |D_x f^n| \cdot (e^{\sum_{k=0}^{n-1} |\ln D_{f^k x} T - \ln D_{f^k z} T|} - 1) \\ &\leqslant |D_x f^n|(e^{C_\# \sum_{k=0}^{n-1} \lambda_\star^{-k}\varepsilon} - 1) \leqslant C_\# |D_x f^n|.\end{aligned}$$

By a similar estimate, $|D_x f^n - D_z f^n| \geqslant C_\# |D_x f^n|$ as well. Moreover,

$$|S_n(x) - S_n(y)| \leqslant \sum_{k=0}^{n-1} |f|_{\mathcal{C}^1} C_\# \lambda_\star^{-k}\varepsilon \leqslant C_\#\varepsilon.$$

We can then write $\cup_l J_l \supset A_{n,I_\delta}$ where J_l are disjoint intervals such that $|f^n J_l| = \varepsilon$. Choosing $\varepsilon \leqslant C_\#\delta$ small enough it follows that the oscillation of S_n on each J_l is smaller than $\delta/2$, hence we can assume $A^+_{n,a+\delta/2} \supset \cup_l J_l$. Moreover

$$\begin{aligned}\|\mathcal{L}^n \mathbb{1}_{J_l}\|_{BV} &= \sup_{|\varphi|_\infty \leqslant 1} \int_{J_l} \varphi' \circ f^n \\ &\leqslant \sup_{|\varphi|_\infty \leqslant 1} \int_{J_l} \frac{d}{dx}\left[(Df^n)^{-1}\varphi \circ f^n\right] + B\,\mathrm{Leb}(J_l) \\ &\leqslant 2 \sup_{x \in J_l} |D_x f^n|^{-1} + B\,\mathrm{Leb}(J_l) \leqslant C_\#\delta^{-1}\,\mathrm{Leb}(J_l).\end{aligned}$$

We can then continue our estimate started in (1.6.18),

$$\begin{aligned}\frac{1}{2} &\leqslant C_\# e^{-(n+m)\ln\alpha_{\bar\lambda_\delta} + n\bar\lambda_\delta(\mu_{\bar\lambda_\delta}(\hat\varphi)+3\delta) + mC_\#} \sum_l \ell_{\bar\lambda_\delta}\left(\mathcal{L}^{n+m}(\mathbb{1}_{J_l})\right) \\ &= C_\# e^{-(n+m)\ln\alpha_{\bar\lambda_\delta} + n\bar\lambda_\delta(\mu_{\bar\lambda_\delta}(\hat\varphi)+3\delta) + mC_\#} \sum_l \ell_{\bar\lambda_\delta}(1)\,\mathrm{Leb}(J_l)(1 + \mathcal{O}(\delta^{-1}\rho^m)) \\ &\leqslant C_\# e^{-n[\ln\alpha_{\bar\lambda_\delta} - \bar\lambda_\delta(\mu_{\bar\lambda_\delta}(\hat\varphi)+3\delta)] - c_\#\ln\delta}\mu(A_{n,I_\delta}),\end{aligned}$$

where we have chosen $m = c_\# \ln \delta^{-1}$. The above computations, together with Equation (1.6.17), imply that

$$\mu(\mathring{A}_{n,a}) \geqslant \mu(A_{I_\delta}) \geqslant C_\# e^{n[\ln \alpha_{\bar\lambda_\delta} - \bar\lambda_\delta(\mu_{\bar\lambda_\delta}(\hat\varphi)+3\delta)] + c_\# \ln \delta}.$$

Next, note that, by hypothesis, there exists $\bar c_a > 0$ such that $\bar\lambda_\delta \leqslant \bar c_a$, thus, recalling that $\bar\lambda_\delta$ has been chosen as the place where $\lambda(a + \frac{3}{2}) - \lambda \ln \alpha_\lambda$ has the maximum, we have

$$\mu(\mathring{A}_{n,a}) \geqslant C_\delta \; e^{-n[\mathbb{J}(a+\frac{3}{2}\delta)+3\bar c_a \delta]}.$$

Next, note that $\frac{d}{da}\mathbb{J}(a) = \bar\lambda_0 \leqslant \bar c_a$. Collecting the above facts yields

$$\mu(\mathring{A}_{n,a}) \geqslant C_\delta \; e^{-n[\mathbb{J}(a)+5\bar c_a \delta]}.$$

It is then sufficient to choose $\delta = \epsilon(5\bar c_a)^{-1}$ and $c_{a,\epsilon} = C_\delta$ to obtain the wanted lower bound

$$\mu(\mathring{A}_{n,a}) \geqslant c_{a,\epsilon} \; e^{-n[\mathbb{J}(a)+\epsilon]}. \tag{1.6.19}$$

To express the rate function in terms of entropy, an extra argument is necessary.

1.6.3 Large deviations. Conclusions

It is possible to give a variational characterization of the rate function in the spirit of general Large deviation theory, Dembo and Zeitouni (2010) and Varadhan (2016).

Lemma 1.46. *Calling $\mathcal{M}_f$ the set of invariant probability measures invariant with respect to f and h_ν the Kolmogorov–Sinai entropy, we have, setting $\mathcal{M}_f(a) = \{\nu \in \mathcal{M}_f \,:\, \nu(\hat\varphi) \geqslant a\}$,*

$$\mathbb{J}(a) = \mathbb{I}(a).$$

Proof. For each $\nu \in \mathcal{M}_f$,

$$\begin{aligned} \ln \alpha_\lambda &= \sup_{\nu \in \mathcal{M}_f} \{h_\nu(f) + \lambda \nu(\hat\varphi) - \nu(\ln |f'|)\} \\ &= h_{\mu_\lambda}(f) + \lambda \mu_\lambda(\hat\varphi) - \mu_\lambda(\ln |f'|). \end{aligned} \tag{1.6.20}$$

The first equality is a formula for the spectral radius (e.g. see Baladi (2000, Remark 2.5)).[30] The second equality is called the *variational principle*. For more information on this and, more generally, on the so called *thermodynamic formalism*, see Keller (1998) for details.

[30] In fact, the remark contains a misprint: the first formula is the logarithm of the spectral radius, not the spectral radius.

Thus, recalling Equation (1.6.14), we can write

$$\begin{aligned}\mathbb{J}(a) &= \sup_{\lambda}(\lambda a - \sup_{\nu\in\mathcal{M}_f}\{h_\nu(f) + \lambda\nu(\hat{\varphi}) - \nu(\ln|f'|)\}) \\ &\leqslant \sup_{\lambda}(\lambda a - \sup_{\nu\in\mathcal{M}_f(a)}\{h_\nu(f) + \lambda\nu(\hat{\varphi}) - \nu(\ln|f'|)\}) \\ &\leqslant -\sup_{\nu\in\mathcal{M}_f(a)}\{h_\nu(f) - \nu(\ln|f'|)\} = \mathbb{I}(a).\end{aligned}.$$

In particular, if $\mathbb{J}(a) = \infty$ then we have $\mathbb{I}(a) = \infty$. We can thus assume $\mathbb{J}(a) < \infty$. Also, note that Equation (1.6.20) implies

$$\mathbb{J}(a) = \sup_{\lambda}\lambda a - \{h_{\mu_\lambda}(f) + \lambda\mu_\lambda(\hat{\varphi}) - \mu_\lambda(\ln|f'|)\}).$$

If for some $\bar{\lambda}$ we have $\mu_{\bar{\lambda}}(\hat{\varphi}) = a$, then

$$\mathbb{J}(a) \geqslant -\{h_{\mu_{\bar{\lambda}}}(f) - \mu_{\bar{\lambda}}(\ln|f'|)\} \geqslant \mathbb{I}(a).$$

Otherwise, recalling Problem 1.45, it means that $a = \sup_\lambda \mu_\lambda(\hat{\varphi})$. Then, since $h_\nu(f) \leqslant h_{top}(f) < \infty$, where h_{top} is the topological entropy (e.g. see Katok and Hasselblatt (1995)) and $\ln|f'| > 0$, $\mathbb{J}(a) < \infty$ implies

$$\sup_{\lambda}\lambda\mu_\lambda(a - \hat{\varphi}) < \infty.$$

Accordingly, if $\lambda \to \infty$, then $\mu_\lambda(a - \hat{\varphi}) \to 0$. By the weak compactness of probability measures we can then choose a sequence λ_j such that $\mu_{\lambda_j} \Longrightarrow \nu_*$, in the sense of weak convergence, and $\lambda_j\mu_{\lambda_j}(a - \hat{\varphi}) \geqslant 0$. Note that it must be that $\nu_*(\hat{\varphi}) = a$. Hence

$$\begin{aligned}\mathbb{J}(a) &\geqslant \lim_{j\to\infty}\left(\lambda_j a - h_{\mu_{\lambda_j}}(f) - \lambda_j\mu_{\lambda_j}(\hat{\varphi}) + \mu_{\lambda_j}(\ln|f'|)\right) \\ &\geqslant \lim_{j\to\infty}\left(-h_{\mu_{\lambda_j}}(f) + \mu_{\lambda_j}(\ln|f'|)\right) = -\left[h_{\nu_*}(f) - \nu_*(\ln|f'|)\right]\end{aligned}$$

where in the last equality we have used that h_ν is (as a function of ν) an upper-semicontinuous function with respect to the weak topology (see Keller (1998, Theorem 4.5.6)). It follows that $\mathbb{J}(a) = \mathbb{I}(a)$. □

Note that Lemma 1.46 and Equations (1.6.16), (1.6.19) conclude the proof of Theorem 1.32.

Remark 1.47. *Using the previous techniques it is possible to obtain much sharper results, see De Simoi and Liverani (2018) for details.*

1.7 Perturbation theory

Another natural question is: how do the statistical properties of a system depend on small changes in the system?

Indeed, in real life situations the dynamics is known only with finite precision, hence it is fundamental to know how small changes in the dynamics affects the asymptotic properties of the system.

To answer such a question we need some type of perturbation theorem. Several such results are available (e.g., see Kifer (1988), Viana (1997) for a review and Baladi and Young (1993) for some more recent results), here we will follow mainly the theory developed in Keller and Liverani (1999) adapted to the special cases at hand.

We will start by considering an abstract family of operators $\mathcal{L}_\varepsilon$ satisfying the following properties.

Hypotheses 1.1. *Given two Banach spaces as in Theorem 1.1, consider a family of linear bounded operators $\mathcal{L}_\varepsilon \in L(\mathcal{B}, \mathcal{B})$, $\varepsilon \in [0,1]$, with the following properties.*

1. *Uniform Lasota–Yorke inequality: There exist $C > 0$, $\lambda_\star > 1$ such that for all $\varepsilon \in [0,1]$*

$$\|\mathcal{L}_\varepsilon^n h\|_{\mathcal{B}} \leqslant C\lambda_\star^{-n}\|h\|_{\mathcal{B}} + C\|h\|_{\mathcal{B}_w}, \quad \|\mathcal{L}_\varepsilon^n h\|_{\mathcal{B}_w} \leqslant C\|h\|_{\mathcal{B}_w};$$

2. *For $L : \mathcal{B} \to \mathcal{B}$ define the norm*

$$|||L||| := \sup_{\|h\|_{\mathcal{B}} \leqslant 1} \|Lf\|_{\mathcal{B}_w},$$

that is the norm of L as an operator from $\mathcal{B} \to \mathcal{B}_w$. Then there exists $D > 0$ such that

$$|||\mathcal{L}_0 - \mathcal{L}_\varepsilon||| \leqslant D\varepsilon.$$

Hypothesis 1.1-(2) specifies in which sense the family $\mathcal{L}_\varepsilon$ can be considered an approximation of the unperturbed operator $\mathcal{L} := \mathcal{L}_0$. Note that the condition is rather weak, in particular the distance between $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ as operators on $\mathcal{B}$ can be always larger than 1. Such a notion of closeness is completely inadequate to apply standard perturbation theory. To obtain some perturbation results it is then necessary to restrict the type of perturbations allowed, this is the content of Hypotheses 1.1-(1) which states that all the approximating operators enjoy properties very similar to $\mathcal{L}$.

To state a precise result consider, for each bounded operator L, the set

$$V_{\delta,r}(L) := \{z \in \mathbb{C} \mid |z| \leqslant r \text{ or } \operatorname{dist}(z, \sigma(L)) \leqslant \delta\}.$$

By $R(z)$ and $R_\varepsilon(z)$ we will mean respectively $(z - \mathcal{L})^{-1}$ and $(z - \mathcal{L}_\varepsilon)^{-1}$.

Theorem 1.48 (Keller and Liverani (ibid.))**.** *Consider a family of operators $\mathcal{L}_\varepsilon : \mathcal{B} \to \mathcal{B}$ satisfying Hypothesis 1.1. Let $V_{\delta,r} := V_{\delta,r}(\mathcal{L})$, $r > \lambda_\star^{-1}$, $\delta > 0$, then there exist $\varepsilon_0, a > 0$ such that, for all $\varepsilon \leqslant \varepsilon_0$, $\sigma(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$ and, for each $z \notin V_{\delta,r}$,*

$$|||R(z) - R_\varepsilon(z)||| \leqslant C\varepsilon^a.$$

A simpler proof, although less optimal, than that given by Keller and Liverani (ibid.) can be found in Appendix C. Actually, in Appendix C it is proven a slightly more complete result and it is also shown how to use it concretely to investigate the spectrum of $\mathcal{L}_\varepsilon$.

The above perturbation theorem has proven rather flexible and able to cover most of the interesting cases, as we show next.

1.8 Stability and computability

1.8.1 Deterministic stability

Let the $\mathcal{L}_\varepsilon$ be Ruelle–Perron–Frobenius (Transfer) operators of maps f_ε which are $\mathcal{C}^1$–close to f, that is $d_{\mathcal{C}^1}(f_\varepsilon, f) = \varepsilon$ and such that $d_{\mathcal{C}^2}(f_\varepsilon, f) \leqslant M$, for some fixed $M > 0$. In this case the uniform Lasota–Yorke inequality is trivial. On the other hand, for all $\varphi \in \mathcal{C}^0$ holds

$$\int (\mathcal{L}_\varepsilon h - \mathcal{L}h)\varphi = \int h(\varphi \circ f_\varepsilon - \varphi \circ f).$$

Now let $\Phi(x) := (D_x f)^{-1} \int_{f(x)}^{f_\varepsilon(x)} \varphi(z)dz$, since

$$\Phi'(x) = -(D_x f)^{-1} D_x^2 f \Phi(x) + D_x f_\varepsilon (D_x f)^{-1} \varphi(f_\varepsilon(x)) - \varphi(f(x)).$$

It follows

$$\int (\mathcal{L}_\varepsilon h - \mathcal{L}h)\varphi = \int h\Phi' + \int h(x)[(D_x f)^{-1} D_x^2 f \Phi(x) + (1 - D_x f_\varepsilon (D_x f)^{-1})\varphi(f_\varepsilon(x))].$$

Given that $|\Phi|_\infty \leqslant \lambda_\star^{-1}\varepsilon|\varphi|_\infty$ and $|1 - D_x f_\varepsilon (D_x f)^{-1}|_\infty \leqslant \lambda_\star^{-1}\varepsilon$, we have

$$\begin{aligned}\int (\mathcal{L}_\varepsilon h - \mathcal{L}h)\varphi &\leqslant \|h\|_{W^{1,1}}\lambda_\star^{-1}|\varphi|_\infty\varepsilon + |h|_{L^1}\lambda_\star^{-1}(B+1)\varepsilon|\varphi|_\infty \\ &\leqslant D\|h\|_{W^{1,1}}\varepsilon|\varphi|_\infty.\end{aligned}$$

Taking the sup on such φ yields the wanted inequality

$$|\mathcal{L}_\varepsilon h - \mathcal{L}h|_{L^1} \leqslant D\|h\|_{W^{1,1}}\varepsilon.$$

We have thus seen that all the required Hypotheses are satisfied. See Keller (1982) for a more general setting including piecewise smooth maps.

1.8.2 Stochastic stability

Next consider a set of maps $\{f_\omega\}$ depending on a parameter $\omega \in \Omega$. In addition assume that Ω is a probability space and P a probability measure on Ω. Consider the process $x_n = f_{\omega_n} \circ \cdots \circ f_{\omega_1} x_0$ where the ω are i.i.d. random variables distributed accordingly to P and let $\mathbb{E}$ be the expectation of such process when x_0 is distributed according to μ. Then, calling $\mathcal{L}_\omega$ the transfer operator associated to f_ω, we have

$$\mathbb{E}(h(x_{n+1}) \mid x_n) = \mathcal{L}_P h(x_n) := \int_\Omega \mathcal{L}_\omega h(x_n) P(d\omega).$$

If, for all $\omega \in \Omega$,

$$|\mathcal{L}_\omega h|_{W^{1,1}} \leqslant \lambda_\omega^{-1}|h|_{W^{1,1}} + B_\omega|h|_{L^1},$$

then integrating yields

$$|\mathcal{L}_P h(x)|_{W^{1,1}} \leqslant \mathbb{E}(\lambda_\omega^{-1})|h|_{W^{1,1}} + \mathbb{E}(B_\omega)|h|_{L^1}.$$

Thus the operator $\mathcal{L}_P$ satisfies a Lasota–Yorke inequality provided that $\mathbb{E}(\lambda_\omega^{-1}) < 1$ and $\mathbb{E}(B_\omega) < \infty$.

In addition, if for some map f and associated transfer operator $\mathcal{L}$,

$$\mathbb{E}(|\mathcal{L}_\omega h - \mathcal{L}h|) \leqslant \varepsilon|h|_{W^{1,1}}$$

then we can apply perturbation theory and obtain stochastic stability.

1.8.3 Computability

If we want to compute exactly the invariant measure and the rate of decay of correlations for a specific system we must reduce the problem to a finite dimensional one that can then be solved numerically. To this end we can introduce the function

$$\phi(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & x \in [-1,0] \\ 1-x & x \in [0,1] \\ 0 & x \geqslant 1. \end{cases}$$

Note that $\sum_{i\in\mathbb{Z}} \phi(x-i) = 1$. We can then introduce the operators

$$P_n h = n \sum_{i=0}^{n-1} \phi(nx-i) \int \phi(ny-i) h(y) dy$$
$$\mathcal{L}_n = P_n \mathcal{L}.$$

Note that $P_n(\mathcal{C}^0) \subset \mathcal{C}^0$ and

$$\|P_n h\|_{L^1} \leqslant \|h\|_{L^1}$$
$$\|P_n h\|_{W^{1,1}} \leqslant \|h\|_{W^{1,1}}$$
$$\|h - P_n h\|_{L^1} \leqslant \frac{1}{n} \|h\|_{W^{1,1}}.$$

So we can again apply Theorem 1.48 to show that the finite dimensional operator $\mathcal{L}_n$ has the peripheral spectrum close to the one of $\mathcal{L}$. The problem is thus reduced to diagonalizing a matrix, which can be done numerically (provided the matrix is not too large). There exists a wide literature on the subject, see Liverani (2001) for more details.

1.8.4 Linear response

Linear response is a theory widely used by physicists. In essence it says the following: consider a one parameter family of systems f_s and the associated (e.g.) invariant measures μ_s, then, for a given observable φ one wants to study the response of the system to a small change in s, and, not surprisingly, one expects $\mu_s(\varphi) = \mu_0(\varphi) + s\nu(\varphi) + o(s)$, for some measure or distribution ν. That is, one

expects differentiability in s, which is commonly called *linear response*. Yet differentiability is not ensured by Theorem 1.48. It is then natural to ask under which conditions linear response holds.

For example linear response holds if the maps are sufficiently smooth and the dependence on the parameter is also smooth in an appropriate sense. These types of results follow from a more sophisticated version of Theorem 1.48 that can be found in Gouëzel and Liverani (2006, Section 8) and Gouëzel (2010, Theorem 3.3). A baby version of such a theory, useful to understand the basic ideas, can be found in Appendix C.

In fact, linear response for certain observables can be obtained even when the map is not very smooth, provided some extra conditions are satisfied, see Keller and Liverani (2009b) for more details.

However, the reader should be aware that there exist natural and relevant cases when linear response fails. See Baladi and Smania (2012) and references therein for an in depth discussion of this issue.

1.9 Piecewise smooth maps

The set of maps treated in the previous sections is rather special. Here we apply similar ideas to piecewise expanding multidimensional maps. We provide only an introduction, see Liverani (2013) and Saussol (2000) for more general results. In fact, similar ideas can be applied even to infinite dimensional expanding systems, Keller and Liverani (2006, 2009a).

More precisely, let $X := [0,1]^d$ together with a (possibly countable) collection of disjoint open sets $\{\Delta_i\}_{i\in\mathcal{I}\subset\mathbb{N}}$ be such that

- $\cup_{i\in\mathcal{I}}\overline{\Delta_i} = X$;
- For each orthogonal basis $E := \{e_i\}$ let $L_k(x,j,E)$ be the number of connected components of $\{x + te_k\}_{t\in[-1,1]} \cap \Delta_j$. Then we assume that $L_j = \inf_E \sup_{x\in\Delta_j} \sup_k L_k(x,j,E) < \infty$.

Next, let $f : X \to X$ be such that, for each $i \in \mathcal{I}$, $f|_{\Delta_j}$ is a $\mathcal{C}^2$ invertible map. Finally we ask that the map be expanding and not too singular

$$\begin{aligned} &\|(D_x f)^{-1}\| \leqslant \lambda_j^{-1} < 1 \quad \text{for all } x \in \Delta_j;\\ &|\nabla(D_x f)^{-1}|_{L^d} < \infty. \end{aligned} \tag{1.9.1}$$

1.9.1 A bit of measure theory

Let us define the following two norms on $\mathcal{M}(X)$:

$$\begin{aligned} |\mu| &:= \sup_{\varphi\in\mathcal{C}^0(X,\mathbb{R})} \frac{\mu(\varphi)}{|\varphi|_\infty} \\ \|\mu\| &:= \sup_{k\in\{1,\dots,d\}} \sup_{\varphi\in\mathcal{C}^1(X,\mathbb{R})} \frac{\mu(\partial_{x_k}\varphi)}{|\varphi|_\infty}. \end{aligned} \tag{1.9.2}$$

Note that, for each $\varphi \in \mathcal{C}^0(X,\mathbb{R})$ and $\varepsilon > 0$ one can find $\varphi_\varepsilon \in \mathcal{C}^1(X,\mathbb{R})$ such that $|\varphi - \varphi_\varepsilon| \leqslant \varepsilon|\varphi|_\infty$, hence

$$\mu(\varphi) \leqslant |\mu|\varepsilon|\varphi|_\infty + \mu(\varphi_\varepsilon) = |\mu|\varepsilon|\varphi|_\infty + \mu\left(\partial_{x_1}\int_0^{x_1}\varphi_\varepsilon\right) \leqslant (|\mu|\varepsilon + \|\mu\|(1+\varepsilon))|\varphi|_\infty.$$

Taking the sup on φ and by the arbitrariness of ε, it follows that

$$|\mu| \leqslant \|\mu\|. \tag{1.9.3}$$

Lemma 1.49. *Let $\mathcal{B} := \{\mu \in \mathcal{M}(X) \,:\, \|\mu\| < \infty\}$. If $\mu \in \mathcal{B}$ then it is absolutely continuous with respect to the Lebesgue measure m. Moreover*

$$\frac{d\mu}{dm} \in L^p(X,m) \quad \textit{for all } p < \frac{d}{d-1}.$$

Proof. Let $\varphi \in \mathcal{C}^0(X,\mathbb{R})$, then for each $\varepsilon \in (0,1)$ there exists $\varphi_\varepsilon \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$, supported in $[-\varepsilon, 1+\varepsilon]^d$, such that $|\varphi - \varphi_\varepsilon|_{\mathcal{C}^0(X,\mathbb{R})} \leqslant \varepsilon$, $|\varphi_\varepsilon|_\infty \leqslant |\varphi|_\infty(1+\varepsilon)$. In addition, if we define

$$\Gamma(\xi) := \begin{cases} -\frac{1}{2}\|\xi\| & \text{if } d = 1 \\ -\frac{1}{2\pi}\ln\|\xi\| & \text{if } d = 2 \\ \frac{1}{d(d-2)\alpha_d\|\xi\|^{d-2}} & \text{if } d \geqslant 3, \end{cases} \tag{1.9.4}$$

where α_d is the d-dimensional volume of the unit ball in $\mathbb{R}^d$, we can define the Newtonian potential $w_\varepsilon(x) = \int_{\mathbb{R}^d} \Gamma(x-z)\varphi_\varepsilon(z)dz$. It is then well known from

potential theory that $\Delta w_\varepsilon = \varphi_\varepsilon$, thus

$$\begin{aligned}\mu(\varphi) &\leqslant \mu(\varphi_\varepsilon) + |\mu|\varepsilon = \sum_{k=1}^{d} \mu(\partial_{x_k}\partial_{x_k} w_\varepsilon) + |\mu|\varepsilon \\ &\leqslant \sum_{k=1}^{d} \|\mu\| \sup_{x\in X} \int \left|\partial_{x_k} \Gamma(x-z)\varphi_\varepsilon(z)dz\right| + |\mu|\varepsilon \\ &\leqslant C \sum_{k=1}^{d} \|\mu\|\, |\varphi_\varepsilon|_{L^q} \left[\int_{[-1,2]^d} \frac{|x_k - z_k|^p}{\|x-z\|^{dp}} dz\right]^{\frac{1}{p}} + |\mu|\varepsilon,\end{aligned}$$

where $q^{-1} + p^{-1} = 1$. Since the integral in square brackets is finite for $p < \frac{d}{d-1}$, we have, by the arbitrariness of ε,

$$\mu(\varphi) \leqslant C(\|\mu\| + |\mu|)|\varphi|_{L^q}.$$

This means that the linear functional $\mu : \mathcal{C}^0 \to \mathbb{R}$ can be extended to a bounded functional on L^q. Since the dual of L^q is L^p it follows that there exists $h \in L^p$ such that $\mu(\varphi) = \int_X h(x)\varphi(x)dx$. □

Remark 1.50. *In fact it follows from the Gagliardo–Nirenberg–Sobolev inequality that the above Lemma holds also for* $p = \frac{d}{d-1}$.

Problem 1.51. *Show that, for all* $\mu \in \mathcal{B}$, *setting* $h = \frac{d\mu}{dm}$, *holds* $|\mu| = |h|_{L^1}$ *and* $\|\mu\| = |h|_{BV}$.

Remark 1.52. *To connect the present notations with the one of the previous section, recall that if* $d\mu = hdx$, *then* $d(f_*\mu) = (\mathcal{L}h)dx$.

The following characterization will be useful in the following: given $h \in L^1(X, m)$ we define

$$\mathrm{Var}^k(h)(x) = \sup_{\varphi\in\mathcal{C}^1([0,1],\mathbb{R})} \frac{\int_0^1 h(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_d)\varphi'(z)dz}{|\varphi|_\infty}.$$

Lemma 1.53. *For each* $\mu \in \mathcal{B}$, *setting* $h = \frac{d\mu}{dm}$,

$$\|\mu\| = \sup_{k\in\{1,\ldots,n\}} |\mathrm{Var}^k(h)|_{L^1}.$$

Proof. First,

$$\|\mu\| \leqslant \sup_k \sup_{|\varphi|_\infty \leqslant 1} \int h \partial_{x_k} \varphi = \sup_k \sup_{|\varphi|_\infty \leqslant 1} \int \operatorname{Var}^k h \sup_{x_k} |\varphi| \leqslant \sup_k |\operatorname{Var}^k(h)|_{L^1}.$$

For the opposite inequality one needs a bit of preparation.

For each $n \in \mathbb{N}$ and a function $\eta \in \mathcal{C}_0^2([-1,1]^n, \mathbb{R}_+)$, $\int \eta = 1$, let us define $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Then, for each $h \in L^1([0,1]^n, m)$ and $\varphi \in \mathcal{C}_0^1(\mathbb{R}^n, \mathbb{R})$ let $h_\varepsilon(x) = \int dz\, h(z)\, \eta_\varepsilon(x-z)$. Then,

$$\begin{aligned} \int \partial_{x_k} h_\varepsilon(x) \cdot \varphi(x) &= \int h(z) \partial_{x_k} \eta_\varepsilon(x-z) \cdot \varphi(x) \\ &= -\int h(z) \partial_{z_k} \eta_\varepsilon(x-z) \cdot \varphi(x) \leqslant |h|_{BV} |\varphi|_\infty. \end{aligned} \tag{1.9.5}$$

That is $\sup_k |\partial_{x_k} h_\varepsilon|_{L^1} \leqslant |h|_{BV}$. On the other hand, for each $\delta > 0$ and $k \in \{1, \dots, d\}$ there exists $\phi \in \mathcal{C}^1$, $|\phi|_\infty = 1$, such that $|h|_{BV} \leqslant \int h \partial_{x_k} \phi + \delta$. Next, consider a compactly supported extension $\widetilde{\phi} \in \mathcal{C}_0^1$ of ϕ on all $\mathbb{R}^n$ such that $|\widetilde{\phi}|_\infty \leqslant 1 + \delta$ and choose $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\sup_{x \in [0,1]^n} \left| \partial_{x_k} \phi(x) - \int_{\mathbb{R}^n} \eta_\varepsilon(x-z) \partial_{z_k} \widetilde{\phi}(z) dz \right| \leqslant \delta |\mu|^{-1}.$$

Hence,

$$|h|_{BV} \leqslant \int h_\varepsilon \partial_{x_k} \widetilde{\phi} + 2\delta = -\int \partial_{x_k} h_\varepsilon \widetilde{\phi} + 2\delta \leqslant |\partial_{x_k} h_\varepsilon|_{L^1}(1+\delta) + 2\delta.$$

Thus, by the arbitrariness of δ,

$$\liminf_{\varepsilon \to 0} \sup_k |\partial_{x_k} h_\varepsilon|_{L^1} = |h|_{BV}. \tag{1.9.6}$$

Finally, let $\widetilde{\eta} : \mathbb{R} \to \mathbb{R}_+$ and $\eta_\varepsilon(x) = \varepsilon^{-1}\widetilde{\eta}(\varepsilon^{-1}x_k)$, using first (1.9.6) for $n = 1$, then Fatu and finally arguing as in (1.9.5),

$$\begin{aligned} |\operatorname{Var}^k(h)|_{L^1} &= \int dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_d \operatorname{Var}^k h(x) \\ &= \int dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \liminf_{\varepsilon \to 0} \int dx_k |\partial_{x_k} h_\varepsilon(x)| \\ &\leqslant \liminf_{\varepsilon \to 0} |\partial_{x_k} h_\varepsilon|_{L^1} \leqslant \liminf_{\varepsilon \to 0} \sup_{\substack{\varphi \in \mathcal{C}^1 \\ |\varphi|_\infty \leqslant 1}} \int h(x) \partial_{x_k} \varphi_\varepsilon(x) \leqslant |h|_{BV}. \end{aligned}$$

□

This concludes the preliminaries concerning the choice and the properties of the Banach spaces. The next Lemma shows that the Banach spaces have the wanted compactness properties.

Lemma 1.54. *The ball* $B = \{\mu \in \mathcal{B} \ : \ \|\mu\| \leqslant 1\}$ *is relatively compact in* $(\mathcal{M}(X), |\cdot|)$.

Proof. For each $t \in \mathbb{N}$, let us consider a partition $\{A_j\}$ of $[0,1]$ into intervals of size t^{-1} and, for each $k \in \{1, \dots, d\}$, define

$$\begin{aligned} P_{t,k}\varphi(x) &= t \sum_j \mathbb{1}_{A_j}(x_k) \int_{A_j} dz \varphi(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_d) \\ P_t\varphi &= P_{t,1} \cdots P_{t,d}\varphi. \end{aligned} \tag{1.9.7}$$

First of all note that

$$P'_{t,k}\mu(\varphi) := \mu(P_{t,k}\varphi) = \int hP_{t,k}\varphi = \int P_{t,k}h \cdot \varphi.$$

Next, if $j \neq k$

$$P'_{t,k}\mu(\partial_{x_j}\varphi) = \int hP_{t,k}\partial_{x_j}\varphi = \int h\partial_{x_j}P_{t,k}\varphi \leqslant \|\mu\|.$$

and

$$P'_{t,k}\mu(\partial_{x_k}\varphi) = \int hP_{t,k}\partial_{x_k}\varphi = \|\mu\| \left| \int_0^{x_k} dx_k P_{t,k}\partial_{x_k}\varphi \right|_\infty \leqslant 4\|\mu\|.$$

In addition,

$$\mu(P_{i,k}\varphi - \varphi) = \|\mu\| \left| \int_0^{x_k} dx_k (P_{t,k}\varphi - \varphi) \right|_\infty .$$

If $x_k \in A_j = [jt^{-1}, (j+1)t^{-1}]$, then

$$\int_0^{x_k} dx_k (P_{t,k}\varphi - \varphi) = \int_{jt^{-1}}^{x_k} \varphi \leqslant |\varphi|_\infty t^{-1}.$$

Accordingly, $\|P_t'\mu\| \leqslant 4^d\|\mu\|$ and $|P_t'\mu - \mu| \leqslant 4^{d+1}t^{-1}$. In addition, notice that $P_t'\mu = t^d \sum_{i_1,\dots,i_d} \mu(\mathbb{1}_{A_{i_1}} \cdots \mathbb{1}_{A_{i_d}}) m_{A_1\times\cdots\times A_{i_d}}$, where $t^{-d} m_{A_1\times\cdots\times A_{i_d}}$ is the Lebesgue measure restricted to the set $A_1 \times \cdots \times A_{i_d}$. In other words the range of P_t' is a finite dimensional space. This implies that if $\{\mu_j\} \subset B$, then $\{P_t'\mu_j\}$ lives in a finite dimensional bounded set, hence it is compact. Thus there exists μ_t and n_j such that $\lim_{j\to\infty} \|P_t'\mu_{n_j} - \mu_t\| = 0$. In addition, for $t' \geqslant t$,

$$|\mu_t - \mu_{t'}| \leqslant |\mu_t - P_t'\mu_{n_j}| + |\mu_t - P_{t'}'\mu_{n_j}| + |P_t'\mu_{n_j} - P_{t'}'\mu_{n_j}| \leqslant Ct^{-1}$$

provided one chooses j large enough. It follows that there exists a sequence t_j and a measure μ such that $\lim_{j\to\infty} |\mu - P_{t_j}\mu_{n_j}| = 0$. □

1.9.2 Dynamical inequalities (Lasota–Yorke)

There exists $C > 0$ such that for each $\alpha \in (0,1)$, $\varepsilon > 0$ and $i \in \mathcal{I}$, there are smooth functions ϕ_i^ε supported in a $\alpha^{-i}\lambda_i^{-1}L_i\varepsilon$-neighborhood[31] of Δ_i and such that $|\phi_i^\varepsilon|_\infty = 1$, $|\phi_i^\varepsilon|_{C^1} \leqslant C\alpha^i\varepsilon^{-1}\lambda^i L_i^{-1}$ and $\phi_i^\varepsilon(x) = 1$ for all $x \in \Delta_i$. Let us define

$$\sigma' := \lim_{\varepsilon\to 0} \left| \sum_{i\in\mathcal{I}} \phi_i^\varepsilon \lambda_j L_j \right|_\infty .$$

We shall adopt the following complexity assumption on the map f:

$$\sigma' < 1 .$$

Note that, in the simple case in which the partition $\{\Delta_i\}$ is finite and can be chosen (eventually by refining it), such that $L_j = 1$, and if $\lambda = \lambda_i$, then $\sigma' = C_\Delta\lambda^{-1}$ where C_Δ is the complexity of the partition:

$$C_\Delta := \sup_{x\in X} \#\{i \in \mathcal{I} \,:\, x \in \overline{\Delta}_i\}.$$

If this is not satisfied by the map f, it will be satisfied by a higher iterate f^{n_0} if the complexity of the map grows at a subexponential rate. In this case, we would replace f by f^{n_0} in the following.

Remark 1.55. *Note that, in the following, we find more convenient to iterate measures rather than densities, even though Lemma 1.49 ensures that the measures we are interested in are indeed absolutely continuous. Recall that the relation between the pushforward f_* and the transfer operator $\mathcal{L}$ (used in the previous sections) is given by $d(f_*\mu) = (\mathcal{L}h)dx$, if $d\mu = h\,dx$.*

[31] Recall that L_i is defined just before (1.9.1).

Lemma 1.56 (Lasota–Yorke inequality)**.** *For each $\sigma \in (\sigma', 1)$ there exists a constant $B > 0$ such that, for each $\mu \in \mathcal{B}$, holds*

$$|f_*\mu| \leqslant |\mu|$$
$$\|f_*\mu\| \leqslant \sigma\|\mu\| + B|\mu|.$$

Proof. First of all notice that, if $\mu \in \mathcal{B}$, then (Remembering Lemma 1.49 and Problem 1.51)

$$|f_*\mu| = \sup_{|\varphi|_{\mathcal{C}^0}\leqslant 1} \mu(\varphi \circ f) \leqslant |\mu|.$$

Next, for all $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \leqslant 1$ and $k \in \{1, \dots, d\}$ we have

$$\begin{aligned} f_*\mu(\partial_{x_k}\varphi) &= \sum_{i\in\mathcal{I}} \mu(\mathbb{1}_{\Delta_i}(\partial_{x_k}\varphi)\circ f) \\ &= \sum_{i\in\mathcal{I}}\sum_{j=1}^{d} \mu(\mathbb{1}_{\Delta_i}\partial_{x_j}((Df)^{-1}_{kj}\varphi\circ f)) - \sum_{i\in\mathcal{I}}\sum_{j=1}^{d}\mu(\mathbb{1}_{\Delta_i}\varphi\circ f\,\partial_{x_j}((Df)^{-1}_{kj})). \end{aligned}$$

Setting $h = \frac{d\mu}{dm}$ and $\psi_{kj} = (Df)^{-1}_{kj}\varphi \circ f$, note that $\sum_j |\psi_{kj}|_\infty \leqslant \lambda_i^{-1}$. Moreover, we can rotate the coordinates as is most convenient (by redefining ψ_{kj} as well), such that

$$\begin{aligned} \mu(\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) &= \mu(\phi_i^\varepsilon \mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) \\ &\leqslant \int h(x)\partial_{x_j}\left[\phi_i^\varepsilon \int_0^{x_j} [\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}](x_1,\dots,x_{j-1},z,x_{j+1},\dots,x_d)dz\right] \\ &\quad + \lambda_i^{-1}L_i|\mu||\phi_i|_{\mathcal{C}^1}. \end{aligned}$$

Hence, remembering the hypotheses on f,

$$\begin{aligned} f_*\mu(\partial_{x_k}\varphi) &= \int \mathrm{Var}^k h \left|\sum_{i\in\mathcal{I}}\phi_i^\varepsilon\lambda_i^{-1}L_i\right|_\infty + \sum_{i\in\mathcal{I}}\lambda_i^{-1}L_i|\mu||\phi_i|_{\mathcal{C}^1} + C\mu(\|\nabla(Df)^{-1}\|) \\ &\leqslant \|\mu\|\sigma + B|\mu| + (\sigma-\sigma')\|\mu\|. \end{aligned}$$

□

1.9.3 Peripheral spectrum

It is then natural to start looking at the eigenvalues of modulus one. By the usual facts about the spectral decomposition of the operators (see Kato (1995) for the

general theory or look at Lemma A.24 and subsequent problems for the minimal facts needed here) it follows that there exists a finite set $\Theta \subset [0, 2\pi)$ such that we can write[32]

$$f_* = \sum_{\theta \in \Theta} e^{i\theta} \Pi_\theta + R$$

where Π_θ are finite rank projectors and the spectral radius of R is strictly smaller than one. Moreover, $\Pi_\theta \Pi_{\theta'} = \delta_{\theta\theta'} \Pi_\theta$, $\Pi_\theta R = R \Pi_\theta = 0$. It follows that, for each $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-ik\theta} (f_*)^k = \begin{cases} \Pi_\theta & \text{if } \theta \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Also, by Lemma 1.56 it follows that $\|\Pi_\theta \mu\| \leqslant C|\mu|$. Since Π_θ is a finite rank projector, there must exist $\mu_{\theta,l} \in \mathcal{B}$, $\ell_{\theta,l} \in \mathcal{B}^*$ such that $\Pi_\theta = \sum_l \mu_{\theta,l} \otimes \ell_{\theta,l}$, moreover $f_* \mu_{\theta,l} = e^{i\theta} \mu_{\theta,l}$ and $\ell_{\theta,l}(f_*\mu) = e^{i\theta} \ell_{\theta,l}(\mu)$ for all $\mu \in \mathcal{B}$. Hence, it must be $|\ell_{\theta,l}(\mu)| \leqslant C|\mu| = C \int |h_\mu| dm$. Since $L^\infty(X, m)$ is the dual of L^1, it follows that there exists $\bar{\ell}_{\theta,l} \in L^\infty(X, m)$ such that

$$\ell_{\theta,l}(\mu) = \int \bar{\ell}_{\theta,l} h_\mu = \mu(\bar{\ell}_{\theta,l}).$$

Hence, for each $\mu \in \mathcal{B}$,

$$\mu(\bar{\ell}_{\theta,l}) = \ell_{\theta,l}(\mu) = e^{-i\theta} \ell_{\theta,l}(f_*\mu) = e^{-i\theta} f_*\mu(\bar{\ell}_{\theta,l}) = e^{-i\theta} \mu(\bar{\ell}_{\theta,l} \circ f).$$

The above implies that $\bar{\ell}_{\theta,l} \circ f = e^{-i\theta} \bar{\ell}_{\theta,l}$ Lebesgue a.s..
Let us set $\mu_* := \Pi_0 m$.

Lemma 1.57. *For each $\ell \in L^\infty(X, m)$ such that $\ell \circ f = \ell$, m-a.s., if we define the measure $\mu(\varphi) := \mu_*(\ell\varphi)$, then μ is invariant and $\mu \in \mathcal{B}$.*

Proof. First of all notice that $f_*\mu(\varphi) = \mu_*(\ell \cdot \varphi \circ f) = \mu_*((\ell\varphi) \circ f) = \mu_*(\ell\varphi) = \mu(\varphi)$, that is μ is an invariant measure. Next, for each $\varepsilon > 0$ there exists $\ell_\varepsilon \in \mathcal{C}^1$ such that $|\ell_\varepsilon|_\infty \leqslant 2|\ell|_\infty$ and $\mu_*(|\ell - \ell_\varepsilon|) + m(|\ell - \ell_\varepsilon|) \leqslant \varepsilon$. Then, setting $\mu_\varepsilon(\varphi) := \mu_*(\ell_\varepsilon\varphi)$

$$|(f_*)^n \mu(\varphi) - (f_*)^n \mu_\varepsilon(\varphi)| \leqslant \varepsilon |\varphi|_\infty$$

[32] Remark that there cannot be Jordan blocks with eigenvector of modulus one, since this would imply that $\|(f_*)^n\|$ grows polynomially, contrary to Lemma 1.56.

implies

$$|\Pi_0\mu_\varepsilon - \mu| \leqslant \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{-ik\theta} (f_*)^k (\mu_\varepsilon - \mu) \right| \leqslant \varepsilon$$

Hence, for each $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \leqslant 1$,

$$\mu(\partial_{x_k}\varphi) = \lim_{\varepsilon\to 0} \Pi_0\mu_\varepsilon(\partial_{x_k}\varphi) \leqslant \lim_{\varepsilon\to 0} \|\Pi_0\mu_\varepsilon\| \leqslant C \lim_{\varepsilon\to 0} |\mu_\varepsilon| \leqslant C.$$

□

Thus, for each $p \in \mathbb{N}$ and $\theta \in \Theta$, the measure $\mu_{p,\theta}(\varphi) := \mu_*(\bar{\ell}^p_{\theta,i}\varphi)$ is in $\mathcal{B}$ and $f_*\mu_{p,\theta} = e^{ip\theta}\mu_{p,\theta}$. But this implies that $\{p\theta\}_{p\in\mathbb{N}} \subset \sigma_{\mathcal{B}}(f_*) \cap \{|z| = 1\}$ and since the latter is finite it must be $\theta = 2\pi\frac{s}{t}$ for some $s, t \in \mathbb{N}$. We have just proven the following

Lemma 1.58. *The peripheral spectrum of f_*, $\sigma_{\mathcal{B}}(f_*) \cap \{|z| = 1\}$, is the finite union of cyclic groups.*

1.9.4 Statistical properties

Lemma 1.59. *If the map f is topologically transitive then* 1 *is a simple eigenvalue for f_*. If all the powers of f are topologically transitive, then $\{1\}$ is the entire peripheral spectrum.*

Proof. We do the proof only for $d = 1$, as in higher dimension it is more complex (see footnote below). If one is not simple, then there exists an invariant set A, $\mu_*(A) \notin \{0, 1\}$. But then $\mathbb{1}_A \in BV$ which implies that A contains an open set, and the same applies to A^c (this is true only for $d = 1$).[33] But then, by topological transitivity, there is an orbit that visits both these open sets, hence the sets are not invariant. The same argument applied to f^n concludes the Lemma. □

[33] In higher dimensions one can have a Cantor like set with characteristic function in BV. Hence one must either use a different functional space (a convenient one in this respect has been introduced in Saussol (2000)) or use explicitly the dynamics: for example note the one can easily bound the ε neighborhood of the boundary of the partition and this, by a commonly used argument, implies that there is a large measure of points with an open neighborhood whose preimages are all bounded away from singularities. One can then proceed to prove that on such open sets the density must be continuous, showing that any invariant set must contain an open set.

In conclusion, we have obtained conditions under which the system has a unique invariant measure μ_* absolutely continuous w.r.t. Lebesgue. In addition, there exists $\rho > 0$ such that for each $\mu \in \mathcal{B}$ we have

$$\|(f_*)^n \mu - \mu_*\| \leqslant C \|\mu\| e^{-\rho n}.$$

1.9.5 Birkhoff averages

From now on we assume that one is simple and is the only eigenvalue of modulus one. Let $\varphi \in L^\infty(X, m)$, and let $\hat{\varphi} = \varphi - \mu_*(\varphi)$, then

$$m(\hat{\varphi}_n^2) = \frac{1}{n^2}\left[\sum_{k=0}^{n-1} m(\hat{\varphi}^2 \circ f^k) + 2 \sum_{j>k=0}^{n-1} m(\hat{\varphi} \circ f^j \hat{\varphi} \circ f^k)\right] \leqslant C n^{-1} |\varphi|_\infty.$$

By Chebyshev's inequality, we have

$$m(\{x \ : \ |\hat{\varphi}_n| \leqslant L^{-1}\}) \leqslant C \frac{L^2}{n}.$$

The above, by Borel–Cantelli, implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) = \mu_*(\varphi) \quad m\text{-almost surely.}$$

Actually one must apply Borel–Cantelli with some care (but this is a quite standard general strategy):

Consider the set $\mathcal{N} := \{4^k + j2^k \ : \ k \in \mathbb{N}\,, \ j < 3 \cdot 2^k\}$. Then

$$\sum_{l\in\mathcal{N}} m(\{x \ : \ |\hat{\varphi}_l| \leqslant L^{-1}\}) \leqslant C L^2 \sum_{k=0}^{\infty} \sum_{j=0}^{3\cdot 2^k} 4^{-k} \leqslant C L^2 \sum_{k=0}^{\infty} 3 \cdot 2^{-k} < \infty.$$

Hence Borel–Cantelli implies that every infinite sequence in $\mathcal{N}$ converges. Next notice that

$$|\hat{\varphi}_n - \hat{\varphi}_{n+m}| \leqslant |f|_\infty \frac{m}{n}$$

which readily implies the wanted result.

In conclusion, μ_* is a physical measure (also SRB) and the unique one. In fact one can obtain much sharper results on the behavior of the $\hat{\varphi}_n$.

2 Contracting maps

Having illustrated the power of the transfer operator approach in the expanding case, it is natural to investigate to which extent it can be generalized. A first remark is that, when it works, it automatically implies that the system either does not mix or mixes exponentially fast. Accordingly, the direct application of the above strategy is ill suited to cases in which the decay of correlations is only polynomial (although one can still apply it after inducing).

On the contrary, when the decay of correlations is expected to be exponential one can reasonably try to implement a transfer operator approach directly. In particular, it is natural to investigate the possibility to apply it to uniformly hyperbolic systems and partially hyperbolic systems. To this end there are several technical difficulties, some of them still outstanding.

Clearly the first obstacle is the existence of contracting directions. Hence, our first question is: can we find appropriate Banach spaces for which the transfer operator of a contracting map has good spectral properties? The answer is yes. In fact, again, there exist several possibilities. They all have the same flavour, although they might be quite different in the details.

For the contracting case the following choices may not be the best, e.g. see Araújo, Galatolo, and Pacifico (2014) and Blank (2001), for interesting alternatives. Yet, we present them because they serve as a stepping stone for the more

general cases treated in the following chapters.

2.1 Smooth maps

In this section we illustrate the simplest possible case: let $f \in \mathcal{C}^3(\mathbb{T},\mathbb{T})$ be an orientation preserving diffeomorphism with two fixed points, one attracting and one repelling. Without loss of generality we can assume that zero is the attracting fixed point. Let $\psi \in \mathcal{C}^2(\mathbb{T},\mathbb{R})$ be a positive function such that $\psi = 1$ in a neighbourhood of zero and $\psi = 0$ in a neighbourhood of the repelling fixed point. Also let us assume that the support of ψ be small enough so that

$$\|\psi f'\|_{\mathcal{C}^0} \leqslant \lambda^{-1} < 1.$$

Consider the transfer operator $\mathcal{L}h = (\psi h[f']^{-1})\circ f^{-1}$. For a measure $d\mu = h\,dx$ we have

$$\int \varphi \mathcal{L}h\,dx = \int \varphi\ d\,[f_*(\psi\mu)].$$

Hence $\mathcal{L}$ is the restriction to L^1 of the operator $\mu \to f_*(\psi\mu)$. In other words $\mathcal{L}$ can be naturally extended to the space of measures; abusing notation, we will still call $\mathcal{L}$ such an extension. With such a notation we have

$$\sup_{|\varphi|_{\mathcal{C}^0}\leqslant 1}\left|\int \varphi d(\mathcal{L}\mu)\right| = \sup_{|\varphi|_{\mathcal{C}^0}\leqslant 1}\left|\int \varphi\circ f\,\psi d\mu\right| \leqslant \sup_{|\varphi|_{\mathcal{C}^0}\leqslant 1}\left|\int \varphi d\mu\right|.$$

Moreover, $\mathcal{L}\delta_0 = \delta_0$, thus the spectral radius of $\mathcal{L}$, when acting on the space of measures $\mathcal{C}^0(\mathbb{T},\mathbb{R})^*$, is one. However, as in the previous example, to obtain a Lasota–Yorke inequality we need to consider the operator acting on a different space. This time the space cannot be $\mathcal{C}^1$ otherwise we would obtain a spectral radius larger than one. We need an idea.

Idea:[1] let $\mathcal{L}$ act on $(\mathcal{C}^1)^*$, the dual of $\mathcal{C}^1$.[2] For each $\varphi \in \mathcal{C}^1$, $\|\varphi\|_{\mathcal{C}^1} \leqslant 1$, we use the following notation[3]

$$\mathcal{L}h(\varphi) = \int \varphi\mathcal{L}h = \int \varphi\circ f\,\psi h = h(\varphi\circ f\,\psi),$$

[1] The following idea is more natural than it may look at first sight: the dual of $\mathcal{L}$ is, essentially, the composition with f, a contractive map. We have seen that, in such a case, looking at the action on $\mathcal{C}^1$ is a good idea. This suggests that we consider $\mathcal{L}$ acting on the dual of $\mathcal{C}^1$.

[2] Recall that $(\mathcal{C}^r)^*$ is the set of continuous linear functionals from $\mathcal{C}^r$ to $\mathbb{C}$ (or $\mathbb{R}$ if one wants to restrict to real functions) and it is a complete Banach space when equipped with the norm $\|\ell\|_{(\mathcal{C}^r)^*} = \sup_{\|\varphi\|_{\mathcal{C}^r}\leqslant 1}|\ell(\varphi)|$.

[3] This is equivalent to using the same notation for a measure and its density.

which is particularly useful when $h \in L^1 \subset (\mathcal{C}^1)^*$. Note that $\|\varphi \circ f\psi\|_{\mathcal{C}^0} \leqslant \|\varphi\|_{\mathcal{C}^0}$ while $\|(\varphi \circ f\psi)'\|_{\mathcal{C}^0} \leqslant \lambda^{-1}\|\varphi'\|_{\mathcal{C}^0} + C_\#\|\varphi\|_{\mathcal{C}^0}$. The above gives a promising estimate for the derivative but not enough to establish a Lasota–Yorke type inequality. To this end note that, for each $\varepsilon > 0$ there exists $\varphi_\varepsilon \in \mathcal{C}^2$ such that $\|\varphi_\varepsilon\|_{\mathcal{C}^1} \leqslant 1$ and $\|\varphi - \varphi_\varepsilon\|_{\mathcal{C}^0} \leqslant \varepsilon$.[4] Then, there exists $B_0 > 0$ such that

$$\begin{aligned}\left|\int \varphi \mathcal{L}h\right| &\leqslant \int |(\varphi - \varphi_\varepsilon) \circ f\psi h| + \left|\int \varphi_\varepsilon \circ f\psi h\right| \\ &\leqslant 2\lambda^{-1}\|h\|_{(\mathcal{C}^1)^*} + B_0\|h\|_{(\mathcal{C}^2)^*}\end{aligned} \tag{2.1.1}$$

where we have chosen ε small enough.

Problem 2.1. *Use computations similar to the above to show that there exists $C, B > 0$ such that, for all $n \in \mathbb{N}$ and $h \in (\mathcal{C}^1)^*$,*

$$\begin{aligned}\|\mathcal{L}^n h\|_{(\mathcal{C}^2)^*} &\leqslant C\|h\|_{(\mathcal{C}^2)^*} \\ \|\mathcal{L}^n h\|_{(\mathcal{C}^1)^*} &\leqslant C\lambda^{-n}\|h\|_{(\mathcal{C}^1)^*} + B\|h\|_{(\mathcal{C}^2)^*}.\end{aligned} \tag{2.1.2}$$

Problem 2.2. *Prove that the unit ball $\{h \in (\mathcal{C}^1)^* \ : \ \|h\|_{(\mathcal{C}^1)^*} \leqslant 1\}$ is relatively compact in $(\mathcal{C}^2)^*$.*

Problems 2.1 and 2.2 and Theorem 1.1 imply that $\mathcal{L}$, when acting on $(\mathcal{C}^1)^*$, has spectral radius one and essential spectral radius bounded by λ^{-1}. We have already seen that one belongs to the spectrum. Suppose that $e^{i\theta}$ is in the spectrum, then there exists $h_\theta \in (\mathcal{C}^1)^*$ such that, for all $\varphi \in \mathcal{C}^1$ and $n \in \mathbb{N}$,

$$\int e^{i\theta n} h_\theta \varphi = \int \mathcal{L}^n h_\theta \varphi = \int h_\theta \left[\prod_{k=0}^{n-1} \psi \circ f^k\right] \varphi \circ f^n.$$

Note that, if $\operatorname{supp}\varphi \cap \{0\} = \emptyset$, then there exists n large enough so that $\psi \cdot \varphi \circ f^n = 0$. By density this implies that $\operatorname{supp} h_\theta = \{0\}$, that is $\int h_\theta \varphi = a\varphi(0) + b\varphi'(0)$. But then $\mathcal{L}h_\theta = e^{i\theta}h_\theta$ implies, for all $\varphi \in \mathcal{C}^1$,

$$e^{i\theta}[a\varphi(0) + b\varphi'(0)] = a\varphi(0) + b\varphi'(0)f'(0)$$

which has a solution only for $\theta = 0$ and $b = 0$. In other words, one is the only eigenvalue of modulus one and it is a simple eigenvalue. It follows that

[4] Simply use a mollifier.

the only invariant measure supported outside the repelling fixed point is the delta function at zero. In addition, such a measure is exponentially mixing, that is, any measure converges (in the $(\mathcal{C}^1)^*$ topology), to such an invariant measure.[5] In the present simple situation the above fact can be proven with much simpler geometric arguments. However, we just showed that the convergence takes place also in the space of distributions, and this is a useful fact that is a bit harder to prove.

The reader who is asking herself how convergence in $(\mathcal{C}^1)^*$ relates to the more usual convergence for measures can gain some intuition by solving these exercises. First, we recall the notion of *coupling*.

Remark 2.3. *Given a compact metric space X and two Borel probability measures μ, ν a* coupling *of the two measures is a probability measure G on X^2 such that*

$$\int_{X^2} \varphi(x) G(dx, dy) = \int_X \varphi(x)\mu(dx) \quad \text{and} \quad \int_{X^2} \varphi(y) G(dx, dy) = \int_X \varphi(y)\nu(dy).$$

Let $\mathcal{G}(\mu, \nu)$ be the set of couplings of μ and ν. We can then introduce the Kantorovič (sometimes called Wasserstein) distances: for each $p \geqslant 1$,

$$d_p(\mu, \nu) = \left[\inf_{G \in \mathcal{G}(\mu,\nu)} \int_{X^2} d(x, y)^p G(dx, dy) \right]^{\frac{1}{p}}.$$

In the following we will be mostly concerned with d_1.

Problem 2.4. *Let X be a compact metric space and let $\mathcal{M}_1(X)$ denote the set of Borel probability measures. Show that d_p defines a distance on $\mathcal{M}_1(X)$.*

It is worth mentioning an important representation theorem, which we state below, but not in its most general form.

Theorem 2.5 (Kantorovič and Rubinšteĭn (1958))**.** *Let X be a compact metric space and $\mu, \nu \in \mathcal{M}_1(X)$, then*

$$d_1(\mu, \nu) = \sup \left\{ \int_X \varphi(x)(\mu - \nu)(dx) \ : \ \varphi \in \mathcal{C}^0(X, \mathbb{R}), \mathrm{Lip}(\varphi) \leqslant 1 \right\}$$

where $\mathrm{Lip}(\varphi)$ *denotes the minimal Lipschitz constant for φ.*

Problem 2.6. *Show that if X is a compact manifold, then on $\mathcal{M}_1(X)$, d_1 is equivalent to the distance $d(\mu, \nu) = \|\mu - \nu\|_{(\mathcal{C}^1)^*}$.*

Finally, all the transfer operator theory previously developed can be applied to this situation. Indeed it is a good exercise to do so.

[5] To be precise it is exponentially mixing for observables that are supported away from the expanding fixed point. Given the above estimates, it is a simple exercise to study what happens to a general observable.

2.2 Piecewise smooth maps

Next, we treat contracting piecewise smooth maps. Let $M \subset \mathbb{R}^d$ be an open set and $\mathcal{P} = \{P_i\}_{i=1}^N$ be a partition of M. That is, the P_i are disjoint open sets such that $\cup_{i=1}^N \overline{P}_i \supset M$. Finally, we consider a map $f : M \to M$ such, for each $i \in \{1, \dots, N\}$, $f|_{P_i} \in \mathcal{C}^3(P_i)$ and $\|Df\|_\infty \leqslant \lambda^{-1} < \frac{1}{2}$.[6]

Remark 2.7. *Note that if $\|Df\|_\infty < 1$, we can always achieve $\|Df\|_\infty < \frac{1}{2}$ by considering f^n, instead of f, for n large enough. We use the condition $\|Df\|_\infty < \frac{1}{2}$ only to simplify the exposition.*

If we set $\Lambda_n = \overline{f^n(M)}$, then $\Lambda_{n+1} \subset \Lambda_n$. Hence, it is well defined and not empty

$$\Lambda = \cap_{n\in\mathbb{N}} \Lambda_n.$$

The study of the general case is subtle due to the presence of discontinuities. Since we are treating this problem only for didactical purposes we are going to introduce a simplifying assumption.[7] Let $\partial\mathcal{P} = \cup_{i=1}^N \partial P_i$, then we assume

$$\Lambda \cap \partial\mathcal{P} = \emptyset. \tag{2.2.1}$$

In particular the above condition implies that, if $x \in \Lambda$, then $x \in \overline{f^n(M)}$ for each $n \in \mathbb{N}$, and $x \notin \partial\mathcal{P}$, hence $f(x) \in \overline{f^{n+1}(M)}$. Thus

$$f(\Lambda) \subset \Lambda. \tag{2.2.2}$$

To study the statistical properties of our map we would like to define a suitable transfer operator. To this end it would be convenient to avoid the discontinuities of the map. This is possible thanks to condition (2.2.1). Indeed, we can consider a function $\psi \in \mathcal{C}^\infty$ such that $\psi(x) = 1$ for all $x \in \Lambda$ while $\psi(x) = 0$ and $\nabla\psi(x) = 0$ for all $x \in \partial\mathcal{P}$. We can then define the transfer operator

$$\mathcal{L}h(x) = \sum_{y\in f^{-1}(x)} \frac{\psi(y)}{|\det(D_y f)|} h(y). \tag{2.2.3}$$

[6] The derivative is meant only for points outside the boundaries of the P_i.

[7] Such an hypothesis is likely generic in a reasonable topology, but we are not aware of such a result in the literature.

Next, we define the norms

$$\begin{aligned} \|h\|_w &:= \sup_{\|\varphi\|_{\mathcal{C}^2(M,\mathbb{C})}\leqslant 1} \int_M h\varphi \\ \|h\| &:= \sup_{\|\varphi\|_{\mathcal{C}^1(M,\mathbb{C})}\leqslant 1} \int_M h\varphi. \end{aligned} \tag{2.2.4}$$

Problem 2.8. *Prove that the closure of $\mathcal{C}^\infty$ in the $\|\cdot\|$ norm is the space of distributions $(\mathcal{C}^1)^*$, while the closure in the $\|\cdot\|_w$ norm is the space $(\mathcal{C}^2)^*$.*

The first step is to check that the above norms satisfy a Lasota–Yorke like inequality.

Lemma 2.9. *There exists a constant $B > 1$ such that, for each $h \in L^\infty$,*

$$\begin{aligned} \|\mathcal{L}h\|_w &\leqslant B\|h\|_w \\ \|\mathcal{L}h\| &\leqslant 2\lambda^{-1}\|h\| + B\|h\|_w. \end{aligned}$$

Proof.

$$\int_M \varphi\mathcal{L}h = \int_M \varphi\circ f\,\psi h.$$

Note that $\|\varphi\circ f\,\psi\|_\infty \leqslant \|\varphi\|_\infty$. Moreover, $\varphi\circ f\,\psi \in \mathcal{C}^2$ since ψ and $\nabla\psi$ are zero on the discontinuities of f. Hence,

$$\begin{aligned} &\|\nabla(\varphi\circ f\,\psi)\|_\infty \leqslant \lambda^{-1}\|\nabla\varphi\|_\infty + \|\varphi\|_\infty\|\nabla\psi\|_\infty \\ &\|D^2(\varphi\circ f\,\psi)\|_\infty \leqslant \lambda^{-2}\|D^2\varphi\|_\infty + 2\lambda^{-1}\|\nabla\varphi\|_\infty\|\nabla\psi\|_\infty + |\varphi\|_\infty\|D^2\psi\|_\infty. \end{aligned}$$

The first inequality of the statement follows. To prove the second, let $\varphi_\varepsilon \in \mathcal{C}^2$ be such that $\|\varphi - \varphi_\varepsilon\|_\infty \leqslant \varepsilon$, $\|\nabla\varphi_\varepsilon\|_\infty \leqslant \|\nabla\varphi\|_\infty$ and $\|D^2\varphi_\varepsilon\|_\infty \leqslant B\varepsilon^{-1}$. Then, write $\varphi\circ f\,\psi = (\varphi - \varphi_\varepsilon)\circ f\,\psi + \varphi_\varepsilon\circ f\,\psi =: \phi_1 + \phi_2$. It follows

$$\begin{aligned} &\|\phi_1\|_\infty \leqslant \varepsilon \\ &\|\phi_2\| \leqslant (1+\varepsilon)\|\varphi\|_\infty \\ &\|\nabla\phi_1\|_\infty \leqslant 2\lambda^{-1}\|\nabla\varphi\|_\infty + \|\varphi\|_\infty\|\nabla\psi\|_\infty. \end{aligned}$$

Hence, choosing $\varepsilon \leqslant 2\lambda^{-1}$, we have, for all $\varphi \in \mathcal{C}^1$,

$$\left|\int_M \varphi\mathcal{L}h\right| \leqslant \left|\int_M \phi_1 h\right| + \left|\int_M \phi_2 h\right| \leqslant 2\lambda^{-1}\|h\| + B\|h\|_w.$$

□

Note that the second inequality of Lemma 2.9 can be iterated yielding, for each $n \in \mathbb{N}$,

$$\|\mathcal{L}^n h\| \leqslant \left(\frac{2}{\lambda}\right)^n \|h\| + \frac{B^n}{1-2\lambda^{-1}}\|h\|_w.$$

Recalling Problem 2.2, Hennion's Theorem 1.1 implies that the spectral radius of $\mathcal{L}$, when acting on $(\mathcal{C}^1)^*$ is B and the essential spectral radius is at most $2\lambda^{-1} < 1$.

Lemma 2.10. *The spectral radius of $\mathcal{L}$, when acting on $(\mathcal{C}^1)^*$, is one. In addition, the peripheral eigenvalues have no Jordan blocks and $1 \in \sigma(\mathcal{L})$.*

Proof. Suppose that $\nu \in \sigma(\mathcal{L})$, with $|\nu| \geqslant 1$ is a maximal eigenvalue. Then, we have the spectral decomposition

$$\mathcal{L} = \sum_{i=0}^{N} \left(e^{i\theta_i}\nu\Pi_i + K_i\right) + Q$$

where $\theta_0 = 0$, N is the number of maximal eigenvalues, Π_i are projectors, K_i are nilpotent operators, $\Pi_i\Pi_j = \delta_{ij}\Pi_i$, $[\Pi_i, K_j] = 0$, $[\Pi_i, Q] = [K_i, Q] = 0$ and there exists $C > 0$ and $\sigma < |\nu|$ such that, for all $n \in \mathbb{N}$, $\|Q^n\| \leqslant C\sigma^n$. Suppose that $K_0^l \leqslant 0$ for $l < d$ while $K_i^d = 0$ for all $i \in \{0, \ldots, N\}$. Then

$$\begin{aligned}
\lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} n^{-d+1}\nu^{-n}\mathcal{L}^n &= \lim_{m\to\infty} \sum_{i=0}^{N} m^{-1} \sum_{n=0}^{m-1} n^{-d+1}\left(e^{i\theta_i}\Pi_i + \nu^{-1}K_i\right)^n \\
&= \lim_{m\to\infty} \sum_{i=0}^{N} m^{-1} \sum_{n=0}^{m-1} \sum_{l=0}^{d-1} \binom{n}{l} n^{-d+1} e^{i\theta_i(n-l)}\nu^{-l}\Pi_i K_i^l \\
&= \lim_{m\to\infty} \sum_{i=0}^{N} m^{-1} \sum_{n=0}^{m-1} \frac{1}{(d-1)!} e^{i\theta_i(n-d+1)}\Pi_i K_i^{d-1}.
\end{aligned}$$

On the other hand, if $\theta \neq 2\pi k$, $k \in \mathbb{N}$, we have

$$\left| m^{-1} \sum_{n=0}^{m-1} e^{i\theta n} \right| = m^{-1} \left| \frac{1-e^{i\theta m}}{1-e^{i\theta}} \right| \leqslant Cm^{-1}.$$

Thus,

$$\lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} n^{-d+1}\nu^{-n}\mathcal{L}^n = \frac{1}{(d-1)!}\Pi_i K_i^{d-1}.$$

On the other hand, let $h \in \mathcal{C}^0$ and $\varphi \in \mathcal{C}^1$ be such that $\int_M \varphi \Pi_i K_i^{d-1} h \neq 0$, then we have

$$\begin{aligned} 0 \neq \left| \int_M \varphi \Pi_i K_i^{d-1} h \right| &\leqslant \lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} n^{-d+1} |\nu|^{-n} \left| \int_M \varphi \mathcal{L}^n h \right| \\ &\leqslant \lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} n^{-d+1} |\nu|^{-n} \int_M |\varphi \circ f^n| \, |h| \\ &\leqslant C \lim_{m\to\infty} m^{-1} \sum_{n=0}^{m-1} n^{-d+1} |\nu|^{-n} \|h\|_\infty \|\varphi\|_\infty = 0. \end{aligned}$$

It follows that the maximal eigenvalue must have modulus one with $d = 1$, otherwise the above equation yields a contradiction. Finally, if h is a probability measure supported on Λ, then, recalling Equation (2.2.2) and the definition of ψ,

$$\int_M \mathcal{L}^n h = \int_M \psi \circ f^{n-1} \cdot \psi \circ f^{n-2} \cdots \psi \cdot h = \int_M h$$

which implies that the spectral radius cannot be smaller than one. To conclude, note that for any measure h supported in Λ we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \int_M \varphi \mathcal{L}^k h = \frac{1}{n} \sum_{k=0}^{n-1} \int_M \varphi \circ f^k h = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k h(\varphi).$$

Thus, by the weak compactness of measures, there exists a weak accumulation point h_* such that $f_* h_* = h_*$. Obviously such a measure is also supported on Λ. This implies that $\mathcal{L} h_* = h_*$, thus $1 \in \sigma(\mathcal{L})$, which concludes the Lemma. □

Note that in this case it is possible to have complex eigenvalues. For example, see the next problem.

Problem 2.11. *Suppose that there exists $x \in \Lambda$ such that $f^j(x) \neq x$, for $j < p$, and $f^p(x) = x$, that is $\{x, f(x), \ldots, f^{p-1}(x)\}$ is a periodic orbit of period p. Define*

$$\mu = \sum_{k=0}^{p-1} e^{2\pi i k/p} \delta_{f^k(x)}.$$

Show that $\mathcal{L}\mu = e^{-2\pi i/p} \mu$.

3 Toral automorphisms

The next step is to treat higher dimensional systems in which both contraction and expansion are present. The simplest such case is the *uniformly hyperbolic* case in which only expanding and contraction directions are present. Before describing some elements of the general theory we discuss in detail the simplest possible example: Toral automorphisms. For such simple systems we will discuss three different approaches that illustrate the basis of three different general theories used to investigate the statistical properties of dynamical systems.

Let us consider the map from $\mathbb{T}^2$ to itself defined by

$$f(x) = Ax \mod 1,$$

with $A \in SL(2,\mathbb{Z})$. Also, for simplicity, let us assume that $A^t = A$ and $A_{i,j} > 0$. In analogy with the previous section we can define the operator $\mathcal{L}h = h \circ f^{-1}$. Note that

$$\int_{\mathbb{T}^2} \varphi \mathcal{L} h = \int_{\mathbb{T}^2} \varphi \circ f \cdot h.$$

Simplifying even further, the reader can consider, as a concrete example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that the Lebesgue measure is invariant since $\det(A) = 1$. Moreover $\mathrm{Tr}(A) > 2$. Accordingly, the characteristic polynomial reads $t^2 + \mathrm{Tr}(A)t + 1$ and has roots λ, λ^{-1}, for some $\lambda > 1$. We call v^u, v^s the two normalized vectors such that

$$\begin{aligned} Av^u &= \lambda v^u \\ Av^s &= \lambda^{-1} v^s. \end{aligned} \tag{3.0.1}$$

Note that, since the matrix is assumed to be symmetric, $\langle v^u, v^s \rangle = 0$. We have thus a natural reference measure. In fact, $(f, \mathbb{T}^2, \mathrm{Leb})$ turns out to be *mixing*, that is: for each $h, \varphi \in \mathcal{C}^0$

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} h(x)\varphi(f^n(x))dx = \int_{\mathbb{T}^2} h(x)dx \int_{\mathbb{T}^2} \varphi(x)dx.$$

Alternatively, the mixing can be stated in the following equivalent way: for each probability measure μ such that $\frac{d\mu}{d\,\mathrm{Leb}} = h \in L^1$ and, for each $\varphi \in \mathcal{C}^0$,[1]

$$\lim_{n\to\infty} f_*^n \mu(\varphi) = \mathrm{Leb}(\varphi). \tag{3.0.2}$$

This is a very relevant property from the applied point of view: it says that asymptotically our system is described by the Lebesgue measure regardless of the initial distribution (provided the initial condition was distributed according to a measure absolutely continuous with respect to Lebesgue).

Of course, property (3.0.2) is truly useful only if the speed in the convergence to the limit is fast enough. From this consideration follows the basic question that we want to address in the following:
What is the speed of convergence in the limit (3.0.2) ?

3.1 Standard pairs

The first technique that we are going to illustrate is based on the idea of *coupling* in probability. This is a widely used tool to study the convergence to equilibrium of Markov chains. A similar technique was previously used in abstract ergodic theory under the name of *joining*. The form we are going to describe was introduced in smooth ergodic theory by Young (1999) and further developed by Dolgopyat.

The basic idea is to consider a special class of measures that behave under push-forward in a manner similar to that encountered in expanding maps. Such a

[1] Recall that $\mu(\varphi) = \int_{\mathbb{T}^2} \varphi(x)h(x)dx$ and $f_*\mu(\varphi) = \mu(\varphi \circ f)$.

class of measures has a long history (e.g. from Pesin and Sinai (1982) to Liverani (1995a)), but they have been systematically developed and used by Dolgopyat under the name of *standard pairs*, Dolgopyat (2004a,b). Fix some $a > 1$ and define

$$D_a = \left\{ h \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+) \ : \ \forall t, s \in \mathbb{R}, \ \frac{h(t)}{h(s)} \leqslant e^{a|t-s|} \right\}.$$

Also, for each $b \in \mathbb{R}_+$, $x \in \mathbb{T}^2$ and $h \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}_+)$, $\int_{-b}^{b} h = 1$, define the measure on $\mathbb{T}^2$ (*standard pair*)

$$\mu_{b,x,h}(\varphi) = \int_{-b}^{b} h(t)\varphi(x + tv^u)dt.$$

The collection of standard pairs will be designated by

$$S_a = \left\{ \mu_{b,x,h} \ : \ b \in [1/2, 1], \ x \in \mathbb{T}^2, \ h \in D_a, \ \int_{-b}^{b} h = 1 \right\}.$$

The above are our building blocks. Let us see what we can construct with them. First of all, we can take the convex hull: for each finite set $\{p_i\}$ of positive numbers such that $\sum_i p_i = 1$ and set $\{\mu_i\} \subset S_a$ we can consider the probability measure

$$\mu = \sum_i p_i \mu_i, \tag{3.1.1}$$

where the p_i are called the *masses* of the standard pairs. The set $\{\mu_i, p_i\}$ is called a *standard family* and is often confused with the measure it defines via (3.1.1). Note however that the representation of a measure by a standard family, if it exists, is far from being unique. We will call $\mathcal{S}_a$ the set of all standard families. The first important fact is the following.

Lemma 3.1. *The Lebesgue measure belongs to the weak closure of* $\mathcal{S}_a$.[2]

Proof. Letting $v^u = (1 + u^2)^{-\frac{1}{2}}(1, u)$, for each $\varphi \in \mathcal{C}^0$,

$$\mathrm{Leb}(\varphi) = \int_0^1 dt \int_0^1 ds\varphi(t, s + ut) = \int_0^1 ds \int_0^{\sqrt{1+u^2}} dt\varphi(se_2 + tv^u).$$

Note that the the second integral can be written as the convex combination of finitely many standard pairs. The result follows since the first integral is the limit of finite sums. □

[2]Recall that μ_n converges weakly to μ if, for all $\varphi \in \mathcal{C}^0$, we have $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$.

Next we want to know how the standard pairs behave under push forward.

Lemma 3.2. *For each $n \in \mathbb{N}$ and $\mu \in \mathcal{S}_a$ it holds true that $f_*^n \mu \in \mathcal{S}_{\lambda^{-n}a}$.*

Proof. It suffices to prove that if $\mu \in S_a$ then $f_*^n \mu \in \mathcal{S}_{\lambda^{-n}a}$. Then, recalling (3.0.1),

$$f_*^n \mu_{b,x,h}(\varphi) = \int_{-b}^{b} h(t)\varphi(f^n(x) + t\lambda^n v^u) = \int_{-\lambda^n b}^{\lambda^n b} h(t\lambda^{-n})\varphi(f^n(x) + tv^u).$$

Next, let $\delta \in [1/2, 1]$ and $K \in \mathbb{N}$ such that $\lambda^n b = 2K\delta$ and define $t_i = -\lambda^n b + (2i+1)\delta$. We can then write

$$\begin{aligned} f_*^n \mu_{b,x,h}(\varphi) &= \sum_{i=0}^{K-1} p_i \int_{-\delta}^{\delta} h_i(t)\varphi([f^n(x) + t_i v^u] + tv^u)dt \\ p_i &= \int_{-\delta}^{\delta} h(\lambda^{-n}(t_i + t))dt \\ h_i(t) &= p_i^{-1} h(\lambda^{-n}(t_i + t)). \end{aligned}$$

Accordingly, the Lemma is proven provided $h_i \in D_{\lambda^{-n}a}$. This follows from

$$\frac{h_i(t)}{h_i(s)} = \frac{h(\lambda^{-n}(t_i + t))}{h(\lambda^{-n}(t_i + s))} \leqslant e^{a\lambda^{-n}|t-s|}.$$

□

Remark 3.3. *Note that the unbounded parameter contraction proven in the previous Lemma is a peculiarity of the linear systems we are studying. However in the nonlinear case a fixed contraction still takes place (provided a is large enough) and this is all we will use in the following.*

To continue, we call two standard pairs $\mu_1 = \mu_{b,x,h}$ and $\mu_2 = \mu_{b,x+sv^s,h}$, $s \in [1, 2]$, *matching*, while we call *prematching* two standard pairs of the form $\mu_1 = \mu_{b,x,h_1}$, $\mu_2 = \mu_{b,x+sv^s,h_2}$. The basic fact underlying our strategy is the following:

Lemma 3.4. *Let μ_1, μ_2 be two matching standard pairs, then, for each $\varphi \in \mathcal{C}^1$,*[3]

$$|f_*^n \mu_1(\varphi) - f_*^n \mu_2(\varphi)| \leqslant 2be^{ab} \|\partial_s \varphi\|_\infty \lambda^{-n}.$$

[3]We are using the notation $\partial_s \varphi = \langle v^s, \nabla \varphi \rangle$.

Proof. It follows by a direct computation:

$$\begin{aligned}\left|f_*^n\mu_1(\varphi) - f_*^n\mu_2(\varphi)\right| &= \left|\int_{-b}^{b} h(t)[\varphi(f^n(x) + \lambda^{-n}sv^s + \lambda^n tv^u) - \varphi(f^n(x) + \lambda^n tv^u)]\right| \\ &\leqslant 2be^{ab}\|\partial_s\varphi\|_\infty\lambda^{-n}.\end{aligned}$$

□

The above Lemma is really a *coupling* between the two measures, see Remark 2.3. The Lemma shows that the convenient topology in which to study the convergence of the push-forward of standard pairs is $(\mathcal{C}^1)^*$. In other words, it suggests that it is natural to consider *distributions* rather than measures. Indeed, this is consistent with our discussion of the contracting case in Chapter 2.

Remark 3.5. *The following is a coupling between two matching standard pairs $\mu_1 = \mu_{b,x,h}$ and $\mu_2 = \mu_{b,x+sv^s,h}$:*

$$G(\varphi) = \int_{[-b,b]^2} \varphi(x + tv^u, x + sv^s + tv^u)h(t)dt.$$

Using such a coupling we can reinterpret the proof of Lemma 3.4 to obtain,[4] *recalling Remark 2.3,*

$$d_1(f_*^n\mu_1, f^*\mu_2) = \inf_{G'\in\mathcal{G}(f_*^n\mu_1, f_*^n\mu_2)} \int_{\mathbb{T}^4} d(x,y)G'(dx,dy) \leqslant 2be^{ab}\lambda^{-n},$$

where $d(x,y) = \inf_{k\in\mathbb{T}^2}\|x - y + k\|$. Also it is not hard to prove that in this case the topology associated to the distance d_1 is the weak topology.

With these definitions in place we are now ready to argue: given two standard pairs μ_1, μ_2, we know that $f_*^n\mu_1$, $f_*^n\mu_2$ are standard families in $\mathcal{S}_{\lambda^{-n}a}$. Note that there is some freedom in how to divide a segment of length $\lambda^n b$ in segments of length between 1 and 2. In particular one can check that, if n is large enough, one can make the division so that the two families contain two prematching standard pairs. That is, there exists a standard pair in the first family supported on $\{y +$

[4] Indeed, for the stated coupling G of $f_*^n\mu_1$, $f_*^n\mu_2$,

$$\left|f_*^n\mu_1(\varphi) - f_*^n\mu_2(\varphi)\right| = \left|\int_{\mathbb{T}^4}[\varphi(x) - \varphi(y)]G(dx,dy)\right| \leqslant \|\partial_s\varphi\|_\infty d_1(f_*^n\mu_1, f_*^n\mu_2).$$

$tv^u\}_{t\in[-b,b]}$ and a standard pair, in the second family, supported on $\{y + sv^s + tv^u\}_{t\in[-b,b]}$ for some $b \in [1/2, 1]$, $s \in [1, 2]$ and $y \in \mathbb{T}^2$. This is a consequence of the fact that the flow $\phi_t(y) = y + tv^u$ is ergodic, since the ratio of the components of v^u is irrational.

Accordingly, for n large enough, $\lambda^n > 2$ and there exist prematching standard pairs for any initial couple of standard pairs. Let n_0 be the smallest such n. Also, we call the two prematching standard pairs $\widetilde{\mu}_{0,1}$ and $\widetilde{\mu}_{0,2}$ respectively. Thus we can write[5]

$$f_*^{n_0}\mu_1(\varphi) - f_*^{n_0}\mu_2(\varphi) = \\ = \sum_{j=1}^{m_1} \widetilde{p}_{j,1}\widetilde{\mu}_{j,1}(\varphi) - \sum_{j=1}^{m_1} \widetilde{p}_{j,2}\widetilde{\mu}_{j,2}(\varphi) + \widetilde{p}_{0,1}\widetilde{\mu}_{0,1}(\varphi) - \widetilde{p}_{0,2}\widetilde{\mu}_{0,2}(\varphi)$$

for some weights $\widetilde{p}_{j,i} \geqslant 0$ and standard pairs $\widetilde{\mu}_{j,i} \in \mathcal{S}_{\lambda^{-n_0}a}$. Note that, if $\widetilde{p}_{j,i} \neq 0$, then $\widetilde{p}_{j,i} \geqslant (2\lambda^{n_0}e^{2a})^{-1}$ by construction. Also we know that

$$\widetilde{\mu}_{0,1}(\varphi) = \int_{-b_0}^{b_0} h_{0,1}(t)\varphi(y + tv^u)dt$$
$$\widetilde{\mu}_{0,2}(\varphi) = \int_{-b_0}^{b_0} h_{0,2}(t)\varphi(y + sv^s + tv^u)dt$$

for some $b_0 \in [1, 2]$, $y \in \mathbb{T}^2$ and $h_{0,i} \in D_{\lambda^{-n_0}a}$.

To obtain a convergence to equilibrium we want to show that some part of the push-forward measures behaves similarly. The tool to do so will be to use Lemma 3.4. To this end we have to exhibit matching standard pairs.

The idea to construct matching standard pairs is to single out a common part of the density by using the fact that $h_{0,i} \geqslant e^{-2\lambda^{-n_0}a}b_0^{-1}$. Of course we want to still have standard pairs, hence a small computation is called for. For each $c > 0$

[5] We can always arrange it so that the two standard families obtained by pushing forward have the same number of elements m_1, for example by allowing some of the $\widetilde{p}_{j,i}$ to be zero or by duplicating the same standard pair giving half of the mass to each copy.

small enough,

$$\begin{aligned}\frac{h_{0,i}(t)-\frac{c}{2b_0}}{h_{0,i}(s)-\frac{c}{2b_0}} &\leqslant \frac{h_{0,i}(s)e^{\lambda^{-n_0}a|t-s|}-\frac{c}{2b_0}}{h_{0,i}(s)-\frac{c}{2b_0}}\\ &\leqslant e^{\lambda^{-n_0}a|t-s|}\frac{h_{0,i}(s)-\frac{c}{2b_0}e^{-\lambda^{-n_0}a|t-s|}}{h_{0,i}(s)-\frac{c}{2b_0}}\\ &\leqslant e^{\lambda^{-n_0}a|t-s|}\left[1+c\frac{1-e^{-\lambda^{-n_0}a|t-s|}}{2e^{-2\lambda^{-n_0}a}-c}\right]\\ &\leqslant e^{\lambda^{-n_0}a|t-s|}\left[1+c\frac{\lambda^{-n_0}a|t-s|}{2e^{-2\lambda^{-n_0}a}-c}\right].\end{aligned}$$

Finally we choose c so small that

$$\gamma=\frac{c}{2e^{-2\lambda^{-n_0}a}-c}\leqslant 1.$$

Hence

$$\frac{h_{0,i}(t)-\frac{c}{2b_0}}{h_{0,i}(s)-\frac{c}{2b_0}}\leqslant e^{\lambda^{-n_0}a(1+\gamma)|t-s|}\leqslant e^{a|t-s|}.$$

This means that we can write

$$\begin{aligned}\tilde{\mu}_{0,1}(\varphi)-\tilde{\mu}_{0,2}(\varphi)&=c\int_{-b_0}^{b_0}\frac{1}{2b_0}[\varphi(y+tv^u)-\varphi(y+sv^s+tv^u)]dt\\ &+(1-c)\left[\int_{-b_0}^{b_0}\frac{h_{0,1}(t)-\frac{c}{2b_0}}{1-c}\varphi(y+tv^u)-\int_{-b_0}^{b_0}\frac{h_{0,2}(t)-\frac{c}{2b_0}}{1-c}\varphi(y+sv^s+tv^u)dt\right].\end{aligned}$$

Note that we have constructed two matching standard pairs with mass c.

We are almost done. The only remaining problem is that the two prematching standard pairs come with different masses. To take care of this we have to rearrange a bit the standard families. Unfortunately the notation is rather unpleasant but if the reader manages to see through the notation she will realise that the strategy is the obvious one.

Let $p_* = \min\{\widetilde{p}_{0,1}, \widetilde{p}_{0,2}\}$, $p_{0,i} = \widetilde{p}_{0,i} - p_* c$ and define

$$
\begin{aligned}
p_{0,i} &= \frac{\widetilde{p}_{0,i} - p_* c}{1 - p_* c}\,; \quad p_{j,i} = \frac{\widetilde{p}_{j,i}}{1 - p_* c} \quad \forall j \in \{1, \dots, m_1\} \\
\mu_{0,1}(\varphi) &= \int_{-b_0}^{b_0} \frac{\widetilde{p}_{0,1} h_{0,1}(t) - \frac{p_* c}{2b_0}}{\widetilde{p}_{0,1} - p_* c} \varphi(y + t v^u) dt \\
\mu_{0,2}(\varphi) &= \int_{-b_0}^{b_0} \frac{\widetilde{p}_{0,2} h_{0,2}(t) - \frac{p_* c}{2b_0}}{\widetilde{p}_{0,2} - p_* c} \varphi(y + s v^s + t v^u) dt \\
\mu^*_{0,1}(\varphi) &= \int_{-b_0}^{b_0} \frac{1}{2b_0} [\varphi(y + t v^u) - \varphi(y + t v^u)] dt \\
\mu^*_{0,2}(\varphi) &= \int_{-b_0}^{b_0} \frac{1}{2b_0} [\varphi(y + t v^u) - \varphi(y + s v^s + t v^u)] dt \\
\mu_{j,i} &= \widetilde{\mu}_{j,i} \quad \forall j \in \{1, \dots, m_1\}.
\end{aligned}
$$

The $\mu^*_{0,i}$ are matching standard pairs, $\mu_{0,i}$ are standard pairs, $\sum_{j=0}^{m_1} p_{j,i} = 1$ and

$$
f_*^{n_0} \mu_i(\varphi) = c p_* \mu^*_{0,i}(\varphi) + (1 - c p_*) \sum_{j=0}^{m_1} p_{j,i} \mu_{j,i}(\varphi).
$$

Then, for each $n \geqslant n_0$, by Lemma 3.4 we have

$$
\begin{aligned}
&\left| f_*^n \mu_1(\varphi) - f_*^n \mu_2(\varphi) - (1 - p_* c) \left[\sum_{j=0}^{m_1} p_{j,1} f_*^{n-n_0} \mu_{j,1}(\varphi) - \sum_{j=0}^{m_1} p_{j,2} f_*^{n-n_0} \mu_{2,1}(\varphi) \right] \right| \\
&\qquad \leqslant c p_* 4 b e^{ab} \|\partial_s \varphi\|_\infty \lambda^{-n+n_0}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left| f_*^n \mu_1(\varphi) - f_*^n \mu_2(\varphi) - (1 - p_* c) \sum_{j,k=0}^{m_1} p_{j,1} p_{k,2} \left[f_*^{n-n_0} \mu_{j,1}(\varphi) - f_*^{n-n_0} \mu_{k,2}(\varphi) \right] \right| \\
&\qquad \leqslant c p_* 2 b e^{ab} \|\partial_s \varphi\|_\infty \lambda^{-n+n_0}.
\end{aligned}
$$

To conclude it suffices to iterate the above formula, applying it to each pair of

standard pairs $\mu_{j,1}, \mu_{k,2}$. Let $n = \ell n_0$, then

$$\begin{aligned}|f_*^n\mu_1(\varphi) - f_*^n\mu_2(\varphi)| &\leqslant 2(1-p_*c)^\ell\|\varphi\|_\infty + \\ &\quad + \sum_{k=0}^{\ell-1} cp_*2be^{ab}\|\partial_s\varphi\|_\infty(1-p_*c)^k\lambda^{-n+(k+1)n_0} \\ &\leqslant C\nu^n(\|\varphi\|_\infty + \|\partial_s\varphi\|_\infty)\end{aligned}$$

for some $C > 0$ and $\nu = \max\{(1-p_*c)^{1/n_0}, \lambda^{-1}\}$. The same estimate carries over to standard families and hence to the weak closure of $\mathcal{S}_a$. The reader can check, arguing similarly to Lemma 3.1, that the above implies that for each $h \in \mathcal{C}^1$,

$$\left|\int_{\mathbb{T}^2} h(x)\varphi\circ T^n(x)dx - \int_{\mathbb{T}^2}\varphi(x)dx\right| \leqslant C(\|h\|_\infty+\|\partial_u h\|_\infty)(\|\varphi\|_\infty+\|\partial_s\varphi\|_\infty)\nu^n.$$

We have thus established that the map is mixing and that the speed of mixing is exponential with a prefactor depending on the smoothness of h along the unstable direction and the smoothness of φ along the stable direction.

3.2 Fourier transform

The standard pairs method is very flexible and can be adapted to a large range of situations. Yet, since the maps we are presently studying are linear, a much more powerful tool is available: Fourier series. Indeed, for each $k \in \mathbb{Z}^2$,

$$\begin{aligned}(\widehat{\mathcal{L}^n h})_k &= \int_{\mathbb{T}^2} e^{2\pi i\langle k,x\rangle}\mathcal{L}^n h(x)dx = \int_{\mathbb{T}^2} e^{2\pi i\langle k,A^n x\rangle}h(x)dx \\ &= \int_{\mathbb{T}^2} e^{2\pi i\langle A^n k,x\rangle}h(x)dx = \hat{h}_{A^n k}.\end{aligned} \tag{3.2.1}$$

Accordingly, for each $h, \varphi \in \mathcal{C}^r$,

$$\begin{aligned}\left|\int_{\mathbb{T}^2}\varphi\mathcal{L}^{2n}h - \int\varphi\right| &\leqslant \sum_{k\in\mathbb{Z}^2/\{0\}}|\hat{\varphi}_k\hat{h}_{A^{2n}k}| \\ &\leqslant \sum_{k\in\mathbb{Z}^2/\{0\}}\frac{\|h\|_{\mathcal{C}^r}\|\varphi\|_{\mathcal{C}^r}}{(\|A^{2n}k\|+1)^r(\|k\|+1)^r} \\ &\leqslant \sum_{k\in\mathbb{Z}^2/\{0\}}\frac{\|h\|_{\mathcal{C}^r}\|\varphi\|_{\mathcal{C}^r}}{(\|A^n k\|+1)^r(\|A^{-n}k\|+1)^r}.\end{aligned}$$

For each $k \in \mathbb{R}^2$, we write $k = av^u + bv^s$ (recall (3.0.1)). It follows that $A^n k = a\lambda^n v^u + b\lambda^{-n} v^s$ and $A^{-n}k = a\lambda^{-n} v^u + b\lambda^n v^s$. Thus

$$\|A^{-n}k\|^2 + \|A^n k\|^2 \geqslant (b^2 + a^2)\lambda^{2n} = \|k\|^2 \lambda^{2n}.$$

Accordingly,

$$(\|A^n k\| + 1)(\|A^{-n}k\| + 1) \geqslant \|k\|\lambda^n.$$

We can thus conclude, for all $r > 2$,

$$\left| \int_{\mathbb{T}^2} \varphi \mathcal{L}^{2n} h - \int \varphi \right| \leqslant \sum_{k \in \mathbb{Z}^2/\{0\}} \frac{\|h\|_{\mathcal{C}^r} \|\varphi\|_{\mathcal{C}^r}}{\|k\|^r} \lambda^{-nr} \leqslant C_r \|h\|_{\mathcal{C}^r} \|\varphi\|_{\mathcal{C}^r} \lambda^{-nr},$$

for some constant C_r independent on h and φ.

We have thus proven, again, that toral automorphisms enjoy exponential decay of correlation but we have also uncovered a new phenomenon: the speed of decay depends very much on the smoothness of the functions.

Yet, there are also reasons of unhappiness: the requirement on the smoothness of the functions (more than $\mathcal{C}^2$) is stronger than the one obtained by using standard pairs. In addition our argument does not look very dynamical and seems to take too much advantage of the special features of the example at hand. What to do with a nonlinear map is highly non-obvious.

It would then be very desirable to obtain the above results via a different, more dynamical, strategy. In particular it would be nice if we could find a Banach space on which it is possible to study the spectrum of the operator $\mathcal{L}$ and such that the above properties can be understood as consequences of the spectral picture.

This can be done in various ways. Let us start with a possibility still based on the Fourier transform.

3.3 A simple class of *Sobolev like* norms

To define a Banach space we can first define a norm on $\mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$ and then we obtain the Banach space by completing $\mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$ with respect to such a norm. The usual Sobolev norms are $\|h\|_p^2 = \sum_{k \in \mathbb{Z}^2} \langle k \rangle^p |\hat{h}_k|^2$ where $\langle k \rangle = 1 + \|k\|^2$ and $p \in \mathbb{R}$. If $p > 0$ then a finite norm implies some regularity while if $p < 0$ also distributions can have a finite norm. However we have learned that hyperbolic dynamics have very different behaviour depending on the direction. Typically $\mathcal{L}^n h$

will be a function regular in the unstable directions but with very wild oscillations in the stable directions. Hence along the stable directions we can have convergence only in a weak sense: in the sense of distributions. To handle this problem different strategies have been proposed. The simplest one is to consider anisotropic Sobolev spaces, that is spaces defined by a norm of the type

$$\|h\|^2_{p\alpha} = \sum_{k\in\mathbb{Z}^2} \langle k\rangle^{p\alpha(\hat{k})} |\hat{h}_k|^2 \tag{3.3.1}$$

where $p \in \mathbb{R}_+$, $\hat{k} = (k_1 : k_2)$ is the projectivization of $k = (k_1, k_2)$, that is the equivalence class containing k with respect to the equivalence relation defined by $k \sim k'$ iff there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $k = \lambda k'$. Finally, $\alpha \in \mathcal{C}^0(\mathbf{P}^1(\mathbb{R}), [-1, 1])$. In other words α depends only on the direction of the vector k. In the following, to simplify notation, we will write $\alpha(\hat{k})$ as $\alpha(k)$.

We have seen that the action of the dynamics in Fourier coefficients is also given by Ak. It is then natural to consider the dynamics in the projective space $\mathbf{P}^1(\mathbb{R})$. Obviously there are two fixed points, v^u and v^s (or, rather, their equivalence classes); the first is attracting while the second is repelling. Fix $\nu \in (\lambda^{-1}, 1)$. It is easy to check that in $\mathbf{P}^1(\mathbb{R})$ there exist intervals $I_+ \ni v^u$, $I_- \ni v^s$ and a constant $K > 0$ such that[6]

$$\begin{aligned} \langle Av\rangle &\geqslant \nu^{-2}\langle v\rangle \quad \text{for all } v \in I_+,\ \|v\| \geqslant K \\ \langle Av\rangle &\leqslant \nu^{2}\langle v\rangle \quad \text{for all } v \in I_-,\ \|v\| \geqslant K. \end{aligned}$$

Let $\hat{I}_\pm = A^{\pm 1} I_\pm \subset I_\pm$. We choose then an α with value 1 in $\hat{I}_+$, value -1 in $\hat{I}_-$ and strictly monotone in between (it is possible to be more explicit about α and optimize it in various ways, but we think it is more important to point out that the above qualitative properties suffice). Note that in $\mathbf{P}^1(\mathbb{R}) \setminus (\hat{I}_+ \cup \hat{I}_-)$ we have that $d(v, Av) \geqslant c$ for some fixed constant c,[7] thus there exists $\gamma > 0$ such that

$$\alpha(v) - \alpha(A^{-1}v) \geqslant \gamma \quad \text{for all } v \notin I_+ \cup I_-. \tag{3.3.2}$$

This defines the norm.

Problem 3.6. *Let, $\alpha, \beta \in \mathcal{C}^0(\mathbf{P}^1(\mathbb{R}), \mathbb{R})$ and $c > 0$ such that $\beta(k) + c \leqslant \alpha(k)$. Prove that the set $\{h : \|h\|_\alpha \leqslant 1\}$ is weakly compact in the norm $\|\cdot\|_\beta$.*

[6] Of course, I_+, I_- correspond to cones in the vector space $\mathbb{R}^2$. We will abuse notation and use I_+, I_- also for the cones of the vectors whose equivalence class belongs to I_+, I_-, respectively.

[7] The definition of the distance is not really important, for example the angle between the two vectors will do.

From equation (3.2.1) it follows that, for all $p \in \mathbb{R}_+$,

$$\|\mathcal{L}h\|_{p\alpha}^2 = \sum_{k\in\mathbb{Z}^2} \langle k\rangle^{p\alpha(\hat{k})} |\hat{h}_{Ak}|^2 = \sum_{k\in\mathbb{Z}^2} \left[\frac{\langle A^{-1}k\rangle^{\alpha(A^{-1}k)}}{\langle k\rangle^{\alpha(k)}}\right]^p \langle k\rangle^{p\alpha(k)} |\hat{h}_k|^2.$$

If $k \in \hat{I}_+$ and $\|v\| \geqslant K$ then

$$\frac{\langle A^{-1}k\rangle^{\alpha(A^{-1}k)}}{\langle k\rangle^{\alpha(k)}} \leqslant \frac{\langle A^{-1}k\rangle}{\langle k\rangle} \leqslant \nu^2.$$

If $k \in \hat{I}_-$ and $\|v\| \geqslant K$, then $Ak \in \hat{I}_-$ and

$$\frac{\langle A^{-1}k\rangle^{\alpha(A^{-1}k)}}{\langle k\rangle^{\alpha(k)}} = \frac{\langle k\rangle}{\langle A^{-1}k\rangle} \leqslant \nu^2.$$

If $k \notin \hat{I}_- \cup \hat{I}_+$ then, setting $B = \|A^{-1}\|$ and recalling (3.3.2),

$$\frac{\langle A^{-1}k\rangle^{\alpha(A^{-1}k)}}{\langle k\rangle^{\alpha(k)}} \leqslant \frac{\langle A^{-1}k\rangle^{\alpha(k)-\gamma}}{\langle k\rangle^{\alpha(k)}} \leqslant B\langle k\rangle^{-\gamma}.$$

It is then natural to consider the set[8]

$$\Gamma = \{k \in \mathbb{Z}^2 \ : \ \langle k\rangle \leqslant \max\{[\nu^{-2}B]^{1/\gamma}, K\} =: L\}.$$

Hence,

$$\sup_{k\notin\Gamma} \frac{\langle A^{-1}k\rangle^{\alpha(A^{-1}k)}}{\langle k\rangle^{\alpha(k)}} \leqslant \nu^2\,.$$

Define the weak norm,

$$\|h\|_w^2 = \sum_{k\in\Gamma} |\hat{h}_k|^2.$$

We can then write

$$\|\mathcal{L}h\|_{p\alpha} \leqslant \sqrt{\nu^{2p}\|h\|_{p\alpha}^2 + B\|h\|_w^2} \leqslant \nu^p\|h\|_{p\alpha} + B^{2p}L^p\|h\|_w. \qquad (3.3.3)$$

Problem 3.7. *Use equation* (3.3.3) *to obtain a Lasota–Yorke type inequality and deduce the quasi compactness of* $\mathcal{L}$ *(recall Remark 1.2).*

[8]Note that Γ is a finite set.

For the reader's amusement, let us deduce quasi-compactness by an alternative argument. Note that setting $Ph(x) = \sum_{k\in\Gamma} \hat{h}_k e^{2\pi\langle k,x\rangle}$ we have

$$\|\mathcal{L}(1-P)h\|_{p\alpha} \leqslant \nu^p \|h\|_{p\alpha}.$$

We can then set $A = \mathcal{L}P$ and $Q = \mathcal{L}(1-P)$, then, for each $\mu > \nu^p$, we can write

$$(\mu\mathbb{1} - \mathcal{L}) = (\mathbb{1}\mu - Q)^{-1}(\mathbb{1} - A(\mathbb{1}\mu - Q)^{-1}).$$

The claim follows then by the Analytic Fredholm alternative. We then conclude that the essential spectrum of $\mathcal{L}$ when acting on the Banach space obtained by closing $\mathcal{C}^\infty$ with respect to the norm $\|\cdot\|_{p\alpha}$ is contained in the set $\{z \in \mathbb{C} \,:\, |z| \leqslant \nu^p\}$. To study the discrete spectrum and obtain independently that it consists only of $\{1\}$ requires a little extra argument that we postpone to the end of Section 3.4, see Lemma 3.12 if you cannot contain your curiosity.

The above is not as precise as our explicit computation (also due to the choice to reduce the technicalities to a bare minimum) but it provides the main idea for a much more far reaching approach.

3.4 A simple class of *geometric* norms

We have seen how the anisotropy of the dynamics can be reflected by the norms using a weight (at one time called *escape function*) in the Fourier transform. Here we present (always in a simplified manner, adapted to the special case at hand) a different, more geometric, approach that has both advantages (it has been adapted to more general systems, e.g. Baladi, Demers, and Liverani (2018)) and disadvantages (for example, the dual of the space is not a space of the same type). The presentation is a bit more detailed than the one in Section 3.3 as we will use it as the basis for further generalizations, see Chapter 4.

Let $\partial_u\varphi = \langle v^u, \nabla\varphi\rangle$, fix $\delta > 0$, $\varphi \in \mathcal{C}_0^\infty([-\delta,\delta],\mathbb{C})$ and $h \in \mathcal{C}^\infty(\mathbb{T}^2,\mathbb{C})$. Define,[9]

$$\begin{aligned}
|\varphi|_q &= \sup_{q'\leqslant q}\sup_{t\in\mathbb{R}} |\varphi^{(q')}(t)| \\
B_q &= \{\varphi \in \mathcal{C}_0^\infty([-\delta,\delta],\mathbb{C}) \,:\, |\varphi|_q \leqslant 1\} \\
\|h\|_{p,q} &= \sup_{x\in\mathbb{T}^2}\sup_{p'\leqslant p}\sup_{\varphi\in B_q} \int_{-\delta}^{\delta} (\partial_u^{p'} h)(x + t v^s)\cdot\varphi(t)dt.
\end{aligned} \tag{3.4.1}$$

[9] We use the notation $\varphi^{(q)}(t) = \frac{d^q}{dt^q}\varphi(t)$.

We will call $\mathcal{B}^{p,q}$ the completion of $\mathcal{C}^\infty$ with respect to the above norm. The first thing we want to understand is which kinds of objects we obtain by this completion. The next Lemma shows that we are inside the usual space of distributions.

Lemma 3.8. *For each $p, q \in \mathbb{N}$, $p > 0$, we have $\boldsymbol{i} : \mathcal{B}^{p,q} \to \mathcal{C}^q(\mathbb{T}^2, \mathbb{C})^*$, where i is bounded and one-to-one.*

Proof. As usual, define $\boldsymbol{i} : \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C}) \to \mathcal{C}^q(\mathbb{T}^2, \mathbb{C})^*$ by $\boldsymbol{i}(h)(\varphi) = \int_{\mathbb{T}^2} \varphi h$.

Let $\{\phi_j\}_{j=1}^N$ be a smooth partition of unity such that $\operatorname{supp} \phi_j$ is contained in a ball of radius $\delta/2$ with centre x_j. Let $h \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$. For each $\varphi \in \mathcal{C}^q(\mathbb{T}^2, \mathbb{C})$ we have

$$\begin{aligned}
|\boldsymbol{i}(h)(\varphi)| &= \left| \int_{\mathbb{T}^2} h\varphi \right| \leqslant \sum_j \left| \int_{\mathbb{T}^2} h\varphi\phi_j \right| \\
&\leqslant \sum_j \int_{-\delta}^{\delta} ds \left| \int_{-\delta}^{\delta} dt h(x_j + sv^s + tv^u)(\varphi\phi_j)(x_j + sv^s + tv^u) \right| \\
&\leqslant 2\delta \|h\|_{0,q} \sum_j |\varphi\phi_j|_{\mathcal{C}^q} \leqslant C_{\delta,q} \|h\|_{p,q} |\varphi|_{\mathcal{C}^q}.
\end{aligned}$$

From which it follows that $\boldsymbol{i}$ is bounded and can be extended to $\mathcal{B}^{p,q}$.

Fix $g \in \mathcal{C}_0^\infty([-1, 1], \mathbb{R}_+)$, $\int g = 1$. For each $x \in \mathbb{T}^2$, $\varphi \in \mathcal{C}_0^\infty([-\delta, \delta], \mathbb{C})$ and $\varepsilon > 0$ define

$$\varphi_\varepsilon(y) = \varphi(\langle y - x, v^s \rangle) g(\langle y - x, v^u \rangle \varepsilon^{-1}) \varepsilon^{-1}.$$

Then, for $h \in \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$ we have

$$\begin{aligned}
\int h\varphi_\varepsilon &= \int ds g(s\varepsilon) \varepsilon^{-1} \int dt h(x + sv^u + tv^s) \varphi(t) \\
&= \int dt\, h(x + tv^s) \varphi(t) + \mathcal{O}(\varepsilon \|h\|_{1,q}).
\end{aligned}$$

Finally, suppose $\boldsymbol{i}(h) = 0$ for some $h \in \mathcal{B}^{p,q}$. Let $h_n \subset \mathcal{C}^\infty$ such that $h_n \to h$ in $\mathcal{B}^{p,q}$, then

$$\begin{aligned}
0 = \boldsymbol{i}(h)(\varphi_\varepsilon) &= \lim_{n \to \infty} \int h_n \varphi_\varepsilon \\
&= \lim_{n \to \infty} \int dt\, h_n(x + tv^s) \varphi(t) + \mathcal{O}(\varepsilon \|h_n\|_{1,q}) \\
&= \int dt\, h(x + tv^s) \varphi(t) + \mathcal{O}(\varepsilon \|h\|_{1,q}).
\end{aligned}$$

Taking the limit $\varepsilon \to 0$ we obtain

$$0 = \int dt\, h(x + tv^s)\varphi(t).$$

Also, since $\boldsymbol{i}(h)(\partial_u^{p'}\varphi_\varepsilon) = 0$, arguing as before and integrating by parts yields, for all $p' \leqslant p$,

$$0 = \int dt\, \partial_u^{p'} h(x + tv^s)\varphi(t).$$

Taking the sup on x we obtain $\|h\|_{p,q} = 0$. Hence $\boldsymbol{i}$ is injective. □

Before continuing it is convenient to make sure that the derivative acts in the natural way on the spaces $\mathcal{B}^{p,q}$.

Lemma 3.9. *For each $p, q \in \mathbb{N}$ the operator ∂_u is bounded as an operator from $\mathcal{B}^{p+1,q}$ to $\mathcal{B}^{p,q}$ and ∂_s is bounded as an operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{p,q+1}$. Moreover, their kernels consist of the constant functions.*

Proof. The boundedness follows immediately from the definition of the norms (and integration by parts in the case of ∂_s).

Next, for each $h \in \mathcal{C}^\infty$, $x \in \mathbb{T}^2$ and $\varphi \in \mathcal{C}_0^{q+1}([-\delta, \delta], \mathbb{C})$ let us define

$$h_\varphi(x) = \int_{-\delta}^{\delta} h(x + tv^s)\varphi(t)dt.$$

Then

$$\partial_u h_\varphi(x) = \int_{-\delta}^{\delta} \partial_u h(x + tv^s)\varphi(t)dt$$
$$\partial_s h_\varphi(x) = \int_{-\delta}^{\delta} \frac{d}{dt} h(x + tv^s)\varphi(t)dt = -\int_{-\delta}^{\delta} h(x + tv^s)\varphi'(t)dt.$$

It follows that $\|\nabla h_\varphi\|_\infty \leqslant \|h\|_{1,q}|\varphi|_{q+1}$. Hence, for $h \in \mathcal{B}^{p,q}$ and $\varphi \in \mathcal{C}^{q+1}$ we have that h_φ is Lipschitz (which follows by density of $\mathcal{C}^\infty$ in $\mathcal{B}^{p,q}$).

We can now study the equation

$$\partial_u h = 0$$

for $h \in \mathcal{B}^{p+1,q}$. Let $\varphi \in \mathcal{C}^\infty$. Then we have $h_\varphi \in \mathcal{C}^1$ and $\partial_u h_\varphi = 0$. This implies $h_\varphi = \text{const}$. Accordingly, for each set $Q_{x,\delta} = \{x + sv^s + tv^u \ : \ t, s \in [-\delta, \delta]\}$ and $\varphi \in \mathcal{C}_0^\infty(Q_{x,\delta}, \mathbb{C})$,

$$\begin{aligned}\int_{\mathbb{T}^2} h\varphi &= \int_{-\delta}^{\delta} dt \int_{-\delta}^{\delta} ds\, h(x + tv^u + sv^s)\varphi(x + tv^u + sv^s) \\ &= \int_{-\delta}^{\delta} dt \int_{-\delta}^{\delta} ds\, h(x + sv^s)\varphi(x + tv^u + sv^s).\end{aligned}$$

We can then set $\widetilde{\varphi}_x(s) = \int_{-\delta}^{\delta} dt \varphi(x + tv^u + sv^s)$ and obtain

$$\begin{aligned}\int_{\mathbb{T}^2} h\varphi = h_{\widetilde{\varphi}_x}(x) &= \int_{\mathbb{T}^2} h_{\widetilde{\varphi}_x}(y)dy = \int_{\mathbb{T}^2} dy \int_{-\delta}^{\delta} ds\, h(y + sv^s)\widetilde{\varphi}_x(s) \\ &= \int_{\mathbb{T}^2} h \int_{\mathbb{T}^2} \varphi.\end{aligned}$$

This shows that $h - \int h$ is zero as a distribution, but then, by Lemma 3.8 it is zero in $\mathcal{B}^{p+1,q}$, thus the Lemma. Similar arguments hold for the study of the kernel of ∂_s. □

Lemma 3.10. *For each $p, q \in \mathbb{N}$ we have that $\mathcal{B}^{p+1,q-1}$ embeds compactly in $\mathcal{B}^{p,q}$.*

Proof. Since the spaces are separable, it suffices to prove that each sequence $\{h_n\} \subset \mathcal{C}^\infty(\mathbb{T}^2, \mathbb{C})$, $\|h_n\|_{p+1,q-1} \leqslant 1$, admits a convergent subsequence. Using the language of Lemma 3.9, for each $\varepsilon > 0$, let $\{x_i\}_{i \in I_\varepsilon}$ be a finite ε dense set. Then for each $h \in \mathcal{C}^\infty$, $\varphi \in B_{q+1}$ there exists x_i such that $\|x - x_i\| \leqslant \varepsilon$ and

$$|h_\varphi(x) - h_\varphi(x_i)| \leqslant \varepsilon \|\nabla h_\varphi\|_\infty \leqslant \varepsilon \|h\|_{1,q}.$$

On the other hand, if $|\varphi - \widetilde{\varphi}|_q \leqslant \varepsilon$, then

$$|h_\varphi(x_i) - h_{\widetilde{\varphi}}(x_i)| \leqslant \varepsilon \|h\|_{0,q}.$$

Finally, since the set B_{q+1} is compact in B_q, there exists a finite set $\{\varphi_j\}_{j \in J_\varepsilon} \subset B_{q+1}$ such that, for all $\varphi \in B_{q+1}$, $\inf_j |\varphi - \varphi_j|_{\mathcal{C}^q} \leqslant \varepsilon$. Accordingly,

$$\|h\|_{p,q+1} \leqslant \sup_{(i,j) \in I_\varepsilon \times J_\varepsilon} |h_{\varphi_j}(x_i)| + \varepsilon \|h\|_{p+1,q}.$$

We can then conclude by the usual diagonal trick: Note that, for each $\varepsilon > 0$, the set $\{(h_n)_{\varphi_j}(x_i)\}$ is bounded, thus contained in a compact set, hence it is possible to extract a subsequence $\{h_{n_k}\}$ such that each sequence $(h_{n_k})_{\varphi_j}(x_i)$ is convergent. Accordingly, we can set $\varepsilon_m = 2^{-m}$, and construct recursively the sequences $\{h_{n_{m,k}}\} \subset \{h_{n_{m-1,k}}\}$, $\{h_{n_{0,k}}\} = \{h_k\}$ such that for each m there exists $K_m \in \mathbb{N}$ such that, for all $k, k' \geqslant K_m$,

$$\|h_{n_{m,k}} - h_{n_{m,k'}}\| \leqslant 2\varepsilon_m.$$

We can then choose the sequence $\widetilde{h}_m = h_{n_{m,K_m}}$, it is easy to check that this is a convergent subsequence. □

Having thus described the Banach space, it is now time to study how the transfer operator acts on it.

Lemma 3.11 (Lasota–Yorke type inequality)**.** *For each $h \in \mathcal{C}^\infty$ and $p, q \in \mathbb{N}$ we have*

$$\begin{aligned}\|\mathcal{L}^n h\|_{p,q} &\leqslant C_\# \|h\|_{p,q}\\ \|\mathcal{L}^n h\|_{p,q} &\leqslant C_\# \lambda^{-\min\{p,q\}n} \|h\|_{p,q} + C_\# \|h\|_{p-1,q+1}.\end{aligned}$$

Proof. Let $h \in \mathcal{C}^\infty$ and $\varphi \in \mathcal{C}_0^q([-a,a],\mathbb{C})$, then

$$\begin{aligned}\int_{-a}^{a} (\mathcal{L}^n h)(x + tv^s) h\varphi(t)dt &= \int_{-a}^{a} h(x + t\lambda^n v^s)\varphi(t)dt\\ &= \lambda^{-n} \int_{-\lambda^n a}^{\lambda^n a} h(x + tv^s)\varphi(\lambda^{-n}t)dt.\end{aligned}$$

Next, we consider a $\mathcal{C}^\infty$ partition of unity $\{\phi_i\}$ of $\mathbb{R}$ such that the elements have support of size δ and $\|\phi_i\|_{\mathcal{C}^{q+1}} \leqslant C$, for some fixed $C > 0$. Clearly $[-\lambda^n a, \lambda^n a]$ intersects, at most, $4\lambda^n + 1 \leqslant 5\lambda^n$ such elements. Let t_i belong to the support of ϕ_i. Then

$$\begin{aligned}\left|\int_a^a (\mathcal{L}^n h)(x + tv^s) h\varphi(t)dt\right| &\leqslant \sum_i \lambda^{-n} \left|\int_{t_i-\delta}^{t_i+\delta} h(x + tv^s)\varphi(\lambda^{-n}t)\phi_i(t)dt\right|\\ &= \sum_i \lambda^{-n} \|h\|_{0,q} \leqslant 5\|h\|_{0,q}.\end{aligned} \tag{3.4.2}$$

This proves the first inequality of the Lemma for $p = 0$. To treat $p > 0$ define $\varphi_i(t) = \sum_{j=0}^{q-1} \frac{\varphi^j(\lambda^{-n}t_i)}{j!}\lambda^{-nj}(t-t_i)^j$ and redo the above computation as follows:

$$\begin{aligned}\int_a^a (\mathcal{L}^n h)(x+tv^s)h\varphi(t)dt &= \sum_i \lambda^{-n}\int_{t_i-\delta}^{t_i+\delta} h(x+tv^s)\varphi(\lambda^{-n}t)\phi_i(t)dt \\ &= \sum_i \lambda^{-n}\int_{t_i-\delta}^{t_i+\delta} h(x+tv^s)\left[\varphi(\lambda^{-n}t)-\varphi_i(t)\right]\phi_i(t)dt \\ &\quad + \sum_i \lambda^{-n}\int_{t_i-\delta}^{t_i+\delta} h(x+tv^s)\varphi_i(t)\phi_i(t)dt.\end{aligned}$$

To continue notice that

$$\left|\int_{t_i-\delta}^{t_i+\delta} h(x+tv^s)\varphi_i(t)\phi_i(t)dt\right| \leqslant C\,|\varphi|_q\|h\|_{0,q+1},$$

and

$$|\varphi(\lambda^{-n}\cdot)-\varphi(\lambda^{-n}t_i)|_q \leqslant C\,|\varphi|_q\lambda^{-nq}.$$

The above yields

$$\|\mathcal{L}^n h\|_{0,q} \leqslant C\lambda^{-nq}\|h\|_{0,q} + C\,\|h\|_{0,q+1}.$$

Next, notice that

$$\int_{-a}^a \partial_u^p(\mathcal{L}^n h)(x+tv^s)h\varphi(t)dt = \lambda^{-np}\int_{-a}^a (\mathcal{L}^n[\partial_i^p h])(x+tv^s)h\varphi(t)dt$$

which, remembering Equation (3.4.2), implies

$$\|h\|_{p,q} \leqslant 5\lambda^{-np}\|h\|_{p,q} + C\sum_{i=0}^{p-1}\lambda^{n(p-i+q)}\|\partial_u^i h\|_{0,q} + C\,\|h\|_{p-1,q+1}$$

which proves the Lemma. □

The above, together with Lemma 3.10, allows us to apply Theorem 1.1 and conclude that the essential spectrum of $\mathcal{L}$, when acting on $\mathcal{B}^{p,q}$, is bounded by $\lambda^{-\min\{p,q\}}$. To complete our alternative derivation of the results obtained by Fourier Transform we need to understand the discrete spectrum.

Lemma 3.12. *For each* $p, q \in \mathbb{N}$ *we have* $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}) \cap \{z \in \mathbb{C} \,:\, |z| > \lambda^{-p}\} = \{1\}$.

Proof. Suppose that $\mathcal{L}h = \mu h$, $|\mu| > \lambda^{-p}$. Then

$$\mu \partial_u h = \partial_u \mathcal{L} h = \lambda^{-1} \mathcal{L} \partial_u h.$$

Thus $\partial_u h \in \mathcal{B}^{p-1,q}$ is an eigenvector of $\mathcal{L}$ with eigenvalue $\lambda\mu$. Doing it p times we have that $\partial_u^p h \in \mathcal{B}^{0,q}$ is an eigenvector with eigenvalue $\lambda^p \mu$, but $|\lambda^p \mu| > 1$ while the spectral radius of $\mathcal{L}$ is bounded by one, hence it must be $\partial_u^p h = 0$. But then Lemma 3.9 implies that $\partial_u^{p-1} h$ is constant. Integrating we see that the constant is zero. Iterating this argument p times we have $h = \text{const}$, but then $\mu = 1$. □

4 Uniformly hyperbolic maps and Banach spaces

In this section we build on what we have learned in the previous sections to treat the general non-linear case in which expanding and contracting directions are both present simultaneously but there is no neutral direction.

The goal is to develop Banach spaces on which the transfer operator has nice properties. This can be done in various ways: Baladi and Tsujii (2007, 2008), Blank, Keller, and Liverani (2002), Faure, Roy, and Sjöstrand (2008), and Gouëzel and Liverani (2006, 2008). Here we will describe the so called *geometric approach* which generalizes the construction detailed in Section 3.4. Alternative approaches are the *Sobolev space approach* and the (similar) *semiclassical approach*, which generalise the norms detailed in Section 3.3. The description below is intended as an introduction, see Gouëzel and Liverani (2006, 2008) for more details and Baladi (2018) for a much more in-depth discussion of all the different functional spaces.

In the geometrical approach one would like to divide the stable and unstable directions in such a way that one can integrate along the stable direction, similarly to what we did in Section 3.4. The simplest possible generalization would be to integrate on pieces of stable manifold (as in Section 3.4). This is possible (it was

indeed the case in the first successful attempts to construct such spaces Blank, Keller, and Liverani (2002)) but it has the drawback that the Banach space depends strongly on the map. Such a feature is very inconvenient if one wants to study an open set of maps, a necessity when investigating the dependence of the SRB measure on some parameter or in the study of random maps. The construction described in the following avoids such a problem, at the price of some extra work.

The type of result that can be obtained with the machinery described in this chapter are as follows:[1]

Theorem 4.1. *If M is a compact Riemannian manifold and $f \in \mathrm{Diff}^r(M)$ is Anosov and topologically transitive,[2] then there exist a unique measure (the Sinai–Ruelle–Bowen measure) μ and $\gamma > 0$, such that, for each $\varphi, h \in cC^\alpha$, $\alpha > 0$,*

$$\left| \int_M \varphi \circ f^n h - \int_M \varepsilon d\mu \int_M h \right| \leqslant C_\# \|\varphi\|_{C^\alpha} \|h\|_{C^\alpha} e^{-\gamma n}.$$

Our strategy is to prove Theorem 4.1 using Hennion's Theorem Theorem B.14. To this end we need a Lasota–Yorke type inequality and a compactness result.

The actual details depend on the choice of the Banach spaces. For example, Lemma 4.9 and Lemma 4.16 will do. Given this ingredients the proof of Theorems like Theorem 4.1 are standard and we leave to the reader to fill the details in complete analogy with what we have done before.

Also we do not provide a detailed description of the statistical and stability properties that can be derived with the present approach (a part form a brief discussion in Section 4.5) as they are either totally general facts (as the ones discussed in Appendix C) or can be obtained in complete analogy with the arguments used in the first chapters.

4.1 Anosov maps

Let us define more precisely the class of maps we want to study: $\mathcal{C}^r$ Anosov maps, $r \geqslant 2$. A diffeomorphism $f \in \mathrm{Diff}^r(M, M)$,[3] where M is a d-dimensional compact Riemannian manifold, is called an Anosov map if there exist two uniformly

[1] By $\mathrm{Diff}^r(M)$ we mean the set of diffeomorphisms of M, r times differentiable with continuity.

[2] A map is called *topologically transitive* if for every pair of non-empty open sets U and V, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.

[3] In fact endomorphisms can be treated in the same way, but let us keep things simple.

transversal closed, continuous cones fields $C^u(x), C^s(x) \subset T_xM$ and $\lambda > 1$ such that $D_x f C^u(x) \subset \operatorname{int} C^u(f(x)) \cup \{0\}$, $D_x f^{-1} C^s(x) \subset \operatorname{int} C^s(f^{-1}(x)) \cup \{0\}$ and

$$\begin{aligned} &\|D_x f v\| > \lambda \|v\| \quad \forall\ v \in C^u(x) \\ &\|D_x f^{-1} v\| > \lambda \|v\| \quad \forall\ v \in C^s(x). \end{aligned} \tag{4.1.1}$$

Note that in higher dimensions, cones can have a variety of shapes.[4] We ask that for each $v \in C^u(x)$ there exists a d^u dimensional subspace E of T_xM such that $v \in E \subset C^u(x)$, and for each $v \in C^s(x)$ there exists a d^s dimensional subspace E of T_xM such that $v \in E \subset C^s(x)$.[5]

It is well known that the above cone invariant and contracting properties are equivalent to the existence of two invariant distributions, Katok and Hasselblatt (1995). More precisely: at each point $x \in M$ there exist two transversal subspaces $E^s(x) \subset C^s(x)$ and $E^u(x) \subset C^u(x)$ such that $Df E^{u/s}(x) = E^{u/s}(f(x))$ and, in addition, $E^{u/s}(x)$ vary in a Hölder continuous way with respect to x.

It is possible to choose an atlas $\{U_i\}_{i=1}^N$ so that for each U_i there exists a special point $x_i \in U_i$, call it the *centroid*, such that $D_{x_i}\phi_i E^s(x_i) = \{(\xi, 0) \ : \ \xi \in \mathbb{R}^{d_s}\}$ and $D_{x_i}\phi_i E^u(x_i) = \{(0, \eta) \ : \ \eta \in \mathbb{R}^{d_u}\}$. Also, without loss of generality, we can assume that $\phi_i(x_i) = 0$ and $\phi_i(U_i) = B_{d_s}(0, r_i) \times B_{d_u}(0, r_i)$ where, for all $d' \in \mathbb{N}$ and $z \in \mathbb{R}^{d'}$, $B_{d'}(z, r) = \{x \in \mathbb{R}^{d'} \ : \ \|z - x\| < r\}$. Clearly, there exists $\delta > 0$ such that $M = \cup_i \phi_i^{-1}(B_{d_s}(0, r_i - 2\delta) \times B_{d_u}(0, r_i - 2\delta)) =: \cup_i \widehat{U}_i$. In other words, a small shrinking $\{(\widehat{U}_i, \phi_i)\}_{i=1}^N$ of the charts still forms an atlas. Finally, we can always arrange so that (4.1.1) holds with respect to the Euclidean norm in the charts for vectors in $\{(0, \eta) \ : \ \eta \in \mathbb{R}^{d_u}\}$ and $\{(\xi, 0) \ : \ \eta \in \mathbb{R}^{d_s}\}$, respectively.[6]

By the continuity of the distributions and the contraction of the cones it follows that, provided the r_i are chosen small enough, the constant cones $C^s_* = \{(\xi, \eta) \in \mathbb{R}^d \ : \ \|\eta\| \leqslant \|\xi\|\}$ and $C^u_* = \{(\xi, \eta) \in \mathbb{R}^d \ : \ \|\xi\| \leqslant \|\eta\|\}$, are invariant. That is, when the composition makes sense,

$$\begin{aligned} &D\phi_j \, Df \, D\phi_i^{-1} C^u_* \subset \operatorname{int} C^u_* \cap \{0\} \\ &D\phi_j \, Df^{-1} \, D\phi_i^{-1} C^s_* \subset \operatorname{int} C^s_* \cap \{0\}. \end{aligned} \tag{4.1.2}$$

[4] A cone is a subset C of a real vector space such that if $v \in C$, then $\lambda v \in C$ for each $\lambda \in \mathbb{R}$.

[5] The sophisticated reader will recognise that it might be more elegant to define the cones as subsets of the Grassmannian.

[6] For example one can use the exponential map at x_i composed with a linear coordinate change to define the chart.

Remark 4.2. *Maps for which there exist cones $C_*^{u/s}$ that satisfy* (4.1.2) *and the equivalent of* (4.1.1)*, with respect to the Euclidean norm in the charts, are called* cone hyperbolic*. Note that if the map is smooth we just argued that cone hyperbolic is equivalent to Anosov. Yet, the notion of cone hyperbolicity applies more generally, for example to piecewise smooth maps, see Baladi and Gouëzel (2010).*

Remark 4.3. *Note that if f is cone hyperbolic, then there exists a neighbourhood $\mathcal{U} \subset \mathcal{C}^1$ such that each $\widetilde{f} \in \mathcal{U}$ is cone hyperbolic with respect to the same cones.*[7]

4.1.1 Transfer operator

Let us compute the Transfer operator. A change of variables yields[8]

$$\int_M h \cdot \varphi \circ f = \int_M h \circ f^{-1} |\det Df|^{-1} \circ f^{-1} \varphi.$$

It is then natural to define, for each $h \in \mathcal{C}^0$, the transfer operator

$$\mathcal{L}h(x) = (h|\det Df|^{-1}) \circ f^{-1}(x). \tag{4.1.3}$$

The reader can easily check that

$$\mathcal{L}^n h = (h|\det Df^n|^{-1}) \circ f^{-n}.$$

Since

$$\int_M |\mathcal{L}h| = \int_M \mathcal{L}|h| \cdot 1 = \int_M |h| \cdot 1 \circ f = \int_M |h|,$$

$\mathcal{L}$ is a contraction in the L^1 norm. Hence we would like to define, as in the previous chapter, a norm for which the spectral radius is one and the essential spectral radius is strictly smaller. In other words, we would like a Banach space on which $\mathcal{L}$ has spectral radius one and is quasi-compact.

[7] It follows from a standard compactness argument.

[8] Unless stated otherwise the integrals are always meant with respect to the volume form associated to the metric.

4.2 A set of almost stable manifolds

By the general theory of hyperbolic systems, Katok and Hasselblatt (1995), a less local statement also holds for Anosov diffeomorphisms: there exist two invariant foliations, the stable and unstable foliations. More precisely, at each point $x \in M$ there exist two local $\mathcal{C}^r$-manifolds $W^s(x)$, $W^u(x)$, of fixed size, such that $W^s(x) \cap W^u(x) = \{x\}$ and, for each $y \in W^{s/u}(x)$, $E^{s/u}(y)$ is the tangent space to $W^{s/u}(x)$ at y. The invariance means that $fW^u(x) \supset W^u(f(x))$ and $fW^s(x) \subset W^s(f(x))$.

Clearly the above foliations yield a natural candidate for the direction on which to integrate and indeed this was the original approach in Blank, Keller, and Liverani (2002). However, as already mentioned, such a choice has at least two drawbacks: first, although the manifolds are as regular as the map, the foliation is, in general, only Hölder, Katok and Hasselblatt (1995). Second, if one would like to have a Banach space in which to analyze not just one map but an open set of maps, then it is necessary to integrate on manifolds that are relatively independent of the map. Both problems have been solved in Gouëzel and Liverani (2006), the idea being to introduce an "invariant" set of manifolds rather than an invariant distribution (in some sense, the equivalent of an invariant cone, see Remark 4.3).

To make precise the above idea it is more transparent to work in charts. Let, $\delta > 0$ be small enough and define

$$\Sigma_i^r = \left\{ G \in \mathcal{C}^r(\mathbb{R}^{d_s}, \mathbb{R}^{d_u}) \ : \|G\|_{\mathcal{C}^0} \leqslant r_i; \ \|DG\|_r^* \leqslant 1 \right\},$$

where $\|\cdot\|_r^*$ is equivalent to the $\|\cdot\|_{\mathcal{C}^{r-1}}$ norm and will be defined in Equation (4.2.2).

Given $G \in \Sigma_i^r$ we have $(y, G(y)) \in B_{d_s}(0, r_i) \times B_{d_u}(0, r_i)$ for all $y \in B_{d_s}(0, r_i)$, thus the manifolds

$$W_{i,z,G} = \{\phi_i^{-1}(y, G(y))\}_{y \in B_{d_s}(z,\delta)} \ ; \quad \widetilde{W}_{i,z,G} = \{\phi_i^{-1}(y, G(y))\}_{y \in B_{d_s}(z,2\delta)} \tag{4.2.1}$$

are well defined d_s dimensional $\mathcal{C}^r$ submanifolds of M for any $i \in \{1, \dots, N\}$, $z \in B_{d_s}(0, r_i - 2\delta)$ and $G \in \Sigma_i^r$. We finally define the announced set of manifolds:

$$\Sigma^r = \bigcup_{i=1}^{N} \bigcup_{z \in B_{d_s}(0, r_i - 2\delta)} \bigcup_{G \in \Sigma_i^r} W_{i,z,G}. \tag{4.2.2}$$

Given $W = W_{i,z,G} \in \Sigma^r$ we will call $\widetilde{W} = \widetilde{W}_{i,z,G}$ its *enlargement*.

The above set of manifolds will play the role of the invariant stable foliation (but it is much more flexible) as is illustrated by the next Lemma.

Lemma 4.4. *For each Anosov map $f \in \mathrm{Diff}^r(M)$ there exist norms $\|\cdot\|_{\mathcal{C}^r}$ and $\|\cdot\|_r^*$, constants $\delta > 0$ and $\bar{n} \in \mathbb{N}$ such that for all $W \in \Sigma^r$ and $n \geqslant \bar{n}$ there exist $m \in \mathbb{N}$ and a collection $\{W_i\}_{i=1}^m \subset \Sigma^r$ such that,*[9]

$$\overline{f^{-n}W} \subset \bigcup_{i=1}^{m} W_i \subset f^{-n}(\widetilde{W}).$$

Moreover, there exists a constant $C_\delta > 0$, depending only on δ, and a partition of unity of $f^{-n}\widetilde{W}$, subordinated to $\{W_i\}_{i=1}^m \cup \{f^{-n}\widetilde{W} \setminus \overline{f^{-n}W}\}$, with $\mathcal{C}^r$ norm bounded by C_δ. That is, a set $\{\boldsymbol{\varphi}_i\}_{i=1}^m$ of functions, from $f^{-n}\widetilde{W}$ to $[0,1]$, such that $\operatorname{supp}\boldsymbol{\varphi}_i \subset W_i$, $\sup_i \|\boldsymbol{\varphi}_i\|_{\mathcal{C}^r(W_i,\mathbb{R})} \leqslant C_\delta$, and $\sum_{i=1}^m \boldsymbol{\varphi}_i(x) = 1$ for each $x \in \overline{f^{-n}W}$.

Proof. Since we will need to control high derivatives it is convenient to use the fact that, for each finite dimensional Banach algebra $\mathbb{A}$, $\mathcal{C}^k(\mathbb{R}^d, \mathbb{A})$ is a Banach algebra as well, provided we choose the right weighted norm. For example

$$\begin{aligned} \|g\|_{\mathcal{C}^0} &= \sup_{x\in\mathbb{R}^d} \|g(x)\| \\ \|g\|_{\mathcal{C}^{k+1}} &= \sup_i \|\partial_{x_i} g\|_{\mathcal{C}^k} + a\|g\|_{\mathcal{C}^k} \end{aligned} \tag{4.2.3}$$

for $a \geqslant 2$ will do. Note that this implies[10]

$$\|g\|_{\mathcal{C}^k} = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} \sup_{|\alpha|=j} \|\partial^\alpha g\|_\infty. \tag{4.2.4}$$

From now on we use such a norm with an a that will be chosen shortly.

Let $W \in \Sigma^r$ and $n \in \mathbb{N}$ large enough. Then $f^{-n}W$ will be a larger manifold and the distance between the boundaries $\partial f^{-n}W$ and $\partial f^{-n}\widetilde{W}$ will be (in charts) larger than 2δ due to the backward expansion in the stable cone. First of all note that, for each point $x \in \overline{f^{-n}W}$ there exist $j_x \in \{1, \dots N\}$, $z_x \in B_{d_s}(0, r_{j_x} -$

[9] With a bit more work one can prove it for each $\bar{n} \in \mathbb{N}$, but let us keep it simple.

[10] Here we use the usual PDE notation in which $\alpha = (i_1, \cdots, i_k)$ is a multiindex, $|\alpha| = k$, and $\partial^\alpha = \partial_{x_{i_1}} \dots \partial_{x_{i_k}}$.

2δ) and $G_x \in \mathcal{C}^r(\mathbb{R}^{d_s}, \mathbb{R}^{d_u})$, with $x = \phi_{j_x}^{-1}(z_x, G_x(z_x))$ and $\|G_x\|_\infty \leqslant r_{j_x} - 2\delta$, such that $\widetilde{W}_{j_x,z_x,G_x} \subset f^{-n}\widetilde{W}$. Then $\{W_{j_{x_k},z_{x_k},G_{x_k}}\}$ covers the closure of a δ neighbourhood of $f^{-n}W$ in $f^{-n}\widetilde{W}$. Accordingly, we can extract a finite covering $\{W_k\}_{i=1}^m := \{W_{j_{x_k},z_{x_k},G_{x_k}}\}$ of $\overline{f^{-n}W}$ by compactness. The existence of a partition of unity with the wanted properties and subordinated to the covering is a standard fact, see Hörmander (1990, Theorem 1.4.10).

To conclude it remains to show that $G_{x_k} \in \Sigma^r_{j_{x_k}}$. Note that, by hypothesis,

$$D_{f(x)}\phi_j\, D_x f^{-\bar{n}} D_{\phi_i(x)}\phi_i^{-1} = \begin{pmatrix} A^{i,j}(x) & B^{i,j}(x) \\ C^{i,j}(x) & D^{i,j}(x) \end{pmatrix} =: \Xi^{i,j}(x)$$

where, by construction, if $f^{\bar{n}}(x_i) = x_j$, then

$$\Xi^{i,j}(x_i) = \begin{pmatrix} A_*^{i,j} & 0 \\ 0 & D_*^{i,j} \end{pmatrix} \tag{4.2.5}$$

with $\|(A_*^{i,j})^{-1}\| \leqslant \lambda^{-\bar{n}}$ and $\|D_*^{i,j}\| \leqslant \lambda^{-\bar{n}}$. Thus, by continuity, for each $\gamma > 0$ we can write

$$\Xi^{i,j} = \Xi_*^{i,j} + \Delta^{i,j},$$

where $\Xi_*^{i,j}$ is a constant matrix with the same properties of $\Xi^{i,j}(x_i)$ in (4.2.5) and

$$\|\Delta^{i,j}\|_\infty \leqslant \gamma, \tag{4.2.6}$$

provided the $r_i \geqslant 2\delta$ have been chosen small enough.

If $W_{j,\xi,H} \subset f^{-\bar{n}}W_{i,z,G}$, then setting $F(x) = \phi_j \circ f^{-\bar{n}} \circ \phi_i^{-1}$ we have that there exist $\alpha \in \mathcal{C}^r(D, B_{d_s}(0, r_j))$, $D \subset B_{d_s}(0, r_i)$, such that

$$F(x, G(x)) = (\alpha(x), H(\alpha(x))). \tag{4.2.7}$$

Hence, for each $\xi \in \mathbb{R}^{d_s}$,

$$(D\alpha\xi, DH \circ \alpha D\alpha\xi) = \Xi^{i,j}(\xi, DG\xi) = \begin{pmatrix} A^{i,j} & B^{i,j} \\ C^{i,j} & D^{i,j} \end{pmatrix} (\xi, DG\xi)$$

which implies

$$\begin{aligned} D\alpha &= A^{i,j} + B^{i,j}DG \\ DH &= \left\{(C^{i,j} + D^{i,j}DG)(A^{i,j} + B^{i,j}DG)^{-1}\right\} \circ \alpha^{-1} \\ &= \left\{(C^{i,j} + D^{i,j}DG)(\mathbb{1} + (A^{i,j})^{-1}B^{i,j}DG)^{-1}(A^{i,j})^{-1}\right\} \circ \alpha^{-1}. \end{aligned} \tag{4.2.8}$$

To estimate the higher order derivatives it is convenient to consider $\Xi^{i,j}$ (and its block constituents) as an operator mapping a vector filed in the chart i to a vector field in the chart j. The norm of such an operator is naturally defined to be[11]

$$\|\Xi\|_r^* = \sup_{\|v\|_{\mathcal{C}^r}\leq 1} \|\Xi v\|_{\mathcal{C}^r}.$$

To estimate such a norm it is helpful the following results.

Sub-lemma 4.5. *For each $r \in \mathbb{N}$ and $\Xi \in \mathcal{C}^r(\mathbb{R}^d, GL(\mathbb{R}^d, \mathbb{R}^d))$*

$$\sup_{|\alpha|\leq r} a^{-|\alpha|}\|\partial^\alpha \Xi\|_\infty \leq \|\Xi\|_r^* \leq e^r (r!)^2 \sup_{|\alpha|\leq r} a^{-|\alpha|}\|\partial^\alpha \Xi\|_\infty. \tag{4.2.9}$$

Proof. Remembering (4.2.4) we have

$$\begin{aligned}
\|\Xi v\|_{\mathcal{C}^r} &= \sum_{k=0}^{r} \binom{r}{k} a^{r-k} \sup_{|\alpha|=k} \|\partial^\alpha(\Xi v)\|_\infty \\
&\leq \sum_{k=0}^{r} \binom{r}{k} a^{r-k} \sum_{|\alpha|+|\beta|=k} \binom{k}{|\beta|} \|\partial^\alpha \Xi\|_\infty \|\partial^\beta v\|_\infty \\
&\leq \sum_{|\beta|=0}^{r} \sum_{k=|\beta|}^{r} \binom{r}{|\beta|} \frac{a^{r-k} r^{k-|\beta|}}{(k-|\beta|)!} \|\partial^{k-|\beta|}\Xi\|_\infty \|\partial^\beta v\|_\infty \\
&\leq \sum_{|\alpha|=0}^{r} \frac{a^{-|\alpha|} r^{|\alpha|} r!}{|\alpha|!} \|\partial^\alpha \Xi\|_\infty \|v\|_{\mathcal{C}^r} \\
&\leq e^r (r!)^2 \sup_{|\alpha|\leq r} a^{-|\alpha|}\|\partial^\alpha \Xi\|_\infty \|v\|_{\mathcal{C}^r}.
\end{aligned}$$

That is

$$\|\Xi\|_r^* \leq e^r (r!)^2 \sup_{|\alpha|\leq r} a^{-|\alpha|}\|\partial^\alpha \Xi\|_\infty.$$

On the other hand, if we restrict to v that are constant vector fields with $\|v\| = 1$ we have, for each $|\alpha| \leq r$,

$$\|\Xi\|_r^* \geq a^{-|\alpha|} \binom{r}{|\alpha|} \sup_{\|v\|=1} \|(\partial^\alpha \Xi) v)\|_\infty \geq a^{-|\alpha|}\|\partial^\alpha \Xi\|_\infty.$$

□

[11] Note that, by definition, $\|AB\|_r^* \leq \|A\|_r^* \|B\|_r^*$.

From Sub-lemma 4.5 and equation (4.2.6) it follows that, by choosing a large enough (depending on γ and $\bar{n}$),

$$\|\Delta^{i,j}\|_r^* \leqslant C_r \gamma.$$

Accordingly, for each constant $C_{r,d} > 1$, choosing γ small enough and $\bar{n}$ large enough, we obtain

$$\begin{aligned} &\sup_{i,j} \|B^{i,j}\|_r^* + \|C^{i,j}\|_r^* \leqslant \frac{1}{2C_{r,d}} \\ &\sup_{i,j} \|(A^{i,j})^{-1}\|_r^* \leqslant \frac{1}{2C_{r,d}} \\ &\sup_{i,j} \|D^{i,j}\|_r^* \leqslant \frac{1}{2C_{r,d}}. \end{aligned} \tag{4.2.10}$$

From the above and equation (4.2.8) it follows

$$\begin{aligned} \|(D\alpha)^{-1}\|_r^* &= \|(\mathbb{1} + (A^{i,j})^{-1} B^{i,j} DG)^{-1} (A^{i,j})^{-1}\|_r^* \\ &\leqslant \frac{1}{2C_{r,d}} \sum_{k=0}^{\infty} (\|(A^{i,j})^{-1} B^{i,j} DG)^{-1}\|_r^*)^k \leqslant \frac{2}{3C_{r,d}}. \end{aligned} \tag{4.2.11}$$

Note that, by similar arguments, we can prove

$$\|((D\alpha)^t)^{-1}\|_r^* \leqslant \frac{2}{3C_{r,d}}, \tag{4.2.12}$$

where A^t is the transpose of the matrix A.

Unfortunately, to estimate (4.2.8) we need to control the norm of $\Xi \circ \alpha^{-1}$ rather than simply the norm of Ξ. To this end we need another technical Lemma.

Sub-lemma 4.6. *For each $k \in \mathbb{N}$ and $\mathcal{C}^k$ function g, we have*

$$\|g \circ \alpha^{-1}\|_{\mathcal{C}^k} \leqslant \|g\|_{\mathcal{C}^k}.$$

Moreover

$$\|\Xi \circ \alpha^{-1}\|_*^r \leqslant C_r \|\Xi\|_*^r.$$

Proof. By equations (4.2.3) the Lemma is true for $k = 0$. Moreover we can write

$$\|g \circ \alpha^{-1}\|_{\mathcal{C}^{k+1}} = \sup_i \|\partial_{x_i}(g \circ \alpha^{-1})\|_{\mathcal{C}^k} + a\|g \circ \alpha^{-1}\|_{\mathcal{C}^k}.$$

We can thus argue by induction and, remembering (4.2.12), conclude

$$\begin{aligned}
\|g \circ \alpha^{-1}\|_{\mathcal{C}^{k+1}} &\leqslant \sup_i \| \left[(\partial_{x_j} g)[(D\alpha)^{-1}]_{j,i}\right] \circ \alpha^{-1}\|_{\mathcal{C}^k} + a\|g\|_{\mathcal{C}^k} \\
&\leqslant \|(D\alpha)^t)^{-1}\nabla g\|_{\mathcal{C}^k} + a\|g\|_{\mathcal{C}^k} \\
&\leqslant \|((D\alpha)^t)^{-1}\|_*^r \|\nabla g\|_{\mathcal{C}^k} + a\|g\|_{\mathcal{C}^k} \\
&\leqslant \frac{2d}{3C_{r,d}} \sup_j \|(\partial_{x_j} g)\|_{\mathcal{C}^k} + a\|g\|_{\mathcal{C}^k} \leqslant \|g\|_{\mathcal{C}^{k+1}},
\end{aligned}$$

provided we have chosen $C_{r,d}$ large enough.

To conclude, recalling (4.2.3), (4.2.4) and Sub-lemma 4.5,

$$\begin{aligned}
\|\Xi \circ \alpha^{-1}\|_r^* &\leqslant e^r (r!)^2 \sup_{|\alpha| \leqslant r} a^{-|\alpha|} \|\partial^\alpha (\Xi \circ \alpha^{-1})\|_\infty \\
&\leqslant e^r (r!)^2 \sup_{|\alpha| \leqslant r} a^{-|\alpha|} \|\Xi \circ \alpha^{-1}\|_{\mathcal{C}^{|\alpha|}} \\
&\leqslant e^r (r!)^2 \sup_{|\alpha| \leqslant r} a^{-|\alpha|} \|\Xi\|_{\mathcal{C}^{|\alpha|}} \\
&\leqslant e^r (r!)^2 \sup_{|\alpha| \leqslant r} \sum_{j=0}^{|\alpha|} \binom{|\alpha|}{j} \sup_{|\beta|=j} a^{-|\beta|} \|\partial^\beta \Xi\|_\infty \\
&\leqslant e^r 2^r (r!)^2 \|\Xi\|_r^*.
\end{aligned}$$

□

Applying Sub-lemma 4.6 to formula (4.2.8) and recalling (4.2.10), (4.2.11) yields

$$\begin{aligned}
\|DH\|_r^* &\leqslant \| \left\{ (C^{i,j} + D^{i,j} DG)(\mathbb{1} + (A^{i,j})^{-1} B^{i,j} DG)^{-1} (A^{i,j})^{-1} \right\} \circ \alpha^{-1} \|_r^* \\
&\leqslant C_r \|(C^{i,j} + D^{i,j} DG)(\mathbb{1} + (A^{i,j})^{-1} B^{i,j} DG)^{-1} (A^{i,j})^{-1}\|_r^* \\
&\leqslant \frac{2C_r}{6C_{r,d}^3} (1 + \|DG\|_r^*) \leqslant \frac{2}{3} < 1,
\end{aligned}$$

provided, again, we have chosen $C_{r,d}$ large enough. This concludes the Lemma.

□

Remark 4.7. *Note that, given $f_0 \in \mathcal{C}^r$ and norms $\|\cdot\|_{\mathcal{C}^r}$, $\|\cdot\|_r^*$ for which Lemma 4.4 holds, there exists a neighbourhood $\mathcal{U} \subset \mathcal{C}^r$ of f_0 such that Lemma 4.4 holds, with the same norms, for each $f \in \mathcal{U}$. This is the equivalent of Remark 4.3.*

4.3 High regularity norms

If $W = W_{i,z,G} \in \Sigma_i^r$ and $\varphi \in \mathcal{C}_0^k(W, \mathbb{C})$, we define

$$|\varphi|_{\mathcal{C}^k} = \|\varphi \circ \phi_i^{-1} \circ \mathbb{G}\|_{\mathcal{C}^k(B_{ds}(z,\delta),\mathbb{C})}$$

where, again, $\mathbb{G}(x) = (x, G(x))$. We are finally ready to define the relevant norms.

For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$ and $h \in \mathcal{C}^r(M, \mathbb{C})$ let[12]

$$\|h\|_{p,q} = \sup_{|\alpha| \leqslant p} \sup_{W \in \Sigma^r} b^{|\alpha|} \sup_{\substack{\varphi \in \mathcal{C}_0^{q+|\alpha|}(W,\mathbb{C}) \\ |\varphi|_{\mathcal{C}^{q+|\alpha|}} \leqslant 1}} \int_W [\partial^\alpha h] \cdot \varphi, \tag{4.3.1}$$

where, for $W = W_{i,z,G} \in \Sigma_i^r$ and $g \in \mathcal{C}^0(W, \mathbb{C})$ we define

$$\int_W g = \int_{B_{ds}(z,\delta)} g \circ \phi_i^{-1}(x, G(x))dx,$$

and b will be chosen later. $\mathcal{B}^{p,q}$ is the completion of $\mathcal{C}^r(M, \mathbb{C})$ with respect to $\|\cdot\|_{p,q}$.

The above norms have been introduced in Gouëzel and Liverani (2006) and are a generalization of the norms (3.4.1). They allow us to prove that the transfer operator is quasi-compact with essential spectral radius smaller than $\lambda^{-\min\{p,q\}}$.

Here, to simplify the presentation, we discuss only the case $p \leqslant 1 \leqslant q$ and we do not attempt to obtain sharp bounds. We refer to Gouëzel and Liverani (ibid.) for the general case and more precise estimates.

Remark 4.8. *From now on we consider δ fixed once and for all, hence we will often not mention the fact that several constants depend on δ.*

Lemma 4.9. *For each $q \in (0, r-2)$, $p \in \{0, 1\}$ and $\nu \in (\lambda^{-\min\{1,q\}}, 1)$ there exists $C, B > 0$ such that, for all $h \in \mathcal{C}^r(M, \mathbb{C})$ and $n \in \mathbb{N}$,*

$$\|\mathcal{L}^n h\|_{0,q} \leqslant C\|h\|_{0,q}$$
$$\|\mathcal{L}^n h\|_{p,q} \leqslant C\nu^n\|h\|_{p,q} + B\|h\|_{0,q+1}.$$

[12] Since, by definition, W belongs to one chart we can define $\partial_{x_j} h := (\partial_{x_j}(h \circ \phi_i^{-1})) \circ \phi_i$.

Proof. By a change of variables we have

$$\int_W \mathcal{L}^n h\varphi = \int_{f^{-n}W} h \,|\det Df^n| J_W f^n \cdot \varphi \circ f^n$$

where $J_W f^n$ is the Jacobian of the change of variables.[13] We can then use Lemma 4.4 to write

$$\begin{aligned}\left|\int_W \mathcal{L}^n h\varphi\right| &\leqslant \sum_{j=1}^m \left|\int_{W_j} h \,|\det Df^n|^{-1} J_W f^n \cdot \varphi \circ f^n \boldsymbol{\varphi}_j\right| \\ &\leqslant \|h\|_{0,q} \sum_{j=1}^m \left|\,|\det Df^n|^{-1} J_W f^n \cdot \varphi \circ f^n \boldsymbol{\varphi}_j\right|_{\mathcal{C}_0^q(W_j)},\end{aligned}$$

where $W_j = W_{k_j, z_j, G_j}$.

Remembering Sub-lemma 4.6 and equation (4.2.7) we can write

$$\begin{aligned}\left|\,|\det Df^n|^{-1} J_W f^n \cdot \varphi \circ f^n \boldsymbol{\varphi}_j\right|_{\mathcal{C}_0^q(W_j)} \leqslant & C_\delta \left|\,|\det Df^n|^{-1}\right|_{\mathcal{C}^q(W_j)} \\ & \times |J_W f^n|_{\mathcal{C}^q(W_j)} \,|\varphi|_{\mathcal{C}_0^q(W_j)}.\end{aligned}$$

To estimate the above integral we need a technical distortion Lemma.

Sub-lemma 4.10 (Gouëzel and Liverani (2006, Lemma 6.2))**.** *There exists constants $C_\delta > 0$ such that, for each $n \in \mathbb{N}$ and $q \leqslant r-1$, it holds*

$$\sum_{i=1}^m \left|\,|\det Df^n|^{-1}\right|_{\mathcal{C}^q(W_j)} \cdot |J_W f^n|_{\mathcal{C}^q(W_j)} \leqslant C_\delta.$$

Remark 4.11. *We refer to Gouëzel and Liverani (ibid., Lemma 6.2) for the proof; however, let me give some intuition about this estimate. If λ_u^n, λ_s^n are, roughly, the expansion and contraction in the unstable and stable directions, respectively, then $|\det Df^n|^{-1} \sim \lambda_u^{-n}\lambda_s^{-n}$ while $J_W f^n \sim \lambda_s^n$. Hence the summands are roughly equal to λ_u^{-n}. However, if we consider a thickening of size λ_u^{-n}, in the unstable directions, of each W_i, then it corresponds to the image of a thickening of size one of W under f^{-n}. Since the map is a diffeomorphism, this implies that all such regions are disjoint, thus their total volume (essentially $\sum_j \lambda_u^{-n}\delta^{d_s}$) is uniformly bounded by the total volume of M, hence the Lemma. The above argument is essentially correct, apart from some standard distortion estimates.*

[13] Note that we are changing variables on a submanifold, hence the Jacobian differs from $|\det Df^n|$ which corresponds to a change of variables on the full manifold.

Hence we have the first inequality in the statement of the Lemma:[14]

$$\|\mathcal{L}^n h\|_{0,q} \leqslant C\,\|h\|_{0,q}. \tag{4.3.2}$$

To prove the second inequality we first consider the case $p = 0$. We can write[15]

$$\int_W \mathcal{L}^n h\varphi = \int_{\widetilde{W}} \mathcal{L}^n h\varphi = \int_{\widetilde{W}} \mathcal{L}^n h\varphi_\varepsilon + \int_{\widetilde{W}} \mathcal{L}^n h(\varphi - \varphi_\varepsilon).$$

where $|\varphi_\varepsilon - \varphi|_{\mathcal{C}^{q-1}} \leqslant \varepsilon|\varphi|_{\mathcal{C}^q}$, $|\varphi - \varphi_\varepsilon|_{\mathcal{C}^q} \leqslant C_\#$ and $|\varphi_\varepsilon|_{\mathcal{C}^{q+1}} \leqslant C_\#\varepsilon^{-1}$. It follows[16]

$$\begin{aligned}|(\varphi - \varphi_\varepsilon)\circ f^n|_{\mathcal{C}^q} &\leqslant |(\partial^q\varphi - \partial^q\varphi_\varepsilon)\circ f^n \cdot (\partial_x f^u)^q|_{\mathcal{C}^0} + C_\#|(\varphi - \varphi_\varepsilon)\circ f^n|_{\mathcal{C}^{q-1}}\\ &\leqslant C_\# \max\{\varepsilon, \lambda^{-qn}\}.\end{aligned}$$

Arguing as before, and choosing $\varepsilon = \lambda^{-qn}$, the above considerations yield

$$\|\mathcal{L}^n h\|_{0,q} \leqslant C_\#\lambda^{-qn}\|h\|_{0,q} + C_n\|h\|_{0,q+1}. \tag{4.3.3}$$

To continue we must compute

$$(\partial_{x_k}(\mathcal{L}^n h \circ \phi_i^{-1}))\circ\phi_{k_j}(x).$$

To this end we must exchange the order of ∂_{x_k} and $\mathcal{L}^n$. Unfortunately, doing so will produce a multiplicative factor larger than one due to the contracting directions. A natural idea to overcome this problem is to decompose the vector fields ∂_{x_k} into a vector field along the manifold W, that can then be integrated by parts without the need of commuting it with $\mathcal{L}^n$, and a vector field in the unstable direction that, upon exchanging the order of ∂_{x_k} and $\mathcal{L}^n$ will produce a contracting multiplicative factor. The obstacle to this strategy is that the unstable vector field is, in general, only Hölder, and hence a vector field along the unstable direction cannot have the required regularity.

To deal with this last problem we will use an approximation instead of the real unstable direction. Indeed, what is really necessary is that the vector field contracts,

[14] Recall that δ has been fixed and its choice depends only on f and M, hence we will no longer keep track of the dependence of the constants on δ. Also we will use, as before, $C_\#$ to designate a generic constant depending only on f and M.

[15] E.g., given a mollifier j_ε having support $\varepsilon \leqslant \delta/2$, define $\bar{\varphi}_\varepsilon = \int j_\varepsilon(x-y)\varphi\circ\phi_i^{-1}\circ\mathbb{G}(y)dy$ and $\varphi_\varepsilon(z) = \bar{\varphi}_\varepsilon\circ\pi\circ\phi_i(z)$, where $\pi(x_s, x_u) = x_s$.

[16] We use $\partial_x f^n$ to mean $\partial_x(\phi_i\circ f^n\circ\phi_{k_j}\circ\mathbb{G}_j)$. This is nothing other than the contraction of the dynamics in the stable direction.

while being pushed backward, only for a time n. If $E = \{(0,\eta) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u}\}$, then

$$E_n(x) = D_{\phi_i \circ f^{-n} \circ \phi_{k_j}^{-1}(x)}(\phi_{k_j} \circ f^n \circ \phi_i^{-1})E = \{(U_n(x)\eta, \eta)\}_{\eta \in \mathbb{R}^{d_u}} \quad (4.3.4)$$

is a $\mathcal{C}^r$ approximation of the unstable direction with the required property.

Sub-lemma 4.12 (Gouëzel and Liverani (2006, Appendix A))**.** *Given the decomposition* (4.3.4)*, we have*

$$\|U_n \circ \phi_i \circ f^n \circ \phi_{k_j}^{-1} \circ \mathbb{G}_j\|_{\mathcal{C}^r(B_{d_s}(z_j,\delta),\mathbb{R}^d)} \leqslant C_{\#}.$$

Remark 4.13. *The Lemma is technical and the proof is rather uneventful, so we refer to Gouëzel and Liverani (ibid., Appendix A) for the details. However, the reader unwilling to look at another paper can simply carry out a proof by herself using the analogues of* (4.2.8) *and* (4.2.10) *in the future rather than the past.*

Sub-lemma 4.14. *For each $k \in \{1,\dots,d\}$, $n \in \mathbb{N}$ and $z \in W \in \Sigma^r$ we can write*

$$e_k = v(z) + w(z)$$

where $v(z) \in T_z W$, $w(z) \in E_n(\phi_i(z))$ and such that

$$|v \circ f^n|_{\mathcal{C}^r(f^{-n}W,\mathbb{R}^d)} + |w \circ f^n|_{\mathcal{C}^r(f^{-n}W,\mathbb{R}^d)} \leqslant C_{\#}.$$

Proof. Since $T_z W$ and $E_n(\phi_i(z))$ are transversal (the first belong to the stable cone while the second to the unstable one), we can uniquely decompose a vector field along two such subspaces and the decomposed vector field will have uniformly bounded $\mathcal{C}^0$ norm. It remains only to check is that the decomposition has the required regularity. Since W is a regular manifold, the issue is reduced to analysing $E_n(\phi_i(z))$. The result follows then from Sub-lemma 4.12. Indeed, the computation boils down to computing the norms of $(\mathbb{1} - DGU_n)^{-1} \circ \phi_i \circ f^n$ and $(\mathbb{1} - U_n DG)^{-1} \circ \phi_i \circ f^n$. These are uniformly bounded in $\mathcal{C}^0$, since $\|U_n\|_\infty \|DG\|_\infty < 1$ (provided we have chosen the r_i small enough), and the $\mathcal{C}^k$ norm can be computed by induction recalling the definition (4.2.3). □

Accordingly, for each $k \in \{1,\dots,d\}$,

$$\int_W \varphi \partial_{x_k} \mathcal{L}^n h = \int_W \varphi \langle w, \nabla \mathcal{L}^n h\rangle + \varphi \langle v, \nabla \mathcal{L}^n h\rangle. \quad (4.3.5)$$

By construction and Sub-lemma 4.14 there exists $\widetilde{w}$, $\|\widetilde{w}\|_{\mathcal{C}^{1+q}(\mathbb{R}^d,\mathbb{R}^{d_s})} \leqslant C_\#$, such that $(\varphi w)\circ\phi_i^{-1}\circ\mathbb{G} = D\phi_i^{-1}D\mathbb{G}\widetilde{w}$. Hence

$$\begin{aligned}\int_W \langle w,\nabla\rangle\mathcal{L}^n h &= \int_{B_{d_s}(z,\delta)} \langle D\phi_i^{-1}D\mathbb{G}\widetilde{w}, \left[\nabla\mathcal{L}^n h\right]\circ\phi_i^{-1}(\mathbb{G}(x))\rangle dx\\ &= \int_{B_{d_s}(z,\delta)} \langle \widetilde{w}, \nabla\left[(\mathcal{L}^n h)\circ\phi_i^{-1}\circ\mathbb{G}\right]\rangle dx\\ &= -\int_{B_{d_s}(z,\delta)} (\operatorname{div}\widetilde{w})\left[\mathcal{L}^n h\right]\circ\phi_i^{-1}(\mathbb{G}(x))dx\\ &= \int_W \bar{\varphi}\mathcal{L}^n h\end{aligned}$$

where $\bar{\varphi} = [\operatorname{div}\widetilde{w}]\circ\pi\circ\phi_i$, $\pi(x,y)=x$. Since $|\bar{\varphi}|_{\mathcal{C}^q}\leqslant C_\#$ by (4.3.2) it follows

$$b\left|\int_W \langle w,\nabla\rangle\mathcal{L}^n h\right| \leqslant C_\# b\|h\|_{0,q} \leqslant C_\# b\|h\|_{1,q}. \tag{4.3.6}$$

To conclude we must analyse the second term on the right hand side of equation (4.3.5). Recalling (4.1.3) we can write

$$\begin{aligned}\int_W \varphi\langle v,\nabla\mathcal{L}^n h\rangle &= \int_W \varphi\langle v,\nabla\left[(h|\det Df^n|^{-1})\circ f^{-n}\right]\rangle\\ &= \int_W \varphi\langle Df^{-n}v, \left[\nabla(h|\det Df^n|^{-1})\right]\circ f^{-n}\rangle\\ &= \int_W \langle\bar{v},\mathcal{L}^n\nabla h\rangle + \int_W \bar{\varphi}\mathcal{L}^n h,\end{aligned}$$

where $\bar{v} = \varphi Df^{-n}v$ and $\bar{\varphi} = \varphi\langle Df^{-n}v, \left[\nabla(|\det Df^n|^{-1})\right]\circ f^{-n}\rangle$.

By construction we have $\|\bar{v}\|_\infty \leqslant C_\#\lambda^{-n}$, and the usual distortion estimate yields $\|\bar{v}\|_{\mathcal{C}^{1+q}} \leqslant C_\#\lambda^{-n}$. We can then use (4.3.2) and the obvious inequality $b\|\partial_{x_j}h\|_{0,q+1}\leqslant\|h\|_{1,q}$ to write

$$b\left|\int_W \varphi\langle v,\nabla\mathcal{L}^n h\rangle\right| \leqslant C_\#\lambda^{-n}\|h\|_{1,q} + C_n b\|h\|_{q+1}. \tag{4.3.7}$$

Collecting equations (4.3.3), (4.3.5), (4.3.6) and (4.3.7) yields

$$\|\mathcal{L}^n h\|_{1,q} \leqslant C_*\max\{\lambda^{-q}, b^{1/n}, \lambda^{-1}\}^n\|h\|_{1,q} + (b+1)C_n\|h\|_{0,q+1},$$

for some constant C_*. We are almost done, the only remaining source of unhappiness is that the constant in front of the weak norm seems to depend on n. Also, we have still to choose b.

Let us first choose the smallest $\bar{n}$ such that at $C_*\lambda^{-\bar{n}\min\{q,1\}} \leqslant \nu^{\bar{n}}$. Then we choose

$$b = \nu^{\bar{n}} C_*^{-1}.$$

At last, for each $n \in \mathbb{N}$ we write $n = k\bar{n} + m$, $m < \bar{n}$, and

$$\begin{aligned}
\|\mathcal{L}^n h\|_{1,q} &\leqslant \nu^{\bar{n}} \|\mathcal{L}^{n-\bar{n}} h\|_{1,q} + 2C_{\bar{n}} \|\mathcal{L}^{n-\bar{n}} h\|_{0,q+1} \\
&\leqslant \nu^{\bar{n}} \|\mathcal{L}^{n-\bar{n}} h\|_{1,q} + C_\# \|h\|_{0,q+1} \\
&\leqslant \nu^{k\bar{n}} \|\mathcal{L}^m h\|_{1,q} + C_\# \sum_{j=0}^{k-1} \nu^{j\bar{n}} \|h\|_{0,q+1} \\
&\leqslant C_\# \nu^n \|h\|_{1,q} + C_\# \|h\|_{0,q+1}.
\end{aligned}$$

This concludes the Lemma. □

Remark 4.15. *Note that the Lasota–Yorke inequality is proven in Lemma 4.9 only for $h \in \mathcal{C}^r$. However by density it follows immediately that it holds for all $h \in \mathcal{B}^{p,q}$.*

The last ingredient of the argument is the compactness of $\mathcal{L}$.

Lemma 4.16. *For each $q > q' > 0$ the operator $\mathcal{L} : \mathcal{B}^{1,q'} \to \mathcal{B}^{0,q}$ is compact.*

Proof. The proof proceeds along the same lines as Lemma 3.10 and is left to the reader as a useful exercise. □

Lemmas 4.9 and 4.16, together with Theorem 1.1, imply that $\mathcal{L}$ has spectral radius one and essential spectral radius bounded by ν.

4.4 Low regularity norms

Here we consider norms adapted to maps with minimal regularity. Such norms are inspired by Demers and Liverani (2008) (of which they constitute a simplification) where they have been developed to treat maps with singularities. Subsequently they have been modified to study the statistical properties of billiards by Baladi, Demers, and Liverani (2018) and Demers and H.-K. Zhang (2011, 2013, 2014).

However, such norms turn out to be useful also in treating $\mathcal{C}^{1+\alpha}$ maps, with $\alpha \in (0,1)$.

The problem with handling $f \in \mathcal{C}^{1+\alpha}$, $\alpha \in (0,1)$, comes from the fact that $p \in \mathbb{N}$, thus the minimal, non trivial, allowed p is 1 while the arguments of the previous section need, at least, that $p \leqslant \alpha$. To overcome this limitation one must introduce the equivalent of a Hölder or Sobolev norm in the unstable direction. This can be done in many ways, the one proposed in Demers and Liverani (2008) being the most geometrical.

The basic idea is that any distribution h that can be integrated along a stable curve naturally gives rise to a function

$$\Psi(h) : \Omega_q = \{(W,\varphi) \ : \ W \in \Sigma^{1+\alpha}, \|\varphi\|_{\mathcal{C}^q_0(W,\mathbb{C})} \leqslant 1\} \to \mathbb{C}$$

defined as

$$\Psi(h)(W,\varphi) := \int_W h\varphi.$$

Thus it suffices to define a distance on Ω_q and impose Hölder regularity on $\Psi(h)$ with respect to such a distance. Since we find it convenient to work in charts we will define a distance in each $\Omega_{i,q} = \{(W,\varphi) \ : \ W \in \Sigma_i^{1+\alpha}, \|\varphi\|_{\mathcal{C}^q_0(W,\mathbb{C})} \leqslant 1\}$. Note that the sets $\Omega_{i,q}$ are not disjoint, yet we will consider their disjoint union, so an object with two different representations will be treated as two different objects. Then, for each $(W_{i,z,G},\varphi), (W_{i,z',G'},\varphi') \in \Omega_{i,q}$ we define

$$\begin{aligned} d((W_{i,z,G},\varphi),(W_{i,z',G'},\varphi')) &= \|z-z'\| + \|G\circ\tau_z - G'\circ\tau_{z'}\|_{\mathcal{C}^0(B_{d_s}(0,2\delta))} \\ &\quad + \|\varphi\circ\phi_i^{-1}\circ\mathbb{G}\circ\tau_z - \varphi'\circ\phi_i^{-1}\circ\mathbb{G}'\circ\tau_{z'}\|_{\mathcal{C}^q_0(B_{d_s}(0,\delta))} \end{aligned} \tag{4.4.1}$$

where $\tau_z(x) = x+z$ and $\mathbb{G}(x) = (x, G(x))$. The reader can easily check that the above is a semi-metric in $\Omega_{i,q}$. Indeed, two curves with the same centre that differ only outside a ball of radius 2δ have zero distance. This is reasonable as the value of G outside such a ball is totally irrelevant and we defined G on the whole space just for convenience, while the introduction of enlarged manifolds was simply a device to avoid invoking some fancy extension theorem to enlarge our manifolds when needed. Thus, it is natural to consider the equivalence classes with respect to the equivalence relation $W \sim W'$ iff $d(W,W') = 0$. In the following we will do so without further mention. We have thus defined a metric and we can now define,

for each $p < q < \alpha$, and $a > 0$, to be chosen later,

$$\|h\|_{p,q} = a\|h\|_{0,q-p} + \sup_i \sup_{\substack{(W,\varphi),(W',\varphi')\in\Omega_i^q \\ d((W,\varphi),(W',\varphi'))\leqslant\delta/4}} \frac{\left|\int_W h\varphi - \int_{W'} h\varphi'\right|}{d((W,\varphi),(W',\varphi'))^p}.$$

Once the norms are defined we can again complete the $\mathcal{C}^{1+\alpha}$ functions with respect to the norms $\|\cdot\|_{0,q}$ and $\|\cdot\|_{p,q}$ to obtain the spaces $\mathcal{B}^{0,q}$ and $\mathcal{B}^{p,q}$, respectively. Next, we need to prove the Lasota–Yorke inequalities.

Lemma 4.17. *For each $1 > \alpha > q > p > 0$ and $\nu \in (\lambda^{-\min\{p,q-p\}}, 1)$ there exist $C, B > 0$ such that, for all $h \in \mathcal{C}^\alpha(M, \mathbb{C})$ and $n \geqslant 0$,*

$$\|\mathcal{L}^n h\|_{0,q} \leqslant C\|h\|_{0,q}$$
$$\|\mathcal{L}^n h\|_{p,q} \leqslant C\nu^n\|h\|_{p,q} + B\|h\|_{0,q}.$$

Proof. The first inequality has been proven in Lemma 4.9. In addition, by (4.3.3),[17]

$$\|\mathcal{L}^n h\|_{0,q-p} \leqslant C_\#\lambda^{-(q-p)n}\|h\|_{0,q-p} + C_n\|h\|_{0,q}. \tag{4.4.2}$$

For the second, let $(W,\varphi) = (W_{i,z,G},\varphi)$, $(W',\varphi') = (W_{i,z',G'},\varphi') \in \Omega_i^q$ and recall from the beginning of the proof of Lemma 4.9 that

$$\int_{W_{i,z,G}} \mathcal{L}^n h\varphi = \sum_{j=1}^m \int_{W_{k_j,z_j,G_j}} h|\det Df^n|^{-1} J_W f^n \cdot \varphi\circ f^n \boldsymbol{\varphi}_j.$$

Let $\widehat{W}_{k_j,z_j,G_j} = \phi_{k_j}^{-1}(\{\mathbb{G}_j(x)\}_{x\in B_{d_S}(z_j,\delta/2)})$ be the restriction of W_{k_j,z_j,G_j}. Since the construction of the decomposition holds for any choice of δ, we can arrange that $\operatorname{supp}\boldsymbol{\varphi}_j \subset \widehat{W}_{k_j,z_j,G_j}$ and that $\cup_j \widehat{W}_{k_j,z_j,G_j} \supset f^{-n}\widetilde{W}$. Let G_j' be the function describing the part of the graph of $f^{-n}W'$ in the chart U_{k_j} which is $C_\# d(W,W')\lambda^{-n}$ close to W_{k_j,z_j,G_j}. Then $\{W_{k_j,z_j,G_j'}\}$ is a covering of $f^{-n}W'$. Next we define $\psi_j : W_{k_j,z_j,G_j'} \to W_{k_j,z_j,G_j}$ as

$$\psi_j(\zeta) = \phi_{k_j}^{-1}\circ\mathbb{G}_j\circ\pi\circ\phi_{k_j}(\zeta),$$

where $\pi(x,y) = x$. Setting $\boldsymbol{\varphi}_j' = \boldsymbol{\varphi}_j\circ\psi_j$ we have

$$\boldsymbol{\varphi}_j'\circ\phi_{k_j}^{-1}\circ\mathbb{G}_j'(x) = \boldsymbol{\varphi}_j\circ\phi_{k_j}^{-1}\circ\mathbb{G}_j(x).$$

[17] Since $\|h\|_{0,q'} \leqslant \|h\|_{0,q''}$ for all $q'' \leqslant q'$ and $q - p + 1 > q$.

If $I_\zeta = \{j \ : \ \boldsymbol{\varphi}_j \circ f^{-n}(\zeta) > 0\}$, then, by definition, $\sum_{j\in I(\zeta)} \boldsymbol{\varphi}_j \circ f^{-n}(\zeta) = 1$. For all $j, j' \in I(\zeta)$, we have $d(\psi_j(\zeta), \psi_{j'}(\zeta)) \leqslant C_\# \lambda^{-n} d(W, W')$.[18] Accordingly,

$$\left| \sum_j \boldsymbol{\varphi}'_j - 1 \right|_{\mathcal{C}^1} \leqslant C_\# d(W, W'). \tag{4.4.3}$$

Next we set

$$\begin{aligned}
&Z_j = |\,|\det Df^n|^{-1} J_W f^n\,|_{\mathcal{C}^q(W)}\,; \quad Z'_j = |\,|\det Df^n|^{-1} J_{W'} f^n\,|_{\mathcal{C}^q(W')}\\
&\gamma_j = Z_j^{-1} |\det Df^n|^{-1} J_W f^n\,; \quad \gamma'_j = (Z'_j)^{-1} |\det Df^n|^{-1} J_{W'} f^n\\
&\varphi_j = \varphi \circ f^n\,; \quad \varphi'_j = \varphi' \circ f^n\\
&\bar\varphi_j = \varphi \circ f^n \circ \psi_j.
\end{aligned}$$

By the usual distortion arguments if follows that

$$|Z'_j \gamma'_j - Z_j \gamma_j \circ \psi_j|_{\mathcal{C}^{\alpha-p}} \leqslant C_\# d(W, W')^p Z_j. \tag{4.4.4}$$

In addition,

$$\varphi'_j \circ \phi_{k_j}^{-1} \circ \mathbb{G}'_j(x) - \bar\varphi_j \circ \phi_{k_j}^{-1} \circ \mathbb{G}'_j(x) = \varphi'_j \circ \phi_{k_j}^{-1} \circ \mathbb{G}'_j(x) - \varphi_j \circ \phi_{k_j}^{-1} \circ \mathbb{G}_j(x)$$

hence, recalling Sub-lemma 4.6 and definition (4.4.1),

$$|\varphi'_j - \bar\varphi_j|_{\mathcal{C}^{q-p}} \leqslant C_\# d((W, \varphi), (W', \varphi'))^p. \tag{4.4.5}$$

Then, recalling (4.4.3) and Sub-lemma 4.10,

$$\left| \int_{W_{i,z',G'}} \mathcal{L}^n h\varphi' - \sum_{j=1}^m \int_{W_{k_j,z_j,G'_j}} h |\det Df^n|^{-1} J_W f^n \cdot \varphi' \circ f^n \boldsymbol{\varphi}'_j \right| \leqslant C_\# \|h\|_{0,q-p} d(W, W').$$

Moreover, by (4.4.4) and (4.4.5),

$$\left| \int_{W_{i,z',G'}} \mathcal{L}^n h\varphi' - \sum_{j=1}^m Z_j \int_{W_{k_j,z_j,G'_j}} h\gamma_j \circ \psi_j \cdot \bar\varphi_j \boldsymbol{\varphi}'_j \right| \leqslant C_\# \|h\|_{0,q-p} d((W, \varphi), (W', \varphi'))^p.$$

[18] Indeed, $\psi_j(\zeta) \neq \psi_{j'}(\zeta)$ only if $k_j \neq k_{j'}$. In such a case the vertical movement in the chart $k_{j'}$ will correspond to a movement in a different vertical direction in the chart k_j (but always inside the unstable cone). Since the manifolds W_{k_j,z_j,G_j} and W_{k_j,z_j,G'_j} are at a distance less than $C_\# \lambda^{-n} d(W, W')$, it follows that the point can move horizontally by at most $C_\# \lambda^{-n} d(W, W')$.

We can finally compute

$$\left|\int_{W_{i,z,G}} \mathcal{L}^n h\varphi - \int_{W_{i,z',G'}} \mathcal{L}^n h\varphi'\right| \leqslant \sum_{j=1}^{m} Z_j \left|\int_{W_{k_j,z_j,G_j}} h\gamma_j\varphi_j\boldsymbol{\varphi}_j - \int_{W_{k_j,z_j,G'_j}} h\gamma_j\circ\psi_j\bar{\varphi}_j\boldsymbol{\varphi}'_j\right| + C_\#\|h\|_{0,q-p}d((W,\varphi),(W',\varphi'))^p.$$

At last notice that, recalling (4.4.1),

$$d((W_{k_j,z_j,G_j},\gamma_j\varphi_j\boldsymbol{\varphi}_j),(W_{k_j,z_j,G'_j},\gamma_j\circ\psi_j\bar{\varphi}_j\boldsymbol{\varphi}'_j)) \leqslant C_\#\lambda^{-n}d(W,W').$$

Taking the sup on the manifolds and test functions and recalling (4.4.2) yields

$$\|\mathcal{L}^n h\|_{p,q} \leqslant C_* \max\{\lambda^{-np}, a^{-1}, \lambda^{(p-q)n}\}\|h\|_{p,q} + C_n\|h\|_{0,q},$$

for some constant $C_* > 0$. To conclude we choose $\bar{n}$ such that

$$\nu^{\min\{p,q-p\}} \geqslant \left[C_* \max\{\lambda^{-np}, \lambda^{(p-q)n}\}\right]^{1/\bar{n}},$$

and then choose $a = \nu^{-\bar{n}}C_*$. The Lemma follows arguing exactly as at the end of Lemma 4.9. □

We leave to the reader the (simple) proof that the unit ball of $\mathcal{B}^{p,q}$ is weakly compact in $\mathcal{B}^{0,q}$ for each $q \in (0,\alpha)$ and $p \in (0,q)$. Hence the transfer operator is compact as an operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{0,q}$. We obtain thus the quasi compactness also in this case. Note however that, due to the low regularity of the map, the essential spectral radius is rather large and it cannot be shrunk by using smaller Banach spaces since on them the transfer operator is not well defined.

Remark 4.18. *The above discussion proves that the essential spectral radius of $\mathcal{L}$ can be made arbitrarily close to $\lambda^{-\alpha/2}$. The factor $1/2$ in the exponent first appeared in the pioneering work of Kitaev (1999) and is most likely unavoidable.*

4.5 Decay of correlations and limit theorems

In Section 4.3 we have seen that $\mathcal{L}$ is quasi compact, hence it has only finitely many eigenvalues of modulus one. Moreover, since $\mathcal{L}$ is a positive operator (it sends positive functions to positive functions) it is possible to prove that the spectrum on the unit circle forms a group under multiplication. In addition, the operator is

power bounded and hence it cannot have Jordan blocks, thus the geometric and algebraic multiplicity of the peripheral spectrum are the same. Hence, since one is an eigenvalue, the dimension of the eigenspace associated to the eigenvalue one corresponds to the number of SRB measures. This is quite a bit of information; however, the fine structure of the spectrum is not known in general.

In particular, it is not known if Anosov maps always have a unique SRB measure. This depends on global topological properties that are not easily read from the study of the transfer operator. If the map has a unique SRB measure, then there is a dichotomy: either the map is not mixing (there are other eigenvalues, besides one, on the unit circle) or it mixes exponentially fast (one is the only eigenvalue on the unit circle and hence the operator has a spectral gap).

Accordingly, if the system is mixing, then the rate of mixing is determined by the eigenvalues of the point spectrum of $\mathcal{L}$. In particular, if an observable belongs to the kernel of the spectral projection of the largest eigenvalues, then it will mix faster.

Without entering into any detail let us conclude by pointing out that we have now the technology to upgrade all the results of Chapter 1 to the case of uniformly hyperbolic maps. In particular, we can study operators with a smooth potential hence obtain the CLT, Local CLT and large deviation estimates. Also the perturbation theory of Appendix C applies and we can prove stochastic and deterministic stability. Moreover, the slightly more general perturbation theorem in Gouëzel and Liverani (2006, Section 8) implies linear response. In addition, using weighted operators one can construct manifold invariant measures and use the thermodynamic formalism to estimate the Hausdorff dimension of many dynamically relevant sets. There is however an issue that we have not discussed: if one wants to study, e.g., the measure of maximal entropy, then one has to consider a transfer operator with a weight given by the expansion in the stable direction. This, unfortunately, is (in general) only Hölder even in the case of very regular maps. Of course one could study such a situation using the norms detailed in Section 4.4, however the question remains whether it is possible to shrink the essential spectral radius or whether one has to live with a very large essential spectral radius also for very regular maps. The answer is that the essential spectrum can be shrunk exactly as in Section 4.3. In order to do so however, it is necessary to consider slightly more general Banach spaces; the details can be found, e.g., in Gouëzel and Liverani (2008).

4.6 A comment on the discontinuous case

Another case in which a map has low regularity is when it is only *piecewise smooth*. This requires a new idea.

Up to now in the definition of the norms we used manifolds of a fixed, possibly small, size (δ) and the test functions were always of compact support. If the map is discontinuous, then $f^{-1}W$ will be cut by the dynamics into several pieces and hence one cannot avoid arbitrarily small manifolds and test functions that are different from zero at the boundary of the manifold. We are thus forced to include in the set of allowed manifolds Σ arbitrarily small manifolds and for $W \in \Sigma$ to consider $\varphi \in \mathcal{C}^q(W, \mathbb{C})$ rather than $\varphi \in \mathcal{C}^q_0(W, \mathbb{C})$.

This implies that we cannot integrate by parts (otherwise we would produce boundary terms that we do not know how to estimate). Hence we are limited to $p < 1$, even if the map is very regular away from the discontinuities.

Luckily a second look at Section 4.4 shows that we never integrated by parts, thus we could have worked with $\mathcal{C}^q(W, \mathbb{C})$ as well.[19] However, a quick inspection of the previous arguments shows that they do not work for arbitrarily small manifolds, as the constants in the Lasota–Yorke inequality depend on δ. It is necessary to treat small manifolds differently.

A possible solution to this problem, first implemented in Demers and Liverani (2008) and inspired by Liverani (1995a), is to add to the strong norm a term of the form

$$\sup_{(W,\varphi)\in\Omega^q} \frac{1}{|W|^\alpha} \int_W h\varphi,$$

for some $\alpha \in (0, 1)$. This means that the integral of h on small manifolds is small, but not proportional to the volume of the piece, hence h is not necessarily a function and it can have very wild behaviour on small scales.

This idea has enabled the extension of this approach to piecewise hyperbolic maps, as already mentioned, as well as to dispersing billiard maps and their perturbations by Demers and H.-K. Zhang (2011, 2013, 2014), including a weakened form to treat the measure of maximal entropy by Baladi and Demers (2020), and to billiard flows by Baladi, Demers, and Liverani (2018).

[19] Indeed, there was no need to restrict to functions vanishing at the boundary of the manifold.

5 Statistical properties of uniformly hyperbolic contact flows

In this chapter we turn our attention to one of the simplest types of partially hyperbolic systems: uniformly hyperbolic flows. The flow direction is neutral and does not enjoy the contracting and expanding properties in the stable and unstable directions that we have exploited in previous chapters when studying the transfer operator for hyperbolic systems. Our goal in this chapter will be to describe how to modify the anisotropic Banach spaces successfully implemented for hyperbolic maps to the case when the flow preserves a contact form. Here we restrict our exposition to the smooth case. In the next chapter, we will address the changes necessary for implementation in the presence of billiard-type singularities.

5.1 Setting

For ease of exposition and to more clearly identify the key features of the techniques we shall present, we will limit our setting to that of a 3-dimensional man-

ifold. This will suffice for the purposes of explaining the main ideas of this technique, as well as its eventual application to dispersing planar billiards.

Let Ω be a 3-dimensional compact, smooth Riemannian manifold, and let $\Phi_t : \Omega \to \Omega$ be a C^2 Anosov flow. By this, we mean that $\{\Phi_t\}_{t\in\mathbb{R}}$ is a family of C^2 diffeomorphisms of Ω satisfying the group properties: (a) $\Phi_0 = Id$; (b) $\Phi_t \circ \Phi_s = \Phi_{t+s}$, for all $s, t \in \mathbb{R}$.

Moreover, at each $x \in \Omega$, there is a $D\Phi_t$-invariant splitting of the tangent space, $T_x\Omega = E^s(x) \oplus E^c(x) \oplus E^u(x)$, continuous in x, such that the angles between $E^s(x)$, $E^u(x)$ and $E^c(x)$ are uniformly bounded away from 0 on Ω. $E^c(x)$ is the flow direction at $x \in \Omega$. We assume there exist constants $C, C' > 0$, $\Lambda > 1$, such that for all $x \in \Omega$ and $t \geqslant 0$,

$$\|D\Phi_t(x)v\| \leqslant C\Lambda^{-t}\|v\| \quad \forall\ v \in E^s(x) \tag{5.1.1}$$

$$\|D\Phi_t(x)v\| \geqslant C'\Lambda^{t}\|v\| \quad \forall\ v \in E^u(x) \tag{5.1.2}$$

We shall assume throughout that our Anosov flow is *contact*, i.e. it preserves a contact form on Ω. More precisely, we assume there exists a C^2 one-form ω on Ω such that $\omega \wedge d\omega$ is nowhere zero. We assume that Φ_t preserves ω:

$$\omega(\Phi_t(x), D\Phi_t(x)v) = \omega(x, v), \qquad \forall x \in \Omega, v \in T_x\Omega. \tag{5.1.3}$$

It is clear from the invariance described by (5.1.3) that $\ker(\omega) = E^s(x) \oplus E^u(x)$. It follows that if $v_0 \in E^c(x)$ is a unit vector in the flow direction, then $\omega(v_0) \neq 0$. Thus replacing ω by $\omega/\omega(v_0)$, we may assume without loss of generality that $\omega(v_0) = 1$ and that the contact volume $\omega \wedge d\omega$ coincides with the Riemannian volume on Ω. It follows from these considerations that the Jacobian of the flow is identically equal to 1, i.e. $J\Phi_t = 1$, and that the flow preserves the Riemannian volume on Ω, which we shall denote by m.

5.2 Decay of correlations

The main question we shall address in these notes is that of the rate of decay of correlations of the contact Anosov flow defined in the previous section. For $\alpha > 0$ and $\varphi, \psi \in C^\alpha(\Omega)$, define the *correlation function*,

$$C_t(\varphi, \psi) = \left| \int_\Omega \varphi\, \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right|.$$

If $C_t(\varphi, \psi) \to 0$ as $t \to \infty$ for all Hölder continuous functions φ and ψ, then we say the flow is mixing. The question then becomes, at what rate? The main result that we shall establish in these notes is the following.

Theorem 5.1. *Let Φ_t be a C^2 Anosov flow of a smooth, compact 3-dimensional Riemannian manifold Ω preserving a C^2 contact form ω. Then for each $\alpha > 0$, there exists $\eta = \eta(\alpha)$, and $C > 0$ such that for all $\varphi, \psi \in C^\alpha(\Omega)$ and all $t \geqslant 0$,*

$$\left| \int_\Omega \varphi \, \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right| \leqslant C |\varphi|_{C^\alpha(\Omega)} |\psi|_{C^\alpha(\Omega)} e^{-\eta t}.$$

This is a special case of a more general result proved for any odd-dimensional manifold by Liverani (2004). We will limit our exposition to three dimensions in order to maintain the focus on the essential elements of the technique.

From the definition of the correlation function, one can see immediately that, due to the invariance of the measure m, a simple change of variables yields,

$$\int_M \varphi \, \psi \circ \Phi_t \, dm = \int_M \varphi \circ \Phi_{-t} \, \psi \, dm = \int_M \mathcal{L}_t \varphi \, \psi \, dm, \tag{5.2.1}$$

where for each t, $\mathcal{L}_t \varphi := \varphi \circ \Phi_{-t}$ is the *transfer operator*, or *Ruelle–Perron–Frobenius operator* associated with Φ_t, defined pointwise, for example, on continuous functions. From this change of variables, it follows that the rate of decay of correlations is tied to the spectral properties of the semi-group $\{\mathcal{L}_t\}_{t \geqslant 0}$. This is the perspective that we will continue to develop in this chapter.

5.3 Some history and present approach

The proof of exponential decay of correlations for some classes of uniformly hyperbolic flows has proved to be much more subtle than the analogous proof for hyperbolic diffeomorphisms. For uniformly hyperbolic diffeomorphisms, there is a type of dichotomy: either the map is exponentially mixing on smooth observables, or it is not mixing at all. This does not hold for uniformly hyperbolic flows. In Ruelle (1983), a class of Axiom A suspension flows with piecewise constant roof function were constructed that mix at a polynomial rate. Pollicott (1985) then generalized this class of examples to obtain polynomial decay of correlations of any power, indeed even logarithmically slow decay.

Some early success in proving exponential decay for geodesic flows on manifolds of constant negative curvature in 2 and 3 dimensions was achieved by Moore (1987), Ratner (1987) and Pollicott (1992), and certain perturbations were considered by Collet, Epstein, and Gallavotti (1984), but the techniques were algebraic and did not generalize to manifolds of variable curvature.

The first dynamical proof of decay of correlations for Anosov flows was given by Chernov (1998), who exploited the 'twist' provided by the contact form in order to estimate a key quantity, the temporal distance function (see (5.7.14) and Remark 5.24), yet he was only able to obtain a stretched exponential bound using Markov partitions. Next, Dolgopyat (1998) was the first to prove *exponential* decay of correlations for Anosov flows, using an assumption of C^1 stable and unstable foliations to estimate a crucial oscillatory integral (see Lemma 5.30). This work was further extended by Liverani (2004), who proved exponential decay for contact Anosov flows by combining a functional analytic approach with the ideas of Dolgopyat and Chernov. These ideas were then adapted to piecewise cone hyperbolic flows by Baladi and Liverani (2012), and finally[1] to some dispersing billiard flows by Baladi, Demers, and Liverani (2018). It is this line of argument that we shall follow in the present chapter, and we shall limit our discussion primarily to the smooth, Anosov case, in order to present the key ideas most clearly.[2]

Given this approach, several choices are available with regards to the functional analytic framework in which to view the transfer operator.

(1) The approach via Markov partitions used by Dolgopyat (1998).

(2) The norms originally used by Liverani (2004), which define norms integrating over the entire phase space of the flow. These were based on the paper by Blank, Keller, and Liverani (2002), which introduced a set of Banach spaces for Anosov diffeomorphisms and subsequently inspired a series of papers constructing norms for hyperbolic maps from several points of view (see Baladi (2017) for a recent survey of these approaches, and Baladi (2000) for a more in-depth treatment).

(3) The Sobolev-type spaces used by Baladi and Liverani (2012) for piecewise cone hyperbolic contact flows. These norms use Fourier transforms and were based on work of Baladi and Tsujii (2007) and Baladi and Gouëzel (2010) who constructed the analogous norms for diffeomorphisms.

(4) The 'geometric' approach of Gouëzel and Liverani (2006), which modified the norms of Blank, Keller, and Liverani (2002) to integrate over cone-stable curves only. This modification turned out to be essential for the adaptability

[1] In the meantime, Chernov (2007) and Melbourne (2007) had proved a stretched exponential bound for dispersing billiard flows using the techniques adapted from Chernov (1998) and Dolgopyat (1998).

[2] A different mechanism for exponential mixing has been proved in the recent work of Tsujii (2018) and Tsujii and Z. Zhang (2020), but this lies outside the scope of the present notes.

of this method to piecewise hyperbolic maps requiring only Hölder continuity in the unstable direction by Demers and Liverani (2008) and finally to dispersing billiards by Demers and H.-K. Zhang (2011, 2013, 2014). Importantly for these notes, it was recently extended to prove exponential decay of correlations for the finite horizon Sinai billiard flow by Baladi, Demers, and Liverani (2018).

In the present chapter, we will define a functional analytic setup for contact Anosov flows which follows the technique described in (4) above. As a result, our exposition and some proofs will differ from Liverani's published proof (Liverani (2004)). Yet we choose this method since it combines a relatively simple exposition with a flexible framework. To date, the geometric version of norms integrating over stable curves has proved to be the most versatile in terms of its applicability to a wide range of hyperbolic systems with discontinuities.

We provide a brief organizational outline of the chapter for the reader's convenience. In Section 5.4, we introduce necessary definitions and define the Banach spaces on which our transfer operators and resolvents will act. We also outline some properties of these spaces regarding embeddings and compactness. Unfortunately, Proposition 5.8 does not provide true Lasota–Yorke inequalities for our semi-group $\{\mathcal{L}_t\}_{t\geq 0}$, so in Section 5.5 we introduce the generator of the semi-group X and the related resolvent $R(z)$, $z \in \mathbb{C}$. As evidenced by Proposition 5.13 and Corollary 5.14, we are able to prove quasi-compactness for $R(z)$, and so obtain useful information about the spectrum of X (Proposition 5.17). In Section 5.6, we introduce an improved estimate on the spectral radius of $R(z)$ when $|\mathrm{Im}(z)|$ is large, which implies a spectral gap for X, and leads to the proof of Theorem 5.1. This in turn is reduced to a Dolgopyat-type estimate, Lemma 5.22, which is proved in Section 5.7. In Chapter 6, we briefly sketch some modifications needed to generalize the present approach to dispersing billiards, as carried out by Baladi, Demers, and Liverani (2018).

5.4 Functional analytic framework

In order to define the Banach spaces on which our transfer operator will act, we first extend its definition from acting on continuous functions introduced in Section 5.2 to acting on spaces of distributions.

For $\alpha \in (0, 1]$, and W a smooth submanifold of Ω, define the C^α-norm for

functions on W by

$$|\varphi|_{C^\alpha(W)} := \sup_{x\in W} |\varphi(x)| + H^\alpha_W(\varphi), \tag{5.4.1}$$

$$H^\alpha_W(\varphi) := \sup_{x\neq y\in W} |\varphi(x)-\varphi(y)| d_W(x,y)^{-\alpha}, \tag{5.4.2}$$

where $d_W(\cdot,\cdot)$ is the Riemannian metric restricted to W. Notice with this definition that $C^1(W)$ is the set of Lipschitz functions on W.

Since the flow is C^2, if $\psi \in C^1(\Omega)$, then $\psi \circ \Phi_{-t} \in C^1(\Omega)$. Thus we may define $\mathcal{L}_t$ acting on $(C^1(\Omega))^*$, the dual of $C^1(\Omega)$, by

$$\mathcal{L}_t f(\psi) = f(\psi \circ \Phi_t), \quad \text{for all } \psi \in C^1(\Omega),\ f \in (C^1(\Omega))^*.$$

If $f \in L^1(m)$, then we identify f with the measure $f dm \in (C^1(\Omega))^*$. With this identification, $\mathcal{L}_t$ has the pointwise definition stated earlier, $\mathcal{L}_t f = f \circ \Phi_{-t}$, and its action is consistent with (5.2.1).

5.4.1 Admissible curves

Due to the uniform hyperbolicity of Φ_t given by (5.1.1), we define stable and unstable cones $C^s(x), C^u(x) \subset E^s(x) \oplus E^u(x)$, lying in the kernel of the contact form. The cones satisfy the strict invariance condition,

$$D\Phi_{-t} C^s(x) \subset C^s(\Phi_{-t}x), \quad D\Phi_t C^u(x) \subset C^u(\Phi_t x), \quad \text{for all } t > 0. \tag{5.4.3}$$

Note that, in contrast to the families of cones used for hyperbolic maps throughout Chapter 4, these cones are 'flat' since they lie in the plane $E^s(x) \oplus E^u(x)$, and have empty interior in $T_x\Omega$. We may choose these cones so that they are continuous and uniformly transverse on Ω. Moreover, the uniform contraction and expansion given by (5.1.1) extends to all vectors in $C^s(x)$ and $C^u(x)$, respectively, with possibly slightly weaker constants C, C' and Λ.

Let $d_0 > 0$ denote the minimal length of a closed geodesic on Ω.

Definition 1. *We define a family of* admissible cone-stable curves, $\mathcal{W}^s = \mathcal{W}^s(\delta_0, C_0)$, *in Ω satisfying:*

(W1) for all $W \in \mathcal{W}^s$ and $x \in W$, the unit tangent vector to W at x belongs to $C^s(x)$;

(W2) there exists $\delta_0 \in (0, d_0/2)$ such that $|W| \leqslant \delta_0$ for all $W \in \mathcal{W}^s$;

(W3) there exists $C_0 > 0$ such that the curvature of W is bounded by C_0.

For brevity, we refer to $W \in \mathcal{W}^s$ simply as stable curves. A family of admissible cone-unstable curves $\mathcal{W}^u$ (referred to simply as unstable curves) is defined similarly.

Due to the strict invariance of the cones, we have $\Phi_{-t}\mathcal{W}^s \subseteq \mathcal{W}^s$, $t \geqslant 0$, up to subdivision of curves longer than length δ_0. Similarly, $\Phi_t \mathcal{W}^u \subseteq \mathcal{W}^u$, $t \geqslant 0$.

In order to compare different curves in $\mathcal{W}^s$, we will introduce a notion of distance between them, reminiscent of (4.4.1), but with the added consideration that we only want to measure distance transverse to the flow direction. To do this, we place finitely many local sections Σ_i in Ω, which are smooth surfaces with piecewise smooth boundary, such that

(a) there exists $\tau_0 \in (0, d_0/2)$, such that each $W \in \mathcal{W}^s$ projects as a smooth, connected curve onto at least one Σ_i under $\{\Phi_t\}_{0 \leqslant t \leqslant \tau_0}$;

(b) each Σ_i is uniformly transverse to the flow direction;

(c) for each i, there exists a common family of stable and unstable cones for all $x \in \Sigma_i$.

On each section, we distinguish a point $\bar{x}_i$ in the approximate center of Σ_i, and define local coordinates $(\bar{x}^s, \bar{x}^u)$ with $\bar{x}_i$ at the origin, and the $\bar{x}^s$ ($\bar{x}^u$) axis tangent to $E^s(\bar{x}_i)$ ($E^u(\bar{x}_i)$) at $\bar{x}_i$. We may construct the Σ_i so that they are approximately rectangular in these coordinates: $\Sigma_i = \{(\bar{x}^s, \bar{x}^u) : \bar{x}^s \in I_i^s, \bar{x}^u \in I_i^u\}$, where I_i^s and I_i^u are two intervals centered at 0.

On each domain[3] of the form $D_i = \{\Phi_{-t}(\Sigma_i)\}_{0 \leqslant t \leqslant \tau_0}$, let P_i^+ denote the projection onto Σ_i, defined at $x \in D_i$ as the first intersection of $\Phi_t(x)$ with Σ_i, for $t \geqslant 0$. For $W \in \mathcal{W}^s$, if $P_i^+ W$ is defined, then we may view it as the graph of a function $G_{i,W} : I_{i,W} \to I_i^u$, where $I_{i,W} \subset I_i^s$, in case the curve W is very short.

Now if $W_1, W_2 \in \mathcal{W}^s$, we define a notion of distance between them as follows. If there exists $U \in \mathcal{W}^u$ such that $U \cap W_1 \neq \emptyset$ and $U \cap W_2 \neq \emptyset$ and at least one i such that $P_i^+ W_1$ and $P_i^+ W_2$ are both defined, then

$$d_{\mathcal{W}^s}(W_1, W_2) := \min_i \{|I_{i,W_1} \triangle I_{i,W_2}| + |G_{i,W_1} - G_{i,W_2}|_{C^1(I_{i,W_1} \cap I_{i,W_2})}\}. \quad (5.4.4)$$

Otherwise,[4] set $d_{\mathcal{W}^s}(W_1, W_2) = \infty$.

[3] Note that these domains may overlap for different i.

[4] That is, if W_1 and W_2 do not project onto a common Σ_i, or if there is no $U \in \mathcal{W}^u$ with the required property.

The purpose of requiring the existence of $U \in \mathcal{W}^u$ intersecting both curves is to ensure that they are sufficiently close in the flow direction (since the distance in (5.4.4) only quantifies the distance between projected curves in Σ_i, which quotients out the flow direction).

Remark 5.2. *The choice to compare curves on sections rather than directly on the manifold Ω may seem unnecessarily awkward at this stage. Yet, it simplifies certain norm calculations considerably by introducing a convenient set of local coordinate systems. In addition, it allows for an immediate generalization to billiards since then one can simply take the sections Σ_i to correspond to the smooth parts of the boundary of the billiard table.*

A second point to notice is that the distance defined by (5.4.4) *does not define a metric, or even a pseudo-metric since it does not satisfy the triangle inequality (compare with* (4.4.1) *which does not contain the term $|I_{i,W_1} \triangle I_{i,W_2}|$ and does satisfy the triangle inequality). This does not affect our analysis at all since the norms we define will satisfy the triangle inequality, and this is sufficient for our purposes.*

For two curves $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) < \infty$, we can use the same coordinate system to define a notion of distance between test functions supported on these curves. Let $\psi_i \in C^0(W_i)$, $i = 1, 2$. Define

$$d_0(\psi_1, \psi_2) = \min_i \{|\psi_1 \circ G_{i,W_1} - \psi_2 \circ G_{i,W_2}|_{C^0(I_{i,W_1} \cap I_{i,W_2})}\},$$

where the minimum is taken over all i such that both $P_i^+(W_1)$ and $P_i^+(W_2)$ are both defined.

5.4.2 Definition of the norms and Banach spaces

Given $\alpha \in (0, 1)$ and $W \in \mathcal{W}^s$, define $C^\alpha(W)$ to be the closure[5] of $C^1(W)$ in the $C^\alpha(W)$ norm, defined by (5.4.1). This definition of $C^\alpha(W)$ guarantees that the embedding of our strong space into our weak space is injective (see Lemma 5.4).

Now fix $\alpha \in (0, 1]$. Given $f \in C^1(\Omega)$, define the *weak norm* of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leq 1}} \int_W f \, \psi \, dm_W,$$

[5] $C^\alpha(W)$ is strictly smaller than the set of functions with finite $|\cdot|_{C^\alpha(W)}$ norm, yet it contains all functions with finite $|\cdot|_{C^{\alpha'}(W)}$ norm for all $\alpha' > \alpha$.

where m_W is arc length measure along W.

By contrast, our strong norm will have three components, one for each of the stable, unstable and neutral directions. Choose $1 < q < \infty$, $\beta \in (0, \alpha)$ and $0 < \gamma \leqslant \min\{\alpha - \beta, 1/q\}$.

For $f \in C^1(\Omega)$, define the *strong stable norm* of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta(W)} \leqslant |W|^{-1/q}}} \int_W f\, \psi\, dm_W.$$

Define the *neutral norm* of f by

$$\|f\|_0 = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leqslant 1}} \int_W \frac{d}{dt}(f \circ \Phi_t)|_{t=0}\, \psi\, dm_W.$$

And finally, define the *unstable norm* of f by

$$\|f\|_u = \sup_{\varepsilon > 0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leqslant \varepsilon}} \sup_{\substack{\psi_i \in C^\alpha(W_i) \\ |\psi_i|_{C^\alpha(W_i)} \leqslant 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f\, \psi_1, dm_{W_1} - \int_{W_2} f\, \psi_2\, dm_{W_2} \right|.$$

Define the *strong norm* of f by

$$\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_0 + c_u \|f_u\|,$$

where $c_u > 0$ is a constant to be chosen later.

Now our weak space $\mathcal{B}_w$ is defined as the completion of $C^2(\Omega)$ in the $|\cdot|_w$ norm, while our strong space $\mathcal{B}$ is defined as the completion of $C^2(\Omega)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

Remark 5.3. *The restrictions on the parameters are placed due to the following considerations. That $\beta < \alpha$ is required for compactness (Lemma 3.10). Then $\gamma \leqslant \alpha - \beta$ is required when adjusting test functions for the unstable norm estimate* (5.4.10)*, while $\gamma \leqslant 1/q$ allows us to account for short unmatched pieces due to our use of sections in the same estimate. Finally, $q > 1$ is required to obtain contraction in the strong stable norm estimate* (5.4.9)*. For a C^2 flow, one may take $\alpha = 1$.*

In order to use the Dolgopyat estimate (Lemma 5.22) to prove Proposition 5.20, we shall introduce additional restrictions on the parameters when applying the mollification lemma (Lemma 5.23). For this proof, we shall need β to be sufficiently small and q sufficiently close to 1 so that $(1 + \beta - 1/q)/\gamma < \gamma_0$, where γ_0 is from Lemma 5.22.

5.4.3 Properties of the Banach spaces

The spaces $\mathcal{B}$ and $\mathcal{B}_w$ are spaces of distributions, and the following lemma describes some important relations with more familiar spaces.

For notational convenience, for $\psi \in C^\alpha(\Omega)$ and $f \in (C^\alpha(\Omega))^*$, we denote by $f(\psi)$ the action of f on ψ. Any $f \in C^0(\Omega)$ can be identified with an element of $(C^\alpha(\Omega))^*$ (which we still denote by f) via the equality $f(\psi) = \int_\Omega \psi f \, dm$.

Lemma 5.4. *The following set of inclusions are continuous, and the first two are injective,*

$$C^1(\Omega) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^\alpha(\Omega))^*.$$

Indeed, there exists $C > 0$ such that for all $f \in C^1(\Omega)$, we have

$$|f|_w \leqslant \|f\|_{\mathcal{B}} \leqslant C|f|_{C^1(\Omega)}. \tag{5.4.5}$$

Moreover,

$$\begin{aligned} |f(\psi)| &\leqslant C|f|_w|\psi|_{C^\alpha(\Omega)} \quad \forall\, f \in \mathcal{B}_w, \\ |f(\psi)| &\leqslant C\|f\|_s|\psi|_{C^\beta(\Omega)} \quad \forall\, f \in \mathcal{B}. \end{aligned} \tag{5.4.6}$$

Proof. The bounds in (5.4.5) are clear from the definitions of the norms, proving the continuity of the first two inclusions. Moreover the injectivity of the first inclusion is obvious, while that of the second follows from the fact that $C^1(W)$ is dense in both $C^\alpha(W)$ and $C^\beta(W)$ because of the way we have defined these spaces of test functions.

It remains to prove the inequalities in (5.4.6), which in turn imply the continuity of the last inclusion. We prove the first inequality in (5.4.6), since the proof of the second is similar.

Let $f \in C^2(\Omega)$, $\psi \in C^\alpha(\Omega)$. We subdivide Ω into a finite number of boxes B_i and foliate each box by a smooth foliation of stable curves $\{W_\xi\}_{\xi\in\Xi_i}$. To see that this is possible, we can choose each box B_i to lie inside one of the domains D_i corresponding to surface Σ_i. Choosing a smooth family of stable curves intersecting Σ_i, we can simply flow it to fill B_i.

Now on each B_i, we disintegrate the measure m into conditional measures $\rho_\xi dm_{W_\xi}$ on each W_ξ and a factor measure $\hat{m}_i$ on the index set Ξ_i. Since the foliation is smooth, we have $|\rho_\xi|_{C^1(W_\xi)} \leqslant C_1$ for some $C_1 > 0$ and all $\xi \in \Xi_i$.

Then,

$$|f(\psi)| = \left| \int_\Omega f\ \psi\ dm \right| \leqslant \sum_i \int_{\Xi_i} \left| \int_{W_\xi} f\ \psi\ \rho_\xi\, dm_{W_\xi} \right| d\hat{m}_i$$
$$\leqslant \sum_i \int_{\Xi_i} |f|_w |\psi|_{C^\alpha(W_\xi)} |\rho_\xi|_{C^\alpha(W_\xi)} d\hat{m}_i \leqslant C |f|_w |\psi|_{C^\alpha(\Omega)}.$$

Since this bound holds for all $f \in C^2(\Omega)$, by density it holds for all $f \in \mathcal{B}_w$. □

Remark 5.5. *The third inclusion in Lemma 5.4 can be made injective as well by adding a factor $|W|^{-1/q'}$ to the weak norm for some $q' > q$, and requiring that $\alpha < 1/q'$. This is done, for example, in Demers and H.-K. Zhang (2014, Lemma 3.8), but we omit this added factor in the present setting since the injectivity is irrelevant for our purposes.*

Lemma 5.6. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. The compactness follows from two important points: the compactness of the unit ball of $C^\alpha(W)$ in $C^\beta(W)$ for each $W \in \mathcal{W}^s$; and the compactness in the C^1 norm of the set of graphs $G_{i,W}$ with C^2 norm bounded by C_0 on each section Σ_i. This allows us to prove that for all $\varepsilon > 0$, there exists a finite set of linear functionals $\ell_{j,k}$ on $\mathcal{B}$, with $\ell_{j,k}(f) = \int_{W_j} f\ \psi_k\, dm_{W_j}$, $W_j \in \mathcal{W}^s$, $\psi_k \in C^\alpha(W_j)$, such that

$$\min_{j,k} \left(|f|_w - \ell_{j,k}(f) \right) \leqslant C \varepsilon^\gamma \|f\|_{\mathcal{B}}, \tag{5.4.7}$$

for a uniform constant $C > 0$ and all $f \in \mathcal{B}$. This implies the required compactness. For the details of the approximation needed to carry out the above estimate, see Demers and H.-K. Zhang (2011, Lemma 3.10) or Baladi, Demers, and Liverani (2018, Lemma 3.10). □

Problem 5.7. *Assume that* (5.4.7) *holds. Show that it implies that the unit ball of $\mathcal{B}$ is compact in $\mathcal{B}_w$.*

5.4.4 Lasota–Yorke type inequalities for the semi-group $\{\mathcal{L}_t\}_{t\geq 0}$

The semi-group of transfer operators $\{\mathcal{L}_t\}_{t\geq 0}$ satisfies the following set of dynamical inequalities, often called Lasota–Yorke, or Doeblin–Fortet, inequalities following their seminal role in the development of the spectral theory of transfer operators Doeblin and Fortet (1937) and Lasota and Yorke (1973).

Proposition 5.8. *There exists $C > 0$ such that for all $f \in \mathcal{B}$ and $t \geqslant 0$,*

$$
\begin{aligned}
|\mathcal{L}_t f|_w &\leqslant C|f|_w && (5.4.8)\\
\|\mathcal{L}_t f\|_s &\leqslant C(\Lambda^{-\beta t} + \Lambda^{-(1-1/q)t})\|f\|_s + C|f|_w && (5.4.9)\\
\|\mathcal{L}_t f\|_u &\leqslant C\Lambda^{-\gamma t}\|f\|_u + C\|f\|_0 + C\|f\|_s && (5.4.10)\\
\|\mathcal{L}_t f\|_0 &\leqslant C\|f\|_0. && (5.4.11)
\end{aligned}
$$

If $\mathcal{L}_t$ were the transfer operator for a hyperbolic diffeomorphism of a 2-dimensional manifold, the inequalities (5.4.8) - (5.4.10) would be the traditional Lasota–Yorke inequalities (there would be no neutral direction), and we would conclude that $\mathcal{L}_t$ is quasi-compact with spectral radius 1, and essential spectral radius strictly smaller than 1. Unfortunately, in the case of a flow, we are left with the inequality (5.4.11) for the neutral norm, due to the lack of hyperbolicity in the flow direction. Thus the above inequalities do not represent a true set of Lasota–Yorke inequalities since the strong norm does not contract. So we do not prove that $\mathcal{L}_t$ is quasi-compact on $\mathcal{B}$.

Before proceeding to the next step in the argument, which is the introduction of the resolvent and the generator of the semi-group, we prove several items of the proposition, to give a flavor for the estimates required (which are in the spirit of the one in Section 4.4).

A full proof of analogous inequalities in a variety of settings can be found in, for example, Gouëzel and Liverani (2006) for Anosov diffeomorphisms, Demers and H.-K. Zhang (2011) for dispersing billiard maps, or Baladi, Demers, and Liverani (2018) for some dispersing billiard flows.

Proof of Proposition 5.8. Due to the density of $C^2(\Omega)$ in $\mathcal{B}$, it suffices to prove the inequalities for $f \in C^2(\Omega)$. We first prove (5.4.9).

When we flow a stable curve $W \in \mathcal{W}^s$ backwards, $\Phi_{-t}W$ may grow to have length greater than δ_0. If so, we subdivide it into a finite collection $\mathcal{G}_t(W) = \{W_i\}_i \subset \mathcal{W}^s$ so that each W_i has length between $\delta_0/2$ and δ_0, and $\cup_i W_i = \Phi_{-t}W$.

Let $f \in C^2(\Omega)$, $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \leqslant |W|^{-1/q}$. We must estimate, for $t \geqslant 0$,

$$
\int_W \mathcal{L}_t f\ \psi\, dm_W = \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\ \psi \circ \Phi_t\ J_{W_i}\Phi_t\, dm_{W_i}, \qquad (5.4.12)
$$

where we have changed variables and subdivided the integral on $\Phi_{-t}W$ into a sum of integrals over the $W_i \in \mathcal{G}_t(W)$. The function $J_{W_i}\Phi_t$ denotes the Jacobian of Φ_t along the curve W_i. Due to (5.1.1), this is a contraction.

Case I. $|\Phi_{-t}W| > \delta_0$.

For each i, define $\overline{\psi}_i$ to be the average value of $\psi \circ \Phi_t$ on W_i. Then subtracting the average on each W_i, we can rewrite (5.4.12) as,

$$\begin{aligned}\int_W \mathcal{L}_t f\ \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\ (\psi \circ \Phi_t - \overline{\psi}_i)\, J_{W_i}\Phi_t\, dm_{W_i} \\ &\qquad + \overline{\psi}_i \int_{W_i} f\ J_{W_i}\Phi_t\, dm_{W_i} \\ &\leqslant \sum_i \|f\|_s |\psi \circ \Phi_t - \overline{\psi}_i|_{C^\beta(W_i)} |W_i|^{1/q} |J_{W_i}\Phi_t|_{C^\beta(W_i)} \\ &\qquad + |f|_w |\overline{\psi}_i|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)},\end{aligned} \tag{5.4.13}$$

where we have applied the strong stable norm to the first set of terms and the weak norm to the second set.

The C^β norm of $\psi \circ \Phi_t - \overline{\psi}_i$ is easy to estimate using the uniform hyperbolicity of Φ_t given by (5.1.1), as well as the fact that we have defined stable curves which are transverse to the flow direction, and whose tangent vector lie exactly in the plane where the hyperbolicity of the flow dominates. Thus for $x, y \in W_i$,

$$|\psi \circ \Phi_t(x) - \psi \circ \Phi_t(y)| \leqslant H^\beta_W(\psi) d(\Phi_t(x), \Phi_t(y))^\beta \leqslant C\Lambda^{-\beta t} d(x,y)^\beta. \tag{5.4.14}$$

This, together with the fact that $\overline{\psi}_i = \psi \circ \Phi_t(y)$ for some $y \in W_i$ yields,

$$|\psi \circ \Phi_t - \overline{\psi}_i|_{C^\beta(W_i)} \leqslant C\Lambda^{-\beta t} |\psi|_{C^\beta(W)} \leqslant C\Lambda^{-\beta t} |W|^{-1/q}. \tag{5.4.15}$$

Then, since $\overline{\psi}_i$ is constant, $|\overline{\psi}_i|_{C^\alpha(W_i)} \leqslant |\psi|_{C^0(W)} \leqslant |W|^{-1/q}$.

In order to complete the estimate on the strong stable norm, we need the following lemma.

Lemma 5.9. *Let $W \in \mathcal{W}^s$, $t \geqslant 0$, and suppose $\Phi_{-t}W = \{W_i\}_i \subset \mathcal{W}^s$.*

(a) There exists $C_d > 0$, independent of W and t, such that for all W_i and $x, y \in W_i$,

$$\left| \frac{J_{W_i}\Phi_t(x)}{J_{W_i}\Phi_t(y)} - 1 \right| \leqslant C_d\, d(x,y).$$

(b) $|J_{W_i}\Phi_t|_{C^1(W_i)} \leqslant (1 + C_d)|J_{W_i}\Phi_t|_{C^0(W_i)}$.

(c) There exists $\bar{C}$, independent of W and $t \geqslant 0$, such that

$$\sum_i |J_{W_i}\Phi_t|_{C^0(W_i)} \leqslant \bar{C}.$$

Proof. Item (a) is a standard distortion bound in hyperbolic dynamics. It can be proved, for example, by choosing $\tau_1 > 0$ and subdividing $[0,t]$ into $[t/\tau_1]$ intervals of length τ_1, plus a last one of length $s \leqslant \tau_1$. Then using again (5.1.1)

$$\begin{aligned}
\log \frac{J_{W_i}\Phi_t(x)}{J_{W_i}\Phi_t(y)} &\leqslant \sum_{j=1}^{[t/\tau_1]} |\log J_{\Phi_{j\tau_1}W_i}\Phi_{\tau_1}(\Phi_{j\tau_1}(x)) - \log J_{\Phi_{j\tau_1}W_i}\Phi_{\tau_1}(\Phi_{j\tau_1}(y))| \\
&\quad + |\log J_{\Phi_{t-s}W_i}\Phi_s(\Phi_{t-s}(x)) - \log J_{\Phi_{t-s}W_i}\Phi_s(\Phi_{t-s}(y))| \\
&\leqslant \sum_{j=1}^{[t/\tau_1]} C d(\Phi_{j\tau_1}(x), \Phi_{j\tau_1}(y)) + C d(\Phi_{t-s}(x), \Phi_{t-s}(y)) \\
&\leqslant C' \sum_{j=1}^{[t/\tau_1]} \Lambda^{-j\tau_1} d(x,y) + \Lambda^{-(t-s)} d(x,y) \leqslant C'' d(x,y),
\end{aligned}$$

where C'' depends on the maximum C^2 norm of Φ_s, $0 \leqslant s \leqslant \tau_1$.

Item (b) is an immediate consequence of (a).

Item (c) also follows from (a). To see this, note that if $\Phi_{-t}W$ has length less than δ_0, then there is only a single W_i, and the fact that the Jacobian along stable curves is a contraction implies the inequality. If $\Phi_{-t}W$ has length longer than δ_0, then each W_i has length at least $\delta_0/2$. Thus using bounded distortion from (a) yields,

$$\begin{aligned}
\sum_i |J_{W_i}\Phi_t|_{C^0(W_i)} &\approx \sum_i \frac{|\Phi_t(W_i)|}{|W_i|} \leqslant 2\delta_0^{-1} \sum_i |\Phi_t(W_i)| \\
&\leqslant 2\delta_0^{-1}|W| \leqslant 2.
\end{aligned} \tag{5.4.16}$$

□

The items of the lemma allow us to complete the proof of (5.4.9). Recalling (5.4.13), and using (5.4.15) and Lemma 5.9(b) yields,

$$\int_W \mathcal{L}_t f\ \psi\ dm_W \leqslant \sum_i C\Lambda^{-\beta t}\|f\|_s \frac{|W_i|^{1/q}}{|W|^{1/q}} |J_{W_i}\Phi_t|_{C^0(W_i)} + C|f|_w |W|^{-1/q} |J_{W_i}\Phi_t|_{C^0(W_i)}.$$

The first sum is uniformly bounded in t and W by Lemma 5.9(a),(c) and a Hölder inequality,

$$\sum_i \frac{|W_i|^{1/q}}{|W|^{1/q}} |J_{W_i}\Phi_t|_{C^0(W_i)} \leqslant \left(\sum_i (1+C_d)\frac{|\Phi_t(W_i)|}{|W|}\right)^{1/q} \left(\sum_i |J_{W_i}\Phi_t|_{C_0(W_i)}\right)^{1-1/q} \leqslant (1+C_d)^{1/q}\bar{C}^{1-1/q}.$$

The second sum is bounded uniformly in t and W since by an estimate similar to (5.4.16),

$$\sum_i |W|^{-1/q} |J_{W_i}\Phi_t|_{C^0(W_i)} \leqslant 2\delta_0^{-1}|W|^{1-1/q}.$$

Putting these estimates together yields,

$$\int_W \mathcal{L}_t f\ \psi\ dm_W \leqslant C\Lambda^{-\beta t}\|f\|_s + C|f|_w. \tag{5.4.17}$$

Case II. $|\Phi_{-t}W| \leqslant \delta_0$.

In this case,[6] we do not subtract an average for the test function, and there is simply one term in (5.4.12), to which we apply the strong stable norm,

$$\int_W \mathcal{L}_t f\ \psi\ dm_W \leqslant \|f\|_s \frac{|\Phi_{-t}(W)|^{1/q}}{|W|^{1/q}} |J_{\Phi_{-t}(W)}\Phi_t|_{C^0},$$

where again, we have used (5.4.14) and Lemma 5.9 to estimate the norms of the test functions. By bounded distortion, $|J_{\Phi_{-t}(W)}\Phi_t|_{C^0} \approx \frac{|W|}{|\Phi_{-t}(W)|}$, so that

$$\int_W \mathcal{L}_t f\ \psi\ dm_W \leqslant C\|f\|_s \frac{|W|^{1-1/q}}{|\Phi_{-t}(W)|^{1-1/q}} \leqslant C\|f\|_s \Lambda^{-(1-1/q)t}.$$

[6] This case can be eliminated entirely by requiring that curves in $\mathcal{W}^s$ have a minimum length of say, $\delta_0/2$. Then Case I would suffice to estimate all curves, and (5.4.9) would simplify to $\|\mathcal{L}_t f\|_s \leqslant C\Lambda^{-\beta t}\|f\|_s + C|f|_w$. Since we are interested in presenting norms which can be applied to discontinuous maps and flows, we do not place this additional restriction on curves in $\mathcal{W}^s$.

Putting Cases I and II together and taking the supremum over W and ψ proves (5.4.9).

The proof of (5.4.8) follows more simply since the weak norm needs no contraction so we do not subtract the average value of the test function on each curve. Also, there is no weight of the form $|W|^{-1/q}$ since for the weak norm, the test function $\psi \in C^\alpha(W)$ satisfies $|\psi|_{C^\alpha(W)} \leqslant 1$. Thus following (5.4.12) and applying the weak norm to each term yields,

$$\begin{aligned}\int_W \mathcal{L}_t f\ \psi\ dm_W &\leqslant \sum_i |f|_w |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)} \\ &\leqslant \sum_i C\,|f|_w |J_{W_i}\Phi_t|_{C^0(W_i)} \leqslant C'|f|_w,\end{aligned}$$

where again we have used Lemma 5.9.

The proof of the neutral norm bound (5.4.11) is similarly straightforward. Using the group property of Φ_t, we have,

$$\frac{d}{ds}\big((\mathcal{L}_t f)\circ\Phi_s\big)|_{s=0} = \lim_{s\to 0}\frac{(f\circ\Phi_s - f)\circ\Phi_{-t}}{s} = \frac{d}{ds}(f\circ\Phi_s)|_{s=0}\circ\Phi_{-t}. \tag{5.4.18}$$

Taking $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$, we use (5.4.18) and change variables as in (5.4.12),

$$\begin{aligned}\int_W \frac{d}{ds}\big((\mathcal{L}_t f)\circ\Phi_s\big)|_{s=0}\,\psi\,dm_W &= \sum_i \int_{W_i} \frac{d}{ds}(f\circ\Phi_s)|_{s=0}\,\psi\circ\Phi_t\,J_{W_i}\Phi_t\,dm_{W_i} \\ &\leqslant \sum_i \|f\|_0 |\psi\circ\Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)},\end{aligned}$$

and the sum is uniformly bounded in t and W, again using Lemma 5.9.

The proof of (5.4.10) is more lengthy, and uses a graph transform-type argument to show that if $d_{\mathcal{W}^s}(W^1, W^2) \leqslant \varepsilon$, then $\Phi_{-t}(W^1)$ and $\Phi_{-t}(W^2)$ can be (mostly) decomposed into matched pieces W_j^1, W_j^2 such that $d_{\mathcal{W}^s}(W_j^1, W_j^2) \leqslant C\Lambda^{-n}\varepsilon$. Unfortunately, to obtain this strict contraction, we compare distances on the sections Σ_i and so this decomposition will also create (short) unmatched pieces which must be estimated using the strong stable norm, taking advantage of the weight $|W|^{1/q}$. To avoid cumbersome technicalities, we shall omit the proof in these notes. We refer the interested reader to Demers and H.-K. Zhang (2011) for the map version or Baladi, Demers, and Liverani (2018) for the flow version. □

5.5 The generator and the resolvent

The novel idea introduced by Liverani (2004) was to shift attention away from the semi-group of transfer operators, and onto the generator of the semi-group, and the associated resolvent. Indeed, the path we shall follow to prove Theorem 5.1 will be to prove a spectral gap for the generator.

For $f \in C^1(\Omega)$, define

$$Xf = \lim_{t \to 0^+} \frac{\mathcal{L}_t f - f}{t}.$$

The operator X is called the *generator* of the semi-group $\{\mathcal{L}_t\}_{t \geqslant 0}$. Since Φ_t is invertible, in fact $\{\mathcal{L}_t\}_{t \in \mathbb{R}}$ is a group when acting pointwise on functions; however, since we are interested in its action on the Banach space $\mathcal{B}$, we consider only the semi-group. This is because the dynamical properties of $\mathcal{L}_t$ for $t < 0$ will not preserve the norms: the roles of the stable and unstable directions are exchanged, and so the definition of the anisotropic spaces would also need to be changed in order to study $t < 0$.

Remark that if $f \in C^2(\Omega)$, then $Xf \in C^1(\Omega)$, so $Xf \in \mathcal{B}$ by Lemma 5.4. By definition, this implies that the domain of X is dense in $\mathcal{B}$.

The following lemma provides additional information about the behavior of $\mathcal{L}_t$ for small t.

Lemma 5.10. *There exists $C > 0$ such that for all $f \in \mathcal{B}$,*

(a) $\lim_{t \to 0^+} \|\mathcal{L}_t f - f\|_{\mathcal{B}} = 0$;

(b) $|\mathcal{L}_t f - f|_w \leqslant C t \|f\|_{\mathcal{B}}$, $t \geqslant 0$.

Proof. For the proof of (a), see Baladi, Demers, and Liverani (2018, Lemma 4.6). We prove (b).

Let $f \in C^2(\Omega)$, $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$. Then using

(5.4.18), we estimate

$$\begin{aligned}\int_W (\mathcal{L}_t f - f)\,\psi\, dm_W &= \int_W \int_0^t \frac{d}{ds}(f\circ\Phi_{-s})\,\psi\, ds\, dm_W \\ &= \int_0^t \int_W \frac{d}{dr}(f\circ\Phi_r)|_{r=0}\circ\Phi_{-s}\,\psi\, dm_W\, ds \\ &= \int_0^t \sum_i \int_{W_i} \frac{d}{dr}(f\circ\Phi_r)|_{r=0}\,\psi\circ\Phi_s\, J_{W_i}\Phi_s\, dm_{W_i}\, ds \\ &\leqslant \int_0^t \|f\|_0 \sum_i |\psi\circ\Phi_s|_{C^\alpha(W_i)}|J_{W_i}\Phi_s|_{C^\alpha(W_i)} \\ &\leqslant Ct\|f\|_0,\end{aligned}$$

where we have changed variables in the third line, and used Lemma 5.9 in the fourth. Taking the supremum over ψ and W proves (b). □

Remark 5.11. *Item (a) of Lemma 5.10 implies that the semi-group $\{\mathcal{L}_t\}_{t\geqslant 0}$ acting on $\mathcal{B}$ is strongly continuous. This in turn implies that X is closed as an operator on $\mathcal{B}$, with a dense domain, Davies (2007).*

Next, for $z\in\mathbb{C}$, we define the *resolvent* $R(z):\mathcal{B}\to\mathcal{B}$ by

$$R(z) = (zI - X)^{-1}. \tag{5.5.1}$$

When $\mathrm{Re}(z) > 0$, $R(z)$ has the following representation,

$$R(z)f = \int_0^\infty e^{-zt}\mathcal{L}_t f\, dt. \tag{5.5.2}$$

The importance of (5.5.2) is that the operator $R(z)$ integrates out time, and so eliminates the neutral direction. This will be the key point that enables the subsequent analysis.

Problem 5.12. *Use the definition of X to verify that $R(z)$ defined by (5.5.2) satisfies $R(z)Xf = -f + zR(z)f$. This implies that $R(z)$ satisfies (5.5.1).*

5.5.1 Quasi-compactness of $R(z)$

Define $\lambda = \max\{\Lambda^{-\beta}, \Lambda^{-\gamma}, \Lambda^{-(1-1/q)}\} < 1$.

Proposition 5.13. *There exists $C \geqslant 1$ such that for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) =: a > 0$, and all $f \in \mathcal{B}$ and $n \geqslant 0$,*

$$
\begin{aligned}
|R(z)^n f|_w &\leqslant C a^{-n} |f|_w \,, && (5.5.3)\\
\|R(z)^n f\|_s &\leqslant C(a - \log\lambda)^{-n} \|f\|_s + C a^{-n} |f|_w \,, && (5.5.4)\\
\|R(z)^n f\|_u &\leqslant C(a - \log\lambda)^{-n} \|f\|_u + C a^{-n} (\|f\|_s + \|f\|_0) \,, && (5.5.5)\\
\|R(z)^n f\|_0 &\leqslant C a^{1-n} (1 + |z|/a) |f|_w \,. && (5.5.6)
\end{aligned}
$$

Due to the integration over time provided by (5.5.2), Proposition 5.13 represents an essential improvement over Proposition 5.8. The key improvement is the weak norm $|f|_w$ appearing on the right hand side of (5.5.6) in place of the neutral norm $\|f\|_0$ which appeared on the right hand side of (5.4.11). This permits the following corollary.

Corollary 5.14. *Let $z = a + ib \in \mathbb{C}$ with $a > 0$. The spectral radius of $R(z)$ on $\mathcal{B}$ is at most a^{-1}. For any $\sigma > (1 - a^{-1}\log\lambda)^{-1}$, we may choose $c_u > 0$ such that the essential spectral radius is at most σa^{-1}. Thus the spectrum of $R(z)$ outside the disk of radius σa^{-1} is finite-dimensional, and if it is nonempty, then $R(z)$ is quasi-compact as an operator on $\mathcal{B}$.*

Proof. Using the definition of the strong norm, we estimate,

$$
\begin{aligned}
a^n \|R(z)^n f\|_{\mathcal{B}} &= a^n \|R(z)^n f\|_s + c_u a^n \|R(z)^n f\|_u + a^n \|R(z) f\|_0 \\
&\leqslant C[(1 - a^{-1}\log\lambda)^{-n} + c_u] \|f\|_s + C c_u (1 - a^{-1}\log\lambda)^{-n} \|f\|_u \\
&\quad + C c_u \|f\|_0 + C(1 + a + |z|) |f|_w .
\end{aligned}
$$

Now choose $\sigma \in ((1 - a^{-1}\log\lambda)^{-1}, 1)$ and $N > 0$ so large that $\sigma^N/2 > C(1 - a^{-1}\log\lambda)^{-N}$. Finally, choose $c_u > 0$ so small that $C c_u < \sigma^N/2$. Then the above estimate yields,

$$
a^N \|R(z)^N f\|_{\mathcal{B}} \leqslant \sigma^N \|f\|_{\mathcal{B}} + C(a + |z| + 1) |f|_w , \qquad (5.5.7)
$$

which is the traditional Lasota–Yorke inequality. Since this can be iterated, it follows from a classical result of Hennion (1993), together with the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ (Lemma 5.6), that the essential spectral radius of $R(z)$ on $\mathcal{B}$ is at most σa^{-1}. □

The following two facts will be useful for proving Proposition 5.13.

Problem 5.15. *Starting from* (5.5.2)*, prove by induction that*

$$R(z)^n f = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\, \mathcal{L}_t f \, dt \,.$$

Problem 5.16. *Let $z = a + ib$ with $a > 0$. Show that*

$$\left| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\, dt \right| \leqslant \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-at}\, dt \leqslant a^{-n}, \text{for all } n \geqslant 1.$$

Proof of Proposition 5.13. As usual, by density, it suffices to prove the inequalities for $f \in C^2(\Omega)$. We begin by proving the weak norm estimate (5.5.3).

Let $W \in \mathcal{W}^s s$, $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$. Then for $n \geqslant 1$,

$$\begin{aligned} \left| \int_W R(z)^n f\, \psi \, dm_W \right| &= \left| \int_0^\infty \int_W \mathcal{L}_t f\, \psi \, dm_W \, \frac{t^{n-1}}{(n-1)!} e^{-zt}\, dt \right| \\ &\leqslant \int_0^\infty |\mathcal{L}_t f|_w \, \frac{t^{n-1}}{(n-1)!} e^{-at}\, dt \leqslant C|f|_w a^{-n}, \end{aligned} \tag{5.5.8}$$

where in the first line we have used Problem 5.15 and reversed the order of integration since the integral of $\mathcal{L}_t f$ on W is uniformly bounded in t; in the second line we have used (5.4.8) and Problem 5.16 to complete the estimate. Taking the appropriate supremum over W and ψ proves (5.5.3).

The proof of (5.5.4) is similar, except that we take advantage of the extra contraction provided by (5.4.9). Taking $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \leqslant |W|^{-1/q}$, we estimate for $n \geqslant 1$, following (5.5.8),

$$\begin{aligned} \left| \int_W R(z)^n f\, \psi \, dm_W \right| &\leqslant \int_0^\infty \|\mathcal{L}_t f\|_s \, \frac{t^{n-1}}{(n-1)!} e^{-at}\, dt \\ &\leqslant \int_0^\infty \Big[C\|f\|_s \frac{t^{n-1}}{(n-1)!} e^{-(a-\log\lambda)t} + \\ &\qquad + C|f|_w \frac{t^{n-1}}{(n-1)!} e^{-at} \Big] dt \\ &\leqslant C(a-\log\lambda)^{-n}\|f\|_s + Ca^{-n}|f|_w, \end{aligned}$$

where again we have used Problems 5.15 and 5.16, as well as (5.4.9).

The estimate for (5.5.5) is again similar, now using (5.4.10).

Finally, we prove (5.5.6). This differs from the previous estimates since we will not simply apply (5.4.11), which would result in no improvement over Proposition 5.8. Rather we first integrate by parts in order to use the weak norm. Now taking $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$, we estimate,

$$\begin{aligned}
&\int_W \frac{d}{ds}((R(z)^n f)\circ \Phi_s)|_{s=0}\,\psi\, dm_W \\
&\quad = \int_W \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\,\frac{d}{ds}\big((\mathcal{L}_t f)\circ\Phi_s\big)|_{s=0}\, dt\,\psi\, dm_W \\
&\quad = -\int_W \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\,\frac{d}{dt}(\mathcal{L}_t f)\, dt\,\psi\, dm_W \\
&\quad = \int_W \int_0^\infty \left(\frac{t^{n-2}}{(n-2)!}e^{-zt} - ze^{-zt}\frac{t^{n-1}}{(n-1)!}\right)\mathcal{L}_t f\, dt\,\psi\, dm_W \\
&\quad = \int_0^\infty \left(\frac{t^{n-2}}{(n-2)!} - \frac{zt^{n-1}}{(n-1)!}\right) e^{-zt}\int_W \mathcal{L}_t f\,\psi\, dm_W\, dt.
\end{aligned}$$

Now we use the triangle inequality, apply the weak norm estimate (5.4.8) to the integral over W, and Problem 5.16 to both terms integrated over t to obtain,

$$\|R(z)^n f\|_0 \leqslant C|f|_w\big(a^{1-n} + |z|a^{-n}\big),$$

which proves (5.5.6). □

5.5.2 Initial results on the spectrum of X

Proposition 5.13 and Corollary 5.14 provide useful information about the spectrum of X, which we denote by $\sigma(X)$. First notice that since $\|\mathcal{L}_t\|_{\mathcal{B}}$ is uniformly bounded in t by Proposition 5.8, the spectrum of X on $\mathcal{B}$ is entirely contained in the left half-plane, $Re(z) \leqslant 0$. Moreover, the invariant measure m, identified with the constant function 1 according to our convention, is an eigenvector with eigenvalue 0 for X.

Proposition 5.17. *The spectrum of X on $\mathcal{B}$ is contained in $Re(z) \leqslant 0$. The intersection $\sigma(X) \cap \{z \in \mathbb{C} : \log\lambda < Re(z) \leqslant 0\}$ consists of at most countably many isolated eigenvalues of finite multiplicity. The spectrum of X on the imaginary axis contains only an eigenvalue at 0 of multiplicity 1.*

We will not present a formal proof of Proposition 5.17, which by now is standard. We refer the interested reader to Baladi and Liverani (2012, Lemma 3.6,

Corollary 3.7) or Baladi, Demers, and Liverani (2018, Corollary 5.4). However, we discuss the main ideas, which are essential for what comes next.

The proof of the proposition relies on the observation that for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we have,

$$\bar{\rho} \in \mathrm{sp}(R(z)) \quad \text{if and only if} \quad \bar{\rho} = (z-\rho)^{-1}, \quad \text{where } \rho \in \mathrm{sp}(X). \tag{5.5.9}$$

Here $R(z)$ and X are understood as operators on $\mathcal{B}$. The proof of this is classical, see for example Davies (2007, Lemma 8.1.9). Furthermore, the following fact holds.

Problem 5.18. *Suppose $\rho \in sp(X)$ and $\bar{\rho} = (z-\rho)^{-1} \in sp(R(z))$. Show that for any $k \geqslant 1$ and $f \in \mathcal{B}$, we have $(R(z)-\bar{\rho})^k f = 0$ if and only if $(X-\rho)^k f = 0$. This implies that $\bar{\rho}$ is an eigenvalue of $R(z)$ of multiplicity k if and only if ρ is an eigenvalue of X of multiplicity k.*

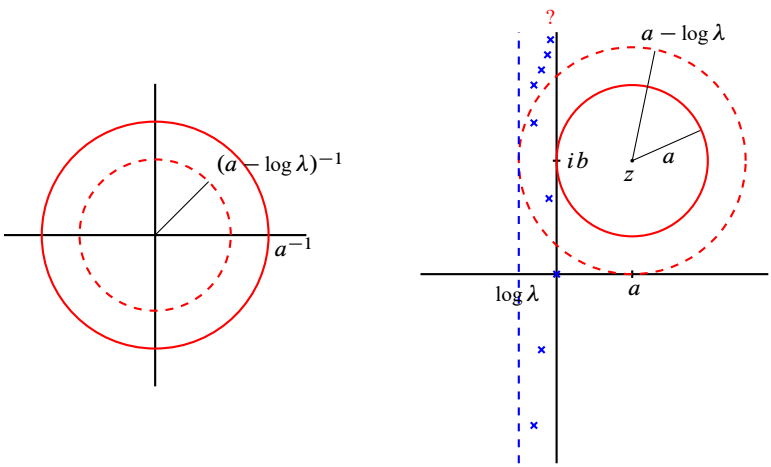
Figure 5.1 summarizes this relationship. By fixing $a > 0$ and considering the family of parameters $\{z = a + ib : b \in \mathbb{R}\}$, we see that the essential spectrum of X is contained in the half plane $\{Re(w) \leqslant \log\lambda\}$, and so is bounded away from the imaginary axis.

Since the spectrum of $R(z)$ in the annulus $\{(a - \log\lambda)^{-1} < |w| \leqslant a^{-1}\}$ contains only finitely many eigenvalues of finite multiplicity by Corollary 5.14, it follows that for each $b_0 > 0$ the intersection of $\sigma(X)$ with the rectangle $\{\mathrm{Re}(w) \in (\log\lambda, 0], |\mathrm{Im}(w)| \leqslant b_0\}$ contains only finitely many eigenvalues of finite multiplicity. Once this identification is made, the fact that the imaginary axis contains only the simple eigenvalue at 0 follows from the fact that contact Anosov flows are mixing, see Katok (1994, Theorem 3.6), together with the classical Hopf argument as in Liverani and Wojtkowski (1995).

5.6 A spectral gap for X

Unfortunately, Proposition 5.17 is not sufficient to prove the desired result on decay of correlations that is the goal of these notes. The problem is that although the spectrum of X in each rectangle $\{w \in \mathbb{C} : \mathrm{Re}(w) \in (\log\lambda, 0], |\mathrm{Im}(w)| \leqslant b_0\}$ is finite dimensional, and so the minimum distance from an eigenvalue $\rho \neq 0$ in this rectangle to the imaginary axis is positive, it may happen that a sequence of eigenvalues $\rho = u + iv$ approaches the imaginary axis as $|v| \to \infty$.

In order to conclude exponential mixing, we will show that in fact, X has a spectral gap.



(a) (b)

Figure 5.1: (a) The spectrum of $R(z)$ is contained in a disk of radius a^{-1} (solid red circle), and its essential spectrum is contained in a disk of radius $(a-\log\lambda)^{-1}$ (dashed red circle).
(b) The red circles are the images of the corresponding circles in (a) under the transformation $w \mapsto z - w^{-1}$. Due to (5.5.9), the spectrum of X lies outside the solid red circle, and its essential spectrum must lie outside the dashed red circle. This forces the strip between the dashed blue line ($\mathrm{Re}(w) = \log\lambda$) and the imaginary axis to contain only isolated eigenvalues of finite multiplicity. The x's are possible eigenvalues of X, which may accumulate on the imaginary axis as $|\mathrm{Im}(w)| \to \infty$.

Theorem 5.19. *There exists $\nu > 0$ such that*

$$\sigma(X) \cap \{w \in \mathbb{C} : -\nu < Re(w) \leqslant 0\} = 0.$$

Theorem 5.19 in turn will follow from the following proposition.

Proposition 5.20. *There exist $\bar{\nu} > 0$, $\bar{C} > 0$ and $b_0 > 0$ such that for all $z = a + ib$ with $1 \leqslant a \leqslant 2$ and $|b| \geqslant b_0$, $\|R(z)^n\|_{\mathcal{B}} \leqslant (a + \bar{\nu})^{-n}$ for all $\bar{C} \log |b| \leqslant n \leqslant 2\bar{C} \log |b|$. Thus the spectral radius of $R(z)$ on $\mathcal{B}$ is at most $(a + \bar{\nu})^{-1}$ for all $1 \leqslant a \leqslant 2$, $|b| \geqslant b_0$.*

Proof of Theorem 5.19, assuming Proposition 5.20. Due to Proposition 5.20 and (5.5.9), the set $\{\mathrm{Re}(w) \in (\bar{\nu}, 0], |\mathrm{Im}(w)| \geqslant b_0\}$ is disjoint from $\sigma(X)$. On the other hand, the set $\{\mathrm{Re}(w) \in (\bar{\nu}, 0], |\mathrm{Im}(w)| \leqslant b_0\}$ contains only finitely many eigenvalues by Proposition 5.17, and 0 is the only eigenvalue on the imaginary axis. The finiteness of this set guarantees a positive minimum distance between the imaginary axis and the closest nonzero eigenvalue. □

5.6.1 Reduction of Proposition 5.20 to a Dolgopyat estimate

Turning our attention to Proposition 5.20, we note that the strength of the claim can be reduced by a couple of straightforward reductions.

The first point to notice is that the constant $|z|$ appearing in (5.5.6) and (5.5.7) ruins the uniformity of our estimates when $|b|$ is large. To compensate for this, we introduce the following modified norm, which depends[7] on $|z|$,

$$\|f\|_{\mathcal{B}}^* = \|f\|_s + \frac{c_u}{|z|}\|f\|_u + \frac{1}{|z|}\|f\|_0. \tag{5.6.1}$$

It suffices to prove Proposition 5.20 for the norm $\|\cdot\|_{\mathcal{B}}^*$, as long as $\bar{C}$ and $\bar{\nu}$ remain independent of $|z|$. For this would imply that the spectral radius of $R(z)$ acting on the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}}^*)$ is at most $(a + \bar{\nu})^{-1}$. And since

$$\|\cdot\|_{\mathcal{B}}^* \leqslant \|\cdot\|_{\mathcal{B}} \leqslant |z|\|\cdot\|_{\mathcal{B}}^*,$$

the two norms are equivalent for each $|z|$, and so the spectral radius of $R(z)$ on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is at most $(a + \bar{\nu})^{-1}$ as well.

[7] Note that $|z| > 1$ since $a \geqslant 1$ in the context of Proposition 5.20.

Problem 5.21. *Show that the same choice of N and c_u as in* (5.5.7) *yield the inequality,*

$$\|R(z)^n f\|_{\mathcal{B}}^* \leqslant \sigma^n a^{-n} \|f\|_{\mathcal{B}}^* + C a^{-n} |f|_w, \quad \forall\ f \in \mathcal{B},$$

for all $n \geqslant N$ and some $\sigma < 1$ and $C > 0$ independent[8] *of z.*

Next, using Problem 5.21 we have the inequality,

$$\|R(z)^{2n} f\|_{\mathcal{B}}^* \leqslant \sigma^n a^{-n} \|R(z)^n f\|_{\mathcal{B}}^* + C a^{-n} |R(z)^n f|_w, \quad \forall\ f \in \mathcal{B}.$$

For the first term on the right hand side, we estimate $\|R(z)^n f\|_{\mathcal{B}}^* \leqslant (1+C) a^{-n} \|f\|_{\mathcal{B}}^*$, again using Problem 5.21 and the bound $|\cdot|_w \leqslant \|\cdot\|_s \leqslant \|\cdot\|_{\mathcal{B}}^*$. Interpolating between σa^{-1} and a^{-1}, and possibly increasing N to overcome the effect of $(1 + C)$, this implies the existence of $\nu > 0$ such that the first term contracts at a rate $(a + \nu)^{-2n} \|f\|_{\mathcal{B}}^*$. Thus to prove Proposition 5.20, it suffices to show that the weak norm decays exponentially at a rate faster than a^{-n}, i.e.

$$|R(z)^n f|_w \leqslant (a + \nu)^{-n} \|f\|_{\mathcal{B}}^* \qquad \forall f \in \mathcal{B}, \tag{5.6.2}$$

for some $\nu > 0$, and z and n as in the statement of the proposition. Due to the density of $C^2(\Omega)$ in $\mathcal{B}$, it suffices to prove (5.6.2) for $f \in C^2(\Omega)$. In fact, we will prove the following key lemma.

Lemma 5.22 (Dolgopyat inequality)**.** *There exists $C_\# > 0$ and for all $0 < \alpha \leqslant 1$, there exists $C_D, \gamma_0, b_0 > 0$ such that for all $f \in C^1(\Omega)$,*

$$|R(z)^{2n} f|_w \leqslant \frac{C_\#}{a^{2n} |b|^{\gamma_0}} \big(|f|_\infty + (1 + a^{-1} \log \Lambda)^{-n} |\nabla f|_\infty \big), \tag{5.6.3}$$

for all $1 < a < 2$, $|b| \geqslant b_0$ and $n \geqslant C_D \ln b$.

Here, $|\cdot|_\infty$ denotes the L^∞ norm of a function.

Equation (5.6.3) is the Dolgopyat-type estimate that will prove the existence of a spectral gap for X. Given (5.6.2), one might expect $\|f\|_{\mathcal{B}}^*$ on the right hand side of (5.6.3) rather than the C^1 norm of f. In fact, the C^1 norm of f can be replaced by the strong norm of f due to the following mollification lemma.

[8] Use the fact that $1 \leqslant a \leqslant 2$ to obtain a choice of σ independent of a. Also, note that $\frac{1+a+|z|}{|z|} \leqslant 3$.

Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative C^∞ function supported on the unit ball in $\mathbb{R}^3$, with $\int \eta \, dm = 1$ and a unique global maximum at the origin. For $\varepsilon > 0$, define $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(x/\varepsilon)$.

For $f \in C^0(\Omega)$ and $\varepsilon > 0$, define the following mollification operator,

$$M_\varepsilon(f)(y) = \int_M \widetilde{\eta}_\varepsilon(y - x) f(x) dm(x), \tag{5.6.4}$$

where $\widetilde{\eta}_\varepsilon$ is the function η_ε in a local chart containing y.

Lemma 5.23. *There exists $C > 0$, such that for all $f \in C^0(\Omega)$ and $\varepsilon > 0$,*

$$|M_\varepsilon(f) - f|_w \leqslant C\varepsilon^\gamma \|f\|_{\mathcal{B}}; \tag{5.6.5}$$

$$|M_\varepsilon(f)|_\infty \leqslant C\varepsilon^{-1-\beta+1/q}\|f\|_s; \tag{5.6.6}$$

$$|\nabla(M_\varepsilon(f))|_\infty \leqslant C\varepsilon^{-2-\beta+1/q}\|f\|_s. \tag{5.6.7}$$

The estimates on the mollification operator are fairly standard, and follow the same lines as the proof of Lemma 5.4: the integral in an ε-neighborhood of a point $x \in \Omega$ is disintegrated using a foliation curves in $\mathcal{W}^s$, and the strong stable norm is applied to the integral on each stable curve. The interested reader is referred to Baladi, Demers, and Liverani (2018, Lemmas 7.3 and 7.4), or Baladi and Liverani (2012, Lemmas 5.3 and 5.4).

Proof of Proposition 5.20 using Lemma 5.22. As already noted, it suffices to show that (5.6.2) holds for all $f \in C^1(\Omega)$. Fix z as in the statement of Proposition 5.20 and without loss of generality, assume $b \geqslant 1$. If necessary, increase b_0 from Lemma 5.22 so that $C_D \log b_0 \geqslant N$. Then for $n \geqslant C_D \log b$ and $\varepsilon > 0$ to be chosen later, we have,

$$\begin{aligned}
|R(z)^{2n} f|_w &\leqslant |R(z)^{2n}(f - M_\varepsilon(f))|_w + |R(z)^{2n} M_\varepsilon(f)|_w \\
&\leqslant Ca^{-2n}\big(|f - M_\varepsilon(f)|_w + b^{-\gamma_0}|M_\varepsilon(f)|_\infty \\
&\qquad + (1 + a^{-1}\log\Lambda)^{-n}|\nabla(M_\varepsilon(f))|_\infty\big) \\
&\leqslant Ca^{-2n}\big(\varepsilon^\gamma\|f\|_{\mathcal{B}} + b^{-\gamma_0}\varepsilon^{-1-\beta+1/q}\|f\|_s \\
&\qquad + b^{-\gamma_0}\varepsilon^{-2-\beta+1/q}(1 + a^{-1}\log\Lambda)^{-n}\|f\|_s\big) \\
&\leqslant Ca^{-2n}\|f\|^*_{\mathcal{B}}\big(\varepsilon^\gamma b + b^{-\gamma_0}\varepsilon^{-1-\beta+1/q} \\
&\qquad + b^{-\gamma_0}\varepsilon^{-2-\beta+1/q}(1 + a^{-1}\log\Lambda)^{-n}\big),
\end{aligned}$$

where for the second inequality we have used (5.5.3) for the first term and Lemma 5.22 for the second, while for the third inequality we have used Lemma 5.23, and for the fourth inequality $\|f\|_{\mathcal{B}} \leqslant |z|\|f\|_{\mathcal{B}}^*$.

Choose $\rho > 1/\gamma$ and set $\varepsilon = b^{-\rho}$. Next, choose β sufficiently small, and $q > 1$ sufficiently close[9] to 1, so that $\rho(1+\beta-1/q) < \gamma_0$. Then,

$$|R(z)^{2n} f|_w \leqslant Ca^{-2n}\|f\|_{\mathcal{B}}^*\big(b^{-\gamma_1} + b^{-\gamma_2} + b^{-\gamma_2} b^{\rho}(1+a^{-1}\log \Lambda)^{-n}\big)\,,$$

where $\gamma_1 = \rho\gamma - 1 > 0$ and $\gamma_2 = \gamma_0 - \rho(1+\beta-1/q) > 0$. Finally, choosing $n \geqslant \frac{\rho \log b}{\log(1+a^{-1}\log \Lambda)}$ implies $b^{\rho}(1+a^{-1}\log\Lambda)^{-n} \leqslant 1$. Putting these estimates together yields,

$$|R(z)^{2n} f|_w \leqslant Ca^{-2n}\|f\|_{\mathcal{B}^*} b^{-\bar{\gamma}}\,,$$

for $\bar{\gamma} = \min\{\gamma_1, \gamma_2\}$, and $n \geqslant \bar{C}\log b := \max\{\frac{\rho}{\log(1+a^{-1}\log\Lambda)}, C_D\}\log b$. Next, choosing b_0 sufficiently large so that $Cb_0^{-\bar{\gamma}/2} \leqslant 1$ eliminates the constant C from the estimate on $|R(z)^{2n} f|_w$. Finally if also $n \leqslant 2\bar{C}\log b$, then $b^{-\bar{\gamma}/2} \leqslant e^{-n\bar{\gamma}/(4\bar{C})}$, and (5.6.2) is proved. □

5.6.2 Corollary of the spectral gap for X: Proof of Theorem 5.1

Using Proposition 5.17 and Theorem 5.19, we apply the results of Butterley (2016) to obtain the following decomposition for $\mathcal{L}_t$. Let ν be as in Theorem 5.19 and $\bar{\nu}$ be as in Proposition 5.20.

There exists a finite set of eigenvalues $\{z_j\}_{j=0}^N = \mathrm{sp}(X) \cap \{w \in \mathbb{C} : \mathrm{Re}(w) \in (-\bar{\nu}, 0]\}$, with $z_0 = 0$ and $Re(z_j) \leqslant -\nu$ for $1 \leqslant j \leqslant N$, a finite rank projector Π, a bounded linear operator P_t on $\mathcal{B}$ satisfying $P_t\Pi = \Pi P_t = 0$, and a matrix $\hat{X} : \Pi(\mathcal{B}) \circlearrowleft$ having $\{z_j\}_{j=0}^N$ as eigenvalues such that

$$\mathcal{L}_t = e^{t\hat{X}}\Pi + P_t, \quad t \geqslant 0.$$

Moreover, for each $\nu_1 < \bar{\nu}$, there exists $C_{\nu_1} > 0$ such that for all $f \in \mathrm{Dom}(X)$,

$$|P_t f|_w \leqslant C_{\nu_1} e^{-\nu_1 t}\|Xf\|_{\mathcal{B}}, \quad \text{for all } t \geqslant 0.$$

Note that according to the above equation, the weak norm of P_t decays on $\mathrm{Dom}(X)$, but not on all of $\mathcal{B}$. Indeed, if $\|P_t f\|_{\mathcal{B}}$ decayed at a uniform exponential rate for

[9] Note that this choice of q does not effect the requirement $\gamma \leqslant 1/q$ from the definition of the norms, since we may safely take $\gamma \leqslant 1/2$, and so make it independent of $1/q$ when q is close to 1.

all $f \in \mathcal{B}$, this would imply a spectral gap for $\mathcal{L}_t$, $t > 0$. The above inequality is significantly weaker, yet sufficient to conclude exponential decay of correlations.

For $f \in \mathcal{B}$, let $\Pi_j f = c_j(f) g_j$ denote the projection onto the eigenvector g_j corresponding to z_j. Note that by conformality of the measure m, for $f \in C^2(\Omega)$, we have $c_0(f) = \int_\Omega f \, dm$.

Now let $\varphi \in C^2(\Omega)$, $\psi \in C^\alpha(\Omega)$. Then

$$\begin{aligned}\int_\Omega \varphi \cdot \psi \circ \Psi_t \, dm &= \int_\Omega \mathcal{L}_t \varphi \cdot \psi \, dm = \int_\Omega P_t \varphi \cdot \psi \, dm + \int_\Omega e^{t\hat{X}}(\Pi f) \cdot \psi \, dm \\ &= \int_\Omega P_t \varphi \cdot \psi \, dm + \int_\Omega \Big(c_0(\varphi) + \sum_{j=1}^{N} e^{t z_j} c_j(\varphi) g_j \Big) \psi \, dm.\end{aligned}$$

Thus recalling (5.4.6),

$$\begin{aligned}&\left| \int_\Omega \varphi \cdot \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right| \\ &\qquad \leqslant C |P_t \varphi|_w |\psi|_{C^\alpha(\Omega)} + \sum_{j=1}^{N} \bar{c}_j \|\varphi\|_{\mathcal{B}} |\psi|_{C^\alpha(\Omega)} e^{-\nu t} \\ &\qquad \leqslant C \Big(e^{-\nu_1 t} \|X\varphi\|_{\mathcal{B}} + e^{-\nu t} \|\varphi\|_{\mathcal{B}} \Big) |\psi|_{C^\alpha(\Omega)} \\ &\qquad \leqslant C e^{-\nu t} |\varphi|_{C^2(\Omega)} |\psi|_{C^\alpha(\Omega)} ,\end{aligned}$$

where we have used the fact that $c_j(\varphi) \leqslant \bar{c}_j \|\varphi\|_{\mathcal{B}}$ for some $\bar{c}_j$ independent of φ, and recalling (5.4.5), that $\|X\varphi\|_{\mathcal{B}} \leqslant C |X\varphi|_{C^1(\Omega)} \leqslant C |\varphi|_{C^2(\Omega)}$.

To complete the proof of Theorem 5.1, it remains only to approximate $\bar{\varphi} \in C^\alpha(\Omega)$ by $\varphi \in C^2(\Omega)$. This is by now a standard approximation, which we recall here for the convenience of the reader.

Let $\bar{\varphi}, \psi \in C^\alpha(\Omega)$ such that $\int_\Omega \psi \, dm = 0$. Given any $\varepsilon > 0$, define $\varphi \in C^2(\Omega)$ such that $|\bar{\varphi} - \varphi|_{L^1(m)} \leqslant \varepsilon |\bar{\varphi}|_{C^\alpha(\Omega)}$ (for example, by using a mollification as in (5.6.4)). One has then that $|\varphi|_{C^2(\Omega)} \leqslant C \varepsilon^{\alpha - 2} |\bar{\varphi}|_{C^\alpha(\Omega)}$. Now for $t \geqslant 0$,

$$\begin{aligned}\int \bar{\varphi} \cdot \psi \circ \Phi_t \, dm &= \int (\bar{\varphi} - \varphi) \, \psi \circ \Phi_t \, dm + \int \varphi \cdot \psi \circ \Phi_t \, dm \\ &\leqslant \varepsilon |\bar{\varphi}|_{C^\alpha(\Omega)} |\psi|_{C^0(\Omega)} + C e^{-\nu t} |\varphi|_{C^2(\Omega)} |\psi|_{C^\alpha(\Omega)} \\ &\leqslant \Big(\varepsilon + C e^{-\nu t} \varepsilon^{\alpha - 2} \Big) |\bar{\varphi}|_{C^\alpha(\Omega)} |\psi|_{C^\alpha(\Omega)} .\end{aligned}$$

Now choosing $\varepsilon = e^{-\nu t / 2}$ completes the proof of Theorem 5.1 with $\eta = \nu \alpha / 2$.

5.7 Dolgopyat estimate: Proof of Lemma 5.22

We conclude this chapter with a proof of the Dolgopyat estimate, which is the content of Lemma 5.22. The reader is advised that this is by far the most technical part of the exposition.

Let $f \in C^1(\Omega)$, $W \in \mathcal{W}^s$, and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$. Let $z = a + ib \in \mathbb{C}$ such that $1 \leqslant a \leqslant 2$ and without loss of generality, take $b \geqslant 1$. For $n \geqslant 0$, we must estimate $\int_W R(z)^n f \, \psi \, dm_W$.

Remark 5.24. *Most of the calculations in this section are made simply in order to arrive at the oscillatory integral appearing in* (5.7.17) *and estimated in Lemma 5.30(c) using the smoothness of the* temporal distance function *established in Lemma 5.30(a) and (b). In order to accomplish this, we will localize in both space and time using partitions of unity in order to exploit the presence of cancellations occurring on small scales according to the oscillation provided by* e^{ibt}.

First, we localize in time. Let $\tau > 0$ be a small time to be chosen later. Let $\widetilde{p} : \mathbb{R} \to \mathbb{R}$ be an even function supported on $(-1, 1)$ with a single maximum at 0, satisfying $\sum_{\ell \in \mathbb{Z}} \widetilde{p}(t - \ell) = 1$ for any $t \in \mathbb{R}$. Define $p(s) = \widetilde{p}(s/\tau)$. Then p and $\widetilde{p}$ both define partitions of unity on $\mathbb{R}$. Next, using Problem 5.15,

$$\begin{aligned} R(z)^n f = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t f \, dt = \int_0^\tau p(s) \frac{s^{n-1}}{(n-1)!} e^{-zs} \, \mathcal{L}_s f \, ds \\ + \sum_{\ell \in \mathbb{N}^*} \int_{-\tau}^\tau p(s) \frac{(s + \ell\tau)^{n-1}}{(n-1)!} e^{-z(s+\ell\tau)} \mathcal{L}_{\ell\tau}(\mathcal{L}_s f) \, ds \, . \end{aligned} \tag{5.7.1}$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. To abbreviate the notation, we introduce the following notation for the kernels,

$$p_{n,\ell,z}(s) := p(s) \frac{(s + \ell\tau)^{n-1}}{(n-1)!} e^{-z(s+\ell\tau)}, \quad \text{for } \ell \geqslant 1,$$

$$\text{and} \quad p_{n,0,z}(s) := p(s) \frac{s^{n-1}}{(n-1)!} e^{-zs} \mathbb{1}_{s \geqslant 0},$$

where $\mathbb{1}_A$ denotes the indicator of a set A.

Using this notation, we write the integral needed to estimate the weak norm

as,

$$\int_W R(z)^n f\ \psi\, dm_W = \sum_{\ell\in\mathbb{N}} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_W \psi\ \mathcal{L}_{\ell\tau}(\mathcal{L}_s f)\, dm_W ds$$
$$= \sum_{\ell\in\mathbb{N}} \sum_{W_j\in\mathcal{G}_{\ell\tau}(W)} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_j} J_{W_j}\Phi_{\ell\tau}\ \psi\circ\Phi_{\ell\tau}\ \mathcal{L}_s f\ dm_{W_j} ds, \tag{5.7.2}$$

where in the first line we have reversed order of integration since the integral in t converges uniformly as x ranges over W, and in the second line we have changed variables for each ℓ, recalling the notation $\mathcal{G}_t(W)$ introduced in the proof of Proposition 5.8.

Next, we introduce partitions of unity in space as well, dividing Ω into 'flow boxes' in which we shall compare integrals on stable curves.

Let $r \in (0, \delta_0)$ and $c > 2$ to be determined below. Set,

$$\tau = r^{1/3}. \tag{5.7.3}$$

At the end of this section, r will be taken sufficiently small with respect to b^{-1}. We choose a finite collection of points x_i so that $\cup_i \mathcal{N}_r(x_i) = M$, where $\mathcal{N}_r(x_i)$ denotes the r-neighborhood of x_i in Ω.

Definition 2 (Darboux coordinates)**.** *Using the fact that Ω and ω are smooth, and the splitting of the tangent space is continuous, we may choose cr sufficiently small, so that the following local coordinates exist in a $3cr$ neighborhood of each x_i: $x = (x^s, x^u, x^0)$, where*

a) $x_i = (0,0,0)$ is placed at the origin;

b) $\{(x^s, 0, 0) : |x^s| \leqslant 2cr\}$ is a stable curve;

c) the tangent vector $(0,1,0)$ at x_i belongs to $E^u(x_i)$;

d) in these local coordinates, the contact form ω is in standard form, $\omega = dx^0 - x^s dx^u$.

The last item (d) in the definition above, distinguishes x^0 as the flow direction. In these local coordinates, define for any $\varepsilon \in (0, cr]$, the *flow box*

$$B_\varepsilon(x_i) = \{y \in \mathcal{N}_{3cr}(x_i) : \max\{|x_i^s - y^s|, |x_i^u - y^u|, x_i^0 - y^0|\} \leqslant \varepsilon\}.$$

Notice that two faces of the box can be obtained by flowing a single stable curve (in our coordinates, this would be the top and bottom faces). We call these the *stable sides* of $B_\varepsilon(x_i)$. Similarly, we define the *unstable sides*, and the remaining two side we call the *flow sides* of each box.

Finally, choose $c > 2$ sufficiently large (depending on the maximum curvature of stable curves in $\mathcal{W}^s$, and maximum width of the stable cone) so that if $W \in \mathcal{W}^s$ intersects $B_r(x_i)$, then $\Phi_s(W)$ does not intersect the stable sides of $B_{cr}(x_i)$ for all $s \in [-cr, cr]$.

Now we return to our required estimate of (5.7.2). We subdivide each curve $W_j \in \mathcal{G}_{\ell\tau}(W)$ into curves $W_{j,i} = W_j \cap B_r(x_i)$, and define

$$A_{\ell,i} = \{j : W_j \in \mathcal{G}_{\ell\tau}(W) \text{ crosses } B_{cr}(x_i) \text{ completely in the stable direction}\}.$$

If $W_j \in G_{\ell\tau}(W)$ intersects $B_r(x_i)$, but does not cross $B_{cr}(x_i)$ completely, then we place $W_{j,i} := W_j \cap B_{cr}(x_i) \in D_\ell$, the set of discarded pieces, and note that

$$\int_{W_j \cap B_{cr}(x_i)} J_{W_j} \Phi_{\ell\tau}\, \psi \circ \Phi_{\ell\tau}\, \mathcal{L}_s f \, dm_{W_j} \leqslant cr |J_{W_j} \Phi_{\ell\tau}|_{C^0(W_j)} |\psi|_\infty |f|_\infty.$$

Then summing over ℓ, we have that the contribution to the integral from discarded pieces is at most,

$$\sum_{\ell \geqslant 0} \sum_{j \in D_\ell} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_j \cap B_{cr}(x_i)} J_{W_j} \Phi_{\ell\tau}\, \psi \circ \Phi_{\ell\tau}\, \mathcal{L}_s f \, dm_{W_j} \leqslant Cr |f|_\infty a^{-n}, \tag{5.7.4}$$

for some $C > 0$.

Problem 5.25. *Prove* (5.7.4)*. Hint: Use the fact that due to the choice of c, there are at most two curves in D_ℓ for each $W_j \in \mathcal{G}_{\ell\tau}(W)$. Then Lemma 5.9(c) and Problem 5.16 complete the argument.*

Next, set $\ell_0 = \frac{n}{ae^2\tau}$. We estimate the contribution from the terms with $\ell < \ell_0$. These are the 'short times' $t \leqslant \frac{n}{ae^2}$ in the integral (5.7.1).

Problem 5.26. *Use Stirling's formula to show that the contribution from terms with $\ell < \ell_0$ is bounded by*

$$\int_0^{\frac{n}{ae^2}} \frac{t^{n-1}}{(n-1)!} e^{-zt} \int_W \mathcal{L}_t f \, \psi \, dm_W \, dt \leqslant C |f|_\infty a^{-n} e^{-n}, \tag{5.7.5}$$

for some $C > 0$ independent of n and a.

Now choose n sufficiently large that

$$\max\{e^{-n}, \Lambda^{-\frac{n}{ae^2}}\} \leqslant r\,. \tag{5.7.6}$$

It remains to estimate terms in the sum (5.7.2) for large times $\ell \geqslant \ell_0$ and components $W_{j,i} \subset W_j \in \mathcal{G}_{\ell\tau}(W)$ that completely cross the box $B_{cr}(x_i)$. Define a partition of unity $\{\phi_{r,i}\}_i$ comprised of C^∞ functions $\phi_{r,i}$ centered at each x_i and supported in $B_r(x_i)$. We may choose this partition such that,

$$\|\nabla\phi_{r,i}\|_{L^\infty} \leqslant Cr^{-1} \quad \text{and} \quad \#\{\phi_{r,i}\}_i \leqslant Cr^{-3}, \tag{5.7.7}$$

for some $C > 0$. Then recalling the definition of $A_{\ell,i}$ together with (5.7.4) and (5.7.5), the sum from (5.7.2) that we must estimate is,

$$\int_W R(z)^n f\,\psi\,dm_W = \sum_{\ell\geqslant\ell_0}\sum_i\sum_{j\in A_{\ell,i}}\int_{-\tau}^{\tau} p_{n,\ell,z}(s)\int_{W_{j,i}} J_{W_j}\Phi_{\ell\tau}\,\psi\circ\Phi_{\ell\tau}\,\phi_{r,i}\,\mathcal{L}_s f\,dm_{W_j}ds, \\ + \mathcal{O}(a^{-n}r|f|_\infty)\,.$$

We would like to use the oscillation in the kernel $p_{n,\ell,z}$ to create cancellation in the integrals against Lipschitz functions. Unfortunately, our integrands are not Lipschitz, but only Hölder continuous. To correct for this, define

$$\overline{\psi}_{j,i} = |W_{j,i}|^{-1}\int_{W_{j,i}} \psi\circ\Phi_{\ell\tau}\,dm_{W_{j,i}}$$

and

$$J_{\ell,j,i} = |W_{j,i}|^{-1}\int_{W_{j,i}} J_{W_j}\Phi_{\ell\tau}\,dm_{W_{j,i}}\,.$$

Due to the regularity of ψ and $J_{W_j}\Phi_{\ell\tau}$, in particular (5.4.15) and Lemma 5.9(a), we have

$$|\overline{\psi}_{j,i}J_{\ell,j,i} - \psi\circ\Phi_{\ell\tau}\,J_{W_j}\Phi_{\ell\tau}|_{C^0(W_{j,i})} \leqslant Cr^\alpha J_{\ell,j,i}\,,$$

for some $C > 0$. Then summing over ℓ and using the fact that $|\overline{\psi}_{j,i}|_\infty \leqslant 1$, we must estimate,

$$\int_W R(z)^n f\,\psi\,dm_W = \sum_{\ell\geqslant\ell_0}\sum_i\sum_{j\in A_{\ell,i}} J_{\ell,j,i}\int_{-\tau}^{\tau} p_{n,\ell,z}(s)\int_{W_{j,i}}\phi_{r,i}\,\mathcal{L}_s f\,dm_{W_j}ds, \\ + \mathcal{O}(a^{-n}r^\alpha|f|_\infty)\,. \tag{5.7.8}$$

Now for each $W_{j,i}$, define $W^0_{j,i} = \{\Phi_s W_{j,i}\}_{s\in(-cr,cr)} \cap B_r(x_i)$ to be the weak stable surface[10] containing $W_{j,i}$. In the local coordinates in $B_r(x_i)$, we view $W^0_{j,i}$ as the graph of the function

$$\mathbb{W}^0_j(x^s, x^0) = \mathbb{W}_j(x^s) + (0, 0, x^0),$$

where

$$\mathbb{W}_j(x^s) = (x^s, E_j(x^s), F_j(x^s)), \quad |x^s|, |x^0| \leqslant r, \tag{5.7.9}$$

and E_j, F_j are uniformly C^2 functions. Due to the contact form $\omega = dx^0 - x^s dx^u$ in the local coordinates, it follows that $F'_j(x^s) = x^s E'_j(x^s)$.

On each $B_r(x_i)$, we use these functions to change variables in each integral on the domain $S_r = \{(x^s, x^0) : |x^s| \leqslant r, |x^0| \leqslant r\}$. Thus,

$$\int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_{j,i}} \phi_{r,i}\, \mathcal{L}_s f \, dm_{W_j} ds = \int_{S_r} p_j\, \phi_{r,j}\, f_j \, dx^s \, dx^0, \tag{5.7.10}$$

where

$$p_j(x^s, x^0) = p_{n,\ell,z}(-x^0), \quad \phi_{r,j}(x^s, x^0) = \phi_{r,i} \circ \mathbb{W}^0_j(x^s, x^0) \cdot \|\mathbb{W}'_j(x^s)\|,$$
$$f_j(x^s, x^0) = f \circ \mathbb{W}^0_j(x^s, x^0)\,.$$

At this point, given two curves, $W_{j,i}, W_{k,i} \in A_{\ell,i}$, we would like to slide these two curves to the same reference weak stable surface in $B_r(x_i)$. Let us define this surface to be

$$W^0_i = \{(x^s, 0, x^0) : |x^s|, |x^0| \leqslant r\}\,,$$

which, by choice of coordinates, is precisely the surface obtained by flowing the stable curve through x_i given by $\{(x^s, 0, 0) : |x^s| \leqslant r\}$, according to Definition 2(b).

In order to carry out this sliding, we will use a local foliation of real strong unstable manifolds[11] in $B_r(x_i)$.

[10] If $W_{j,i}$ is a local strong stable manifold, then $W^0_{j,i}$ is the corresponding local weak stable manifold.

[11] For systems with discontinuities such as billiards, the real unstable manifolds do not create a nice foliation of $B_r(x_i)$, so a smooth local foliation of unstable curves lying in the kernel of the contact form must be constructed. This is quite laborious and outside the scope of these notes. The interested reader should refer to Baladi, Demers, and Liverani (2018, Section 6) for the details of the construction.

Definition 3 (Unstable foliation)**.** *For each i, define a foliation $\mathbb{F}$ on $B_r(x_i)$, such that for all $x^0 \in [-cr/2, cr/2]$,*

$$\mathbb{F}(x^s, x^u) = \{(G(x^s, x^u), x^u, H(x^s, x^u) + x^0) : |x^s|, |x^u| \leqslant cr/2\},$$

and each curve $x^u \mapsto \gamma^u_{x^s}(x^u) = (G(x^s, x^u), x^u, H(x^s, x^u) + x^0)$ is a local unstable manifold through $(x^s, 0, 0)$. Moreover, for all $x^s \in [-cr/2, cr/2]$,

(i) $\partial_{x^u} H = G$*, so that $\gamma^u_{x^s}$ lies in the kernel of ω;*

(ii) $G(x^s, 0) = x^s$*,* $H(x^s, 0) = 0$*;*

(iii) $\Phi_{-s}(\gamma^u_{x^s}) \in \mathcal{W}^u$*, for all $s \geqslant 0$;*

(iv) there exists $C > 0$, independent of x^s, such that $C^{-1} \leqslant \|\partial_{x^s} G\|_\infty \leqslant C$, (and so by (i), $\|\partial_{x^s}\partial_{x^u} H\|_{L^\infty} \leqslant C$);

(v) $\|\partial_{x^u}\partial_{x^s} G\|_{C^\eta} \leqslant C$*, for some $\eta > 0$ and $C > 0$ independent of x^s;*

(vi) $\|\partial_{x^s} H\|_{C^0} \leqslant Cr$*,* $\|\partial_{x^s} H\|_{C^\eta} \leqslant C$*.*

Remark 5.27. *We list properties (i)-(vi) for the convenience of the reader: it is known that the foliation by local strong unstable manifolds enjoys these properties for Anosov flows (see, for example Liverani (2004, Appendix B) for the Anosov case or Baladi and Liverani (2012, Appendix D) for the piecewise Anosov case). Indeed, item (i) is immediate since unstable manifolds lie in the kernel of the contact form; (ii) is simply a normalization that we take, choosing our parametrization to be the identity on the stable manifold of x_i; (iii) holds due to the invariance of unstable manifolds.*

To justify the estimates in (iv)-(vi), we present the following suggestive calculation, which while not a complete proof, does give a flavor for the estimates involved. We consider the 2-dimensional case on one of the sections Σ_i defined in Section 5.4.1. On such a section, we adopt local coordinates $(\bar{x}^s, \bar{x}^u)$.

For $\xi \in [-r, r]$, let $V_\xi = \{(\bar{x}^s, \bar{x}^u) : \bar{x}^u = \xi)\}$ denote a stable curve in Σ_i. We project the foliation $\mathbb{F}$ onto Σ_i and normalize $\bar{G}(\bar{x}^s, \bar{x}^u)$ so that $\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, 0) = 1$. Define

$$h_{\xi,0} : V_\xi \to V_0$$

to be the holonomy map along the projected unstable foliation. It follows that the Jacobian $Jh_{\xi,0}$ satisfies the following relation,

$$Jh_{\xi,0} = \frac{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, 0)}{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, \xi)} = \frac{1}{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, \xi)},$$

so that $\partial_{\bar{x}^s}\bar{G}$ can be expressed in terms of the Jacobian of the holonomy map, which is known to be Hölder continuous. This is the content of (iv).

Moreover, using the invariance (ii),

$$Jh_{\xi,0}(x) = \prod_{\ell=1}^{\infty} \frac{J_{\Phi_{-\ell}V_0}\Phi_1(\Phi_{-\ell}(x))}{J_{\Phi_{-\ell}V_\xi}\Phi_1(\Phi_{-\ell}(h_{\xi,0}(x)))},$$

and taking $\partial_{\bar{x}^u}$ of this product converges since the unstable direction is the contracting direction for $\Phi_{-\ell}$. This is the main idea behind (v).

Lifting these calculations to the flow yields (iv) and (v) for G. Item (vi) follows from the normalization (ii) together with (iv).

Having defined our foliation, for $j \in A_{\ell,i}$, we consider the associated holonomy map $h_{j,i} : W_{j,i} \to W_i^0$. As a function of x^s, we have,

$$h_{j,i} \circ \mathbb{W}_j(x^s) =: (h_j^s(x^s), 0, h_j^0(x^s)). \tag{5.7.11}$$

This yields in particular that $\mathbb{F}(h_j^s(x^s), E_j(x^s), h_j^0(x^s)) = \mathbb{W}_j(x^s)$, so that,

$$G(h_j^s(x^s), E_j(x^s)) = x^s \quad \text{and} \quad H(h_j^s(x^s), E_j(x^s)) + h_j^0(x^s) = F_j(x^s)\,. \tag{5.7.12}$$

On S_r, define

$$K_{\ell,n,i,j}(x^s, x^0) = \frac{p(x^0)(\ell\tau - x^0)^{n-1}}{|W_{j,i}|(n-1)!} e^{-z\ell\tau} e^{ax^0} \phi_{r,j}(x^s, x^0).$$

Then (5.7.10) yields,

$$\begin{aligned}\int_{S_r} p_j\, \phi_{r,j}\, f_j\, dx^s\, dx^0 &= |W_{j,i}| \int_{S_r} K_{\ell,n,i,j}(x^s, x^0) f(\mathbb{W}_j(x^s) + (0,0,x^0)) e^{ibx^0}\, dx^s\, dx^0 \\ &= |W_{j,i}| \int_{S_r} K_{\ell,n,i,j}(x^s, x^0) f(h_j^s(x^s), 0, h_j^0(x^s) + x^0) e^{ibx^0}\, dx^s\, dx^0 \\ &\quad + |W_{j,i}| \mathcal{O}(|\partial^u f|_\infty r^2) \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau},\end{aligned}$$

where $\partial^u f$ denotes the derivative of f in the unstable direction. Changing variables twice, first $x^0 \mapsto x^0 - h_j^0(x^s)$, and then $x^s \mapsto (h_j^s)^{-1}(x^s)$, results in the

following,

$$\begin{aligned}\int_{S_r} p_j\,\phi_{r,j}\,f_j\,dx^s\,dx^0 &= |W_{j,i}|\int_{S_r}\frac{K^*_{\ell,n,i,j}(x^s,x^0)}{|(h^s_j)'\circ(h^s_j)^{-1}(x^s)|}f(x^s,0,x^0)e^{ib(x^0-\Delta_j(x^s))}\,dx^s\,dx^0\\ &\quad+|W_{j,i}|\mathcal{O}(|\partial^u f|_\infty r^2)\frac{(\ell\tau)^{n-1}}{(n-1)!}e^{-a\ell\tau}\\ &= |W_{j,i}|\int_{S_r}K^*_{\ell,n,i,j}(x^s,x^0)f(x^s,0,x^0)e^{ib(x^0-\Delta_j(x^s))}\,dx^s\,dx^0\\ &\quad+|W_{j,i}|\mathcal{O}(|\partial^u f|_\infty+|f|_\infty)r^2\frac{(\ell\tau)^{n-1}}{(n-1)!}e^{-a\ell\tau}\,,\end{aligned}\tag{5.7.13}$$

where

$$\begin{aligned}&K^*(x^s,x^0)=K((h^s_j)^{-1}(x^s),x^0-\Delta_j(x^s))\quad\text{and}\\ &\Delta_j(x^s)=h^0_j\circ(h^s_j)^{-1}(x^s),\end{aligned}\tag{5.7.14}$$

and in the second line we have used the fact that $(h^s_j)'\approx 1+r$ due to items (ii) and (iv) of Definition 3 (see also the proof of Sub-lemma 5.31). The function Δ_j is the so-called *temporal distance function* alluded to in Remark 5.24.

Next we use (5.7.13) to sum over ℓ, i and j in (5.7.8).

$$\begin{aligned}&\int_W R(z)^n f\,\psi\,dm_W\\ &\quad=\sum_{\ell,i}\sum_{j\in A_{\ell,i}}J_{\ell,j,i}|W_{j,i}|\int_{S_r}K^*_{\ell,n,i,j}(x^s,x^0)f(x^s,0,x^0)e^{ib(x^0-\Delta_j(x^s))}\,dx^s\,dx^0\\ &\qquad+\sum_{\ell\geqslant\ell_0}\sum_{i}\sum_{j\in A_{\ell,i}}J_{\ell,j,i}|W_{j,i}|\mathcal{O}(|\partial^u f|_\infty+|f|_\infty)r^2\frac{(\ell\tau)^{n-1}}{(n-1)!}e^{-a\ell\tau}\\ &\qquad+\mathcal{O}(a^{-n}r^\alpha|f|_\infty)\,.\end{aligned}\tag{5.7.15}$$

Problem 5.28. *Reverse order of summation and use bounded distortion to show that*
$\sum_i\sum_{j\in A_{\ell,i}}J_{\ell,j,i}|W_{j,i}|\leqslant C$, *for some* $C>0$ *independent of* W *and* n.

Problem 5.29. *Use* (5.7.3) *to show that*

$$\sum_{\ell\geqslant\ell_0}\frac{(\ell\tau)^{n-1}}{(n-1)!}e^{-a\ell\tau}\leqslant C\tau^{-1}a^{-n}\leqslant Cr^{-1/3}a^{-n},$$

for some constant $C > 0$ independent of ℓ_0 and τ.

Summing over ℓ and using Problems 5.28 and 5.29 yields,

$$\sum_{\ell\geq\ell_0}\sum_{i}\sum_{j\in A_{\ell,i}} J_{\ell,j,i}|W_{j,i}|\mathcal{O}(|\partial^u f|_\infty + |f|_\infty)r^2\frac{(\ell\tau)^{n-1}}{(n-1)!}e^{-a\ell\tau} = \mathcal{O}(a^{-n}r^{5/3})(|\partial^u f|_\infty + |f|_\infty)\,. \tag{5.7.16}$$

Next, we estimate the sums over the integrals in (5.7.15). Setting $Z_{\ell,j,i} = J_{\ell,j,i}|W_{j,i}|$, we have

$$\begin{aligned}
&\sum_{\ell\geq\ell_0}\sum_{i}\int_{S_r}\sum_{j\in A_{\ell,i}} Z_{\ell,j,i}K^*_{\ell,n,i,j} f e^{ib(x^0-\Delta_j(x^s))}\,dx^s\,dx^0\\
&\leq \sum_{\ell\geq\ell_0}\sum_{i}\left(\int_{S_r}\Big|\sum_{j\in A_{\ell,i}} Z_{\ell,j,i}K^*_{\ell,n,i,j}e^{ib(x^0-\Delta_j(x^s))}\Big|^2\right)^{1/2}\Big(\int_{S_r}|f|^2\Big)^{1/2}\\
&\leq \sum_{\ell\geq\ell_0}\sum_{i}|f|_\infty r\left(\sum_{j,k\in A_{\ell,i}} Z_{\ell,j,i}Z_{\ell,k,i}\int_{S_r}K^*_{\ell,n,i,j}\overline{K}^*_{\ell,n,i,k}e^{ib(\Delta_k-\Delta_j)}\right)^{1/2}\\
&\leq \sum_{\ell\geq\ell_0}|f|_\infty r^{-1/2}\left(\sum_{i}\sum_{j,k\in A_{\ell,i}} Z_{\ell,j,i}Z_{\ell,k,i}\int_{S_r}K^*_{\ell,n,i,j}\overline{K}^*_{\ell,n,i,k}\,e^{ib(\Delta_k-\Delta_j)}\right)^{1/2}
\end{aligned} \tag{5.7.17}$$

where in the second line we have used the Cauchy–Schwarz inequality, in the third line we have used that $|\sum_j v_j|^2 = (\sum_j v_j)(\sum_k \overline{v}_k)$ for any set of complex numbers $\{v_j\}_j$, and in the fourth line we have used the Hölder inequality together with the fact that the cardinality of the sum over i is at most Cr^{-3} by (5.7.7).

The last integral remaining in (5.7.17) is the oscillatory integral which has been the object of the rearrangements and changes of variables of this entire section. It is at the heart of the Dolgopyat estimate. Define the flow surface, $W^0_{j,i} = B_r(x_i)\cap(\cup_{s\in[-cr,cr]}\Phi_s(W_{j,i}))$.

Lemma 5.30. *Recalling $\eta > 0$ from Definition 3, there exists $C > 0$, independent of r, n and W, such that:*

a) $\inf_{x^s}|\partial_{x^s}(\Delta_j - \Delta_k)(x^s)| \geq Cd(W^0_{j,i}, W^0_{k,i})$;

b) $|\Delta_j - \Delta_k|_{C^{1+\eta}(S_r)} \leqslant Cr$;

c)

$$\left| \int_{S_r} K^*_{\ell,n,i,j} \overline{K}^*_{\ell,n,i,k} \, e^{ib(\Delta_k - \Delta_j)} dx^s dx^0 \right| \leqslant C \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau} \left[\frac{r}{d(W^0_{j,i}, W^0_{k,i})^{1+\eta} b^\eta} + \frac{r^{-1}}{d(W^0_{j,i}, W^0_{k,i}) b} \right].$$

Proof. Items (a) and (b) are preliminaries needed to establish the estimate (c) on the key oscillatory integral.

We choose a curve $W_{j,i}$ with $j \in A_{\ell,i}$ crossing the box $B_{cr}(x_i)$. Without loss of generality (by flowing it if necessary), we may assume $W_{j,i}$ intersects the x^u axis in the local coordinates. For a fixed $\xi \in (-r, r)$, we consider the closed path starting at $(\xi, 0, 0)$ on W^0_i (i.e. $x^s = \xi$ on the strong stable manifold of x_i), running to x_i along the stable manifold of x_i, and up the coordinate axis of x^u (which lies in $\mathcal{W}^u$) to $W_{j,i}$. From there, the path runs along $W_{j,i}$ until it reaches the point $\mathbb{W}_j((h^s_j)^{-1}(\xi))$, then follows the strong unstable manifold γ^u_ξ (this is an element of the foliation defined in Definition 3) down to W^0_*, and from there follows the flow direction back to $(\xi, 0, 0)$. We call this path $\Gamma(\xi)$. See Figure 5.2.

Recalling (5.7.14), we notice that $\Delta_j(\xi) = h^0_j((h^s_j)^{-1}(\xi))$ is precisely the distance in the flow direction from $(\xi, 0, 0)$ to the point of intersection of γ^u_ξ with W^0_i. In addition, every other smooth component of $\Gamma(\xi)$ lies in the kernel of ω by construction of $\mathcal{W}^s$ and $\mathcal{W}^u$. Since $\omega(v) = 1$ for every unit vector v in the flow direction, and using Stokes' theorem, we have,

$$\Delta_j(\xi) = \int_{\Gamma(\xi)} \omega = \int_{\Sigma_1} d\omega + \int_{\Sigma_2} d\omega \,,$$

where Σ_1 is the 'vertical' surface defined by the part of the foliation $\mathbb{F}$ connecting $W_{j,i}$ to W^0_i, and Σ_2 is the 'horizontal surface' comprised of the part of W^0_i enclosed by $\Gamma(\xi)$ and the curve $h_{j,i}(W_{j,i})$ (remembering (5.7.11). The integral over Σ_2 is 0 since the flow direction lies in the kernel of $d\omega$. Writing the integral over Σ_1 in local coordinates and using (5.7.9) and Definition 3 yields,

$$\Delta_j(\xi) = \int_0^\xi \int_0^{E_j((h^s_j)^{-1}(x^s))} \partial_{x^s} G(x^s, x^u) \, dx^u \, dx^s. \tag{5.7.18}$$

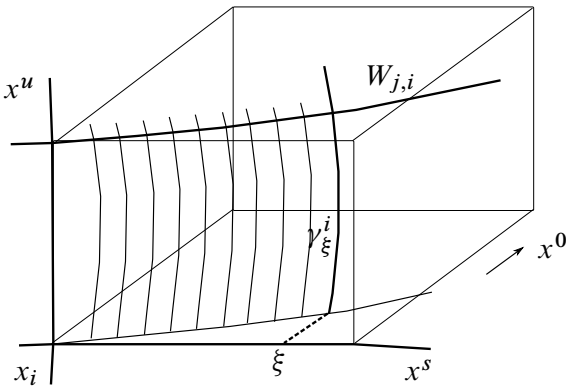


Figure 5.2: Part of a flow box $B_r(x_i)$ with path $\Gamma(\xi)$ and the unstable foliation shown. $\Gamma(\xi)$ starts at ξ, goes along the x^s-axis to x_i, up the x^u-axis to $W_{j,i}$, across $W_{j,i}$ to γ_ξ^i, down γ_ξ^i to the flow surface W_i^0, and then in the flow direction back to ξ. The length of the dotted line is $\Delta_j(\xi)$.

And so, assuming that $W_{k,i}$ with $k \in A_{\ell,i}$ is also in standard position intersecting the x^u axis, we obtain

$$\begin{aligned}
\partial_{x^s}\Delta_k(\xi) - \partial_{x^s}\Delta_j(\xi) &= \int_{E_j((h_j^s)^{-1}(\xi))}^{E_k((h_k^s)^{-1}(\xi))} \partial_{x^s} G(x^u, \xi)\, dx^u \\
&= \int_{E_j((h_j^s)^{-1}(\xi))}^{E_k((h_k^s)^{-1}(\xi))} \Big[1 + \int_0^{x^u} \partial_{x^u}\partial_{x^s} G(u, \xi)\, du\Big] dx^u \\
&= \big[E_k((h_k^s)^{-1}(\xi)) - E_j((h_j^s)^{-1}(\xi))\big](1 + \mathcal{O}(r)) \\
&\geqslant d(W_{j,i}, W_{k,i})(1 + \mathcal{O}(r))\,.
\end{aligned}$$

This proves item (a) of the lemma, and immediately gives the required bound on the C^0 norm for part (b). The bound on the C^η norm follows from the same integral expression for $\partial_{x^s}(\Delta_k - \Delta_j)$, together with property (v) of the foliation.

For item (c) of the lemma, we follow Baladi, Demers, and Liverani (2018, Appendix B). Define

$$L_{j,k}(x^s, x^0) = K^*_{\ell,n,i,j}(x^s, x^0)\overline{K}^*_{\ell,n,i,k}(x^s, x^0) \quad \text{and} \quad \boldsymbol{\Delta}_{j,k} = \Delta_k - \Delta_j\,.$$

We shall need the following preliminary result.

Sub-lemma 5.31. *In the present setting, we have*

$$\left|\tfrac{d}{dx^s}\Delta_j\right| \leqslant Cr \qquad \text{and} \qquad \left|\tfrac{d}{dx^s}(h_j^s)^{-1}\right| = 1 + \mathcal{O}(r)\,.$$

Proof of Sublemma. The first inequality follows from (5.7.18),

$$\partial_{x^s}\Delta_j(x^s) = \int_0^{E_j((h_j^s)^{-1}(x^s))} \partial_{x^s}G(x^s, x^u)\,dx^u \leqslant Cr\,,$$

using Definition 3(iv) and recalling (5.7.9) so that $|E_j((h_j^s)^{-1}(x^s))| \leqslant Cr$ since the foliation is in $B_r(x_i)$.

For the second statement of the lemma, differentiate the first expression in (5.7.12) to obtain,

$$(h_j^s)'(x^s) = \frac{1 - \partial_2 G(h_j^s(x^s), E_j(x^s))\,E_j'(x^s)}{\partial_1 G(h_j^s(x^s), E_j(x^s))}\,.$$

Then we use the fact $\partial_1 G(s,0) = 1$ by Property (ii) and then $\partial_1 G(s,u) = 1 + \mathcal{O}(r)$ by Property (v) of the foliation whenever $|s|, |u| \leqslant cr/2$. This implies that $(h_j^s)'(x^s) = 1 + \mathcal{O}(r)$. Then since h_j^s is invertible, we have $\frac{d}{dx^s}(h_j^s)^{-1}(x^s) = \frac{1}{(h_j^s)'((h_j^s)^{-1}(x^s))} = 1 + \mathcal{O}(r)$ as well. □

Problem 5.32. *Show that there exists $C > 0$, independent of W, n, ℓ, i, j and k, such that*

$$|L_{j,k}|_\infty \leqslant \frac{C(\ell\tau)^{2n-2}}{r^2[(n-1)!]^2}e^{-2a\ell\tau} \quad \text{and} \quad |\partial_{x^s}L_{j,k}|_\infty \leqslant \frac{C(\ell\tau)^{2n-2}}{r^3[(n-1)!]^2}e^{-2a\ell\tau}\,.$$

We define a sequence $\{s_m\}_{m=0}^M \subset \mathbb{R}$ such that $s_0 = -r$, and $\partial_{x^s}\boldsymbol{\Delta}_{j,k}(s_m)\cdot[s_{m+1} - s_m] = 2\pi b^{-1}$, and let $M \in \mathbb{N}$ be such that $s_{M-1} \leqslant r$ and $s_M > r$. Such a finite M exists by part (a) of the lemma. By part (b) of the lemma,

$$|\boldsymbol{\Delta}_{j,k}(x^s) - \boldsymbol{\Delta}_{j,k}(s_m) - \partial_{x^s}\boldsymbol{\Delta}_{j,k}(s_m)[s_{m+1} - s_m]| \leqslant Cr|s_m - x^s|^{1+\eta},$$

for all $x^s \in [s_m, s_{m+1}]$. Moreover, using Problem 5.32, we have

$$|L_{j,k}(x^s, x^0) - L_{j,k}(s_m, x^0)| \leqslant C\delta_m e_{\ell,n} r^{-3},$$

for a uniform $C > 0$, where $\delta_m = s_{m+1} - s_m$ and $e_{\ell,n} = \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau}$. Notice then that by part (a) of the lemma,

$$b\delta_m \leqslant 2\pi d(W^0_{j,i}, W^0_{k,i})^{-1}. \tag{5.7.19}$$

Now we fix x^0 and estimate for each m,

$$\begin{aligned}
&\Big| \int_{s_m}^{s_{m+1}} e^{-ib\boldsymbol{\Delta}_{j,k}(x^s)} L_{j,k}(x^s, x^0)\, dx^s \Big| \\
&\quad = \Big| \int_{s_m}^{s_{m+1}} e^{-ib[\partial_{x^s}\boldsymbol{\Delta}_{j,k}(s_m)[x^s - s_m] + \mathcal{O}(r|x^s - s_m|^{1+\eta})]} \\
&\qquad \times \big(L_{j,k}(s_m, x^0) + \mathcal{O}(r^{-3}\delta_m e_{\ell,n}) \big) dx^s \Big| \\
&\quad \leqslant C \left(b\delta_m^{1+\eta} r^{-1} + r^{-3}\delta_m \right) \delta_m e_{\ell,n} \\
&\quad \leqslant C \left(\frac{r^{-1}}{d(W^0_{j,i}, W^0_{k,i})^{1+\eta} b^\eta} + \frac{r^{-3}}{d(W^0_{j,i}, W^0_{k,i}) b} \right) \delta_m e_{\ell,n},
\end{aligned}$$

where again we have used Problem 5.32 and in the last line we have used (5.7.19). The last integral over the interval $[s_{M-1}, r]$ is trivially bounded by $C r^{-2}\delta_M \leqslant C r^{-2}(bd(W^0_{j,i}, W^0_{k,i}))^{-1}$, again using (5.7.19). Then summing over m yields $\sum_{m=0}^{M-1} \delta_m \leqslant 2r$, and integrating over x^0 yields another factor of r, completing the proof of part (c). □

The bound given by Lemma 5.30(c) is nearly what we need to complete the Dolgopyat estimate. We require one more lemma, which allows us to neglect the contribution from curves in $A_{\ell,i}$ that are too close together.

Lemma 5.33. *There exists $C > 0$ such that for each $\ell \geqslant \ell_0$, $i \in \mathbb{N}$ and $j \in A_{\ell,i}$,*

$$\sum_{\substack{k \in A_{\ell,i} \\ d(W^0_{j,i}, W^0_{k,i}) \leqslant \rho}} Z_{\ell,k,i} \leqslant C\left[r(\rho^{1/2} + \Lambda^{-\ell\tau}) \right].$$

Proof. Let $A(\rho) = \{k \in A_{\ell,i} : d(W^0_{j,i}, W^0_{k,i}) \leqslant \rho\}$. First notice that by bounded distortion,

$$\begin{aligned}
\sum_{k \in A(\rho)} Z_{\ell,k,i} &= \sum_{k \in A(\rho)} |W_{k,i}| |J_{W_{k,i}} \Phi_{\ell\tau}|_{C^0(W_{k,i})} \\
&= C^{\pm 1} \sum_{k \in A(\rho)} |\Phi_{\ell\tau}(W_{k,i})|,
\end{aligned} \tag{5.7.20}$$

where the notation $P = C^{-1}Q$ means $C^{-1}Q \leqslant P \leqslant CQ$ for some $C \geqslant 1$.

Let $W_r^0 = \cup_{s\in[-2r,2r]}\Phi_s(W)$. Fix $\rho^* > 0$, and consider the set of local strong unstable manifolds $\{\gamma_x^u\}_{x\in W_r^0}$ having length ρ^* in both directions, and centered x. Let $G_{i,k}^0 = \{x \in W_r^0 : x \in \Phi_{\ell\tau}(W_{k,i}^0)\}$ and note that the sets $\cup_{x\in G_{i,k}^0}\gamma_x^u$ are disjoint for different k. On the one hand, due to the uniform transversality of E^s, E^u and E^c, we have

$$\sum_{k\in A(\rho)} m(\cup_{x\in G_{i,k}^0}\gamma_x^u) = \mathcal{O}(r\rho^*)\sum_{k\in A(\rho)}|\Phi_{\ell\tau}(W_{k,i})|\,. \tag{5.7.21}$$

On the other hand, for each k, $\Phi_{\ell\tau}(\cup_{x\in G_{i,k}^0}\gamma_x^u)$ is approximately a parallelepiped having length in the flow and stable directions of about r, and having length in the unstable direction at most $2\rho^*\Lambda^{-\ell\tau}$. Moreover, these sets are disjoint for different k and their union lies in a set of length in the unstable direction at most $\rho + 2\rho^*\Lambda^{-\ell\tau}$. Then using the invariance of the measure,

$$\sum_{k\in A(\rho)} m(\cup_{x\in G_{i,k}^0}\gamma_x^u) = \sum_{k\in A(\rho)} m(\Phi_{\ell\tau}(\cup_{x\in G_{i,k}^0}\gamma_x^u)) \leqslant Cr^2(\rho + \rho^*\Lambda^{-\ell\tau})\,. \tag{5.7.22}$$

Using (5.7.20) in (5.7.21) and equating this with (5.7.22) yields,

$$\sum_{k\in A(\rho)} Z_{\ell,k,i} \leqslant Cr(\rho^*)^{-1}(\rho + \rho^*\Lambda^{-\ell\tau})\,,$$

and choosing $\rho^* = \rho^{1/2}$ completes the proof of the lemma. □

We will apply Lemma 5.33 with $\rho = r^2$. For each $j \in A_{\ell,i}$ define $A_{\ell,i,j}^{\text{close}} = \{k \in A_{\ell,i} : d(W_{k,i}^0, W_{j,i}^0) \leqslant r^2\}$, and $A_{\ell,i,j}^{\text{far}} = A_{\ell,i} \setminus A_{\ell,i,j}^{\text{close}}$. Then,

$$\sum_i \sum_{j\in A_{\ell,i}} \sum_{k\in A_{\ell,i,j}^{\text{close}}} Z_{\ell,j,i}Z_{\ell,k,i} \leqslant Cr(r + \Lambda^{-\ell\tau}) \leqslant Cr^2, \tag{5.7.23}$$

remembering (5.7.6) and using Problem 5.28.

Finally, we apply Lemma 5.30(c), summing over $A^{\text{far}}_{\ell,i,j}$,

$$\begin{aligned}
&\Big(\sum_i \sum_{j\in A_{\ell,i}} \sum_{k\in A^{\text{far}}_{\ell,i,j}} Z_{\ell,j,i} Z_{\ell,k,i} \int_{S_r} K^*_{\ell,n,i,j}\, \overline{K}^*_{\ell,n,i,k}\, e^{ib(\Delta_k-\Delta_j)}\Big)^{1/2}\\
&\leqslant \Big(\sum_i \sum_{j\in A_{\ell,i}} \sum_{k\in A^{\text{far}}_{\ell,i,j}} Z_{\ell,j,i} Z_{\ell,k,i} C \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau}\big[r^{-1-2\eta}b^{-\eta} + r^{-3}b^{-1}\big]\Big)^{1/2}\\
&\leqslant C r^{-1/2}[r^{-2\eta}b^{-\eta} + r^{-2}b^{-1}]^{1/2} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau},
\end{aligned} \tag{5.7.24}$$

where again we have used Problem 5.28.

Problem 5.34. *Show that for all ℓ, n, i, j, k,*

$$\left|\int_{S_r} K^*_{\ell,n,i,j}\, \overline{K}^*_{\ell,n,i,k}\, e^{ib(\Delta_k-\Delta_j)}\right| \leqslant C \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau}.$$

Now combining Problem 5.29 and Problem 5.34 with with (5.7.23) and (5.7.24) in (5.7.17) yields,

$$\begin{aligned}
&\sum_{\ell\geqslant\ell_0}\sum_i \int_{S_r} \sum_{j\in A_{\ell,i}} Z_{\ell,j,i} K^*_{\ell,n,i,j} f e^{ib(x^0-\Delta_j(x^s))}\, dx^s\, dx^0\\
&\qquad\leqslant \sum_{\ell\geqslant\ell_0} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} |f|_\infty \left(r^{1/2} + r^{-1}[r^{-2\eta}b^{-\eta} + r^{-2}b^{-1}]^{1/2}\right)\\
&\qquad\leqslant a^{-n}|f|_\infty \left(r^{1/6} + r^{-4/3}[r^{-2\eta}b^{-\eta} + r^{-2}b^{-1}]^{1/2}\right).
\end{aligned} \tag{5.7.25}$$

Now we use (5.7.16) and (5.7.25) in (5.7.15) to estimate,

$$\begin{aligned}
\int_W R(z)^n f\, \psi\, dm_W \leqslant C a^{-n} \Big(|f|_\infty \Big(&r^\alpha + r^{5/3} + r^{1/6}\\
&+ r^{-4/3}[r^{-2\eta}b^{-\eta} + r^{-2}b^{-1}]^{1/2}\Big) + r^{5/3}|\partial^u f|_\infty\Big).
\end{aligned}$$

We can assume without loss of generality that $\eta < 1$ so that the first term in the square root above is the larger of the two. Setting $r = b^{-\frac{\eta}{8+6\eta}}$, bounds the term

with the square root by by $b^{-\eta/3}$. Since all other powers of r are positive, we obtain,

$$\int_W R(z)^n f\, \psi\, dm_W \leqslant Ca^{-n}b^{-\gamma_0}(|f|_\infty + |\partial^u f|_\infty), \tag{5.7.26}$$

for some $\gamma_0 > 0$, and all $b \geqslant b_0$, where b_0 depends only on the maximum size of r determined by Definition 2. As a final step, we apply (5.7.26) to $R(z)^n f$ rather than f.

Problem 5.35. *Use* (5.1.1) *and Problem 5.15 to show that*

$$|\partial^u(R(z)^n f)|_\infty \leqslant C(a + \log \Lambda)^{-n}|\nabla f|_\infty.$$

Now Problem 5.35 together with (5.7.26) and the bound $|R(z)^n f|_\infty \leqslant Ca^{-n}|f|_\infty$ (from Problem 5.16) yield,

$$\begin{aligned}\int_W R(z)^{2n} f\, \psi\, dm_W &\leqslant Ca^{-n}b^{-\gamma_0}\big(|R(z)^n f|_\infty + |\partial^u(R(z)^n f)|_\infty\big)\\ &\leqslant C'a^{-2n}b^{-\gamma_0}\big(|f|_\infty + (1 + a^{-1}\log \Lambda)^{-n}|\nabla f|_\infty\big),\end{aligned}$$

which completes the proof of Lemma 5.22.

6 Dispersing billiards

In this section, we briefly describe some of the ideas needed to adapt the technique and framework presented in these notes to the continuous time billiard flow associated with a dispersing billiard table. This is done in full detail in Baladi, Demers, and Liverani (2018) for the finite horizon periodic Lorentz gas, and we only recall here in broad terms some of the adjustments that must be made. We remark that although presently a proof of exponential decay of correlations exists only in this context, these results are expected to generalize to dispersing billiard tables with corner points, and cusps (the fact that the discrete time billiard map for tables with cusps has a polynomial rate of decay of correlations will not prevent the associated continuous time flow from having an exponential one), and some billiard tables with focusing boundaries, such as those studied in Bálint and Melbourne (2008). The flow associated with the infinite horizon periodic Lorentz gas, however, is known to have decay of correlations at the polynomial rate of $1/t$ (Bálint, Butterley, and Melbourne (2019)).

6.1 The billiard table

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-torus, and place finitely many open convex sets Γ_i, $i = 1, \dots d$, in $\mathbb{T}^2$ so that their closures are pairwise disjoint and the boundary of each set Γ_i is a C^3 curve with strictly positive curvature. We shall call these sets *scatterers* and the billiard table is $Q = \mathbb{T}^2 \setminus (\cup_{i=1}^d \Gamma_i)$.

The *billiard flow* is defined by the motion of a point particle traveling at unit speed in Q and colliding elastically at the boundaries of the scatterers. The particle's velocity changes only at collisions, which are defined when the particle belongs to $\partial\Gamma_i$ for some i. We assume that the table satisfies a *finite horizon* condition: there is a finite upper bound on the time between consecutive collisions in Q.

Define $\Omega_0 = Q \times \mathbb{S}^1 \subset \mathbb{T}^3$. In Ω_0, we may describe the billiard flow in the coordinates (x, y, θ), where $(x, y) \in Q$ denotes position and $\theta \in \mathbb{S}^1$ denotes velocity. Then,

$$\Phi_t(x, y, \theta) = (x + t\cos\theta, y + t\sin\theta, \theta), \tag{6.1.1}$$

between collisions, and at collisions the velocity changes from θ^- (precollision) to θ^+ (post-collision) according to the usual law of reflection. If we identify $(x, y, \theta^-) \sim (x, y, \theta^+)$, then the flow becomes continuous on the phase space $\Omega := \Omega_0/\sim$. We will find it convenient to work in both the spaces Ω_0 and Ω depending on the context.

Analysis of the flow is often aided by appealing to the associated discrete time *billiard map*. This is defined by introducing coordinates to track each collision (r for position on $\partial\Gamma_i$ parametrized by arc length, and φ for the angle the post-collision velocity vector makes with the normal to $\partial\Gamma_i$). The two-dimensional phase space for the map is then a union of cylinders $M = \cup_{i=1}^d \partial\Gamma_i \times [-\pi/2, \pi/2]$ and the billiard map $T(r, \varphi) = (r_1, \varphi_1)$ maps one collision to the next.

6.2 Hyperbolicity and contact structure

In the coordinates described above, the flow preserves the one form defined by,

$$\omega = \cos\theta\, dx + \sin\theta\, dy\,.$$

Between collisions, this is obvious from the definition (6.1.1) since θ is constant except at collisions. That the one form is preserved through collisions is a simple

calculation (see Chernov and Markarian (2006, Section 3.3)). Since $(\cos\theta, \sin\theta)$ is the direction of motion of the particle in the table Q, we see that geometrically, the kernel of the one form is the plane perpendicular to the flow direction in Ω, and $\omega(v) = 1$ for any unit vector $v \in \mathbb{R}^3$ pointing in the flow direction.

Problem 6.1. *Show that* $\omega \wedge d\omega = dx \wedge d\theta \wedge dy$.

Problem 6.1 shows that the contact volume is Lebesgue measure on Ω_0, and this is preserved by the flow. Thus the flow and one form are already normalized according to the requirements of Section 5.1.

Due to the strictly positive curvature of the $\partial\Gamma_i$, both the map and the flow are hyperbolic. Let $\tau_{\min}, \mathcal{K}_{\min} > 0$ denote the minimum time between collisions and the minimum curvature, respectively, and let $\tau_{\max} < \infty$ denote the maximum time between collisions, which is finite due to the finite horizon condition. The constant $\Lambda_0 = 1 + 2\tau_{\min}\mathcal{K}_{\min}$ represents the minimum hyperbolicity constant for the map; then setting $\Lambda = \Lambda_0^{1/\tau_{\max}}$ gives a lower bound on the hyperbolicity constant for the flow satisfying (5.1.1).

The billiard map T preserves the following stable cone on all of M,

$$C^s(r,\varphi) = \{(dr, d\varphi) \in \mathbb{R}^2 : -\mathcal{K}_{\min} \geqslant d\varphi/dr \geqslant -\mathcal{K}_{\max} - \tau_{\min}^{-1}\}, \qquad (6.2.1)$$

and an analogous unstable cone C^u is defined by $\mathcal{K}_{\min} \leqslant d\varphi/dr \leqslant \mathcal{K}_{\max} + \tau_{\min}^{-1}$. Then flowing C^u forward between consecutive collisions and C^s backwards between collisions defines a family of cones in Ω that is invariant under the flow (satisfying (5.4.3)) and lies in the kernel of ω. This family of cones is continuous on each component of Ω_0 that does not cross one of the singularity surfaces (defined below). See Baladi, Demers, and Liverani (2018, Section 2.1).

6.3 Singularities

The singularities for both the map and the flow are created by tangential collisions with the scatterers. For the map, this is the set $\mathcal{S}_0 = \{(r,\varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$. For $n \geqslant 1$, the sets $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$ and $\mathcal{S}_{-n} = \cup_{i=0}^n T^i\mathcal{S}_0$ are the singularity sets for T^n and T^{-n}, respectively. The map T is discontinuous at $\mathcal{S}_1$. Moreover, its derivative satisfies

$$\|DT(z)\| \approx d(z, \mathcal{S}_1)^{-1/2}, \quad \text{for } z = (r,\varphi) \in M,$$

so that the derivative becomes infinite at tangential collisions.

The local sections Σ_i introduced for Anosov flows in Section 5.4.1 can be defined naturally for the billiard flow as the boundaries of the scatterers, $\partial\Gamma_i$. The projections P^+ and P^- are defined for $Z \in \Omega$ as the first intersection of $\Phi_t(Z)$ with one of the Γ_i, for $t > 0$ for P^+ and for $t < 0$ for P^-.

While the flow remains continuous on Ω, its derivative also becomes infinite at tangential collisions (with the same order of magnitude as the map). Thus the flow is only Hölder continuous with exponent $1/2$ due to the tangential collisions. Let $\mathcal{S}_0^+$ denote the surface in Ω_0 created by flowing $\mathcal{S}_0$ forward to its next collision (on $\mathcal{S}_{-1}$). Then the family of unstable cones C^u is continuous in Ω_0 away from the surface $\mathcal{S}_0^+$. Similarly, let $\mathcal{S}_0^-$ denote the surface obtained by flowing $\mathcal{S}_0$ under the inverse flow to $\mathcal{S}_1$. The family of stable cones C^s is continuous in Ω_0 away from $\mathcal{S}_0^-$.

In order to regain control of distortion, one introduces *homogeneity strips*, which are artificial subdivisions of the phase space on which the derivative has comparable rates of expansion and contraction. For the map, the standard choice is to choose $k_0 > 0$ and then define the homogeneity strip

$$\mathbb{H}_k = \{(r,\varphi) : k^{-2} \leqslant \tfrac{\pi}{2} - \varphi \leqslant (k+1)^{-2}\} \quad \text{for } k \geqslant k_0,$$

with a similar definition for $\mathbb{H}_{-k}$ for φ near $-\frac{\pi}{2}$. Since expansion factors for the map are proportional to $1/\cos\varphi_1$ when $T(r,\varphi) = (r_1,\varphi_1)$, these subdivisions of the space imply that the Jacobians of the map satisfy distortion bounds as in Lemma 5.9(a), but with[1] Hölder exponent $1/3$.

Problem 6.2. *Suppose $z, \tilde{z} \in \mathbb{H}_k$ for some $k \in \mathbb{Z}$. Show that*

$$\left|\frac{\cos\varphi(z)}{\cos\varphi(\tilde{z})} - 1\right| \leqslant C d(z,\tilde{z})^{1/3}, \quad \textit{for some } C > 0 \textit{ independent of } k.$$

Here, $\varphi(z)$ denotes the second coordinate of $z = (r,\varphi) \in M$.

One extends this distortion control to the Jacobians of the flow by only comparing derivatives at points whose next collisions lie in the same homogeneity strip under the forward flow (for the unstable Jacobian) or the backward flow (for the stable Jacobian).

[1] The exponent $1/3$ is a simple consequence of defining the homogeneity strips to decay like k^{-2}. If, instead, one chooses a decay rate of k^{-p}, $p > 1$, then the Hölder exponent becomes $1/(p+1)$. Thus it is possible to obtain a Hölder exponent arbitrarily close to $1/2$ by choosing p close to 1. However, $p = 1$ is not an acceptable choice since it ruins the summability of the series and the growth lemma needed for the analogue of Lemma 5.9(c) fails.

6.4 Admissible curves and definition of norms

Since our invariant cones C^u and C^s satisfy (5.4.3), we may define a family of *admissible cone-stable curves* $\mathcal{W}^s$ which is invariant under Φ_{-t}, $t > 0$, and satisfies the requirements of Definition 1. In addition, we require stable curves to be disjoint from $\partial\Omega_0$. Thus if a stable curve is in the midst of a collision, we omit the collision points, and consider each of the two or three connected components as separate stable curves.

Due to our definition of C^s, we have that $P^+(W)$ is a stable curve for the map whenever $W \in \mathcal{W}^s$. Due to our discussion of distortion in Section 6.3, we call a stable curve $W \in \mathcal{W}^s$ *homogeneous* if $P^+(W) \subset \mathbb{H}_k$ for some $k \in \mathbb{Z}$. Similarly, we define an invariant family of unstable curves $\mathcal{W}^u$ and call an unstable curve U homogeneous if $P^-(U) \subset \mathbb{H}_k$ for some $k \in \mathbb{Z}$.

Using the (global) coordinates (r, φ) in M and (6.2.1) allows us to view each map-stable curve $P^+(W)$ as the graph of a function G_W over the r-coordinate. We then use the same definition of distance between stable curves, $d_{\mathcal{W}^s}(W_1, W_2)$, as given in (5.4.4), with the added requirement that $d_{\mathcal{W}^s}(W_1, W_2) = \infty$ unless $P^+(W_1)$ and $P^+(W_2)$ lie in the same homogeneity strip.

With these conventions in place, we may define the weak and strong norms for $f \in C^1(\Omega_0)$ precisely as in Section 5.4.2. Due to Problem 6.2, we choose $\alpha \leqslant 1/3$ in order that the Jacobian along a stable curve may be a viable test function. The other restrictions on the parameters remain the same.

The definitions of the weak and strong Banach spaces are again the closures with respect to $|\cdot|_w$ and $\|\cdot\|_{\mathcal{B}}$, respectively, but now $C^1(\Omega_0)$ is replaced by slightly different function spaces, see Baladi, Demers, and Liverani (2018, Definition 2.12). However, Lemma 5.4 (embedding) and Lemma 5.6 (compactness) continue to hold as stated.

6.5 Lasota–Yorke inequalities and complexity bounds

The Lasota–Yorke inequalities of Proposition 5.8 continue to hold as written as well, except that their proofs change considerably.

As an example, consider the proof of the weak norm inequality, (5.4.8). Fol-

lowing (5.4.12), we write,

$$\begin{aligned}\int_W \mathcal{L}_t f\ \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\ \psi \circ \Phi_t\ J_{W_i}\Phi_t\, dm_{W_i} \\ &\leqslant \sum_{W_i \in \mathcal{G}_t(W)} |f|_w |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)} \qquad (6.5.1) \\ &\leqslant C|f|_w \sum_{W_i \in \mathcal{G}_t(W)} |J_{W_i}\Phi_t|_{C^0(W_i)},\end{aligned}$$

where we have used bounded distortion and the equivalent of (5.4.14) to estimate the Hölder norms of $\psi \circ \Phi_t$ and $J_{W_i}\Phi_t$. However, the counterpart of the bound on the sum over the Jacobians, Lemma 5.9(c), is not immediately available due to the cutting caused by the singularities. Indeed, the set $\mathcal{G}_t(W)$ contains a countably infinite number of stable curves since in order to have bounded distortion for $J_{W_i}\Phi_t$, we must subdivide $\Phi_{-t}W$ so that for each $W_i \in \mathcal{G}_t(W)$, $P^+(\Phi_s W_i)$ lies in a single homogeneity strip for all $s \in [0,t]$.

Despite the countable subdivision of $\Phi_{-t}W$ which defines $\mathcal{G}_t(W)$, one can show that the sum over Jacobians in (6.5.1) remains uniformly bounded in t and $W \in \mathcal{W}^s$. This is an essential property of both the map and the flow: that the hyperbolicity dominates the complexity due to cuts created by singularities, including the countable collection of cuts made by the boundaries of homogeneity strips. The key estimate which encapsulates this property is the *one step expansion* for the map, due to Chernov. Let $\overline{\mathcal{W}}^s$ denote the set of homogeneous stable curves for the map.

Lemma 6.3 (One Step Expansion)**.** *For any $W \in \overline{\mathcal{W}}^s$, let V_i denote the connected homogeneous components of $T^{-1}W$. There exists an adapted metric $\|\cdot\|_*$, equivalent to the Euclidean metric in $\mathbb{R}^2$, such that*

$$\lim_{\delta \downarrow 0} \sup_{\substack{W \in \overline{\mathcal{W}}^s \\ |W| \leqslant \delta}} \sum_i |J_{V_i} T|_* < 1,$$

where $|J_{V_i}T|_*$ is the minimum contraction factor on V_i in the adapted metric $\|\cdot\|_*$.

This is proved, for example, in Chernov and Markarian (2006, Lemma 5.56). The main idea is that on homogeneity strips, the contraction factor is $\approx k^{-2}$, so one can choose k_0 sufficiently large to make the sum $\sum_{k \geqslant k_0} k^{-2}$ as small as one likes. The constant $\Lambda_0 > 1$ defined earlier gives the minimum contraction factor Λ_0^{-1}

in the adapted metric, and then choosing δ small enough guarantees that $T^{-1}W$ can contain at most one component in $M \setminus (\cup_{|k|\geq k_0}\mathbb{H}_k)$, and a bounded number of components[2] that must be divided according to homogeneity strips $\mathbb{H}_k$ with $|k| \geq k_0$.

Then choosing δ_0 in the definition of $\mathcal{W}^s$ (Definition 1) and the analogous map-stable family $\overline{\mathcal{W}}^s$ according to Lemma 6.3, the one-step expansion can be iterated for the map (Demers and H.-K. Zhang (2011, Lemmas 3.1 and 3.2)) and then extended to the flow (Baladi, Demers, and Liverani (2018, Lemma 3.8)), yielding finally that the sum in (6.5.1) is bounded uniformly in t and W, proving (5.4.8) for the billiard flow.

Similar adjustments must be made for the strong norm estimates, with increased complexity due to cutting and distortion control.

6.6 The generator and the resolvent

The definition of the generator X and the resolvent $R(z)$ proceeds as described in Section 5.5. Lemma 5.10 and the Lasota–Yorke inequalities of Proposition 5.13 go through with minor changes. Thus the characterization of the spectra of X and $R(z)$ given by Corollary 5.14 and Proposition 5.17 hold for the billiard flow.

To prove that in fact, X has a spectral gap, one can follow again the path outlined in Section 5.6. The major difference is in the proof of the Dolgopyat estimate, Lemma 5.22. In Section 5.7, we used a local foliation of strong unstable manifolds to compare the integrals on stable curves in the same flow box in Lemma 5.30. Unfortunately, the foliation of unstable manifolds for the billiard flow is only measurable due to the density of the sets $\{\Phi_t(\mathcal{S}_0)\}_{t\in\mathbb{R}}$ in Ω, so that Definition 3 is no longer valid.

Instead, one must construct a foliation of flow-unstable curves, lying in the kernel of the contact form, which approximate the properties enumerated in Definition 3. Since the curves are not real unstable manifolds, in item (iii) of the definition, they only remain invariant for a specified amount of time χ, chosen proportional to $\log|b|$. And due to the singularities, there are gaps in the parts of the foliation that can be mapped backwards for time χ. We must interpolate across these gaps in order to obtain the required smoothness for the foliation. Finally, item (v) of the foliation fails, yet a four-point estimate does hold which suffices to prove the items in Lemma 5.30. The construction of this foliation is carried out in

[2] Indeed, the number of components is at most $\frac{\tau_{\max}}{\tau_{\min}} + 1$.

detail in Baladi, Demers, and Liverani (2018, Section 6), and is one of the most technical parts of that paper.

With the Dolgopyat estimate proved, the proof of Theorem 5.1 can proceed as in Section 5.6.

A Functional analysis: the minimum

The following is really super condensed (although self-consistent). If you want more details see Dunford and Schwartz (1988), Kato (1995), and Reed and Simon (1980) in which you probably can find more than you are looking for.

A.1 Bounded operators

Given a Banach space $\mathcal{B}$ we can consider the set $L(\mathcal{B}, \mathcal{B})$ of the linear bounded operators from $\mathcal{B}$ to itself.[1] We can then introduce the norm

$$\|B\| = \sup_{\|v\| \leqslant 1} \|Bv\|.$$

Problem A.1. *Show that* $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ *is a Banach space. That is that* $\|\cdot\|$ *is really a norm and that the space is complete with respect to such a norm.*

[1] Recall that a Banach space is a complete normed vector space (in the following we will consider vector spaces on the field of complex numbers), that is a normed vector space in which all the Cauchy sequences have a limit in the space. If you are uncomfortable with Banach spaces, in the following read $\mathbb{R}^d$ instead of $\mathcal{B}$ and matrices instead of operators, but be aware that we have to develop the theory without the use of the determinant that, in general, is not defined for operators on Banach spaces.

Problem A.2. *Show that there exists a norm such that the set of $n \times n$ matrices forms a Banach algebra.*[2]

Problem A.3. *Show that $(L(\mathcal{B}, \mathcal{B}), \|\cdot\|)$ forms a Banach algebra.*[3]

To each $A \in L(\mathcal{B}, \mathcal{B})$ are associated two important subspaces: the range

$$R(A) = \{v \in \mathcal{B} \ : \ \exists\, w \in \mathcal{B} \text{ such that } v = Aw\}$$

and the kernel

$$N(A) = \{v \in \mathcal{B} \ : \ Av = 0\}.$$

Problem A.4. *Prove, for each $A \in L(\mathcal{B}, \mathcal{B})$, that $N(A)$ is a closed linear subspace of $\mathcal{B}$. Show that this is not necessarily the case for $R(A)$ if $\mathcal{B}$ is not finite dimensional.*

A very special, but very important, class of operators are the projectors.

Definition 4. *An operator $\Pi \in L(\mathcal{B}, \mathcal{B})$ is called a* projector *iff $\Pi^2 = \Pi$.*

Note that if Π is a projector, so is $\mathbb{1} - \Pi$. We have the following interesting fact.

Lemma A.5. *If $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $N(\Pi) \oplus R(\Pi) = \mathcal{B}$.*

Proof. If $v \in \mathcal{B}$, then $v = \Pi v + (\mathbb{1} - \Pi)v$. Note that $R(\mathbb{1} - \Pi) = N(\Pi)$ and $R(\Pi) = N(\mathbb{1} - \Pi)$. Finally, if $v \in N(\Pi) \cap R(\Pi)$, then $v = 0$, which concludes the proof. □

Another, more general, very important class of operators are the compact ones.

Definition 5. *An operator $K \in L(\mathcal{B}, \mathcal{B})$ is called* compact *iff for any bounded set B the closure of $K(B)$ is compact.*

Remark A.6. *Note that not all the linear operators on a Banach space are bounded. For example consider the derivative acting on $\mathcal{C}^1((0,1), \mathbb{R})$. However, if the operator is linear, continuous and everywhere defined, then it is bounded.*[4]

[2] A Banach algebra $\mathcal{A}$ is a Banach space in which multiplication between elements is defined with the usual properties of an algebra and, in addition, for each $a, b \in \mathcal{A}$ holds $\|ab\| \leqslant \|a\| \cdot \|b\|$.

[3] The multiplication is given by the composition.

[4] Indeed, if A is continuous in zero, then there exists $\delta > 0$ such that, for all $\|x\| \leqslant \delta$ we have $\|Ax\| \leqslant 1$. Then, by continuity, we have, for each x,

$$\|Ax\| \leqslant \delta^{-1}\|x\| \ \|A(\delta\|x\|^{-1}x)\| \leqslant \delta^{-1}\|x\|.$$

A.2 Analytical functional calculus

First of all recall that the Riemannian theory of integration works verbatim for function $f \in \mathcal{C}^0(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is a Banach space. We can thus talk of integrals of the type $\int_a^b f(t)dt$.[5] Next, we can talk of *analytic functions* for functions in $\mathcal{C}^0(\mathbb{C}, \mathcal{B})$: a function is analytic in an open region $U \subset \mathbb{C}$ iff at each point $z_0 \in U$ there exists a neighborhood $B \ni z_0$ and elements $\{a_n\} \subset \mathcal{B}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in B. \tag{A.2.1}$$

Problem A.7. *Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ is analytic in $U \subset \mathbb{C}$, then given any smooth closed curve γ, contained in a sufficiently small disk in U, holds*[6]

$$\int_\gamma f(z)dz = 0 \tag{A.2.2}$$

Then show that the same holds for any piecewise smooth closed curve with interior contained in U, provided U is simply connected.

Problem A.8. *Show that if $f \in \mathcal{C}^0(U, \mathcal{B})$ is analytic in a simply connected open set $U \subset \mathbb{C}$, then given any smooth closed curve γ, with interior contained in U and having in its interior a point z, the following formula holds*

$$f(z) = \frac{1}{2\pi i} \int_\gamma (\xi - z)^{-1} f(\xi) d\xi. \tag{A.2.3}$$

Problem A.9. *Show that if $f \in \mathcal{C}^0(\mathbb{C}, \mathcal{B})$ satisfies* (A.2.3) *for each smooth closed curve in a simply connected open set U, then f is analytic in U.*

A.3 Spectrum and resolvent

Given $A \in L(\mathcal{B}, \mathcal{B})$ we define the *resolvent*, called $\rho(A)$, as the set of the $z \in \mathbb{C}$ such that $(z\mathbb{1} - A)$ is invertible and the inverse belongs to $L(\mathcal{B}, \mathcal{B})$. The *spectrum* of A, called $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$.

[5] This is a special case of the so called Bochner integral, e.g. see Yosida (1995).

[6] Of course, by $\int_\gamma f(z)dz$ we mean that we have to consider any smooth parametrization $g : [a, b] \to \mathbb{C}$ of γ, $g(a) = g(b)$, and then $\int_\gamma f(z)dz := \int_a^b f \circ g(t) g'(t) dt$. Show that the definition does not depend on the parametrization and that one can use piecewise smooth parametrizations as well.

Problem A.10. *Prove that, for each Banach space $\mathcal{B}$ and operator $A \in L(\mathcal{B}, \mathcal{B})$, if $z \in \rho(A)$, then there exists a neighborhood U of z such that $(z\mathbb{1} - A)^{-1}$ is analytic in U.*

From the above exercise follows that $\rho(A)$ is open, hence $\sigma(A)$ is closed.

Problem A.11. *Show that, for each $A \in L(\mathcal{B}, \mathcal{B})$, $\sigma(A) \neq \emptyset$.*

Problem A.12. *Show that if $\Pi \in L(\mathcal{B}, \mathcal{B})$ is a projector, then $\sigma(\Pi) = \{0, 1\}$.*

Up to now the theory for operators seems very similar to the one for matrices. Yet, the spectrum for matrices is always given by a finite number of points while the situation for operators can be very different.

Problem A.13. *Consider the operator $\mathcal{L} : \mathcal{C}^0([0,1], \mathbb{C}) \to \mathcal{C}^0([0,1], \mathbb{C})$ defined by*

$$(\mathcal{L}f)(x) = \frac{1}{2} f(x/2) + \frac{1}{2} f(x/2 + 1/2).$$

Show that $\sigma(\mathcal{L}) = \{z \in \mathbb{C} \ : \ |z| \leqslant 1\}$.

Problem A.14. *Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and p is any polynomial, then for each $n \in \mathbb{N}$ and smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,*

$$p(A) = \frac{1}{2\pi i} \int_\gamma p(z)(z\mathbb{1} - A)^{-1} dz.$$

Problem A.15. *Show that, for each $A \in L(\mathcal{B}, \mathcal{B})$ the limit*

$$r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$$

exists.

The above limit is called the *spectral radius* of A due the the following fact.

Lemma A.16. *For each $A \in L(\mathcal{B}, \mathcal{B})$, $\sup_{z \in \sigma(A)} |z| = r(A)$.*

Proof. Since we can write

$$(z\mathbb{1} - A)^{-1} = z^{-1}(\mathbb{1} - z^{-1}A)^{-1} = z^{-1} \sum_{n=0}^{\infty} z^{-n} A^n,$$

and since the series converges if it converges in norm, from the usual criteria for the convergence of a series follows $\sup_{z \in \sigma(A)} |z| \leqslant r(A)$. Suppose now that the inequality is strict, then there exists $0 < \eta < r(A)$ and a curve $\gamma \subset \{z \in \mathcal{C} \ : \ |z| \leqslant \eta\}$ which contains $\sigma(A)$ in its interior. Then applying Problem A.14 yields $\|A^n\| \leqslant C\eta^n$, which contradicts $\eta < r(A)$. □

Note that if $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is an analytic function in all $\mathbb{C}$ (entire), then we can define, for all $A \in L(\mathcal{B}, \mathcal{B})$,

$$f(A) = \sum_{n=0}^{\infty} f_n A^n.$$

Problem A.17. *Show that, if $A \in L(\mathcal{B}, \mathcal{B})$ and f is an entire function, then for each smooth curve $\gamma \subset \mathbb{C}$, with $\sigma(A)$ in its interior,*

$$f(A) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z\mathbb{1} - A)^{-1} dz.$$

In view of the above fact, the following definition is natural:

Definition 6. *For each $A \in L(\mathcal{B}, \mathcal{B})$, f analytic in a region U containing $\sigma(A)$, then for each smooth curve $\gamma \subset U$, with $\sigma(A)$ in its interior, define*

$$f(A) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z\mathbb{1} - A)^{-1} dz. \tag{A.3.1}$$

Problem A.18. *Show that the above definition does not depend on the curve γ.*

Problem A.19. *For each $A \in L(\mathcal{B}, \mathcal{B})$ and functions f, g analytic on a domain $D \supset \sigma(A)$, show that $f(A) + g(A) = (f + g)(A)$ and $f(A)g(A) = (f \cdot g)(A)$.*

Problem A.20. *In the hypotheses of Definition 6, show that $f(\sigma(A)) = \sigma(f(A))$ and $[f(A), A] = 0$.*

Problem A.21. *Consider $f : \mathbb{C} \to \mathbb{C}$ entire and $A \in L(\mathcal{B}, \mathcal{B})$. Suppose that $\{z \in \mathbb{C} : f(z) = 0\} \cap \sigma(A) = \emptyset$. Show that $f(A)$ is invertible and $f(A)^{-1} = f^{-1}(A)$.*

Problem A.22. *Let $A \in L(\mathcal{B}, \mathcal{B})$. Suppose there exists a semi-line ℓ, starting from the origin, such that $\ell \cap \sigma(A) = \emptyset$. Prove that it is possible to define an operator $\ln A$ such that $e^{\ln A} = A$.*

Remark A.23. *Note that not all the interesting functions can be constructed in such a way. In fact, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is such that $A^2 = -\mathbb{1}$, thus it can be interpreted as a square root of $-\mathbb{1}$ but it cannot be obtained directly by a formula of the type (A.3.1).*

The next result is extremely useful as it allows one to decompose an operator according to its spectrum.

Lemma A.24. *Suppose that $A \in L(\mathcal{B},\mathcal{B})$ and $\sigma(A) = B \cup C$, $B \cap C = \emptyset$. Suppose that the smooth closed curve $\gamma \subset \rho(A)$ contains B, but not C, in its interior. Then*

$$P_B := \frac{1}{2\pi i}\int_\gamma (z\mathbb{1} - A)^{-1} dz \tag{A.3.2}$$

is a projector that does not depend on γ. In addition, $P_B A = A P_B$

Proof. The non dependence on γ is proven as in Problem A.8. A projector is characterized by the property $P^2 = P$. Thus we must compute

$$\begin{aligned} P_B^2 :=& \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2} (z\mathbb{1} - A)^{-1}(\zeta\mathbb{1} - A)^{-1} dz\, d\zeta \\ =& \frac{1}{(2\pi i)^2}\int_{\gamma_1} dz \int_{\gamma_2} d\zeta (z-\zeta)^{-1}\left[(z\mathbb{1} - A)^{-1} - (\zeta\mathbb{1} - A)^{-1}\right]. \end{aligned}$$

If we have chosen γ_1 in the interior of γ_2, then $(z-\zeta)^{-1}(\zeta\mathbb{1} - A)^{-1}$ is analytic in the interior of γ_1, hence the corresponding integral gives zero. The other integral gives P_B, as announced.

The commutation follows from the fact that A commutes with $(z\mathbb{1} - A)^{-1}$ and the integral representation of the projector. □

By the above it follows that $AR(P_B) \subset R(P_B)$ and $AN(P_B) \subset N(P_B)$. Thus $\mathcal{B} = R(P_B) \oplus N(P_B)$ provides an invariant decomposition for A. The next problems make more explicit the announced decomposition.

Problem A.25. *In the hypotheses of Lemma A.24, prove that $A = P_B A P_B + (\mathbb{1} - P_B) A (\mathbb{1} - P_B)$.*

Problem A.26. *In the hypotheses of Lemma A.24, prove that $\sigma(P_B A P_B) = B \cup \{0\}$. Moreover, if $\dim(R(P_B)) = D < \infty$,[7] then the cardinality of B is $\leqslant D$.*

We conclude the section with an easy but useful Lemma.

Lemma A.27. *For each $A \in L(\mathcal{B},\mathcal{B})$ if $\sigma(A) = \{b\} \cup C$, $\{b\} \cap C = \emptyset$, then there exists a projector P such that $\sigma(PAP - bP) = \{0\}$. In addition, if $\dim(R(P)) = D < \infty$, then $P(b\mathbb{1} - A)P$ is Nilpotent.*

[7] Given a vector space $\mathbb{V}$, by $\dim(\mathbb{V})$ we mean the dimension of $\mathbb{V}$.

Proof. Since $\sigma(A)$ is closed, there exists a neighborhood of b disjoint from C. We can thus define $P = P_{\{b\}}$ by Equation (A.3.2). By Lemma A.5 and Problem A.26 we can restrict A to $R(P)$ and have $\sigma(A|_{R(P)}) = \{b\}$, thus $\sigma((bP - A)|_{R(P)}) = \sigma((b\mathbb{1} - A)|_{R(P)}) = \{0\}$. Hence, by Lemma A.5 again,

$$\begin{aligned}\sigma(PAP - bP) &= \sigma((A - b\mathbb{1})P) \\ &= \sigma((A - b\mathbb{1})|_{R(P)}) \cup \sigma((A - b\mathbb{1})P|_{N(P)}) = \{0\} \cup \{0\} = \{0\}.\end{aligned}$$

Next, note that if $\dim(R(P)) = D$, then $(A - b\mathbb{1})|_{R(P)}$ is isomorphic to a D dimensional matrix K.[8] Thus $\sigma(K) = \{0\}$, and since $(z\mathbb{1} - K)^{-1}$ is a rational function (the ratios of polynomials of degree at most D) it follows that it has a pole only at zero. Hence, for all $|z| > \|K\|$,

$$(z\mathbb{1} - K)^{-1} = \sum_{n=1}^{D} B_n z^{-n} = \sum_{k=0}^{\infty} z^{-n-1} K^n.$$

Which implies that $K^n = 0$ for all $n \geqslant D$. This implies that $[P(b\mathbb{1} - A)P]^D = 0$. □

A.4 Perturbations

Let us consider $A, B \in L(\mathcal{B}, \mathcal{B})$ and the family of operators $A_\nu := A + \nu B$.

Lemma A.28. *For each $\delta > 0$ there exists $\nu_\delta \in \mathbb{R}$ such that, for all $|\nu| \leqslant \nu_\delta$, $\rho(A_\nu) \supset \{z \in \mathbb{C} \,:\, d(z, \sigma(A)) > \delta\}$.*

Proof. Let $d(z, \sigma(A)) > \delta$, then

$$(z\mathbb{1} - A_\nu) = (z\mathbb{1} - A)\left[\mathbb{1} - \nu(z\mathbb{1} - A)^{-1}B\right] \tag{A.4.1}$$

Now $\|(z\mathbb{1} - A)^{-1}B\|$ is a continuous function in z outside $\sigma(A)$, moreover it is bounded outside a ball of large enough radius, hence there exists $M_\delta > 0$ such that $\sum_{d(z,\sigma(A))>\delta} \|(z\mathbb{1} - A)^{-1}B\| \leqslant M_\delta$. Choosing $\nu_\delta = (2M_\delta)^{-1}$ yields the result. □

Suppose that $\bar{z} \in \mathbb{C}$ is an isolated point of $\sigma(A)$, that is there exists $\delta > 0$ such that $\{z \in \mathbb{C} \,:\, |z - \bar{z}| \leqslant \delta\} \cap (\sigma(A) \setminus \{\bar{z}\}) = \emptyset$, then the above Lemma shows that, for ν small enough, $\{z \in \mathbb{C} \,:\, |z - \bar{z}| \leqslant \delta\}$ still contains an isolated part of the spectrum of $\sigma(A_\nu)$, let us call it B_ν, clearly $B_0 = \{\bar{z}\}$.

[8] Just, write $A - b\mathbb{1}$ is a base of $R(P)$.

Problem A.29. *Let P_{B_ν} be defined as in Lemma A.24. Prove that, for ν small enough, it is an analytic function of ν.*

Problem A.30. *If P, Q are two projectors and $\|P - Q\| < 1$, then* $\dim(R(P)) = \dim(R(Q))$.

The above two exercises imply that the dimension of the eigenspace $R(P_{B_\nu})$ is constant. Next, we consider the case in which B_0 consists of one point and $\dim(R(P_{B_0})) = 1$. It follows that also B_ν must consist of only one point. Let us set $P_\nu := P_{B_\nu}$.

Lemma A.31. *If* $\dim(R(P_0)) = 1$*, then A_ν has a unique eigenvalue z_ν in a neighborhood of $\bar{z}$, $z_0 = \bar{z}$. In addition z_ν is an analytic function of ν.*

Proof. From the previous exercises it follows that P_ν is a rank one operator which depends analytically on ν. In addition, since P_ν is a rank one projector it must have the form $P_\nu w = v_\nu \ell_\nu(w)$, where $\ell_\nu \in \mathcal{B}^*$.[9] Then $z_\nu P_\nu = P_\nu A_\nu P_\nu$. Next, setting $a(\nu) := \ell_0(P_\nu v_0) = \ell_\nu(v_0)\ell_0(v_\nu)$, we have that a is analytic and $a(0) = 1$. Thus $a \neq 0$ in a neighborhood of zero and $z_\nu = a(\nu)^{-1}\ell_0(P_\nu A_\nu P_\nu v_0)$ is analytic in such a neighborhood. □

Problem A.32. *If* $\dim(R(P_0)) = 1$*, then there exists $h_\nu \in \mathcal{B}$ and $\ell_\nu \in \mathcal{B}^*$ such that $P_\nu f = h_\nu \ell_\nu(f)$ for each $f \in \mathcal{B}$. Prove that h_ν, ℓ_ν can be chosen to be analytic functions of ν.*

Hence in the case of $A \in L(\mathcal{B}, \mathcal{B})$ with an isolated simple[10] eigenvalue $\bar{z}$ we have that the corresponding eigenvalue z_ν of $A_\nu = A + \nu B$, $B \in L(\mathcal{B}, \mathcal{B})$, for ν small enough, depends smoothly on ν. In addition, using the notation of the previous Lemma, we can easily compute the derivative: differentiating $A_\nu v_\nu = z_\nu v_\nu$ with respect to ν and then setting $\nu = 0$, yields

$$Bv + Av_0' = z_0' v + \bar{z} v_0'.$$

But, for all $w \in \mathcal{B}$, we have $Pw = v\ell(w)$, with $\ell(Aw) = \bar{z}\ell(w)$ and $\ell(v) = 1$, thus applying ℓ to both sides of the above equation yields

$$z_0' = \ell(Bv).$$

[9] By $\mathcal{B}^*$, the dual space, we mean the set of bounded linear functionals on $\mathcal{B}$. Verify that $\mathcal{B}^*$ is a Banach space with the norm $\|\ell\| = \sum_{w \in \mathcal{B}} \frac{|\ell(w)|}{\|w\|}$

[10] That is with the associated eigenprojector of rank one.

Problem A.33. *Compute* v_0'.

Problem A.34. *What happens if the eigenspace associated to* $\bar{z}$ *is finite dimensional, but with dimension strictly larger than one?*

B Hennion–Neussbaum Theory

This appendix is devoted to providing a complete proof of Hennion–Neussbaum theory.

While such results are routinely used in many papers devoted to the study of the statistical properties of dynamical systems, as far as we know no elementary complete account of the theory is available. Our goal here is to present such a complete account in a manner that is accessible to a reader with a basic knowledge of functional analysis and reduces the technicalities to a minimum. We start by discussing the definition of *essential spectrum*. In fact, there exist many alternative definitions of essential spectrum; here we use the most convenient for our goals. The reader interested in more details can have a look the first chapter of Edmunds and W. D. Evans (2018).

B.1 Essential Spectrum

Our aim is to divide the spectrum $\sigma(T)$ of a bounded, linear operator T into two parts, $\sigma_d(T)$ and $\sigma_{ess}(T)$. The discrete spectrum of T, $\sigma_d(T)$, consists of isolated points $\lambda \in \sigma(T)$ such that their associated Riesz projector has finite rank and the range of $\lambda - T$ is closed, while the essential spectrum of T, $\sigma_{ess}(T)$, will be the remaining part of the spectrum. This motivates the following definition of the

essential spectrum.

Definition 7 (Browder (1960/61))**.** *Let T be a bounded linear operator on a Banach space X. The (Browder) essential spectrum of T, $\sigma_{ess}(T)$, is the set of $\lambda \in \sigma(T)$, such that at least one of the following conditions holds:*

1) *The range of $\lambda - T$, $R(\lambda - T)$, is not closed;*

2) *$\bigcup_{r\geqslant 1} N(\lambda - T)^r$ is infinite dimensional;*

3) *λ is a limit point of $\sigma(T) \setminus \{\lambda\}$.*

There are many other definitions of the essential spectrum. For example, Wolf's (Wolf (1959)) essential spectrum is the set of those $z \in \mathbb{C}$ such that $z - T$ is not Fredholm. Recall that an operator $T : X \to X$ is Fredholm if $R(T)$ is closed and the dimensions of both $N(T)$ and the quotient $X \diagup R(T)$ are finite.

However, the essential spectral radius of a bounded operator T is the same using all these different definitions, see Edmunds and W. D. Evans (2018, Section 1.4) and subsequent discussion.

B.1.1 Subspaces

Definition 8. *Let $V \subset X$ be a subspace of a normed vector space X. Given $x \in X$, we define the distance to V by:*

$$\operatorname{dist}(x, V) = \inf\{\|x - y\| : y \in V\}.$$

Definition 9. *A subspace V is called a proper subspace of X if it is neither the whole space X nor the zero subspace $\{0\}$.*

Lemma B.1. *Let X be a Banach space, $V \subset X$ a proper closed subspace. Then for every $\varepsilon > 0$ there exists $x_0 \in X$ with $\|x_0\| = 1$ and $\operatorname{dist}(x_0, V) \geqslant 1 - \varepsilon$.*

Proof. Let $x' \in X \setminus V$, then $d = \operatorname{dist}(x', V) > 0$, (since V is closed). For each $\eta > 0$ there exists $y' \in V$ so that $d \leqslant \|x' - y'\| \leqslant d + \eta$. Let $x_0 = \frac{x'-y'}{\|x'-y'\|}$ and $\eta = \frac{\varepsilon d}{1-\varepsilon}$. For any $z \in V$ we have:

$$\|x_0 - z\| = \frac{1}{\|x' - y'\|}\|x' - y' - \|x' - y'\| z\| \geqslant \frac{d}{\|x' - y'\|} \geqslant \frac{d}{d + \eta} = 1 - \varepsilon,$$

since $y' + \|x' - y'\| z \in V$. The result follows since ε is arbitrary. □

Definition 10. *A normed vector space X is locally compact if any bounded sequence in X has a convergent subsequence.*

Theorem B.2. *(S. Banach) Every locally compact Banach space X has finite dimension.*

Proof. Given a set of linearly independent vectors $x_1, \cdots, x_r$ in X of unit norm, let $G_r \subset E$ be the r-dimensional subspace of X spanned by these vectors. Being finite-dimensional, G_r is a closed subspace of X. If it is a proper subspace, by the Lemma B.1 we may find a unit vector $x_{r+1} \in X$ such that $\|x_{r+1} - x_i\| \geqslant \frac{1}{2}, i = 1, \cdots, r$.

If we may do this for each r, we obtain an infinite sequence $(x_r)_{r \geqslant 1}$ of unit vectors satisfying $\|x_p - x_q\| \geqslant \frac{1}{2}$ for each $p \neq q$, in particular admitting no convegent subsequence. This contradicts the assumption that X is locally compact. □

Definition 11. *A continuous map $F : U \subset X \to Y$ between topological spaces is called proper if $F^{-1}(M)$ is compact whenever $M \subset Y$ is compact.*

Let $L(X, Y)$ be the space of bounded linear maps from X to Y.

Lemma B.3. *Let X and Y be complex Banach spaces and $S \in L(X, Y)$. If S restricted to closed, bounded sets is proper then $N(S)$, the null space of S, is finite dimensional and $R(S)$, the range of S, is closed.*

Proof. Since S is proper, $N(S) = S^{-1}(0)$ is locally compact. By Theorem B.2, $N(S)$ is finite dimensional.

Next we prove that $R(S)$ is closed. Let $\{x_n\}$ be a sequence in X such that $\{S(x_n)\}$ is a Cauchy sequence on Y. We need to show that $\{S(x_n)\}$ converges to a point $y \in R(S)$. Since Y is Banach, $\{S(x_n)\}$ is convergent. The set $\{S(x_n)\}$ with its limit is compact so by hypothesis $\{x_i\}$ has a convergent subsequence, let us call x the limit. Since T is continuous, $S(x) = y$. □

B.1.2 Measure of Noncompactness

Let X be a complete Banach space and A a bounded subset of X.

Definition 12. *We define $\gamma(A)$, which we call the measure of noncompactness of A, to be the infimum of $d > 0$ such that there exists a finite number of sets $S_1, \cdots, S_n$ with diameter$(S_i) \leqslant d$ and $A = \bigcup_{i=1}^{n} S_i$.*

Definition 13. *We call the ball measure of noncompactness of A in X, $\widetilde{\gamma}_X(A)$, to be the infimum of $r > 0$ such that there exists a finite number of balls $V_1, \cdots, V_n$ with centers in X and radii r and $A \subset \bigcup_{i=1}^{n} V_i$.*

Definition 14. *If X_1 and X_2 are Banach spaces and $T \in L(X_1, X_2)$, we say that T is a k-set-contraction if for every bounded set $A \subset X_1$,*

$$\gamma_{X_2}(T(A)) \leqslant k\gamma_{X_1}(A).$$

We say that T is a ball-k-set-contraction if

$$\widetilde{\gamma}_{X_2}(T(A)) \leqslant k\widetilde{\gamma}_{X_1}(A)$$

for every bounded set A in X_1.
We define

$$\gamma(T) = \inf\{k > 0 : T \text{ is a } k\text{-set-contraction}\}$$
$$\widetilde{\gamma}(T) = \inf\{k > 0 : T \text{ is a ball-}k\text{-set-contraction}\}.$$

Remark B.4. *The above ideas can also be defined for nonlinear maps between metric spaces Darbo (1955) and Nussbaum (1969).*

Denote the closed ideal of compact linear operators of X into X by K.[1] Let $Z = L(X, X)/K$.

Definition 15. *We define a seminorm $\|T\|_K$ on $L(X, X)$ by*

$$\|T\|_K = \inf_{C \in K} \|T + C\|.$$

Note that $\|T\|_K$ induces a norm on Z with respect to which Z is a complete normed space.

Lemma B.5. *The measure of noncompactness and the ball measure of noncompactness satisfy the following properties:*

a) Let $A \subseteq X$, then $\bar{A}$ is compact if and only if $\widetilde{\gamma}(A) = 0$. Also, $\bar{A}$ is compact if and only if $\gamma(A) = 0$.

b) An operator $T \in L(X, X)$ is compact if and only if $\widetilde{\gamma}(T) = 0$. Also, T is compact if and only if $\gamma(T) = 0$.

[1] Recall that an operator is compact iff the image of a bounded set is relatively compact, that is, if its closure is compact.

c) $\gamma(T) \leqslant \|T\|$.

d) For bounded subsets $A, B \subseteq X$, *we have* $\gamma(A+B) \leqslant \gamma(A) + \gamma(B)$ *and* $\widetilde{\gamma}(A+B) \leqslant \widetilde{\gamma}(A) + \widetilde{\gamma}(B)$.

Proof. a) For $\varepsilon > 0$, since $\bar{A}$ is compact, A can be covered by a finite number of balls of radius ε. Since ε is arbitrary, we have $\widetilde{\gamma}(A) = 0$. Therefore $\gamma(A) = 0$, because $\gamma(A) \leqslant \widetilde{\gamma}(A)$.
Now assume that $\bar{A}$ is not compact, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \bar{A}$ which has no accumulation point in $\bar{A}$. Define $B_\varepsilon(x_n) := \{y \in X : \|x_n - y\| < \varepsilon\}$. Then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ such that for any $i, j \in \mathbb{N}$, $B_\varepsilon(x_{n_i}) \cap B_\varepsilon(x_{n_j}) = \emptyset$, for some $\varepsilon > 0$. If not, then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for any $n, m \geqslant N$, $|x_n - x_m| < 2\varepsilon$. So $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence which is Cauchy and therefore it has an accumulation point in $\bar{A}$, which is in contrary to the assumption. So we conclude that $\widetilde{\gamma}(A) \geqslant \gamma(A) > \varepsilon$.

b) First suppose that T is a compact operator. For any bounded set $A \subseteq X$, $\overline{T(A)}$ is compact. So by (a), $\widetilde{\gamma}(T(A)) = 0$ and $\gamma(T(A)) = 0$. Hence for any $k > 0$, T is a ball-k-set-contraction and a k-set-contraction. So $\widetilde{\gamma}(T) = 0$ and $\gamma(T) = 0$.

Now assume that $\gamma(T) = 0$. Let $A \subseteq X$, be a ball of radius $R > 0$. For $\varepsilon > 0$, we have $\gamma(T) < \frac{\varepsilon}{R}$. Therefore $\gamma(T(A)) < \frac{\varepsilon}{R}\gamma(A) < \varepsilon$. So $\gamma(T(A)) = 0$, then (a) implies $\overline{T(A)}$ is compact. So T is a compact operator. The same proof works for the case $\widetilde{\gamma}(T) = 0$.

c) If $\gamma(A) = r$, then for $\lambda > r$, there is a covering of A by finitely many sets $\{B_i\}_{i=1}^n$ of diameter not greater than λ. So $\{T(B_i)\}_{i=1}^n$ will cover $T(A)$. For any $1 \leqslant i \leqslant n$

$$\operatorname{diam}(T(B_i)) = \sup_{x,y\in B_i} \|Tx - Ty\| \leqslant \|T\| \sup_{x,y\in B_i} \|x - y\| \leqslant \|T\|\lambda,$$

which implies $\gamma(T) \leqslant \|T\|$.
d) Let $\gamma(A) = \alpha$ and $\gamma(B) = \beta$. Then for $r > \alpha$, there is a covering of A by a finite number of sets $\{a_i\}_{i=1}^n$ of diameter not greater than r and for $\rho > \beta$, there is a covering of B by a finite number of sets $\{b_j\}_{j=1}^m$ of diameter not greater than ρ. So $A + B = \{x + y\}_{x\in A, y\in B} \subseteq \cup_{i,j}\{x + y\}_{x\in a_i, y\in b_j}$. For any $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$ and $x, x' \in a_i$, $y, y' \in b_j$ we have

$$\|x + y - x' - y'\| \leqslant \|x - x'\| + \|y - y'\| \leqslant r + \rho.$$

Therefore $\gamma(A+B) \leqslant \gamma(A) + \gamma(B)$.

Now let $\widetilde{\gamma}(A) = \kappa$ and $\widetilde{\gamma}(B) = \lambda$. Then for $\mu > \kappa$, there is a covering of A by a finite number of balls $\{B(a_i, r_i)\}_{i=1}^n$ of radius $r_i \leqslant \mu$ and for $\nu > \lambda$, there is a covering of B by a finite number of balls $\{B(b_j, \rho_j)\}_{j=1}^m$ of radius $\rho_j \leqslant \nu$. So $A + B = \{x+y\}_{x\in A, y\in B} \subseteq \cup_{i,j}\{x+y\}_{x\in B(a_i,r_i), y\in B(b_j,\rho_j)}$. For any $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$ and $x \in B(a_i, r_i)$, $y \in B(b_j, \rho_j)$ we have

$$\|x + y - (a_i + b_j)\| \leqslant \|x - a_i\| + \|y - b_j\| \leqslant \mu + \nu.$$

Therefore $\widetilde{\gamma}(A+B) \leqslant \widetilde{\gamma}(A) + \widetilde{\gamma}(B)$. □

Lemma B.6. *Let X and Y be complex Banach spaces and $T \in L(X,Y)$. Then we have $\gamma(T^*) \leqslant \widetilde{\gamma}(T)$.*[2]

Proof. Suppose T is a ball-k-set-contraction. To show that T^* is a k-set-contraction, it suffices to show that if S is a set of diameter less than or equal to d in Y^*, $T^*(S)$ can be covered by a finite number of sets of diameter less than or equal than $kd + \varepsilon$, for any $\varepsilon > 0$.

Consider $T(B)$, where $B = \{x \in X, \|x\| \leqslant 1\}$. Since $\widetilde{\gamma}(B) \leqslant 1$ and T is a ball-k-set-contraction, $T(B)$ can be covered by a finite number of balls $B_{k+\frac{\varepsilon}{2d}}(y_i)$ in Y, $1 \leqslant i \leqslant n$, with centers at y_i, and radii $k + \frac{\varepsilon}{2d}$. Select M such that $\|y_i\| \leqslant M$, $1 \leqslant i \leqslant n$, and $\|y^*\| \leqslant M$ for all $y^* \in S$. Hence, we have $|y^*(y_i)| \leqslant M^2$ for each $y^* \in S$. Decompose the closed interval $[-M^2, M^2]$ into a union of disjoint intervals Δ_i, $1 \leqslant i \leqslant p$, of length less than $\frac{\varepsilon}{2}$. We consider an equivalence relation as follows: Given y_1^* and $y_2^* \in S$, write $y_1^* \sim y_2^*$ iff for each i, $1 \leqslant i \leqslant n$, $y_1^*(y_i)$ and $y_2^*(y_i)$ lie in the same interval $\Delta_{j(i)}$, $1 \leqslant j(i) \leqslant p$. Then we divide S into equivalence classes S_j, $1 \leqslant j \leqslant q$,

We claim that diameter $(T^*(S_i)) \leqslant kd + \varepsilon$. Take y_1^* and y_2^* in S_i. We have

$$\|T^*(y_1^*) - T^*(y_2^*)\| = \sup_{x\in B} |y_1^*(Tx) - y_2^*(Tx)| = \sup_{y\in T(B)} |y_1^*(y) - y_2^*(y)|.$$

If $y \in T(B)$, we know that $y \in B_{k+\frac{\varepsilon}{2}}(y_i)$ for some i, $1 \leqslant i \leqslant n$. It follows that

$$|y_1^*(y) - y_2^*(y)| \leqslant |y_1^*(y - y_i) - y_2^*(y - y_i)| + |y_1^*(y_i) - y_2^*(y_i)|$$

$$= |(y_1^* - y_2^*)(y - y_i)| + |y_1^*(y_i) - y_2^*(y_i)| \leqslant d(k + \frac{\varepsilon}{2d}) + \frac{\varepsilon}{2} = kd + \varepsilon.$$

[2] By T^* we mean the dual operator: for all continuous linear functional $\ell \in Y'$ we have $T^*\ell \in X'$ where $T^*\ell(x) = \ell(Tx)$.

Thus, for each $\varepsilon > 0$, $\|T^*(y_1^*) - T^*(y_2^*)\| \leqslant kd + \varepsilon$. This shows that diameter $(T^*(S_i)) \leqslant kd + \varepsilon$, and since $T^*(S) \subset \bigcup_{i=1}^{q} T^*(S_i)$, we have covered $T^*(S)$ by a finite number of sets of diameter less than or equal to $kd + \varepsilon$. □

Lemma B.7. *Let X be a complex Banach space and $T \in L(X, X)$. Assume that for some $n \geqslant 1$, $\tilde{\gamma}(T^n) < 1$. Then for any $r \geqslant 1$, $(\mathbb{1} - T)^r$ restricted to closed, bounded sets is proper.*

Proof. Let A be a closed, bounded set in X and M a compact set. We have to show that $M_1 = \{x \in A : (\mathbb{1} - T)x \in M\}$ is compact. By Lemma B.5, in order to show that M_1 is compact it suffices to show that $\tilde{\gamma}(M_1) = 0$. Notice that $\tilde{\gamma}(M_1)$ is defined, since A is bounded. Suppose $x \in M_1$, so that $x = Tx + m$ for some $m \in M$. Substituting for x on the right, $x = T^2x + Tm + m$, and continuing in this way we find

$$x = T^n x + \sum_{i=0}^{n-1} T^i m. \tag{B.1.1}$$

If we write $M_* = \sum_{i=0}^{n-1} T^i(M)$, M_* is compact, since it is the continuous image of a compact set. Furthermore, (B.1.1) implies that $M_1 \subset T^n(M_1) + M_*$, so that $\tilde{\gamma}(M_1) \leqslant \tilde{\gamma}(T^n(M_1))$, by Lemma B.5. Since T^n is a ball-k-set-contraction, $k < 1$, $\tilde{\gamma}(M_1) \leqslant k\tilde{\gamma}(M_1)$. It follows that $\tilde{\gamma}(M_1) = 0$. Hence $\mathbb{1} - T$ is proper.

To show that $(1 - T)^r$, $r > 1$ is proper we proceed by induction. Assume that for $r > 1$, $(1 - T)^{(r-1)}$ is proper, then for compact set M, $(1 - T)^{-(r-1)}(M)$ is compact. So $(1 - T)^{-r}(M) = (1 - T)^{-1}(1 - T)^{-(r-1)}(M)$ is also compact. Therefore $(1 - T)^r$ is proper. □

B.2 Nussbaum formula

In this section, we obtain a characterization of the essential spectral radius $r_e = \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\}$. We essentially follow Nussbaum (1970).

Lemma B.8. *Let X be a Banach space and $T \in L(X, X)$. Let $r'_e := \inf_n(\tilde{\gamma}(T^n))^{\frac{1}{n}}$. Then $\lim_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$ exist and equal r'_e. Furthermore, if $|\lambda| > r'_e$, then $N(\lambda - T)^r$ is finite dimensional for any $r \geqslant 1$ and $R(\lambda - T)$ is closed.*

Proof. We start showing that $\limsup_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}} \leqslant r'_e$.

For any $\varepsilon > 0$, choose m such that $(\widetilde{\gamma}(T^m))^{\frac{1}{m}} \leqslant r'_e + \varepsilon$. For large enough n, write $n = pm + q$ where $0 \leqslant q \leqslant (m-1)$.

For all $S \in L(X,X)$, $A \subseteq X$, we have:

$$\widetilde{\gamma}(S(A)) \leqslant \widetilde{\gamma}(S)\widetilde{\gamma}(A)$$

Hence for all $S, T \in L(X,X)$, $A \subseteq X$

$$\widetilde{\gamma}(ST(A)) \leqslant \widetilde{\gamma}(S)\widetilde{\gamma}(T(A)) \leqslant \widetilde{\gamma}(S)\widetilde{\gamma}(T)\widetilde{\gamma}(A).$$

Therefore $\widetilde{\gamma}$ has the submultiplicative property:

$$\widetilde{\gamma}(ST) \leqslant \widetilde{\gamma}(S)\widetilde{\gamma}(T).$$

Then, by the above fact and $\widetilde{\gamma}(T) \geqslant 0$ for $T \in L(X,X)$, we obtain

$$(\widetilde{\gamma}(T^n))^{\frac{1}{n}} \leqslant (\widetilde{\gamma}(T^m))^{\frac{p}{n}} \cdot (\widetilde{\gamma}(T))^{\frac{q}{n}} \leqslant (r'_e + \varepsilon)^{\frac{pm}{n}} (\widetilde{\gamma}(T))^{\frac{q}{n}}.$$

Since $\frac{pm}{n} \to 1$ and $\frac{q}{n} \to 0$ as $n \to \infty$, we must have $\limsup_{n\to\infty}(\widetilde{\gamma}(T^n))^{\frac{1}{n}} \leqslant r'_e + \varepsilon$. Since ε was arbitrary, we have proved $\limsup_{n\to\infty}(\widetilde{\gamma}(T^n))^{\frac{1}{n}} \leqslant r'_e \leqslant \liminf_{n\to\infty}(\widetilde{\gamma}(T^n))^{\frac{1}{n}}$. Therefore $\lim_{n\to\infty}(\widetilde{\gamma}(T^n))^{\frac{1}{n}}$ exists and equals r'_e. In the exact same way, we can prove that $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$ exists.

Suppose $|\lambda| > r'_e$ and n is such that $(\widetilde{\gamma}(T^n))^{\frac{1}{n}} < |\lambda|$. Consider $T_1 = (\frac{1}{\lambda})T$ and notice that $\widetilde{\gamma}(T_1^n) = (\frac{1}{|\lambda|^n})\widetilde{\gamma}(T^n) = k < 1$. By Lemma B.7, $(\mathbb{1} - T_1)^r$, for any $r \geqslant 1$ is proper on closed, bounded sets. By Lemma B.3, $N(\mathbb{1} - T_1)^r$ is finite dimensional for any $r \geqslant 1$, so $R(I - T_1)$ is closed. □

Lemma B.9. *If $|\lambda_0| > r'_e$, then λ_0 is not a limit point of $\sigma(T) \setminus \{\lambda_0\}$.*

Proof. We show that all points $\lambda \neq \lambda_0$, in some neighborhood of the point λ_0, belong to the resolvent of T and so λ_0 is not a limit point of $\sigma(T)$. The case $\lambda_0 \in \rho(T)$ is trivial. Let $\lambda_0 \in \sigma(T)$. First we prove that either $N(\lambda_0 - T) \neq 0$ or $N(\lambda_0 - T^*) \neq 0$.

Suppose that $N(\lambda_0 - T) = N(\lambda_0 - T^*) = 0$. Then $(\lambda_0 - T)^{-1} : D \to X$ exists on $D = R(\lambda_0 - T)$ which is closed, by Lemma B.8 applied to λ_0. Assume that $D \neq X$, then by Lemma B.1, there is $u \in X$, such that $\|u\| = 1$ and $\|u - w\| \geqslant \frac{1}{2}$ for any $w \in D$. Let $V := span\{u, D\}$, then for any $v \in V$ we can write $v = \alpha u + w$ with $w \in D$. Define $l(v) := \alpha$, then

$$\|v\| = \|\alpha u + w\| = |\alpha|\|u - (-\alpha^{-1}w)\| \geqslant \frac{1}{2}|\alpha| = \frac{1}{2}|l(v)|.$$

So

$$|l(v)| \leqslant 2\|v\|.$$

We can then apply the Hahn–Banach theorem to produce an extension of l on all of X and $l \neq 0$, since $l(u) = 1$. For any $v \in X$, $(\lambda_0 - T^*)l(v) = l\big((\lambda_0 - T)v\big) = 0$. So $(\lambda_0 - T^*)l = 0$. This contradicts $N(\lambda_0 - T^*) = 0$. So $D = X$, which implies that $\lambda_0 - T$ is invertible on X and by the bounded inverse theorem, $(\lambda_0 - T)^{-1}$ is a bounded operator. Therefore $\lambda_0 \notin \sigma(T)$ and this contradicts the assumption.

Suppose that there exists a sequence $\{\widetilde{\lambda}_n\}_{n=1}^{\infty} \subset \sigma(T) \setminus \{\lambda_0\}$ which accumulates to λ_0. Then there are either infinitely many $\widetilde{u}_n \in N(\widetilde{\lambda}_n - T)$ or infinitely many $\widetilde{l}_n \in N(\widetilde{\lambda}_n - T^*)$. For each $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that, for $n > \bar{n}$, $|\widetilde{\lambda}_n - \lambda_0| < \varepsilon|\lambda_0|$.

In the first case, for any $k \in \mathbb{N}$, let M_k be the subspace spanned by the vectors $\widetilde{u}_{\bar{n}}, \cdots, \widetilde{u}_{\bar{n}+k}$. Set $u_k := \widetilde{u}_{\bar{n}+k}$ and $\lambda_k := \widetilde{\lambda}_{\bar{n}+k}$. Since $u_1, u_2, \cdots$ are linearly independent, each M_{k-1} is a closed proper subspace of M_k. So, by Lemma B.1, there exists $v_k \in M_k$, such that $\|v_k\| = 1$ and $d(v_k, M_{k-1}) \geqslant 1 - \varepsilon$.

Note that $v_k = \alpha_k u_k + w_k$ where $\alpha_k \in \mathbb{R}, w_k \in M_{k-1}$. So for $k, r, s \in \mathbb{N}$, such that $s > k$,

$$\|T^r v_s - T^r v_k\| = \|T^r(\alpha_s u_s) + T^r w_s - T^r v_k\| = \|\alpha_s \lambda_s^r u_s + T^r w_s - T^r v_k\|$$

$$= |\lambda_s^r|\|v_s - (w_s - \lambda_s^{-r} T^r w_s + \lambda_s^{-r} T^r v_k)\| \geqslant |\lambda_s^r|(1-\varepsilon) = |(\lambda_s - \lambda_0 + \lambda_0)^r|(1-\varepsilon)$$

$$= |\lambda_0^r|\Big|1 + \frac{\lambda_s - \lambda_0}{\lambda_0}\Big|^r (1-\varepsilon) \geqslant |\lambda_0|^r \Big(1 - \Big|\frac{\lambda_s - \lambda_0}{\lambda_0}\Big|\Big)^r (1-\varepsilon) \geqslant |\lambda_0|^r (1-\varepsilon)^{r+1}.$$

This implies that $T^r\{|v| \leqslant 1\}$ cannot be covered by finitely many sets of diameter $\frac{1}{4}|\lambda_0|^r(1-\varepsilon)^{r+1}$. Therefore, by the arbitrariness of ε, $\widetilde{\gamma}(T^r) \geqslant \gamma(T^r) \geqslant \frac{1}{4}|\lambda_0|^r$.

In the second case, exactly the same argument implies $\gamma(T^{*r}) \geqslant \frac{1}{4}|\lambda_0|^r$. By Lemma B.6, $\widetilde{\gamma}(T^r) \geqslant \frac{1}{4}|\lambda_0|^r$.

Thus in both cases, $r_e' = \inf_n(\widetilde{\gamma}(T^n))^{\frac{1}{n}} \geqslant |\lambda_0|$ which contradicts the assumption. So λ_0 is not a limit point of $\sigma(T)$. □

Corollary B.10. *According to the definition of the essential spectrum, Lemma B.8 and Lemma B.9 imply that* $r_e' \geqslant r_e$.

Lemma B.11. *Let* T *be as above and* $r_e = \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\}$. *Take* $r > r_e$. *Then there exists a finite dimensional linear operator* F *such that* $\sigma(T + F) \subset \{\lambda : |\lambda| \leqslant r\}$.

Proof. Since $\sigma(T) \cap \{\lambda : |\lambda| \geqslant r\}$ is a compact set of isolated points, it consists of a finite number of points $\lambda_1, \cdots, \lambda_n$. Let C_i be a small circle about λ_i, $C_i \cap C_j = \emptyset$ for $i \neq j$ and containing only λ_i from $\sigma(T)$, and $P_i = (\frac{1}{2\pi i}) \int_{C_i} (\lambda - T)^{-1} d\lambda$ be the Riesz projector associated to λ_i. Since λ_i does not belong to the essential spectrum, $R(P_i)$, which is the eigenspace associated to λ_i, is finite dimensional. If we write $P = \sum_{i=1}^{n} P_i$, we therefore see that P is a finite dimensional projection. We take $F = TP$.

Let us write $N = N(P)$, the null space of P, and $R = R(P)$, the range of P, and note that $X = N \oplus R$. Consider $\lambda - T - F$ for $|\lambda| > r$. For $|\lambda| > r$ and $\lambda \neq \lambda_i$, $1 \leqslant i \leqslant n$, we have $\lambda \in \rho(T)$. Then it is clear that $(\lambda - T - F)|_N = (\lambda - T)|_N N$ is a one to one map of N onto N. Furthermore $(\lambda - T - F)|_R = \lambda|_R$, which is clearly one to one and onto for $|\lambda| > r$. Thus $\lambda - T - F$ is a one to one map of X for $|\lambda| > r$. □

The following lemma is not necessary for our applications but we include it for completeness.

Lemma B.12. *Let X be a complex Banach space and $T \in L(X, X)$. Then $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$, $\lim_{n\to\infty}(\widetilde{\gamma}(T^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(\|T^n\|_K))^{\frac{1}{n}}$ are all equal to r_e.*

Proof. We have already seen in Lemma B.8 that

$$\lim_{n\to\infty} (\widetilde{\gamma}(T^n))^{\frac{1}{n}}$$

and $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$ exist and equal r'_e. The same argument as in Lemma B.8 shows that $r''_e := \lim_{n\to\infty} \|T^n\|_K^{\frac{1}{n}}$ exists. For $S \in L(X, X)$ and any compact operator $C \in L(X, X)$, $\gamma(S) = \gamma(S + C) \leqslant \|S + C\|$. Therefore $\gamma(S) \leqslant \|S\|_K$, which implies $r'_e \leqslant r''_e$.

Now we show that $r''_e \leqslant r_e$. Suppose not, so that $r_e < r''_e$, and select $r_e < r < r''_e$. For this r, let F be as in Lemma B.11 and write $T_1 = T + F$. By the ordinary spectral radius theorem we know that $\lim_{n\longrightarrow\infty} \|T_1^n\|^{\frac{1}{n}} \leqslant r$. On the other hand, $\|T^n\|_K \leqslant \|T_1^n\|$, so that we obtain $r''_e = \lim_{n\longrightarrow\infty} \|T^n\|_K^{\frac{1}{n}} \leqslant r$, a contradiction. It follows that $r''_e \leqslant r_e$. Now by Corollary B.10, we have $r_e = r'_e = r''_e$. □

B.3 Hennion's theorem and its generalizations

We start by proving Hennion's theorem and then provide a more recent generalization.

In fact, the next Theorem is itself a small generalization of Hennion (1993), since it allows the weak norm to be just a semi-norm. A similar generalization is contained in Hennion and Hervé (2001, Theorem XIV.3). To this end we need a bit of notation: given a vector space X and a semi-norm $\|\cdot\|_w$ we call $X_{0,w}$ the space X equipped with the topology induced by the semi-norm. Next, we can consider the vector space of the equivalence classes with respect to the semi-norm (i.e. $x \sim y$ iff $\|x-y\|_w = 0$). This yields a metric space X_w. Let $\|\cdot\|'_w$ be the associated norm, and its completion $\bar{X}_w$ is a Banach space.

Definition 16. *A normed space Y and a continuous (w.r.t. the topology induced by the semi-norm) operator $T : Y \to X$ canonically induce an operator $\widetilde{T} : Y \to X_w$. We will say that $T : Y \to X$ is $\|\cdot\|_w$-compact if for each bounded set $B \subset Y$, $\widetilde{T}(B)$ is relatively compact in $\bar{X}_w$.*

Problem B.13. *Show that the above constructions and Definition 16 make sense.*

Theorem B.14 (Hennion (1993))**.** *Let $(X, \|\cdot\|)$ be a Banach space and $T \in L(X,X)$. Assume that there exists a continuous[3] semi-norm $\|\cdot\|_w$ on X, and $M > \theta > 0$, $A, B, C > 0$, such that, for all $n \in \mathbb{N}$ and $f \in X$,*

$$\|T^n f\|_w \leqslant CM^n \|f\|_w; \quad \|T^n f\| \leqslant A\theta^n \|f\| + BM^n \|f\|_w .$$

Then the spectral radius of $T \in L(X,X)$ is bounded by M. If, in addition, T is $\|\cdot\|_w$-compact, then the essential spectral radius of T is bounded by θ.

Proof. Continuity of the semi-norm implies that there exists $C' > 0$ such that $\|f\|_w \leqslant C'\|f\|$ for all $f \in \mathcal{B}$. For if not, then for any $n \in \mathbb{N}$, there must exist $f_n \in \mathcal{B}$ with $\|f_n\| = 1$, but $\|f_n\|_w \geqslant n$. But then $\|\frac{1}{n} f_n\| \to 0$ while $\|\frac{1}{n} f_n\|_w \geqslant 1$, contradicting continuity of the semi-norm.

This fact plus the second assumed inequality yields $\|T^n f\| \leqslant (A + BC')M^n \|f\|$ for all $n \in \mathbb{N}$, $f \in \mathcal{B}$. Using the formula for spectral radius, see Problem A.15, we conclude the spectral radius is bounded by M.

For the second part, by Lemma B.12 we have

$$r_e = \lim_{n\to\infty} \sqrt[n]{\widetilde{\gamma}(T^n)} \leqslant \lim_{n\to\infty} \sqrt[n]{\widetilde{\gamma}(T^n B_1)}$$

[3] By continuous, we mean that if $(f_n)_n \subset \mathcal{B}$ is a sequence such that $\|f_n\| \to 0$, then necessarily $\|f_n\|_w \to 0$.

where $B_1 := \{f \in X \mid \|f\| \leqslant 1\}$.

Now we prove that $T^n B_1$ can be covered by a finite number of balls of radius $C_\# \cdot \theta^n$, which implies that $r_e \leqslant \lim_{n\to\infty} \sqrt[n]{\widetilde{\gamma}(T^n B_1)} \leqslant \lim_{n\to\infty} \sqrt[n]{C_\# \cdot \theta^n} = \theta$.

By hypotheses $\widetilde{T} B_1$ is relatively compact in $\bar{X}_w$. Thus, for each $\varepsilon > 0$ we can extract a finite subcover $\{\widetilde{B}_\varepsilon(\widetilde{f}_i)\}_{i=1}^{N_\varepsilon}$ from the covering $\{\widetilde{B}_\varepsilon(\widetilde{f})\}_{\widetilde{f}\in\widetilde{T}B_1}$, where $\widetilde{B}_\varepsilon(\widetilde{f}) = \{\widetilde{g} \in \bar{X}_w \ : \ \|\widetilde{g} - \widetilde{f}\|'_w < \varepsilon\}$. Then, choosing [4] $f_i \in \widetilde{f}_i \cap TB_1$ and setting $U_\varepsilon(f_i) = \{f \in X \ : \ \|f - f_i\|_w < \varepsilon\} = \{f \in \widetilde{f} \ : \ \widetilde{f} \in \widetilde{B}_\varepsilon(\widetilde{f}_i)\}$ we have a finite covering of TB_1.

Next, if $f = T(g)$, $g \in B_1$, then again using the continuity of the semi-norm, $\|f\| \leqslant \|Tg\| \leqslant A\theta + BC'M$. Accordingly, for each $f \in U_\varepsilon(f_i) \cap TB_1$ we have

$$\begin{aligned}\|T^{n-1}(f - f_i)\| &\leqslant A\theta^{n-1}\|f - f_i\| + BM^{n-1}\|f - f_i\|_w \\ &\leqslant A\theta^{n-1}2(A\theta + BC'M) + BM^{n-1}\varepsilon.\end{aligned}$$

Choosing ε sufficiently small we can conclude that for each $n \in \mathbb{N}$ the set $T^n(B_1)$ can be covered by a finite number of $\|\cdot\|$-balls of radius $C_\# \cdot \theta^n$ centered at the points $\{T^{n-1} f_i\}_{i=1}^{N_\varepsilon}$. □

To conclude we show that the hypotheses of the above theorem can be further weakened to situations in which T is not necessarily continuous with respect to the weak norm.[5]

Theorem B.15 (Bardet, Gouëzel, and Keller (2007))**.** *Let $(X, \|\cdot\|)$ be a Banach space and $T \in L(X, X)$. Assume that there exists a semi-norm $\|\cdot\|_w$ on X such that any bounded sequence in $\|\cdot\|$ contains a Cauchy sequence for $\|\cdot\|_w$. If there exist $n_0 \in \mathbb{N}$ and $\theta, B > 0$ such that,*

$$\|T^{n_0} f\| \leqslant \theta^{n_0}\|f\| + B\|f\|_w, \tag{B.3.1}$$

then the essential spectral radius of T is bounded by θ.

Proof. Note that there must exist $C > 0$ such that $\|f\|_w \leqslant C\|f\|$. If not then there would be a sequence $\{f_n\}$, $\|f_n\| \leqslant 1$, such that $\lim_{n\to\infty} \|f_n\|_w = \infty$, but this contradicts that f_n must have a Cauchy subsequence.

[4] Recall that elements of $\bar{X}_w$ are equivalence classes of elements in X.

[5] Indeed, note that the first displayed inequality in Theorem B.14 amounts simply to the continuity of T in the weak norm.

Let $M = 2\|T\|$, then we can define the new seminorm,

$$\|f\|'_w := (2C)^{-1} \sum_{n=0}^{\infty} M^{-n} \|T^n f\|_w.$$

Note that

$$\begin{aligned} \|f\|'_w &\leqslant \frac{1}{2} \sum_{n=0}^{\infty} M^{-n} \|T^n f\| \leqslant \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \|f\| = \|f\| \\ \|Tf\|'_w &\leqslant (2C)^{-1} \sum_{n=0}^{\infty} M^{-n} \|T^{n+1} f\|_w \\ &= (2C)^{-1} M \sum_{n=1}^{\infty} M^{-n} \|T^n f\|_w \leqslant M \|f\|'_w. \end{aligned} \tag{B.3.2}$$

Thus, if we set $A = M^{n_0} \theta^{-n_0}$, for each $n \in \mathbb{N}$ we can write $n = kn_0 + m$, $m < n_0$, and, iterating Equation (B.3.1),

$$\begin{aligned} \|T^n f\| &\leqslant \theta^{kn_0} M^m \|f\| + \sum_{j=0}^{k-1} B\theta^{(k-1-j)n_0} \|T^{jn_0+m} f\|_w \\ &\leqslant \theta^{kn_0} M^m \|f\| + B \max\{\theta^{(k-1-j)n_0} M^{jn_0+m}\} \|f\|'_w \\ &\leqslant A\theta^n \|f\| + BM^n \|f\|'_w \end{aligned}$$

since it must be that $\theta \leqslant \|T\| = M/2$.

Next, if $\{f_n\}$ is bounded in the $\|\cdot\|$ norm, so are the sequences $T^m f_n$, $m \in \mathbb{N}$. Then, by hypothesis, we can extract a sequence n_j^1 such that $f_{n_j^1}$ is Cauchy in the $\|\cdot\|_w$ norm. From it we can extract a sequence n_j^2, with $n_1^2 = n_1^1$, such that $Tf_{n_j^2}$ is Cauchy in the $\|\cdot\|_w$ norm, and so on. Note that, by construction, $n_j^j = n_j^m$ for $m \geqslant j$. Then the sequence n_j^j is such that $T^m f_{n_j^j}$ is Cauchy in the $\|\cdot\|_w$ norm for all $m \in \mathbb{N}$. Then, for each $\varepsilon > 0$, if $(2C)^{-1} 2^{-L} < \varepsilon/2$, then, by the definition of the norm $\|\cdot\|'_w$, we can write

$$\|f_{n_j^j} - f_{n_k^k}\|'_w \leqslant (2C)^{-1} \sum_{m=0}^{L} M^{-m} \|T^m (f_{n_j^j} - f_{n_k^k})\|_w + \varepsilon/2.$$

It follows that there exists $m \in \mathbb{N}$ such that, if $j, k \geqslant m$, then $\| f_{n_j^j} - f_{n_k^k} \|'_w \leqslant \varepsilon$, i.e. we can extract a Cauchy sequence in the $\| \cdot \|'_w$ norm. So the $\| \cdot \|'_w$ norm has the same property as the $\| \cdot \|_w$ norm. This implies that T is a $\| \cdot \|'_w$-compact operator. The statement follows then from Theorem B.14. □

Bardet, Gouëzel, and Keller (2007) provide an application of Theorem B.15 to prove a local limit theorem for weakly coupled lattices of expanding maps in which the relevant operators are indeed not continuous in the weak norm. For more details, see Bardet, Gouëzel, and Keller (ibid., Section 3).

C More on perturbation theory

This section contains some useful perturbation results. We follow and extend the ideas in Liverani (2003, Theorem 3.2). Several such results are available (e.g., see Kifer (1988), Baladi and Young (1993) or Baladi (2000) for a review). Here we provide a simplification of the theory developed in Gouëzel and Liverani (2006) and Keller and Liverani (1999), see the original works for the full story.

We start by recalling, for the reader's convenience, the setting introduced in Section 1.7.

Hypotheses C.1. *Let $X \subset X_w$ be two Banach spaces, $\|\cdot\|$ and $|\cdot|_w$ being the respective norms, satisfying $|\cdot|_w \leqslant \|\cdot\|$. Also assume that the unit ball of X is weakly compact in X_w. Consider a family of operators $\mathcal{L}_\varepsilon$ with the following properties.*

1. *A uniform Lasota–Yorke inequality: There exist $\lambda_\star > 1$ and $A, B, C > 0$ such that,*

$$\|\mathcal{L}_\varepsilon^n h\| \leqslant A\lambda_\star^{-n}\|h\| + B|h|_w, \quad |\mathcal{L}_\varepsilon^n h|_w \leqslant C|h|_w \ ;$$

2. *For $L : X \to X$ define the norm*

$$|||L||| := \sup_{\|h\|\leqslant 1} |Lf|_w,$$

that is the norm of L as an operator from $X \to X_w$. Then we require that there exists $D > 0$ such that

$$|||\mathcal{L} - \mathcal{L}_\varepsilon||| \leqslant D\varepsilon.$$

To state a precise result consider, for each operator L, the set

$$V_{\delta,r}(L) := \{z \in \mathbb{C} \mid |z| \leqslant r \text{ or } \operatorname{dist}(z, \sigma(L)) \leqslant \delta\}.$$

Since the complement of $V_{\delta,r}(L)$ belongs to the resolvent of L it follows that

$$H_{\delta,r}(L) := \sup\left\{\|(z-L)^{-1}\| \mid z \in \mathbb{C} \setminus V_{\delta,r}(L)\right\} < \infty.$$

By $R(z)$ and $R_\varepsilon(z)$ we will mean respectively $(z-\mathcal{L})^{-1}$ and $(z-\mathcal{L}_\varepsilon)^{-1}$.

Theorem C.1 (Keller and Liverani (1999))**.** *Consider a family of operators $\mathcal{L}_\varepsilon : X \to X$ satisfying Hypothesis C.1. Let $V_{\delta,r} := V_{\delta,r}(\mathcal{L})$, $r > \lambda_\star^{-1}$, $\delta > 0$, then, if $\varepsilon \leqslant \varepsilon_1(\mathcal{L}, r, \delta)$, $\sigma(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$. In addition, if $\varepsilon \leqslant \varepsilon_0(\mathcal{L}, r, \delta)$, there exists $a > 0$ such that, for each $z \notin V_{\delta,r}$,*

$$|||R(z) - R_\varepsilon(z)||| \leqslant C\varepsilon^a.$$

In addition, for each $r > \lambda_\star^{-1}$ and $\delta > 0$ there are constants $a, b > 0$, such that a depends only on r and b depends also on δ, such that, for all $h \in X$ and $\varepsilon \leqslant \varepsilon_0(\mathcal{L}, r, \delta)$,

$$\|R_\varepsilon(z)h\| \leqslant a\|h\| + b|h|_w.$$

Proof.[1] To start with we collect some trivial, but very useful algebraic identities. For each operator $L : X \to X$ and $n \in \mathbb{Z}$ holds

$$\frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}L)^i(z-L) + (z^{-1}L)^n = \mathbb{1} \tag{C.0.1}$$

$$R(z)(z-\mathcal{L}_\varepsilon) + \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L}) + R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L}) = \mathbb{1} \tag{C.0.2}$$

$$(z-\mathcal{L}_\varepsilon)\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right] = \mathbb{1} - (z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z) \tag{C.0.3}$$

$$\left[G_{n,\varepsilon} + (z^{-1}\mathcal{L}_\varepsilon)^n R(z)\right](z-\mathcal{L}_\varepsilon) = \mathbb{1} - (z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L}), \tag{C.0.4}$$

[1] This proof is simpler than the one in Keller and Liverani (1999), yet it gives worse bounds, although sufficient for the present purposes.

where we have set $G_{n,\varepsilon} := \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L}_\varepsilon)^i$.

Let us start applying the above formulae. For each $h \in X$ and $z \notin V_{r,\delta}$, and n large and ε small enough,

$$\begin{aligned}\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\| &\leqslant (r\lambda_\star)^{-n}A\|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h\| \\ &\quad + \frac{B}{r^n}|(\mathcal{L}_\varepsilon - \mathcal{L})R(z)h|_w \\ &\leqslant [(r\lambda_\star)^{-n}A2C_1 + Br^{-n}D\varepsilon]H_{\delta,r}(\mathcal{L})\|h\| < \|h\|\end{aligned}$$

To obtain the last inequality, choose $n \in \mathbb{N}$ such that $n = \lfloor -\frac{\ln\varepsilon}{\ln\lambda_\star}\rfloor$. Then assuming $r < 1$ without loss of generality, we have $r^{-n} \leqslant \varepsilon^{\frac{\ln r}{\ln\lambda_\star}}$, so that both terms are bounded by $C\varepsilon^{1+\frac{\ln r}{\ln\lambda_\star}}$, and $\frac{\ln r}{\ln\lambda_\star} > -1$ since $r\lambda_\star > 1$ by hypothesis. The claimed inequality follows for $\varepsilon > 0$ sufficiently small.

Thus $\|(z^{-1}\mathcal{L}_\varepsilon)^n(\mathcal{L}_\varepsilon - \mathcal{L})R(z)\| < 1$ and the operator on the right hand side of (C.0.3) can be inverted by the usual Neumann series. Accordingly, $(z - \mathcal{L}_\varepsilon)$ has a well defined right inverse. Analogously,

$$\begin{aligned}\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\| &\leqslant (r\lambda_\star)^{-n}A\|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\| \\ &\quad + Br^{-n}|R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h|_w.\end{aligned}$$

This time to continue we need some information on the X_w norm of the resolvent. For $g \in X$ equation (C.0.1) yields

$$\begin{aligned}|R(z)g|_w &\leqslant \frac{1}{r}\sum_{i=0}^{n-1}|(z^{-1}\mathcal{L})^i g|_w + \|R(z)(z^{-1}\mathcal{L})^n g\| \\ &\leqslant \frac{C}{r^n(1-r)}|g|_w + H_{\delta,r}(\mathcal{L})A(r\lambda_\star)^{-n}\|g\| + H_{\delta,r}(\mathcal{L})Br^{-n}|g|_w \\ &\leqslant r^{-n}(H_{\delta,r}(\mathcal{L})B + C(1-r)^{-1})|g|_w + H_{\delta,r}(\mathcal{L})A(r\lambda_\star)^{-n}\|g\|.\end{aligned} \tag{C.0.5}$$

Substituting, we have

$$\begin{aligned}&\|(z^{-1}\mathcal{L}_\varepsilon)^n R(z)(\mathcal{L}_\varepsilon - \mathcal{L})h\| \leqslant \{(r\lambda_\star)^{-n}AH_{\delta,r}(\mathcal{L})2C_1[1 + Br^{-n}] \\ &+ Br^{-2n}[H_{\delta,r}(\mathcal{L})B + (1-r)^{-1}]D\varepsilon\}\|h\| < 1,\end{aligned}$$

again, provided ε is small enough and choosing n appropriately. Hence the operator on the right hand side of (C.0.4) can be inverted, thereby providing a left inverse for $(z - \mathcal{L}_\varepsilon)$. This implies that z does not belong to the spectrum of $\mathcal{L}_\varepsilon$.

To investigate the second statement note that (C.0.2) implies

$$R(z) - R_\varepsilon(z) = \frac{1}{z}\sum_{i=0}^{n-1}(z^{-1}\mathcal{L})^i(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z) - R(z)(z^{-1}\mathcal{L})^n(\mathcal{L}_\varepsilon - \mathcal{L})R_\varepsilon(z).$$

Accordingly, for each $\varphi \in X$,

$$|R(z)\varphi - R_\varepsilon(z)\varphi|_w \leqslant \{r^{-n}(1-r)^{-1}\varepsilon + H_{\delta,r}(\mathcal{L})(\lambda_\star r)^{-n}2AC_1 + H_{\delta,r}(\mathcal{L})B\varepsilon\}\|R_\varepsilon(z)\varphi\|.$$

To complete the argument, choose $n = \lfloor -\frac{\ln \varepsilon}{\ln \lambda_\star} \rfloor$ as before and note that by our previous bounds on the inverse of $z - \mathcal{L}_\varepsilon$, we have $\|R_\varepsilon(z)\varphi\| \leqslant C_{\varepsilon_0}\|\varphi\|$, for all $\varepsilon \leqslant \varepsilon_0$ and $\varepsilon_0 > 0$ small enough. The first inequality of the theorem follows with $a = 1 + \frac{\ln r}{\ln \lambda_\star}$.

To prove the second inequality, for $|z| = r > \lambda_\star^{-1}$, we use Equation (C.0.1) to write

$$\begin{aligned}\|(z-\mathcal{L}_\varepsilon)^{-1}h\| = &\left\|\sum_{k=0}^{m-1} z^{-k-1}\mathcal{L}_\varepsilon^k + (z^{-1}\mathcal{L}_\varepsilon)^m(z-\mathcal{L}_\varepsilon)^{-1}h\right\| \\ \leqslant &A(1-r^{-1}\lambda_\star^{-1})^{-1}\|h\| + C_{r,m}|h|_w \\ &+ \lambda_\star^{-m}r^{-m}\|(z-\mathcal{L}_\varepsilon)^{-1}h\| + r^{-m}B|(z-\mathcal{L}_\varepsilon)^{-1}h|_w,\end{aligned}$$

for some constant $C_{r,m}$ depending on r and m. We can thus choose m such that $A\lambda_\star^{-m}r^{-m} < \frac{1}{2}$ and, recalling the first inequality of the Theorem, write

$$\|(z-\mathcal{L}_\varepsilon)^{-1}h\| \leqslant C_r\|h\| + C_{r,m}|h|_w + C\varepsilon^a r^{-m}B\|h\| + r^{-m}B|(z-\mathcal{L})^{-1}h|_w.$$

To conclude, we can use Equation (C.0.5) and write, for all $n \in \mathbb{N}$,

$$\|(z-\mathcal{L}_\varepsilon)^{-1}h\| \leqslant C_\#[C_r + \varepsilon^a r^{-m} + AH_{\delta,r}(\mathcal{L})(r\lambda_\star)^{-n}r^{-m}]\|h\| + C_{r,m,n,\delta}|h|_w.$$

Choosing n and ε so that $H_{\delta,r}(\mathcal{L})(r\lambda_\star)^{-n}r^{-m} \leqslant 1$ and $\varepsilon^a r^{-m} \leqslant 1$ yields the statement. □

Theorem C.1 shows that the point spectrum is stable. Yet, in applications it is also important to control the multiplicity of the spectrum. This can be done thanks to the following Lemma.

Lemma C.2. *Consider a family of operators $\mathcal{L}_\varepsilon : X \to X$ satisfying Hypothesis C.1. Let $\nu \in \sigma(\mathcal{L})$, $|\nu| > \lambda_\star$, and let m be the dimension of the eigenspace associated to ν. Then, for each δ small enough there exists $\varepsilon_2(\mathcal{L}, \nu, \delta)$ such that, for all $\varepsilon \leqslant \varepsilon_2(\mathcal{L}, \nu, \delta)$, $\sigma(\mathcal{L}_\varepsilon) \cap \{z \in \mathbb{C} \ : \ |z - \nu| < \delta\}$ contains at most m eigenvalues and the total dimension of their eigenspaces is m.*

Proof. Since $|\nu| > \lambda_\star$, Theorem B.14 implies that ν belongs to the point spectrum. Hence, there exists δ_0 such that $\{z \in \mathbb{C} \ : \ |z - \nu| < \delta_0\} \cap \sigma(\mathcal{L}) = \{\nu\}$. Then Theorem C.1 implies that, for each $\delta < \delta_0/2$ and $\varepsilon \leqslant \varepsilon_0(\mathcal{L}, r, \delta)$, we can split the spectrum as $\sigma(\mathcal{L}_\varepsilon) = \sigma_1 \cup \sigma_2$ where $\sigma_1 \cap \sigma_2 = \emptyset$ and $\sigma_1 \subset \{z \in \mathbb{C} \ : \ |z - \nu| < \delta\}$. Accordingly, by Lemma A.24 we can define the eigenprojectors

$$\Pi_\varepsilon := \frac{1}{2\pi i} \int_{\gamma_\delta} (z\mathbb{1} - \mathcal{L}_\varepsilon)^{-1} dz, \tag{C.0.6}$$

where $\gamma_\delta(t) = \nu + \delta e^{it}$, and $\sigma(\Pi_\varepsilon \mathcal{L}_\varepsilon) = [\sigma(\mathcal{L}_\varepsilon) \cap \{z \in \mathbb{C} \ : \ |z - \nu| < \delta\}] \cup \{0\}$. Note that the first inequality of Theorem C.1 implies, for $\varepsilon \leqslant \varepsilon_0(\mathcal{L}, r, \delta)$, where we can choose $r = \{\lambda_\star^{-1} + |\nu|\}/2$,

$$|(\Pi_\varepsilon - \Pi_0)h|_w \leqslant C_\delta \varepsilon^a \|h\|,$$

for some constant C_δ, depending on the choice of δ. While the second inequality of Theorem C.1 implies that there exist constants a and b_δ, the latter depending on δ, such that

$$\|\Pi_\varepsilon h\| \leqslant a\delta \|h\| + b_\delta |h|_w.$$

Since Π_ε is independent of δ (see Lemma A.24) we have

$$\|\Pi_\varepsilon h\| \leqslant (a\delta_0 + b_{\delta_0}) \|h\| =: c_0 \|h\|.$$

The above inequalities imply

$$\begin{aligned} \|(\Pi_\varepsilon - \Pi_0)^2 h\| &\leqslant 2a\delta \|(\Pi_\varepsilon - \Pi_0)h\| + 2b_\delta \|(\Pi_\varepsilon - \Pi_0)h\| \\ &\leqslant \left[4ac_0\delta + 2b_\delta C_\delta \varepsilon^a\right] \|h\|. \end{aligned}$$

Accordingly, if we choose δ such that $8ac_0\delta \leqslant 1$ and ε_2 such that $2b_\delta C_\delta \varepsilon^a < \frac{1}{2}$, we obtain

$$\|(\Pi_\varepsilon - \Pi_0)^2\| < 1. \tag{C.0.7}$$

This concludes the Lemma due to the following general fact.

Problem C.3. *Let $\Pi_1, \Pi_2 \in L(X, X)$ be two projectors. Assume that*

$$\|(\Pi_1 - \Pi_2)^2\| < 1,$$

then $\dim(\Pi_1(X)) = \dim(\Pi_2(X))$.

□

The above two results are rather effective to study perturbations of transfer operators. The reader can verify this directly by solving the next problem.

Problem C.4. *Consider the maps $f_n : \mathbb{T}^1 \to \mathbb{T}^1$ defined by*

$$f(x) = 2x + \frac{1}{2n} \sin 2\pi \sqrt{n} x \mod 1$$

and use Theorem C.1 and Lemma C.2 to study the spectrum of the operators $\mathcal{L}_n h(x) = \sum_{y \in f_n^{-1}(x)} \frac{h(y)}{f_n'(y)}$, for n large. In particular, show that, for n large enough, $\mathcal{L}_n$ has a spectral gap close to $\frac{1}{2}$.

Given the above results it is natural to ask if the spectral data have some more regular dependence on the change in the operator. These types of questions are related to *linear response*.

Linear Response

In order to have linear response one needs more control on the operators $\mathcal{L}_\varepsilon$ than that provided by Hypothesis C.1. Here we provide the simplest possibility, see Gouëzel and Liverani (2006, Section 8) and Keller and Liverani (2009b) for more details.[2]

Hypotheses C.2. *Let $X_2 \subset X_1 \subset X_0$ be three Banach spaces, equipped with the norms $\|\cdot\|_i$, respectively, satisfying $\|\cdot\|_0 \leqslant \|\cdot\|_1 \leqslant \|\cdot\|_2$. Also assume that the unit ball of X_i is weakly compact in X_{i+1}. Consider a family of operators $\mathcal{L}_\varepsilon$ with the following properties.*

1. *A uniform Lasota–Yorke inequality: There exist $\lambda_\star > 1$ and $A, B, C > 0$ such that,*

$$\|\mathcal{L}_\varepsilon^n h\|_i \leqslant A\lambda_\star^{-n} \|h\|_i + B\|h\|_{i-1}, \quad \text{for } i > 0 \text{ and for all } h \in X_i$$
$$\|\mathcal{L}_\varepsilon^n h\|_i \leqslant C\|h\|_i, \quad \text{for } i \geqslant 0 \text{ and for all } h \in X_i.$$

[2]Note that Gouëzel and Liverani (2006, Section 8) contains an imprecision which is fixed in Gouëzel (2010, Theorem 3.3).

2. *We require that there exists an operator $\mathcal{A} \in L(X_j, X_i)$, for each $j > i$, such that*

$$\begin{aligned}
\|(\mathcal{L}_\varepsilon - \mathcal{L} - \varepsilon\mathcal{A})h\|_0 &\leqslant D\varepsilon\|h\|_1, \quad \text{for all } h \in X_1\\
\|(\mathcal{L}_\varepsilon - \mathcal{L} - \varepsilon\mathcal{A})h\|_1 &\leqslant D\varepsilon\|h\|_2, \quad \text{for all } h \in X_2\\
\|(\mathcal{L}_\varepsilon - \mathcal{L} - \varepsilon\mathcal{A})h\|_0 &\leqslant D\varepsilon^{1+\alpha}\|h\|_2, \quad \text{for all } h \in X_2,
\end{aligned}$$

for some $\alpha > 0$ and each $h \in X_2$.

Remark C.5. *The Hypothesis C.2 are a bit different from the ones in Gouëzel and Liverani (2006). This is made in order to present a simplified proof.*

Remark C.6. *Note that the Hypothesis C.2 imply Hypothesis C.1 for $\mathcal{L}, \mathcal{L}_\varepsilon$ both with respect to the norms $\|\cdot\|_0, \|\cdot\|_1$ and with respect to the norms $\|\cdot\|_1, \|\cdot\|_2$.*

We will need the following well known fact.

Problem C.7. *Prove that for any $A, B \in L(X, X)$ and $z \notin \sigma(A) \cup \sigma(B)$ we have*

$$(z\mathbb{1} - A)^{-1} - (z\mathbb{1} - B)^{-1} = (z\mathbb{1} - A)^{-1}(A - B)(z\mathbb{1} - B)^{-1},$$

which is called the resolvent identity.

Finally, let us define

$$V_{\delta,r}(\mathcal{L}) := \{z \in \mathbb{C} \mid |z| \leqslant r \text{ or } \operatorname{dist}(z, \sigma_{X_1}(\mathcal{L})) \leqslant \delta\},$$

where $\sigma_X(\mathcal{L})$ is the spectrum of $\mathcal{L}$ seen as an operator in $L(X, X)$.

Remark C.8. *Note that $\left[\sigma_{X_2}(\mathcal{L}) \cap \{|z| \geqslant \lambda_\star^{-1}\}\right] \subset \left[\sigma_{X_1}(\mathcal{L}) \cap \{|z| \geqslant \lambda_\star^{-1}\}\right]$ since by Theorem B.14 this part of the spectrum belongs to the point spectrum. Accordingly, if $\nu \in \sigma_{X_2}(\mathcal{L}) \cap \{|z| \geqslant \lambda_\star^{-1}\}$, then there exists $h \in X_2$ such that $\mathcal{L}h = \nu h$ and hence $\nu \in \sigma_{X_1}(\mathcal{L})$.*

We are then ready to provide the last result of this section.

Remark C.9. *Theorem C.10 says that $(z - \mathcal{L}_\varepsilon)^{-1}$, when seen as a function from $\mathbb{R}$ to $L(X_2, X_0)$ is differentiable at zero. But then also the eigenprojectors Π_ε defined in Equation* (C.0.6) *are differentiable and so is $\Pi_\varepsilon\mathcal{L}_\varepsilon$. In particular, if the projector Π_ε is associated with a simple eigenvalue ν_ε, and hence has the form $\Pi_\varepsilon = \ell_\varepsilon \otimes h_\varepsilon$, then $\Pi_\varepsilon\mathcal{L}_\varepsilon = \nu_\varepsilon\Pi_\varepsilon$. It follows that ν_ε is differentiable and $\varepsilon \to h_\varepsilon$ is differentiable as a function from $\mathbb{R}$ to X_0.*

Theorem C.10. *Consider a family of operators $\mathcal{L}_\varepsilon : X_0 \to X_0$ satisfying Hypothesis C.2. Let $r > \lambda_\star^{-1}$ and $\delta > 0$. If $\varepsilon \leqslant \varepsilon_2(\mathcal{L}, r, \delta)$, then $\sigma_{X_1}(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$ and $\sigma_{X_2}(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$. Moreover, there exists $\eta > 0$ such that, for all $z \notin V_{\delta,r}(\mathcal{L})$ and $h \in X_2$,*

$$\|[R(z) - R_\varepsilon(z) - \varepsilon R(z)\mathcal{A}R(z)]h\|_0 \leqslant C_\delta \varepsilon^{1+\eta} \|h\|_2.$$

Proof. The fact that $\sigma_{X_i}(\mathcal{L}_\varepsilon) \subset V_{\delta,r}(\mathcal{L})$ follows from Theorem C.1 and Remark C.8.

Let $\mathcal{Q}_\varepsilon = \mathcal{L}_\varepsilon - \mathcal{L} - \varepsilon\mathcal{A}$ and, as before $R(z) = (z\mathbb{1} - \mathcal{L})^{-1}$ and $R_\varepsilon(z) = (z\mathbb{1} - \mathcal{L}_\varepsilon)^{-1}$. By Problem C.7 we can write

$$R_\varepsilon(z) - R(z) = R_\varepsilon(z)(\mathcal{L}_\varepsilon - \mathcal{L})R(z).$$

Thus if we define $\Xi = R_\varepsilon(z)\mathcal{A}R(z)$, we have that

$$\|(R_\varepsilon(z) - R(z) - \varepsilon\Xi)h\|_0 = \|R_\varepsilon(z)\mathcal{Q}_\varepsilon R(z)h\|_0.$$

Arguing as in Equation (C.0.5), recalling Remark C.6 and the second inequality of Theorem C.1, we can show that there exists $C_{r,\delta} > 0$ such that for all $g \in X_1$,

$$\|R_\varepsilon(z)g\|_0 \leqslant C_{\delta,r}\left[r^{-m}\|g\|_0 + (r\lambda_*)^{-m}\|g\|_1)\right].$$

Accordingly, using Hypothesis C.2-(2) and recalling $\sigma_{X_2}(\mathcal{L}) \subset V_{\delta,r}(\mathcal{L})$, we have, for each $h \in X_2$,

$$\begin{aligned}\|(R_\varepsilon(z) - R(z) - \varepsilon\Xi)h\|_0 &\leqslant C_{\delta,r}\left[r^{-m}\|\mathcal{Q}_\varepsilon R(z)h\|_0 + (r\lambda_\star)^{-m}\|\mathcal{Q}_\varepsilon R(z)h\|_1)\right]\\ &\leqslant C_{\delta,r}D\left[r^{-m}\varepsilon^{1+\alpha} + (r\lambda_\star)^{-m}\varepsilon)\right]\|R(z)h\|_2\\ &\leqslant C'_{\delta,r}\left[r^{-m}\varepsilon^{1+\alpha} + (r\lambda_\star)^{-m}\varepsilon)\right]\|h\|_2\end{aligned}$$

for some constant $C'_{\delta,r}$. Choosing m so that $\varepsilon^\alpha = \lambda_\star^{-m}$, the above implies that, setting $\eta_0 = \alpha(1 - \frac{\ln r^{-1}}{\ln\lambda}) > 0$, we have

$$\|(R_\varepsilon(z) - R(z) - \varepsilon\Xi)h\|_0 \leqslant C_\delta \varepsilon^{1+\eta_0}\|h\|_2.$$

On the other hand, Theorem C.1 implies

$$\begin{aligned}\|\left[R_\varepsilon(z)\mathcal{A}R(z) - R(z)\mathcal{A}R(z)\right]h\|_0 &\leqslant C_\delta\varepsilon^a\|\mathcal{A}R(z)h\|_1\\ &\leqslant C_\delta\varepsilon^a\|R(z)h\|_2 \leqslant C'_\delta\varepsilon^a\|h\|_2.\end{aligned}$$

Which concludes the proof with $\eta = \min\{\eta_0, a\}$. □

D Hilbert metric and Birkhoff theorem

In this section we will see that the Banach fixed point theorem can produce unexpected results if used with respect to an appropriate metric: a projective metric.

As already remarked projective metrics are widely used in geometry, and have imprtant generalizations (e.g. Kobayashi metrics) for the study of complex manifolds, see Isaev and Krantz (2000a).
Here we limit ourselves to a few words on the Hilbert metric, an important tool in hyperbolic geometry. For more details on Hilbert metrics see Birkhoff (1979), and Nussbaum (1988) for an overview of the field.

D.1 Projective metrics

Let $C \subset \mathbb{R}^n$ be a strictly convex compact set. For each pair of points $x, y \in C$ consider the line $\ell = \{\lambda x + (1-\lambda y) \mid \lambda \in \mathbb{R}\}$ passing through x and y. Let $\{u, v\} = \partial C \cap \ell$ and define[1]

$$\Theta(x, y) = \left| \ln \frac{\|x-u\|\|y-v\|}{\|x-v\|\|y-u\|} \right| \tag{D.1.1}$$

[1] Remark that u, v can also be ∞.

(the logarithm of the cross ratio).

Problem D.1. *Prove that Θ defines a metric.*

Note that the distance from an inner point to the boundary is always infinite. One can also check that if the convex set is a disc then the disc with the Hilbert metric is nothing other than the Poincaré disc. This points to the connection with complex geometry that, however, we will not explore further.

The objects that we will use in our subsequent discussion are not convex sets but rather convex cones, yet their projectivization is a convex set and one can define the Hilbert metric on it (whereby obtaining a semi-metric for the original cone). It turns out that there exists a more algebraic way of defining such a metric, which is easier to use in our context. Moreover, there exists a simple connection between vector spaces with a convex cone and vector lattices (in a vector lattice one can always consider the positive cone). This justifies the next digression into lattice theory.[2]

Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[3] We require the partial order to be "continuous," i.e. given $\{f_n\} \in \mathbb{V}$ $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for each n, then $f \succeq g$. We call such vector lattices "integrally closed." [4]

We define the closed convex cone [5] $\mathcal{C} = \{f \in \mathbb{V} \mid f \neq 0,\ f \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ iff there exists $\lambda \in \mathbb{R}^+ \backslash \{0\}$ such that $f = \lambda g$. If we call $\widetilde{\mathcal{C}}$ the quotient of $\mathcal{C}$ with respect to $\sim$, then $\widetilde{\mathcal{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa. The reader must therefore be advised that

[2] For more details see Birkhoff (1957), and Nussbaum (1988) for an overview of the field.

[3] We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $f, g \in \mathbb{V}$ $f \succeq g \iff f - g \succeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+$, $f \succeq 0 \Longrightarrow \lambda f \succeq 0$; for each $f \in \mathbb{V}$, $f \succeq 0$ and $f \preceq 0$ imply $f = 0$ (antisymmetry of the order relation).

[4] To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$ with $\alpha_n \to \alpha$ and $f, g \in \mathbb{V}$, if $\alpha_n f \succeq g$, then $\alpha f \succeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[5] Here, by "cone," we mean any set such that, if f belongs to the set, then λf belongs to it as well, for each $\lambda > 0$.

"$\preceq$" will mean different things in different contexts.

It is then possible to define a projective metric Θ (Hilbert metric),[6] in $\mathcal{C}$, by the construction:

$$\begin{aligned} \alpha(f,\, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\} \\ \beta(f,\, g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\} \\ \Theta(f,\, g) &= \log\left[\frac{\beta(f,\, g)}{\alpha(f,\, g)}\right] \end{aligned} \tag{D.1.2}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The relevance of the above metric in our context is due to the following Theorem by Garrett Birkhoff.

Theorem D.2. *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed real vector lattices;*[7] *$\mathcal{L} : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$ with $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,\, g \in T(\mathcal{C}_1)} \Theta_2(f,\, g)$ we have*

$$\Theta_2(\mathcal{L}f,\, \mathcal{L}g) \leqslant \tanh\left(\frac{\Delta}{4}\right)\Theta_1(f,\, g) \qquad \forall\, f,\, g \in \mathcal{C}_1$$

($\tanh(\infty) \equiv 1$*).*

Proof. The proof is provided for the reader's convenience.

Let $f,\, g \in \mathcal{C}_1$. On the one hand if $\alpha = 0$ or $\beta = \infty$, then the inequality is obviously satisfied. On the other hand, if $\alpha \neq 0$ and $\beta \neq \infty$, then

$$\Theta_1(f,\, g) = \ln\frac{\beta}{\alpha}$$

where $\alpha f \preceq g$ and $\beta f \succeq g$, since $\mathbb{V}_1$ is integrally closed. Notice that $\alpha \geqslant 0$, and $\beta \geqslant 0$ since $f \succeq 0$, $g \succeq 0$. If $\Delta = \infty$, then the result follows from $\alpha\mathcal{L}f \preceq \mathcal{L}g$ and $\beta\mathcal{L}f \succeq \mathcal{L}g$. If $\Delta < \infty$, then, by hypothesis,

$$\Theta_2\left(\mathcal{L}(g - \alpha f),\, \mathcal{L}(\beta f - g)\right) \leqslant \Delta$$

[6] In fact, we define a semi–metric, since $f \sim g \Rightarrow \Theta(f,\, g) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$.

[7] Recall that a topological vector lattice $(\mathbb{V}, \preceq)$ is *integrally closed* if for all sequences $\{f_n\}$, $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for all $n \in \mathbb{N}$, then $f \succeq g$. In fact, this definition is a bit stronger than the usual one, see Footnote 4 of this chapter.

which means that there exist $\lambda,\ \mu \geqslant 0$ such that

$$\lambda \mathcal{L}(g - \alpha f) \preceq \mathcal{L}(\beta f - g)$$
$$\mu \mathcal{L}(g - \alpha f) \succeq \mathcal{L}(\beta f - g)$$

with $\ln \frac{\mu}{\lambda} \leqslant \Delta$. The previous inequalities imply

$$\frac{\beta + \lambda\alpha}{1 + \lambda} \mathcal{L}f \succeq \mathcal{L}g$$
$$\frac{\mu\alpha + \beta}{1 + \mu} \mathcal{L}f \preceq \mathcal{L}g.$$

Accordingly,

$$\begin{aligned}\Theta_2(\mathcal{L}f,\ \mathcal{L}g) &\leqslant \ln \frac{(\beta + \lambda\alpha)(1 + \mu)}{(1 + \lambda)(\mu\alpha + \beta)} = \ln \frac{e^{\Theta_1(f,g)} + \lambda}{e^{\Theta_1(f,g)} + \mu} - \ln \frac{1 + \lambda}{1 + \mu} \\ &= \int_0^{\Theta_1(f,g)} \frac{(\mu - \lambda)e^{\xi}}{(e^{\xi} + \lambda)(e^{\xi} + \mu)} d\xi \leqslant \Theta_1(f,\ g) \frac{1 - \frac{\lambda}{\mu}}{\left(1 + \sqrt{\frac{\lambda}{\mu}}\right)^2} \\ &\leqslant \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\ g).\end{aligned}$$

□

Remark D.3. *If $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$, then it follows that $\Theta_2(\mathcal{L}f,\ \mathcal{L}g) \leqslant \Theta_1(f,\ g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of $\mathcal{L}$ (see Birkhoff (1979) for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the relation of the Hilbert metric compared to other, more usual, metrics and the connection with spectral theory. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

D.2 Hilbert Metric and spectral theory

We start with the relation between the Hilbert metric and norms. The following is Liverani, Saussol, and Vaienti (1998, Lemma 2.2).

Lemma D.4. *Let $\|\cdot\|$ be a semi-norm on the vector lattice $\mathbb{V}$, and suppose that, for each $f, g \in \mathbb{V}$,*

$$-f \preceq g \preceq f \Longrightarrow \|f\| \geqslant \|g\|.$$

Let $\mathcal{C} \subset \mathbb{V}$ and suppose $\rho : \mathcal{C} \to \mathbb{R}_{\geqslant 0}$ is a homogeneous and order preserving function, i.e.

$$\begin{aligned} \forall\, f \in \mathcal{C},\ \forall \lambda \in \mathbb{R}^+ \qquad & \rho(f) = \lambda f \\ \forall\, f, g \in \mathcal{C} \qquad & f \preceq g \implies \rho(f) \leqslant \rho(g)\,. \end{aligned}$$

Then, for all $f, g \in \mathcal{C}$ with $\rho(f) = \rho(g) > 0$,

$$\|f - g\| \leqslant \left(e^{\Theta(f,g)} - 1\right) \min\{\|f\|, \|g\|\}.$$

Proof. We know that $\Theta(f, g) = \ln \frac{\beta}{\alpha}$, where $\alpha f \preceq g$, $\beta f \succeq g$. Since ρ is order preserving, this implies $\alpha\rho(f) \leqslant \rho(g) \leqslant \beta\rho(f)$. Since $\rho(f) > 0$, this implies $\alpha \leqslant 1 \leqslant \beta$. Hence,

$$\begin{aligned} g - f \preceq & (\beta - 1) f \preceq (\beta - \alpha) f \\ g - f \succeq & (\alpha - 1) f \succeq -(\beta - \alpha) f \end{aligned}$$

which implies

$$\|g - f\| \leqslant (\beta - \alpha)\|f\| \leqslant \frac{\beta - \alpha}{\alpha}\|f\| = \left(e^{\Theta(f,g)} - 1\right)\|f\|.$$

Reversing the roles of f and g completes the proof. □

It is possible to take $\rho = \|\cdot\|$ in the above lemma since by assumption, the semi-norm is order preserving. Yet it is convenient in many applications to be able to separate the two. See Appendix D.3 for one such application.

Many normed vector lattices satisfy the hypothesis of Lemma D.4 (e.g. Banach lattices[8]). In particular, it is often possible to construct a standard norm with the wanted properties.

[8] A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\| \, |f| \, \| = \|f\|$ for each $f \in \mathbb{V}$, where $|f|$ is the least upper bound of f and $-f$. For this definition to make sense it is necessary to require that $\mathbb{V}$ is "directed," i.e. any two elements have an upper bound.

We say that $\mathbb{V}$ is Archimedean if there exists $e \in \mathcal{C}$ such that, for all $f \in \mathcal{C}$ there exists $\lambda \in \mathbb{R}$: $f \preceq \lambda e$. For each $f \in \mathbb{V}$ we define

$$\|f\|_\star = \inf\{\lambda \ : \ -\lambda e \preceq f \preceq \lambda e\}. \tag{D.2.1}$$

Lemma D.5. *The function $\|\cdot\|_\star$ is an order preserving norm, that is: $-g \preceq f \preceq g$ implies $\|f\|_\star \leqslant \|g\|_\star$. Moreover, $(\mathbb{V}, \|\cdot\|_\star, \preceq)$ is an integrally closed vector lattice.*

Proof. To start with, note that if $\|f\|_\star = 0$, then there exists $\lambda_n \to 0$ such that $-\lambda_n e \preceq f \preceq \lambda_n e$. It follows that $\lambda_n e - f \in \mathcal{C}$ and $\lambda_n e + f \in \mathcal{C}$, hence $f, -f \in \mathcal{C} \cup \{0\}$, and thus $f = 0$ (since $\mathcal{C} \cap -\mathcal{C} = \emptyset$ by assumption).

Since $f \preceq g$ is equivalent to $\nu f \preceq \nu g$, for $\nu \in \mathbb{R}_+$, it follows immediately that $\|\nu f\|_\star = \nu \|f\|_\star$.

Let $f, g \in \mathbb{V}$, then for each $\varepsilon > 0$ there exist a, b with $a \leqslant \varepsilon + \|f\|_\star$, $b \leqslant \varepsilon + \|g\|_\star$, such that $-ae \preceq f \preceq ae$ and $-be \preceq g \preceq be$. Then

$$-(\|f\|_\star + \|g\|_\star + 2\varepsilon)e \preceq -(a+b)e \preceq f+g \preceq (a+b)e \leqslant (\|f\|_\star + \|g\|_\star + 2\varepsilon)e$$

implies the triangle inequality by the arbitrariness of ε. We have thus proven that $\|\cdot\|_\star$ is a norm.

Next, suppose that $-g \preceq f \preceq g$, then

$$-\|g\|_\star e \preceq -g \preceq f \preceq g \preceq \|g\|_\star e$$

which implies $\|f\|_\star \leqslant \|g\|_\star$. Hence, the norm is order preserving.

To conclude, let us prove that $\mathbb{V}$ is integrally closed. Assume that $\{f_n\}$ converges to f in the $\|\cdot\|_\star$ topology, and $f_n \succeq g$ for all $n \in \mathbb{N}$. Then there exists a sequence $\alpha_n \to 0$ such that $-\alpha_n e \preceq f - f_n \preceq \alpha_n e$. Hence, $f - g + \alpha_n e \succeq f_n - g \succeq 0$ and since the cone is closed it follows that $f - g \succeq 0$. □

Remark D.6. *Note that we can always complete $\mathbb{V}$ with respect to the norm $\|\cdot\|_\star$, whereby obtaining a Banach space. From now on we thus assume that $(\mathbb{V}, \|\cdot\|_\star)$ is a Banach space .*

Among the order preserving norms, the norm $\|\cdot\|_\star$ enjoys a special status, as is illustrated by the next lemma.

Lemma D.7. *If the norm $\|\cdot\|$, on $\mathbb{V}$, is order preserving, then there exists a constant $C > 0$ such that, for all $f \in \mathbb{V}$, we have $\|f\| \leqslant C\|f\|_\star$.*

Proof. By definition we have $-\|f\|_{\star}\mathbb{e} \preceq f \preceq \|f\|_{\star}\mathbb{e}$. By the order preserving property of the norm it follows that

$$\|f\| \leqslant \|\|f\|_{\star}\mathbb{e}\| = \|f\|_{\star}\|\mathbb{e}\|.$$

□

Our last result allows us to link our cone language to spectral theory. In particular we show that a strict cone contraction implies a spectral gap for the operator acting on a Banach space equipped with an order preserving norm.

Theorem D.8. *Let $\mathcal{L} : \mathbb{V} \to \mathbb{V}$ be order preserving, let $\|\cdot\|$ be an order preserving norm, and assume*

$$\Delta = \sup_{f,g\in\mathcal{C}} \Theta(\mathcal{L}f, \mathcal{L}g) < \infty,$$

then, setting $\nu = \rho(\mathcal{L})$,[9] *$\chi := \tanh\left(\frac{\Delta}{4}\right)$, $h \in \mathbb{V}$ and $\ell \in \mathbb{V}^*$ such that $\mathcal{L}(f) = \nu h\ell(f) + Qf$ where $\ell(h) = 1$, $Qh = 0$, $\ell(Qf) = 0$, for all $f \in \mathbb{V}$, and $\|Q^n\| \leqslant \chi^{n-1}\nu^n\Delta$.*

Proof. Since $\mathcal{L}$ is order preserving, for each $f \in \mathbb{V}$, $-\|f\|\mathcal{L}\mathbb{e} \preceq \mathcal{L}f \preceq \|f\|\mathcal{L}\mathbb{e}$. Hence, $\|\mathcal{L}f\| \leqslant \|\mathcal{L}\mathbb{e}\|\|f\|$, that is, $\mathcal{L}$ is bounded. Accordingly, Theorem D.2 implies that, for each $f, g \in \mathcal{C}$,

$$\Theta(\mathcal{L}f, \mathcal{L}g) \leqslant \tanh\left(\frac{\Delta}{4}\right)\Theta(f, g) =: \chi\Theta(f, g), \tag{D.2.2}$$

and note that $\chi < 1$ since $\Delta < \infty$. For $f, g \in \mathcal{C}$, let $\eta_n(f) = \|\mathcal{L}^n f\|^{-1}\mathcal{L}^n f$. Then Lemma D.4 implies, for all $n > m > 0$,

$$\|\eta_n(f) - \eta_m(f)\| \leqslant \left[e^{\Theta(\mathcal{L}^n f, \mathcal{L}^m f)} - 1\right] \leqslant \left[e^{\chi^{m-1}\Delta} - 1\right]$$
$$\|\eta_n(f) - \eta_n(g)\| \leqslant \left[e^{\Theta(\mathcal{L}^n f, \mathcal{L}^n g)} - 1\right] \leqslant \left[e^{\chi^{m-1}\Delta} - 1\right].$$

It follows that $\eta_n(f)$ is a Cauchy sequence, and its limit $h(f) =: h$ does not depend on f. Moreover,

$$\mathcal{L}\eta_n(f) = \frac{\|\mathcal{L}^{n+1} f\|}{\|\mathcal{L}^n f\|}\eta_n(\mathcal{L}f).$$

[9] By $\rho(\mathcal{L})$ we mean the spectral radius of $\mathcal{L}$.

Since $\frac{\|\mathcal{L}^{n+1} f\|}{\|\mathcal{L}^n f\|} \leqslant \|\mathcal{L}\|$, we can choose a subsequence n_j such that

$$\lim_{j\to\infty} \frac{\|\mathcal{L}^{n_j+1} f\|}{\|\mathcal{L}^{n_j} f\|} = \nu$$

for some $\nu > 0$. Then,

$$\mathcal{L}h = \lim_{j\to\infty} \mathcal{L}\eta_{n_j}(f) = \lim_{j\to\infty} \frac{\|\mathcal{L}^{n_j+1} f\|}{\|\mathcal{L}^{n_j} f\|} \eta_{n_j}(\mathcal{L}f) = \nu h.$$

Since by construction $\|h\| = 1$, then $\nu = \|\mathcal{L}h\|$.

Since the cone is closed, $h \in \mathcal{C}$. Thus, $h \in \mathcal{L}\mathcal{C} \subset \mathrm{int}(\mathcal{C})$. Hence, for each $f \in \mathbb{V}$, there exists $\mu > 0$ such that $-\mu\|f\|h \preceq f \preceq \mu\|f\|h$. Thus, since the norm is order preserving, $\|\mathcal{L}^n f\| \leqslant \mu\|f\|\|\mathcal{L}^n h\| \leqslant \mu\|f\|\nu^n$. For each, $f \in \mathcal{C}$ let

$$\ell_0(f) = \limsup_{n\to\infty} \nu^{-n}\|\mathcal{L}^n f\|.$$

Note that ℓ_0 is bounded, homogeneous of degree one and order preserving, moreover it satisfies the triangle inequality, hence it is a seminorm. Since $\ell_0(h) = 1$ and $\ell_0(\nu^{-m}\mathcal{L}^m f) = \ell_0(f)$ we can apply Lemma D.4 to f and $\ell_0(f)h$ and obtain

$$\begin{aligned} \|\mathcal{L}^n(f - \ell_0(f)h)\| &= \|\mathcal{L}^n f - \mathcal{L}^n h\ell_0(f)\| \leqslant C\Theta(\mathcal{L}^n f, \mathcal{L}^n h)\nu^n \\ &\leqslant \chi^{n-1}\nu^n\Delta. \end{aligned} \tag{D.2.3}$$

On the other hand, for each $f \in \mathcal{C}$ and $t^{-1} \geqslant \|f\|$, we have

$$\begin{aligned} \|\mathcal{L}^n f - t^{-1}\{\ell_0(\mathbb{e} + tf) - \ell_0(\mathbb{e})\}h\| \leqslant& t^{-1}\big\{\|\mathcal{L}^n(\mathbb{e} + tf) - \ell_0(\mathbb{e} + tf)h\| \\ &+ \|\mathcal{L}^n\mathbb{e} - \ell_0(\mathbb{e})h\|\big\} \\ \leqslant& 2t^{-1}\chi^{n-1}\nu^n\Delta. \end{aligned}$$

Hence, if $f \in \mathrm{int}(\mathcal{C})$, we have

$$\ell_0(f) = \ell_0(\|f\|\mathbb{e} + f) - \ell_0(\|f\|\mathbb{e}) = \ell_0(t\mathbb{e} + f) - \ell_0(t\mathbb{e})$$

for all $t \geqslant \|f\|$ We can then define, for all $f \in \mathbb{V}$,

$$\ell(f) := \ell_0(\|f\|\mathbb{e} + f) - \ell_0(\|f\|\mathbb{e}).$$

Note that equation (D.2.3) implies

$$0 = \lim_{n\to\infty} \|\nu^{-n}\mathcal{L}^n(f + g + [\|f\| + \|g\|]\mathbb{e}) - \nu^{-n}\mathcal{L}^n(f + \|f\|\mathbb{e}) - \nu^{-n}\mathcal{L}^n(g + \|g\|\mathbb{e})\|$$
$$= \|\{\ell_0(f + g + [\|f\| + \|g\|]\mathbb{e}) - \ell_0(f + \|f\|\mathbb{e}) - \ell_0(f + \|f\|\mathbb{e})\}h\|.$$

Since $\ell_0(f+g+[\|f\|+\|g\|]\mathbb{e})-\ell_0([\|f\|+\|g\|]\mathbb{e}) = \ell_0(f+g+\|f+g\|\mathbb{e})-\ell_0(\|f+g\|\mathbb{e})$ the above implies that ℓ is linear. Hence, we have that $\ell \in \mathbb{V}^*$ and the Theorem. By equation (D.2.3) it follows that ν is a simple eigenvalue, that it equals the spectral radius, and the spectral decomposition claimed by the Lemma follows from spectral theory, see Appendix A.3. □

D.3 A simple application: Perron–Frobenius

Consider a matrix $L : \mathbb{R}^n \to \mathbb{R}^n$ of all strictly positive elements: $L_{ij} \geqslant \gamma > 0$. The Perron–Frobenius theorem states that there exists a unique eigenvector v^+ such that $v_i^+ > 0$, and in addition the corresponding eigenvalue ν is simple, maximal and positive. There quite a few proofs of this theorem. A possible one is based on Birkhoff's theorem. Consider the cone $\mathcal{C}^+ = \{v \in \mathbb{R}^2 \mid v_i \geqslant 0\}$. Then obviously $L\mathcal{C}^+ \subset \mathcal{C}^+$.

Problem D.9. *Show that*

$$\Theta(v, w) = \ln \sup_{ij} \frac{v_i w_j}{v_j w_i}, \tag{D.3.1}$$

where Θ is defined as in (D.1.2).

Then, setting $M = \max_{ij} L_{ij}$, it follows that

$$\Theta(Lv, Lw) \leqslant 2 \ln \frac{M}{\gamma} := \Delta < \infty. \tag{D.3.2}$$

We have then a finite diameter in the Hilbert metric and we can apply the theory previously described.

Theorem D.10 (Perron-Frobenius)**.** *The matrix L has a simple maximal eigenvalue $\nu \in \left[\min_j \sum_i A_{ij}, \max_j \sum_i A_{ij}\right]$, which equals the spectral radius of L, and the associated eigenvector has positive entries. In addition, the other eigenvalues of L have size, at most, $\nu \cdot \frac{M-\gamma}{M+\gamma} < \nu$.*

Proof. Remark that by Equation (D.3.2), $\chi = \tanh(\frac{\Delta}{4}) = \frac{M-\gamma}{M+\gamma}$. Thus the theorem follows directly from Theorem D.8, apart from the estimate of ν. To see that, let v^+ be the corresponding eigenvalue and normalize it so that $\sum_i v_i^+ = 1$. Then

$$\nu = \nu \sum_i v_i^+ = \sum_{i,j} A_{i,j} v_j^+ \geqslant \left[\min_j \sum_i A_{ij} \right] \sum_j v_j^+ = \left[\min_j \sum_i A_{ij} \right].$$

The upper bound follows similarly. □

Remark D.11. *Note that an explicit estimate of the size of the gap (which is larger than $\frac{2\gamma\nu}{M+\gamma}$) is not usually part of the Perron-Frobenius theorem. In this respect, it may be possible to obtain better bounds on the gap by choosing more sophisticated cones, especially in the presence of more information on the structure of the matrix L.*

E Solutions to the problems

Here we provide hints to solving the problems found in the text. We provide some details only for the non trivial ones.

E.1 Problems in Chapter 1

(1.6) Differentiate further Equation (1.2.1) and argue exactly as in the proof of Equation (1.2.2).

(1.7) For each $\alpha > \lambda_\star^{-1}$ we prove that the essential spectral radius, when acting on $\mathcal{C}^p$, is smaller than α^p, the result follows by the arbitrariness of α. Let us start with $\mathcal{C}^0$: there exist a constant $C_\star > 0$ such that

$$\|\mathcal{L}^n h\|_{\mathcal{C}^0} \leqslant \|h\|_{\mathcal{C}^0} \|\mathcal{L}^n 1\|_{\mathcal{C}^0} \leqslant \|h\|_{\mathcal{C}^0} \|\mathcal{L}^n 1\|_{W^{1,1}} \leqslant C_\star \|h\|_{\mathcal{C}^0},$$

where we have used Equation (1.2.2). Then, by Equation (1.2.1) applied to f^m,

$$\|\mathcal{L}^m h\|_{\mathcal{C}^1} \leqslant \lambda_\star^{-m} \|\mathcal{L}^m h\|_{\mathcal{C}^1} + C_m \|h\|_{\mathcal{C}^0} \leqslant C_\star \lambda_\star^{-m} \|h\|_{\mathcal{C}^1}. + C_m \|h\|_{\mathcal{C}^0}$$

Next, choose $m \in \mathbb{N}$ such that $C_\star \lambda_\star^{-m} \leqslant \alpha^m$. We can then iterate (writing $n = km + s$, $s < m$) and obtain

$$\|\mathcal{L}^n h\|_{\mathcal{C}^1} \leqslant C_\# \alpha^{-n} \|h\|_{\mathcal{C}^1} + C_\# \|h\|_{\mathcal{C}^0}.$$

The result then follows from Theorem 1.1 since the unit ball in $\mathcal{C}^1$ is compact in $\mathcal{C}^0$ (by Ascoli–Arzelà). The result with $p > 1$ is more of the same using a derivative of Equation (1.2.1).

(1.8) Note that, for each $h \in L^1$, $\int h = \int \mathcal{L}^n h = \lim_{n\to\infty} \int \mathcal{L}^n h = \int h_* \int h$. Thus $\int h_* = 1$. Also,

$$\mathcal{L} h_* = \lim_{h\to\infty} \mathcal{L}^{n+1} 1 = \lim_{h\to\infty} \mathcal{L}^n 1 = h_*.$$

It follows that, for all $\varphi \in \mathcal{C}^0$,

$$\int \varphi \circ f h_* = \int \varphi \mathcal{L} h_* = \int \varphi h_*,$$

thus $d\mu := h_* dx$ is an invariant measure. By Birkhoff's ergodic theorem the limit of $\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x)$ exists for μ almost all x. Thus, since $h_* > 0$, it exists Lebesgue almost surely. Let φ_+ be the limit. Then φ_+ is an invariant function, hence for each interval $I \subset \mathbb{R}$ the set $A_I = \{x \in \mathbb{T} : : \varphi_+(x) \in I\}$ is Lebesgue almost surely invariant which implies $\mathbb{1}_{A_I} \circ f = \mathbb{1}_{A_I}$ Lebesgue almost surely. Hence

$$\begin{aligned} \int \varphi \circ f \mathbb{1}_{A_I} h_* &= \int \varphi \mathcal{L}(\mathbb{1}_{A_I} h_*) = \int \varphi \mathcal{L}(\mathbb{1}_{A_I} \circ f h_*) \\ &= \int \varphi \mathbb{1}_{A_I} \mathcal{L}(h_*) = \int \varphi \mathbb{1}_{A_I} h_*. \end{aligned}$$

This implies that also $\mathbb{1}_{A_I} h_* dx$ is an invariant measure, but then

$$\mathbb{1}_{A_I} h_* = \mathcal{L}^n \mathbb{1}_{A_I} h_* = \lim_{n\to\infty} \mathcal{L}^n \mathbb{1}_{A_I} h_* = h_* \int \mathbb{1}_{A_I} h_* = h_* \int_{A_I} h_*$$

which means that $\mu(A_I) \in \{0, 1\}$. Accordingly, φ_+ is almost surely constant which implies $\varphi_+ = \int \varphi h_*$ which is the wanted claim.

(1.4) Let $j \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $\operatorname{supp} j \subset [-1,1]$ and $\int_{\mathbb{R}} j = 1$. Next, define $j_\varepsilon(x) = \varepsilon^{-1} j(\varepsilon^{-1}x)$ (this is called a *mollifier*). Note that if $h \in \mathcal{C}^\alpha(\mathbb{T}, \mathbb{C})$, then

$$\begin{aligned} |h(x) - \int_{\mathbb{T}} j_\varepsilon(x-y)h(y)dy| &\leqslant \left| \int_{\mathbb{T}} j_\varepsilon(x-y)(h(x)-h(y))dy \right| \\ &\leqslant C_\# \varepsilon^\alpha \int_{\mathbb{T}} j_\varepsilon(x-y)dy = C_\# \varepsilon^\alpha. \end{aligned}$$

While

$$\begin{aligned} &\int_{\mathbb{T}} \left| \int_{\mathbb{T}} j_\varepsilon(x-y)h(y)dy \right| dx \leqslant \|h\|_{\mathcal{C}^\alpha} \\ &\int_{\mathbb{T}} \left| \frac{d}{dx} \int_{\mathbb{T}} j_\varepsilon(x-y)h(y)dy \right| dx = \varepsilon^{-2} \int_{\mathbb{T}} \int_{\mathbb{T}} \left| j'(\varepsilon^{-1}(x-y))h(y)dy \right| dx \\ &\qquad \leqslant C_\# \varepsilon^{-1} \|h\|_{\mathcal{C}^\alpha}. \end{aligned}$$

It follows that, setting $h_\varepsilon(x) = \int_{\mathbb{T}} j_\varepsilon(x-y)h(y)dy$, $\|h_\varepsilon\|_{W^{1,1}} \leqslant C_\# \varepsilon^{-1}$. Hence,

$$\begin{aligned} \left| \int_{\mathbb{T}} \varphi \circ f^n h - \int_{\mathbb{T}} \varphi h_* \int_{\mathbb{T}} h \right| &\leqslant \left| \int_{\mathbb{T}} \varphi \circ f^n h_\varepsilon - \int_{\mathbb{T}} \varphi h_* \int_{\mathbb{T}} h_\varepsilon \right| \\ &\qquad + C_\# \varepsilon^\alpha \|\varphi\|_{\mathcal{C}^0} \|h\|_{\mathcal{C}^\alpha} \\ &\leqslant \|\varphi\|_{\mathcal{C}^0} \|h\|_{\mathcal{C}^\alpha} \left(\varepsilon^\alpha + C_\# \varepsilon^{-1} e^{-\nu n} \right). \end{aligned}$$

We can then conclude by choosing $\varepsilon = e^{-\frac{\nu}{1+\alpha} n}$ which provides the result with $\nu_\alpha = \frac{\alpha \nu}{1+\alpha}$.

(1.12) (a) Consider $\alpha(g) = \int g(x,y)\mu(dx)\nu(dy)$. Obviously $\alpha \in \mathcal{G}(\mu,\nu)$

(b) First of all $\boldsymbol{d}(\mu,\mu) = 0$ since we can consider the coupling

$$\alpha(g) = \int g(x,x)\mu(dx).$$

Next, note that $\mathcal{G}(\mu,\nu)$ is weakly closed and is a subset of the probability measures on X^2, which is weakly compact. Hence, $\mathcal{G}(\mu,\nu)$ is weakly compact, thus the inf is attained. Accordingly, if $\boldsymbol{d}(\nu,\mu) = 0$,

there exists $\alpha \in \mathcal{G}(\nu, \mu)$ such that $\alpha(d) = 0$. Thus α is supported on the set $D = \{(x, y) \in X^2 \mid x = y\}$. Thus, for each $\varphi \in \mathcal{C}^0(X)$, we have, setting $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$,

$$\mu(\varphi) = \alpha(\varphi \circ \pi_1) = \alpha(\mathbb{1}_D \varphi \circ \pi_1) = \alpha(\mathbb{1}_D \varphi \circ \pi_2) = \alpha(\varphi \circ \pi_2) = \nu(\varphi).$$

The fact that $\boldsymbol{d}(\nu, \mu) = \boldsymbol{d}(\mu, \nu)$ is obvious from the definition. It remains to prove the triangle inequality. It is possible to obtain a fast argument by using the disintegration of the couplings, but here is an elementary proof. Let us start with some preparation. Since X is compact, for each ε we can construct a finite partition $\{p_i\}_{i=1}^{M}$ of X such that each p_i has diameter less than ε (do it). Given two probability measures μ, ν and $\alpha \in \mathcal{G}(\mu, \nu)$ note that if $\mu(p_i) = 0$, then

$$\sum_j \alpha(p_i \times p_j) = \alpha(p_i \times X) = \mu(p_i) = 0.$$

Hence $\alpha(p_i \times p_j) = 0$ for all j. Thus we can define

$$\alpha_\varepsilon(g) = \sum_{i,j} \int_{X^2} g(x, y) \mathbb{1}_{p_i}(x) \mathbb{1}_{p_j}(y) \frac{\alpha(p_i \times p_j)}{\mu(p_i)\nu(p_j)} \mu(dx)\nu(dy),$$

where the sum runs only over the indexes for which the denominator is strictly positive. It is easy to check that $\alpha_\varepsilon \in \mathcal{G}(\mu, \nu)$ and that the weak limit of α_ε is α. Finally, let ν, μ, ζ be three probability measures and let $\alpha \in \mathcal{G}(\mu, \zeta), \beta \in \mathcal{G}(\zeta, \nu)$ such that $\boldsymbol{d}(\mu, \zeta) = \alpha(d)$ and $\boldsymbol{d}(\nu, \zeta) = \beta(d)$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(d) - \alpha_\delta(d)| \leqslant \varepsilon$ and, likewise, $|\beta(d) - \beta_\delta(d)| \leqslant \varepsilon$. We can then define the following measure on X^3

$$\gamma_\delta(g) = \sum_{i,j,k} \int_{X^3} g(x, z, y) \mathbb{1}_{p_i}(x) \mathbb{1}_{p_j}(z) \mathbb{1}_{p_k}(y) \times \frac{\alpha(p_i \times p_j)\beta(p_j \times p_k)}{\mu(p_i)\zeta(p_j)^2\nu(p_k)} \mu(dx)\nu(dy)\zeta(dz),$$

where again the sum is restricted to the indexes for which the denominator is strictly positive. The reader can check that the marginal on

(x, z) is α_δ, and the marginal on (z, y) is β_δ. It follows that the marginal on (x, y) belongs to $\mathcal{G}(\mu, \nu)$. Thus

$$\begin{aligned}\boldsymbol{d}(\mu, \nu) &\leqslant \int_{X^3} d(x, y)\gamma_\delta(dx, dz, dy)\\ &\leqslant \int_{X^2} d(x, z)\alpha_\delta(dx, dz) + \int_{X^2} d(z, y)\beta_\delta(dz, dy)\\ &\leqslant \alpha(d) + \beta(d) + 2\varepsilon = \boldsymbol{d}(\mu, \zeta) + \boldsymbol{d}(\zeta, \nu) + 2\varepsilon.\end{aligned}$$

The result follows by the arbitrariness of ε.

(c) It suffices to prove that μ_n converges weakly to μ if and only if

$$\lim_{n\to\infty} \boldsymbol{d}(\mu_n, \mu) = 0.$$

If it converges in the metric, then, letting $\alpha_n \in \mathcal{G}(\mu_n, \mu)$, for each Lipschitz function φ we have

$$|\mu_n(\varphi) - \mu(\varphi)| \leqslant \int_{X^2} |\varphi(x) - \varphi(y)|\alpha_n(dx, dy) \leqslant L_\varphi \alpha_n(d)$$

where L_φ is the Lipschitz constant. Taking the inf on α_n we have

$$|\mu_n(\varphi) - \mu(\varphi)| \leqslant L_\varphi \boldsymbol{d}(\mu_n, \mu).$$

We have thus that $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$ for each Lipschitz function. The claim follows since the Lipschitz functions are dense by the Stone–Weierstrass Theorem.

If it converges weakly, then, to prove convergence in the metric, we need slightly more sophisticated partitions $\mathcal{P}_\delta$: partitions with the property that $\mu(\partial p_i) = 0$. Note that this implies $\lim_{n\to\infty} \mu_n(p_i) = \mu(p_i)$, Varadhan (2001). Let us construct explicitly such partitions. For each $x \in X$ consider the balls $B_r(x) = \{z \in X : d(x, z) < r\}$. Given $\delta > 0$, let $S_{1,1} = \{x \in \overline{B}_\delta \setminus B_{\frac{3}{4}\delta}\}$ and $S_{1,2} = \{x \in \overline{B}_{\frac{1}{2}\delta} \setminus B_{\frac{1}{4}\delta}\}$. These two spherical shells are disjoint. Let $\sigma(1) \in \{1, 2\}$ be such that $\mu(S_{1,\sigma(1)}) = \min\{\mu(S_{1,1}), \mu(S_{1,2})\}$. Divide again the spherical shell $S_{1,\sigma(1)}$ into three, throw away the middle part and let $S_{2,\sigma(2)}$ be the one with the smaller measure. Continue in this way to obtain a sequence $S_{n,\sigma(n)}$. Note that $\mu(B_\delta(x)) \geqslant 2^n \mu(S_{n,\sigma(n)})$

and $S_{n+1,\sigma(n+1)} \subset S_{n,\sigma(n)}$, thus there exists $r(x) \in [\delta/3, \delta]$ such that $\partial B_{r(x)}(x) = \cap_{n=1}^{\infty} S_{n,\sigma(n)}$ and $\mu(\partial B_{r(x)}(x)) = 0$. Since X is compact we can extract a finite sub cover, $\{B_i\}_{i=1}^{N}$, from the cover $\{B_{r(x)}(x)\}_{x\in X}$. If we consider all the (open) sets $B_i \cap B_j$ they form a mod-0 partition of X. To get a partition $\{p_i\}$ just attribute the boundary points in any (measurable) way you like.[1] Also, for each partition element p_i choose a reference point $x_i \in p_i$.

Having constructed the wanted partition we can discretize any measure ν by associating to it the measure

$$\nu_\delta(\varphi) = \sum_i \varphi(x_i)\nu(p_i).$$

Define also

$$\alpha(\varphi) = \sum_i \int_{p_i} \varphi(x, x_i)\nu(dx)$$

and check that $\alpha \in \mathcal{G}(\nu, \nu_\delta)$, hence

$$\boldsymbol{d}(\nu, \nu_\delta) \leqslant \alpha(d) \leqslant 2\delta.$$

For each $n \in \mathbb{N}$ and $\delta > 0$ we can then write

$$\boldsymbol{d}(\mu, \mu_n) \leqslant \boldsymbol{d}(\mu_\delta, \mu_{n,\delta}) + 4\delta.$$

Next, let $z_{n,i} = \min\{\mu(p_i), \mu_n(p_i)\}$, $Z_n^{-1} = \sum_i z_{n,i}$, and define

$$\begin{aligned}\beta_n(\varphi) =& Z_n \sum_i \varphi(x_i, x_i) z_i \\ &+ \frac{(1 - Z_n)^2}{Z_n^2} \sum_{i,j} \varphi(x_i, x_j)[\mu(p_i) - z_{n,i}] \cdot [\mu_n(p_i) - z_{n,i}]\end{aligned}$$

and verify that $\beta \in \mathcal{G}(\mu_\delta, \mu_{n,\delta})$. In addition, for each $\delta > 0$, we have $\lim_{n\to\infty} z_{n,i} = \mu_n(p_i)$. Hence, $\lim_{n\to\infty} Z_n = 1$. Collecting the above facts, and calling K the diameter of X, yields

$$\lim_{n\to\infty} \boldsymbol{d}(\mu, \mu_n) \leqslant \lim_{n\to\infty} \beta_n(d) + 4\delta \leqslant K \lim_{n\to\infty} \frac{(1 - Z_n)^2}{Z_n^2} + 4\delta = 4\delta.$$

[1] E.g., if C_i are the elements of the partition, you can set $p_1 = \overline{C}_1$, $p_2 = \overline{C}_2 \setminus (\partial C_1)$ and so on.

The result follows by the arbitrariness of δ.

Comment: In the field of *optimal transport* one usually would prove the above facts via the duality relation

$$\boldsymbol{d}(\mu,\nu) = \sup_{\phi\in L_1}\{\mu(\phi)-\nu(\phi)\}$$

where L_1 is the set of Lipschitz functions with Lipschitz constant equal to 1. We refrain from this point of view because, in spite of its efficiency, it requires the development of a little bit of technology outside the scope of these notes. The interested reader can see Viana (1997, Chapter 1) for details.

(d) The first metric gives rise to the usual topology, hence convergence in $\boldsymbol{d}$ is equivalent to the usual weak convergence of measures. The metric d_0 instead give rise to the discrete topology, hence each function in L^∞ is continuous. Hence the convergence in $\boldsymbol{d}$ is equivalent to the usual strong convergence of measures.

(1.13) Just compute.

(1.14) Argue as in Lemma 1.11.

(1.18) Note that

$$\mathcal{L}h(x) = \sum_{z\in f^{-1}(x)} \frac{h(z)}{f'(z)}\,; \quad \mathcal{L}h(y) = \sum_{w\in f^{-1}(y)} \frac{h(w)}{f'(w)}$$

where $|z-w| \leqslant \lambda^{-1}|x-y|$. Hence

$$\mathcal{L}h(x) = \sum_{z\in f^{-1}(x)} \frac{h(w)e^{\lambda^{-1}a|x-y|+D|x-y|}}{f'(w)} \leqslant e^{(\lambda^{-1}a+D)|x-y|}\mathcal{L}h(y).$$

(1.19) Note that if $\varphi \in \mathcal{C}_a$, then $e^{-a}\int_{\mathbb{T}}\varphi \leqslant \varphi(x) \leqslant e^a\int_{\mathbb{T}}\varphi$. Using the Equation (D.1.2) it follows that, for all $\varphi_1,\varphi_2 \in \mathcal{C}_{\sigma a}$ we have

$$\Theta(\varphi_1,\varphi_2) \leqslant \ln\frac{(1+\sigma)^2}{1-\sigma)^2}e^{4a}.$$

(1.34) Set $\varphi(t) = tg(y) + (1-t)g(x) - g(ty + (1-t)x)$. Note that $\varphi(0) = \varphi(1) = 0$. Also $\varphi''(t) = -\langle y - x, D^2 g(ty + (1-t)x)(y-x)\rangle \leqslant 0$. This implies $\varphi(t) \geqslant 0$ for all $t \in [0,1]$. Indeed, the function must have a minimum, but the minimum cannot be in $(0,1)$ otherwise we would have that φ' is increasing, which is not the case. Hence the minimum must be at the extrema, hence the claim. Clearly strict convexity is equivalent to $D^2 g$ being strictly positive.

(1.35) First of all consider $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ convex and bounded. Let $a \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant b$ then

$$\frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \leqslant \frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1} \tag{E.1.1}$$

To see this set $\alpha = \frac{t_2 - t_1}{t_3 - t_1}$. By hypothesis $\alpha \in [0,1]$ and $t_2 = (1-\alpha)t_1 + \alpha t_3$. Thus, by convexity,

$$\varphi(t_2) \leqslant (1-\alpha)\varphi(t_1) + \alpha\varphi(t_3),$$

which implies

$$\varphi(t_2) - \varphi(t_1) \leqslant \alpha(\varphi(t_3) - \varphi(t_1)) = \frac{t_2 - t_1}{t_3 - t_1}(\varphi(t_3) - \varphi(t_1))$$

from which Equation (E.1.1) follows. Similarly one can prove

$$\frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1} \leqslant \frac{\varphi(t_3) - \varphi(t_2)}{t_3 - t_2}. \tag{E.1.2}$$

Next, suppose that $a \leqslant s \leqslant s + h \leqslant t \leqslant t + h' \leqslant b$. Then, using first Equation (E.1.1) and then Equation (E.1.2), we have

$$\frac{\varphi(s+h) - \varphi(s)}{h} \leqslant \frac{\varphi(t+h') - \varphi(s)}{t + h' - s} \leqslant \frac{\varphi(t+h') - \varphi(t)}{h'}. \tag{E.1.3}$$

Accordingly, for $t \in (a,b)$ and $\min\{t-a, b-t\} > h' > 0$. We can then use Equation (E.1.3) and write, for each $h' > h > 0$,

$$\varphi(t+h) - \varphi(t) \leqslant h\frac{\varphi(t+h+h') - \varphi(t+h)}{h'} \leqslant hC_{\#}$$

by the boundedness of φ. Analogously,

$$\varphi(t+h)-\varphi(t) \geqslant h\frac{\varphi(t)-\varphi(t-h')}{h'} \geqslant -hC_{\#}.$$

The above, by the arbitrariness of h, implies the continuity of φ at t. Hence φ is continuous on $[a,b]$. The last step is to extend the results to higher dimensions: for each $x \in D$ and $v \in \mathbb{R}^d$, $\|v\| = 1$, define $\varphi(t) = g(x+tv)$ and note that the above discussion implies that φ is continuous. The statement then follows by the arbitrariness of v and x.

(1.36) Use the fact that the sup of a sum is smaller than the sum of the sups.

(1.37) Since, for all $x, y \in \mathbb{R}^d$, $g^*(y) \geqslant \langle y, x\rangle - g(x)$ we have

$$g^{**}(x) = \sup_y \langle x, y\rangle - g^*(y) \leqslant g(x).$$

(1.38) If g is strictly convex, then the sup is realized at the unique point at which $x = D_y g = h(y)$. Moreover, by Problem 1.37, $D_y^2 g$ is a strictly positive matrix, hence, by the implicit function theorem, $h(y)$ is locally invertible. On the other hand if $h(y) = h(x)$, then set $\varphi(t) = \langle y-x, h(ty+(1-t)x)\rangle$, so $\varphi(0) = \varphi(1)$. It follows that $h'(t) = \langle y-x, D^2 g(y-x)\rangle > 0$ which yields a contradiction.

(1.39) Just compute using Problem 1.38.

(1.40) It follows directly from the definition of g^*.

(1.45) The behaviour for small a can be computed similarly to Lemma 1.28. Note that, recalling Equation (1.6.8). $\frac{d}{dt}(\lambda a - \ln \alpha_\lambda) = a - \mu_\lambda(\hat{\varphi})$. Thus, if $\sup_\lambda \mu_\lambda(\hat{\varphi}) < a$ we have $\mathbb{J}(a) = +\infty$.

(1.51) The fact that $|\mu| = |h|_{L^1}$ follows from $|h|_{L^1} = \sup_{|\varphi|_{L^\infty}\leqslant 1} \int \varphi h$, since $|\int \varphi h| \leqslant |\varphi|_{L^\infty} \int |h|$ and if $\varphi = \operatorname{sign}(h)$, then $\int \varphi h = \int |h|$. The second equality follows from the definition of the BV norm, e.g. see L. C. Evans and Gariepy (2015).

E.2 Problems in Chapter 2

(2.1) Note that

$$\int \varphi \mathcal{L}^n h = \int \varphi \circ f^n \prod_{k=0}^{n-1} \psi \circ f^k h =: \int \theta_n h.$$

It follows that

$$\theta_n' = (f^n)'\varphi' \circ f^n \prod_{k=0}^{n-1} \psi \circ f^k + \sum_{j=0}^{n-1} (f^j)'\psi' \circ f^j \prod_{k\neq j} \psi \circ f^k$$

Taking a further derivative and since in the support of ψ we have $|f'| \leqslant \lambda^{-1}$, we have the first inequality. The second inequality is proven arguing as in Equation (2.1.1).

(2.2) Since the unit ball in $\mathcal{C}^2$ is compact in $\mathcal{C}^1$, then for each $\varepsilon > 0$ there exists a finite set $\{\varphi_i^\varepsilon\} \subset \mathcal{C}^1$ such that, for each $\varphi \in \mathcal{C}^2$, $\|\varphi\|_{\mathcal{C}^2} \leqslant 1$, we have $\inf_i \|\varphi - \varphi_i^\varepsilon\|_{\mathcal{C}^1} \leqslant \varepsilon$. Accordingly, if we have a sequence $\|h_n\|_{(\mathcal{C}^1)^*} \leqslant 1$, then we can consider the sequences $\int \varphi_i^\varepsilon h_n$. Since they are bounded, they admit a convergent subsequence. Hence, there exists a sequence n_j^ε such that $\int \varphi_i^\varepsilon h_{n_j^\varepsilon}$ converges for all i. We can then procede by the usual diagonalization trick: choose a sequence ε_l which converges to zero; from $h_{n_j^{\varepsilon_1}}$ extract a subsequence $n_j^{\varepsilon_2}$ that converges on all the $\{\varphi_i^{\varepsilon_2}\}$ and so on. One can then consider the subsequence $h_{m_s} = h_{n_s^{\varepsilon_s}}$ that converges on all the functions $\{\varphi_i^{\varepsilon_j}\}$. Hence, for each $\epsilon > 0$ we have that, for each $\varphi \in \mathcal{C}^2$, $\|\varphi\|_{\mathcal{C}^2} \leqslant 1$, and $s > s'$ large enough,

$$\left|\int \varphi h_{m_s} - \int \varphi h_{m_s'}\right| \leqslant \left|\int \varphi_i^{\varepsilon_s} h_{m_s} - \int \varphi_i^{\varepsilon_s} h_{m_s'}\right| + 2\varepsilon_s \leqslant \epsilon.$$

That is $\{h_{m_s}\}$ is a Cauchy sequence in $(\mathcal{C}^2)^*$, hence the claim.

(2.4) Clearly, if $d_p(\mu,\nu) = d_p(\nu,\mu)$. Also if $d_p(\mu,\nu) = 0$ then, for any cou-

pling G and $\varphi \in \mathrm{Lip}(X)$, the set of Lipschitz functions,

$$\begin{aligned}\left|\int_X \varphi(x)\mu(dx) - \int_X \varphi(y)\nu(dy)\right| &\leqslant \int_{X^2} |\varphi(x)-\varphi(y)|G(dx,dy)\\ &\leqslant C_\# \int_{X^2} d(x,y)G(dx,dy)\\ &\leqslant C_\# \left[\int_{X^2} d(x,y)^p G(dx,dy)\right]^{\frac{1}{p}} = 0.\end{aligned}$$

Then $\mu = \nu$ follows since $\mathrm{Lip}(X)$ is dense in $\mathcal{C}^0(X)$. Finally, to prove the triangle inequality one can proceed as in Problem 1.12.

(2.6) Since if $\varphi \in \mathcal{C}^1$ then $\varphi \in \mathrm{Lip}$, then Theorem 2.5 implies one inequality. On the other hand, the same theorem implies that, for each $\varepsilon > 0$, there exists $\varphi \in \mathcal{C}^0$, $\mathrm{Lip}(\varphi) \leqslant 1$, such that $\int \varphi(x)(\mu-\nu)(dx) \geqslant d_1(\mu,\nu) - \varepsilon$. Next, note that there exists $K > 0$ such that, for all $\varphi \in \mathcal{C}^0$ such that $\mathrm{Lip}(\varphi) \leqslant 1$, there exists $\varphi_\varepsilon \in \mathcal{C}^1$ such that $\|\varphi_\varepsilon - \varphi\|_{\mathcal{C}^0} \leqslant \varepsilon$ and $\|\varphi\|_{\mathcal{C}^1} \leqslant K$ (for example, define φ_ε by a convolution with a mollifier, using a partition of unity and the charts of the manifold). Thus

$$\begin{aligned}\|\mu-\nu\|_{(\mathcal{C}^1)^*} &\geqslant K^{-1}\int \varphi_\varepsilon(x)(\mu-\nu)(dx)\\ &\geqslant K^{-1}\int \varphi_\varepsilon(x)(\mu-\nu)(dx) - \varepsilon K^{-1} \geqslant K^{-1}d_1(\mu,\nu) - 2\varepsilon K^1.\end{aligned}$$

The result follows by the arbitrariness of ε.

(2.8) More generally consider the norm

$$\|h\|_r^* := \sup_{\|\varphi\|_{\mathcal{C}^r(M,\mathbb{C})}\leqslant 1} \int_M h\varphi$$

(2.11) Simply note that, for all $\varphi \in \mathcal{C}^0$, since the periodic orbit belongs to the attractor and recalling the definition of ψ,

$$\begin{aligned}\mathcal{L}\mu(\varphi) = \mu(\psi\cdot\varphi\circ f) &= \sum_{k=0}^{p-1} e^{2\pi i k/p}\psi(f^k(x))\varphi(f^{k+1}(x))\\ &= \sum_{k=0}^{p-1} e^{2\pi i k/p}\varphi(f^{k+1}(x)) = e^{-2\pi i/p}\mu(\varphi).\end{aligned}$$

E.3 Problems in Chapter 3

(3.6) Let $\{h_n\}$ be a sequence such that $\|h_n\|_\alpha \leqslant 1$. Then, for all $L \in \mathbb{N}$ we that

$$\left\| h_n - \sum_{|k|\leqslant L} (\hat{h}_n)_k e^{2\pi i k\cdot} \right\|_\beta = \sum_{|k|>L} \langle k\rangle^{\beta(k)} |(\hat{h}_n)_k|^2$$
$$\leqslant L^{-c} \sum_{|k|>L} \langle k\rangle^{\alpha(k)} |(\hat{h}_n)_k|^2 \leqslant L^{-c}.$$

On the other hand, for $|k| \leqslant L$, $|(\hat{h}_n)_k|$ is a uniformly bounded sequence and hence it admits a convergent subsequence. By the usual diagonalization trick we can then construct a convergent subsequence.

(3.7) Let $c > 0$ so that $\beta = \alpha - c$ satisfies the same properties as α. Then Equation (3.3.3) implies

$$\|\mathcal{L}h\|_{p\beta} \leqslant C\|h\|_{p\beta}$$

since $\|h\|_w \leqslant C_\#\|h\|_{p\beta}$. Using this last fact again yields

$$\|\mathcal{L}^n h\|_{p\alpha} \leqslant \nu^p \|h\|_{p\alpha} + C\|h\|_{p\beta}$$

which is a proper Lasota–Yorke inequality thanks to Problem 3.6. Then by Theorem 1.1 we know that the essential spectral radius is at most ν^p. However the spectral radius could be C. Yet, let τ be a maximal eigenvalue, and assume that its algebraic and geometric multiplicities are equal (i.e., $\mathcal{L}$ does not have Jordan block). The discussion of the case of a Jordan block is similar and is left to the imagination of the reader. Then, by the spectral decomposition, there exists a smooth function h_0 such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau^{-k} \mathcal{L}^k h_0 = h.$$

But then, if $|\tau| > 1$ we have, for each smooth φ,

$$\left| \int \varphi h \right| = \lim_{n\to\infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} \tau^{-k} \int \varphi \mathcal{L}^k h_0 \right| \leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\tau|^{-k} \int |\varphi \circ f^k| |h_0|$$
$$\leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\tau|^{-k} \|\varphi\|_{\mathcal{C}^0} |h_0|_{L^1} = 0$$

It follows that the spectral radius must be bounded by one. In fact, since $\mathcal{L}^* \,\mathrm{Leb} = \mathrm{Leb}$, it is easy to verify that the spectral radius is exactly one.

E.4 Problems in Chapter 5

(5.7) Let $\mathcal{B}_1$ denote the unit ball of $\mathcal{B}$ in the strong norm. For $\varepsilon > 0$, let $F(\varepsilon) = \{\ell_{j,k} : 1 \leqslant j \leqslant J_\varepsilon, 1 \leqslant k \leqslant K_\varepsilon\}$ be the finite collection of linear functionals given by (5.4.7). We can associate to each $f \in \mathcal{B}_1$, a finite $J_\varepsilon \times K_\varepsilon$ matrix $A(f)$, defined by $A(f)_{j,k} = (\ell_{j,k}(f))$. Note that $|\ell_{j,k}(f)| \leqslant \|f\|_{\mathcal{B}} \leqslant 1$. Thus the map $A : \mathcal{B}_1 \to \mathbb{R}^{J_\varepsilon + K_\varepsilon}$ has a compact image and we can choose a finite set $\{f_\ell\}_{\ell=1}^{L_\varepsilon} \subset \mathcal{B}_1$ such that $\{A(f_\ell)\}_{\ell=1}^{L_\varepsilon}$ forms an ε-cover of $A(\mathcal{B}_1)$. Then for each $f \in \mathcal{B}_1$, there exists $\ell \leqslant L_\varepsilon$ such that $\min_{j,k} |\ell_{j,k}(f) - \ell_{j,k}(f_\ell)| < \varepsilon$ and so by (5.4.7), $|f - f_\ell|_w \leqslant |\min_{j,k} \ell_{j,k}(f - f_\ell)| + C\varepsilon^\gamma \leqslant \varepsilon + C\varepsilon^\gamma$, so that $\{f_\ell\}_{\ell=1}^{L_\varepsilon}$ forms a finite $2C\varepsilon^\gamma$-cover of $\mathcal{B}_1$.

(5.12) Use the fact that as in (5.4.18), the group property implies that $\mathcal{L}_t(Xf) = X(\mathcal{L}_t f) = \frac{d}{dt}(\mathcal{L}_t f)$, and then integrate by parts.

(5.15) The case $n = 1$ is true by definition. Assuming the formula holds for n, we calculate

$$\begin{aligned} R(z)^{n+1} f &= \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t R(z) f \, dt \\ &= \int_0^\infty \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-z(t+s)} \mathcal{L}_{t+s} f \, ds dt \\ &= \int_0^\infty e^{-zu} \mathcal{L}_u f \int_0^u \frac{v^{n-1}}{(n-1)!} dv du = \int_0^\infty \frac{u^n}{n!} e^{-zu} \mathcal{L}_u f \, du \,, \end{aligned}$$

where we have made the substitution $u = t + s$, $v = t$.

(5.16) The first inequality is just the triangle inequality. For the second, if $n > 1$, integrate by parts to obtain,

$$\int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-at} \, dt = a^{-1} \int_0^\infty \frac{t^{n-2}}{(n-2)!} e^{-at} \, dt \,.$$

Then the required identity follows by induction and the fact that for $n = 1$, $\int_0^\infty e^{-at} \, dt = a^{-1}$.

(5.18) Remark that by (5.5.1),

$$\bar\rho R(z)(X-\rho) = \bar\rho R(z)\big(z-\rho-(z-X)\big) = R(z) - \bar\rho\,.$$

So if $(X-\rho)^k f = 0$, then also $(R(z)-\bar\rho)^k f = \bar\rho^k R(z)^k (X-\rho)^k f = 0$. Similarly,

$$\bar\rho^{-1}(z-X)\big(R(z)-\bar\rho\big) = \big(z-\rho-(z-X)\big) = X-\rho\,,$$

which implies the converse statement.

(5.21) Using (5.6.1), we estimate as in the proof of Corollary 5.14,

$$\begin{aligned} a^n \|R(z)^n f\|_{\mathcal B}^* &= a^n \|R(z)^n f\|_s + \frac{c_u a^n}{|z|}\|R(z)^n f\|_u + \frac{a^n}{|z|}\|R(z)^n f\|_0 \\ &\leqslant C\big[(1-a^{-1}\log\lambda)^{-n} + c_u|z|^{-1}\big]\|f\|_s \\ &\quad + Cc_u|z|^{-1}(1-a^{-1}\log\lambda)^{-n}\|f\|_u \\ &\quad + Cc_u|z|^{-1}\|f\|_0 + C(1+a+|z|)|z|^{-1}|f|_w\,. \end{aligned}$$

Since $a\in[1,2]$, we have $(1-a^{-1}\log\lambda)^{-1} \leqslant (1-\frac{\log\lambda}{2})^{-1}$, and $(1+a+|z|)|z|^{-1} \leqslant 3$. Fix $\sigma \in \big((1-\frac{\log\lambda}{2})^{-1}, 1\big)$ (independent of z), and choose $N>0$ such that $\sigma^N/2 > C(1-a^{-1}\log\lambda)^{-N}$. Choose $c_u>0$ such that $Cc_u < \sigma^N/2$. Then using $|z|\geqslant 1$ yields the required inequality for $n\geqslant N$,

$$\begin{aligned} a^n \|R(z)^n f\|_{\mathcal B}^* &\leqslant \sigma^n\|f\|_s + \frac{\sigma^n}{2}\frac{c_u}{|z|}\|f\|_u + \frac{\sigma^n}{2}\frac{1}{|z|}\|f\|_0 + 3C|f|_w \\ &\leqslant \sigma^n\|f\|_{\mathcal B}^* + 3C|f|_w\,. \end{aligned}$$

(5.25) We use the fact that by choice of c, there can be at most two components $W_{j,i}$ per $W_j\in\mathcal G_{\ell\tau}(W)$, and each component has length less than cr. Then, starting from the expression in (5.7.2), we estimate,

$$\begin{aligned} &\sum_{\ell\geqslant 0}\sum_{W_{j,i}\in D_\ell}\int_{-\tau}^{\tau} p_{n,\ell,z}(s)\int_{W_{j,i}} J_{W_j}\Phi_{\ell\tau}\,\psi\circ\Phi_{\ell\tau}\,\mathcal L_s f\,dm_{W_j}\,ds \\ &\quad\leqslant \sum_{\ell\geqslant 0}\int_{-\tau}^{\tau}|p_{n,\ell,z}(s)|\sum_{W_j\in\mathcal G_{\ell\tau}(W)} 2cr|J_{W_j}\Phi_{\ell\tau}|_{C^0(W_j)}|\psi|_\infty|f|_\infty\,ds \\ &\quad\leqslant 2\bar C cr|f|_\infty\int_0^\infty \frac{t^{n-1}}{(n-1)}e^{-at}\,dt \leqslant 2\bar C cr|f|_\infty a^{-n}\,, \end{aligned}$$

where for the second inequality, we have used Lemma 5.9(c) and for the fourth we have applied Problem 5.16.

(5.26) Recall that by Stirling's formula, $n! \geqslant \sqrt{2\pi n}\, n^n e^{-n}$ for all $n \geqslant 1$. Since $t = s + \ell\tau$ and $s \leqslant \tau$ on each interval in the sum, $\ell \leqslant \ell_0 - 1$ implies $t \leqslant \ell_0 \tau = \frac{n}{ae^2}$. Thus,

$$\begin{aligned}\sum_{\ell=0}^{\ell_0-1} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_W \psi \mathcal{L}_{\ell\tau+s} f \, dm_W ds &\leqslant |f|_\infty |\psi|_\infty \int_0^{\frac{n}{ae^2}} \frac{t^{n-1}}{(n-1)!} e^{-at}\, dt \\ &\leqslant |f|_\infty a^{1-n} e^{2(1-n)} \frac{n^{n-1}}{(n-1)!} \int_0^\infty e^{-at} dt \\ &\leqslant |f|_\infty a^{-n} e^{1-n} \frac{n^{n-1}}{(n-1)^{n-1}} \frac{1}{\sqrt{2\pi n}}\,,\end{aligned}$$

which implies the required estimate since $(\frac{n}{n-1})^{n-1} \leqslant e$.

(5.28) By definition, $J_{\ell,j,i}|W_{j,i}| = \int_{W_{j,i}} J_{W_j} \Phi_{\ell\tau}\, dm_{W_{j,i}}$. Then, for each $W_j \in \mathcal{G}_{\ell\tau}(W)$, $\cup_i W_{j,i} \subset W_j$ and the number of overlaps on each subcurve is bounded by $C > 0$ according to (5.7.7). Then using Lemma 5.9(c) we estimate,

$$\sum_i \sum_{j \in A_{\ell,i}} J_{\ell,j,i}|W_{j,i}| \leqslant C \sum_{W_j \in \mathcal{G}_{\ell\tau}(W)} \int_{W_j} J_{W_j} \Phi_{\ell\tau}\, dm_{W_j} \leqslant C\bar{C}|\delta_0|\,.$$

(5.29) We want to compare the sum with the integral of the function

$$g(t) = \frac{(t\tau)^{n-1}}{(n-1)!} e^{-at\tau}$$

The function g is increasing from 0 to $\frac{n-1}{a\tau}$ and decreasing afterwards. Since $\ell_0 = \frac{n}{ae^2\tau}$, this maximum falls within the domain of integration. Still, the sum for $\ell \geqslant \lceil \frac{n-1}{a\tau} \rceil$ is bounded by the integral on $[\lfloor \frac{n-1}{a\tau} \rfloor, \infty)$, and the sum for $\ell \in [\ell_0, \lfloor \frac{n-1}{a\tau} \rfloor]$ is bounded by the integral on $[\ell_0, \lceil \frac{n-1}{a\tau} \rceil]$. Thus,

$$\begin{aligned}\sum_{\ell \geqslant \ell_0} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} &\leqslant 2 \int_{\ell_0}^\infty \frac{(t\tau)^{n-1}}{(n-1)!} e^{-at\tau} dt \\ &\leqslant 2\tau^{-1} \int_0^\infty \frac{s^{n-1}}{(n-1)!} e^{-as} ds \leqslant 2\tau^{-1} a^{-n}\,,\end{aligned}$$

where for the second inequality we have made the substitution $s = t\tau$ and for the third we have applied Problem 5.16. Finally, (5.7.3) completes the required estimate.

(5.32) The bound on $|L_{j,k}|_\infty$ follows from the definition of K^* in (5.7.14) and K just after (5.7.11) since $|W_{j,i}| \geqslant r$ and p, $\phi_{r,j}$ are both bounded by 1. Also, $|x^0| \leqslant r$ on S_r so that $\ell\tau - x^0 \leqslant (\ell+1)\tau$ since $\tau = r^{1/3}$ by (5.7.3). Remark that $\ell \geqslant \ell_0 = \frac{n}{ae^2\tau}$ so that $(\frac{\ell+1}{\ell})^n \leqslant (1 + \frac{ae^2\tau}{n})^n \leqslant e^{2e^2}$ since $a \leqslant 2$ and $\tau < 1$.

The bound on $|\partial_{x^s} L_{j,k}|_\infty$ follows similarly using the fact that

$$\begin{aligned}\partial_{x^s} K^*_{\ell,n,i,j}(x^s, x^0) = {} & \partial_1 K_{\ell,n,i,j}((h^s_j)^{-1}(x^s), x^0 - \Delta_j(x_s)) \cdot ((h^s_j)^{-1})'(x^s) \\ & + \partial_2 K_{\ell,n,i,j}((h^s_j)^{-1}(x^s), x^0 - \Delta_j(x_s)) \cdot \Delta'_j(x^s)\,,\end{aligned}$$

together with Sub-lemma 5.31. The extra factor of r^{-1} comes from the fact that $|\nabla\phi_{r,j}|_\infty \sim r^{-1}$, which term appears in both $\partial_1 K$ and $\partial_2 K$.

(5.34) The required bound follows immediately from the first bound in Problem 5.32 by the triangle inequality, together with the fact that the integral over S_r cancels the factor r^{-2}.

(5.35) Use Problem 5.15 to compute

$$\begin{aligned}\partial^u(R(z)^n f) &= \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\, \partial^u(\mathcal{L}_t f)\, dt \\ &\leqslant C|\nabla f| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-(a+\log\Lambda)t}\, dt \\ &\leqslant C|\nabla f|(a+\log\Lambda)^{-n}\,,\end{aligned}$$

where in the second line we have used (5.1.1) and in the third line we have used Problem 5.16 with a replaced by $a + \log\Lambda$.

E.5 Problems in Chapter 6

(6.1) This is a straightforward calculation, using that by definition of ω, $d\omega = -\sin\theta\, d\theta \wedge dx + \cos\theta\, d\theta \wedge dy$.

(6.2) Remark that if $z, \tilde{z} \in \mathbb{H}_k$, then $\cos\varphi(\tilde{z}) \geqslant ck^{-2}$ and $|\varphi(z)-\varphi(\tilde{z})| \leqslant Ck^{-3}$ for uniform constants $c, C > 0$. Then using the differentiability of $\cos\varphi$,

$$\begin{aligned}\left|\frac{\cos\varphi(z)}{\cos\varphi(\tilde{z})} - 1\right| &= \frac{1}{\cos\varphi(\tilde{z})}\,|\cos\varphi(z) - \cos\varphi(\tilde{z})| \\ &\leqslant c^{-1}k^2|\varphi(z)-\varphi(\tilde{z})| \leqslant c^{-1}k^2(Ck^{-3})^{2/3}|\varphi(z)-\varphi(\tilde{z})|^{1/3} \\ &\leqslant c^{-1}C^{2/3}|\varphi(z)-\varphi(\tilde{z})|^{1/3}\,.\end{aligned}$$

Finally, $|\varphi(z) - \varphi(\tilde{z})| \leqslant d(z,\tilde{z})$ by the triangle inequality since $dz^2 = dr^2 + d\varphi^2$.

E.6 Problems in **Appendix A**

(A.1) The triangle inequality follows trivially from the triangle inequality of the norm of $\mathcal{B}$. To verify the completeness suppose that $\{B_n\}$ is a Cauchy sequence in $L(\mathcal{B},\mathcal{B})$. Then, for each $v \in \mathcal{B}$, $\{B_n v\}$ is a Cauchy sequence in $\mathcal{B}$, hence it has a limit, call it $B(v)$. We have so defined a function from $\mathcal{B}$ to itself. Show that such a function is linear and bounded, hence it defines an element of $L(\mathcal{B},\mathcal{B})$, which can easily be verified to be the limit of $\{B_n\}$.

(A.2) Use the norm $\|A\| = \sup_{v\in\mathbb{R}^n} \frac{\|Av\|}{\|v\|}$.

(A.3) Argue as in Problem A.2.

(A.4) The first part is trivial. For the second one can consider the vector space $\ell^2 = \{x \in \mathbb{R}^{\mathbb{N}} \;:\; \sum_{i=0}^{\infty} x_i^2 < \infty\}$. Equipped with the norm $\|x\| = \sqrt{\sum_{i=0}^{\infty} x_i^2}$ it is a Banach (actually a Hilbert) space. Consider now the vectors $e_i \in \ell^2$ defined by $(e_i) = \delta_{ik}$ and the operator $(Ax)_k = \frac{1}{k}x_k$. Then $R(A) = \{x \in \ell^2 \;:\; \sum_{k=0}^{\infty} k^2 x_k^2 < \infty\}$, which is dense in ℓ^2 but strictly smaller.

(A.7) Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

(A.8) Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

(A.9) Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ works also here.

(A.10) Note that, if $\zeta \in \mathbb{C}$ belongs to a small neighborhood of z,

$$(\zeta \mathbb{1} - A) = (z\mathbb{1} - A - (z-\zeta)\mathbb{1}) = (z\mathbb{1} - A)\left[\mathbb{1} - (z-\zeta)(z\mathbb{1} - A)^{-1}\right]$$

and that if $\|(z-\zeta)(z\mathbb{1}-A)^{-1}\| < 1$ then the inverse of $\mathbb{1}-(z-\zeta)(z\mathbb{1}-A)^{-1}$ is given by $\sum_{n=0}^{\infty}(z-\zeta)^n[(z\mathbb{1}-A)^{-1}]^n$ (the Von Neumann series – which really is just the geometric series).

(A.11) If $\sigma(A) = \emptyset$, then $f(\xi) = (\xi\mathbb{1} - A)^{-1}$ is an entire function, then the Von Neumann series shows that $(\xi\mathbb{1}-A)^{-1} = \xi^{-1}(\mathbb{1}-\xi^{-1}A)^{-1}$ goes to zero for large ξ, and then (A.2.3) shows that $(z\mathbb{1} - A)^{-1} = 0$ which is impossible.

(A.12) Verify that $(z\mathbb{1} - \Pi)^{-1} = z^{-1}\left[\mathbb{1} - (1-z)^{-1}\Pi\right]$.

(A.13) Note that $\|\mathcal{L}f\|_{\mathcal{C}^0} \leqslant \|f\|_{\mathcal{C}^0}$, thus $\sigma(\mathcal{L}) \subset \{z \in \mathbb{C} \ ; \ |z| \leqslant 1\}$. To prove equality the simplest idea is to look for eigenvalues by using Fourier series. Let $f = \sum_{k\in\mathbb{Z}} f_k e^{2\pi i k x}$ and consider the equation $\mathcal{L}f = zf$,

$$\sum_{k\in\mathbb{Z}} f_k \frac{1}{2}\left\{e^{\pi i k x} + e^{\pi i k x + \pi i k}\right\} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi i k x}.$$

Let us then restrict to the case in which $f_{2k+1} = 0$, then

$$\sum_{k\in\mathbb{Z}} f_{2k} e^{2\pi i k x} = z\sum_{k\in\mathbb{Z}} f_k e^{2\pi i k x}.$$

Thus we have a solution provided $f_{2k} = zf_k$, such conditions are satisfied by any sequence of the type

$$f_k = \begin{cases} z^j & \text{if } k = 2^j m,\ j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for $m \in \mathbb{N}$. It remains to verify that $\sum_{j=0}^{\infty} z^j e^{2\pi i 2^j m x}$ belongs to $\mathcal{C}^0$. This is the case if the series is uniformly convergent, which happens for $|z| < 1$. Thus all the points in $\{z \in \mathbb{C} \ : \ |z| < 1\}$ belong to the point spectrum and have infinite multiplicity. Since the spectrum is closed, the statement of the Problem follows.

(A.14) Let $p(z) = z^n$. Then

$$\frac{1}{2\pi i}\int_\gamma z^n(z\mathbb{1} - A)^{-1}dz = A^n + \frac{1}{2\pi i}\int_\gamma (z^n - A^n)(z\mathbb{1} - A)^{-1}dz$$
$$= A^n + \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_\gamma z^k A^{n-k}dz = A^n.$$

The statement for general polynomials follows trivially.

(A.15) Note that $r(A) = e^{\lim_{n\to\infty}\frac{1}{n}\ln\|A^n\|}$. On the other hand $\ln\|A^n\|$ is a sub-additive sequence. This implies the existence of the limit, by a standard argument (e.g. see Katok and Hasselblatt (1995, Proposition 9.6.4)).

(A.17) Approximate by polynomials.

(A.18) Check that the same argument used in the well known case $\mathcal{B} = \mathbb{C}$ also works here.

(A.19) Use the definition.

(A.20) For $z \notin f(\sigma(A))$ the function

$$K(z) := \frac{1}{2\pi i}\int_\gamma (z - f(\zeta))^{-1}(\zeta\mathbb{1} - A)^{-1}\,d\zeta,$$

with γ containing $\sigma(A)$ in its interior, is well defined. By direct computation, using Definition 6, one can verify that $(z\mathbb{1} - f(A))K(z) = \mathbb{1}$, thus $\sigma(f(A)) \subset f(\sigma(A))$. On the other hand, if f is not constant, then for each $z \in \mathbb{C}$, one may define the function $g(\xi) = \frac{f(z)-f(\xi)}{z-\xi}$, $\xi \neq z$, and $g(z) = f'(z)$. Hence, applying Definition 6 and Problem A.19 it follows that $f(z)\mathbb{1} - f(A) = (z - A)g(A)$ which shows that if $z \in \sigma(A)$, then $f(z) \in \sigma(f(A))$ (otherwise $(z - A)\left[g(A)(f(z)\mathbb{1} - f(A))^{-1}\right] = \mathbb{1}$). The commutator follows, again, from Problem A.19.

(A.22) Since one can define the logarithm on $\mathbb{C} \setminus \ell$, one can use Definition 6 to define $\ln A$. It suffices to prove that if $f : U \to \mathcal{C}$ and $g : V \to \mathcal{C}$, with $\sigma(A) \subset U$, $f(U) \subset V$, then $g(f(A)) = g \circ f(A)$. Whereby showing that

the Definition 6 is a reasonable one. Indeed, remembering Problems A.20 and A.21,

$$\begin{aligned} g(f(A)) &= \frac{1}{2\pi i}\int_\gamma g(z)(z\mathbb{1} - f(A))^{-1}dz \\ &= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_\gamma \frac{g(z)}{z - f(\xi)}(\xi\mathbb{1} - A)^{-1}dz\,d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma_1} g(f(\xi))(\xi\mathbb{1} - A)^{-1}d\xi = f\circ g(A). \end{aligned}$$

From this immediately follows $e^{\ln A} = A$.

(A.25) Use the decomposition $\mathcal{B} = R(P_B)\oplus N(P_B)$ and the fact that $(\mathbb{1} - P_B)$ is a projector.

(A.26) The first part follows from the previous decomposition. Indeed, for z large (by Neumann series)

$$(z\mathbb{1} - A)^{-1} = (z\mathbb{1} - P_BAP_B)^{-1} + (z\mathbb{1} - (\mathbb{1} - P_B)A(\mathbb{1} - P_B))^{-1}.$$

Since the above functions are analytic in the respective resolvent sets it follows that $\sigma(A)\subset\sigma(P_BAP_B)\cup\sigma((\mathbb{1} - P_B)A(\mathbb{1} - P_B))$. Next, for $z\notin B$, define the operator

$$K(z) := \frac{1}{2\pi i}\int_\gamma (z-\xi)^{-1}(\xi\mathbb{1} - A)^{-1}\,d\xi,$$

where γ contains B, but no other part of the spectrum, in its interior. By direct computation (using Fubini and the standard facts about residues and integration of analytic functions) verify that

$$(z\mathbb{1} - P_BAP_B)K(z) = P_B.$$

This implies that, for $z\neq 0$, $(z\mathbb{1}-P_BAP_B)(K(z)+z^{-1}(\mathbb{1}-P_B)) = \mathbb{1}$, that is $(z\mathbb{1}-P_BAP_B)^{-1} = K(z)+z^{-1}(\mathbb{1}-P_B)$. Hence $\sigma(P_BAP_B)\subset B\cup\{0\}$. Since P_B has a kernel, zero must be in the spectrum. On the other hand the same argument applied to $\mathbb{1}-P_B$ yields $\sigma((\mathbb{1}-P_B)A)\mathbb{1}-P_B))\subset C\cup\{0\}$. Hence $\sigma(P_BAP_B) = B\cup\{0\}$.

The second property follows from the fact that P_BAP_B, when restricted to the space $R(P_B)$ is described by a $D\times D$ matrix A_B and the equation

$\det(z\mathbb{1} - A_B) = 0$ is a polynomial of degree D in z and hence has exactly D solutions (counted with multiplicity).[2]

(A.29) Use the representation in Lemma A.24 and formula (A.4.1).

(A.30) Note that $Q(\mathbb{1} + P - Q) = QP$, so that $Q = (\mathbb{1} - (Q - P))^{-1}QP$, and hence $\dim(R(P)) \geqslant \dim(R(Q))$. Exchanging the role of P and Q, the result follows.

(A.32) Note that $\ell_\nu(h_\nu) = 1$ since P_ν is a projector, hence they are unique apart from a normalization factor. Then we can chose the normalization $\ell_\nu(h_0) = 1$ for all ν small enough. Thus $P_\nu f = h_\nu$, that is h_ν is analytic. Hence, for each $g \in \mathcal{B}$ and ν small, $\ell_\nu(g)\ell_0(h_\nu) = \ell_0(P_\nu g)$, which implies ℓ_ν analytic for ν small.

(A.34) Think hard.[3]

E.7 Problems in **Appendix B**

(B.13) The fact that $\sim$ is an equivalence relation is obvious. The fact that the equivalence classes form a vector space follows from the triangle inequality. Indeed, given two equivalence classes $\widetilde{f}, \widetilde{g}$ to define their sum let $f \in \widetilde{f}$ and $g \in \widetilde{g}$ and define the sum $\widetilde{f} + \widetilde{g}$ as the equivalence class of $f = g$. This is well defined since if $f' \in \widetilde{f}$ and $g' \in \widetilde{g}$ then

$$\|f + g - (f' + g')\|_w \leqslant \|f - f'\|_w + \|g - g'\|_w = 0$$

[2] This is the real reason why spectral theory is done over the complex rather than the real numbers. You should be well acquainted with the fact that a polynomial p of degree D has D roots over $\mathbb{C}$ but, in case you have forgotten, consider the following: first a polynomial of degree larger than zero must have at least one root, otherwise $\frac{1}{p(z)}$ would be an entire function and hence

$$\frac{1}{p(z)} = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{p(z + re^{i\theta})} = 0.$$

Let z_1 be a root. From the Taylor expansion in z_1, one obtains the decomposition $p(z) = (z - z_1)p_1(z)$ where p_1 has degree $D - 1$. The result follows by induction.

[3] A good idea is to start by considering concrete examples, for instance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

hence $f+g$ and $f'+g'$ define the same equivalence class. Next, define $\|\widetilde{f}\| = \inf_{f\in\widetilde{f}} \|f\|$. It is easy to prove that this is a norm on the vector space of equivalence classes. So we have the announced normed space X_w. Its completion (e.g. one can obtain it by considering the equivalence classes of Cauchy sequences, as in one of the standard constructions of the real numbers) is, by definition, a Banach space.

It remains to prove that $T : Y \to X$ induces a map $\widetilde{T} : Y \to X_w$ in a canonical way. Obviously we define $\widetilde{T}(y)$ as the equivalence class associated to $T(y)$. One can check directly that $\widetilde{T}$ is a bounded linear operator.

E.8 Problems in Appendix C

(C.3) One can try to argue as in Problem A.30. Yet, for the reader's amusement, here is a different proof. Let $\gamma := \|\Pi_2 - \Pi_1\| < 1$. Suppose that $\dim(\Pi_2(X)) > \dim(\Pi_1(X))$, the other case being similar. Then $\dim(\Pi_2\Pi_1(X)) \leqslant \dim(\Pi_1(X)) < \dim(\Pi_2(X))$. Next, by Lemma B.1, there exists $v \in \Pi_2(X)$, $\|v\| = 1$, such that

$$\operatorname{dist}(v, \Pi_2\Pi_1(X)) \geqslant \frac{1+\gamma}{2}.$$

It follows that

$$\|(\Pi_2-\Pi_1)^2 v\| = \|\Pi_2^2 v - \Pi_1\Pi_2 v - \Pi_2\Pi_1 v + \Pi_1^2 v\| = \|v - \Pi_2\Pi_1 v\| > \gamma,$$

contrary to the assumption. Thus $\dim(\Pi_2(X)) = \dim(\Pi_1(X))$.

(C.4) By Equation (1.2.1) we know that, for $n \geqslant 10$,

$$\begin{aligned}\|\mathcal{L}_n h\|_{L^1} &\leqslant \|h\|_{L^1}\\ \|\mathcal{L}_n h\|_{L^1} &\leqslant (2-\pi n^{-\frac{1}{2}})^{-1}\|h\|_{W^{1,1}} + 2\pi^2\|h\|_{L^1}.\end{aligned}$$

where $2-\pi n^{-\frac{1}{2}} > 1$. Moreover, calling $\mathcal{L}$ the transfer operator associated to the map $f(x) = 2x \mod 1$, we want to compute $\|\mathcal{L}_n h - \mathcal{L}h\|_{L^1}$.

Note that each x has two preimages under f and any of the maps f_n. If y is one preimage of x under f, then we call $\widetilde{y}$ the corresponding preimage of x

under f_n. By the implicit function theorem it follows that $|y - \tilde{y}| \leqslant c_\# n^{-1}$. Hence, setting $\alpha(x) = f(\tilde{y})$ we have $|x - \alpha(x)| \leqslant c_\# n^{-1}$. Accordingly,

$$\begin{aligned}\|\mathcal{L}_n h - \mathcal{L}h\|_{L^1} &\leqslant \int_{\mathbb{T}} \sum_{y \in f^{-1}(x)} \frac{|h(y) - h(\tilde{y})|}{f'(y)} + c_\# \|h\|_{L^1} n^{-1} \\ &\leqslant 2 \int_0^{\frac{1}{2}} dx \int_x^{\alpha(x)} dz |h'(z)| + c_\# \|h\|_{L^1} n^{-1} \leqslant c_\# \|h\|_{L^1} n^{-1}\end{aligned}$$

where, in the last line, we have used Fubini to exchange the integrals. We are thus in the situation in which we can apply our general theory. First of all Theorem B.14 implies that the essential spectral radius of $\mathcal{L}$, when acting on $W^{1,1}$ is bounded by $\frac{1}{2}$, while the spectral radius is bounded by one. Analogously, the essential spectral radius of $\mathcal{L}_n$, when acting on $W^{1,1}$, is bounded by $(2 - \pi n^{-\frac{1}{2}})^{-1}$, and the spectral radius is bounded by one.

In addition note that if $\mathcal{L}h = zh$, then taking the derivative we have

$$\frac{1}{4} \sum_{y \in f^{-1}(x)} h'(y) = z h'(x).$$

But then

$$|z| \|h'\|_{L^1} \leqslant \int_{\mathbb{T}} \left| \frac{1}{4} \sum_{y \in f^{-1}(x)} h'(y) \right| dx \leqslant \frac{1}{2} \int_{\mathbb{T}} \mathcal{L}|h'|(x) dx \leqslant \frac{1}{2} \|h'\|_{L^1}.$$

Thus it must be that $|\nu| \leqslant \frac{1}{2}$. Hence, since $\mathcal{L}^* \mathrm{Leb} = \mathrm{Leb}$ and $\mathcal{L}_n^* \mathrm{Leb} = \mathrm{Leb}$, we have that $\sigma(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} \ : \ |z| \leqslant \frac{1}{2}\}$ and $1 \in \sigma(\mathcal{L}_n)$. We can now apply Theorem C.1 to claim that, for each $\tau \in (0, \frac{1}{4})$, there exists $n_R \in \mathbb{N}$ such that, for all $n \geqslant n_R$, we have $\sigma(\mathcal{L}_n) \subset \{z \in \mathbb{C} \ : \ |z - 1| \leqslant \tau\} \cup \{z \in \mathbb{C} \ : \ |z| \leqslant \frac{1}{2} + \tau\}$.

Moreover, Lemma C.2 implies that $\{z \in \mathbb{C} \ : \ |z - 1| \leqslant \tau\}$ contains a simple eigenvalue that must necessarily be 1.

We can then conclude that $\sigma(\mathcal{L}_n) \subset \{1\} \cup \{z \in \mathbb{C} \ : \ |z| \leqslant \frac{1}{2} + \tau\}$.

(C.7) Note that

$$\begin{aligned}(z\mathbb{1} - A)^{-1}(A - B)(z\mathbb{1} - B)^{-1} =& (z\mathbb{1} - A)^{-1}(z\mathbb{1} - B)(z\mathbb{1} - B)^{-1} \\ &- (z\mathbb{1} - A)^{-1}(z\mathbb{1} - A)(z\mathbb{1} - B)^{-1}.\end{aligned}$$

E.9 Problems in **Appendix D**

(D.1) The key fact is that the cross ratio is a projective invariant and, by Equation (D.1.1), the metric is the logarithm of a cross ratio.

By figure Figure E.1 it follows that[4]

$$\Theta(x,y) = \ln \frac{\|\alpha - y\|\,\|x - \beta\|}{\|\alpha - x\|\,\|y - \beta\|}$$
$$\Theta(x,z) = \ln \frac{\|u - z\|\,\|x - v\|}{\|u - x\|\,\|z - v\|}$$
$$\Theta(y,z) = \ln \frac{\|b - y\|\,\|z - a\|}{\|b - z\|\,\|y - a\|}$$

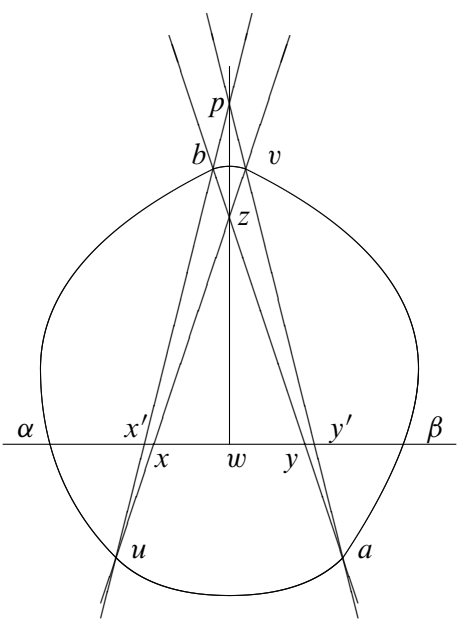


Figure E.1: Hilbert metric

[4]Note that w is the intersection of the line passing through x, y with the perpendicular to such a line passing through z.

On the other hand, if we project, from the point p, the points u, x, z, v to the line passing from α, β, we obtain the points points x', x, w, y'. Hence

$$\Theta(x,z) = \ln \frac{\|u-z\|\|x-v\|}{\|u-x\|\|z-v\|} = \ln \frac{\|x'-w\|\|x-y'\|}{\|x'-x\|\|w-y'\|}.$$

Analogously, projecting, from the point p, the points b, z, y, a to the line passing from α, β, we obtain the points points x', w, y, y'. Hence

$$\Theta(y,z) = \ln \frac{\|b-y\|\|z-a\|}{\|b-z\|\|y-a\|} = \ln \frac{\|x'-y\|\|w-y'\|}{\|x'-w\|\|y-y'\|}.$$

It follows that

$$\Theta(x,z) + \Theta(y,z) = \ln \frac{\|x-y'\|\|x'-y\|}{\|x'-x\|\|y'-y\|}.$$

But

$$\frac{\|x'-y\|}{\|x'-x\|} = \frac{\|\alpha-y\|-\|\alpha-x'\|}{\|\alpha-x\|-\|\alpha-x'\|} \geqslant \frac{\|\alpha-y\|}{\|\alpha-x\|}$$
$$\frac{\|x-y'\|}{\|y'-y\|} = \frac{\|\beta-x\|-\|\beta-y'\|}{\|\beta-y\|-\|\beta-y'\|} \geqslant \frac{\|\beta-x\|}{\|\beta-y\|}$$

which yields the triangle inequality. The other properties needed to show that Θ defines a metric are easily checked.

(D.9) Just apply the definition.

Bibliography

V. Araújo, S. Galatolo, and M. J. Pacifico (2014). "Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors." *Math. Z.* 276.3-4, pp. 1001–1048. MR: 3175169. Zbl: 1337.37019 (cit. on p. 54).

V. Baladi (2000). *Positive transfer operators and decay of correlations*. Vol. 16. Advanced Series in Nonlinear Dynamics. World Scientific Publishing Co., Inc., River Edge, NJ, pp. x+314. MR: 1793194. Zbl: 1012.37015 (cit. on pp. ii, 7, 38, 106, 178).

— (2017). "The quest for the ultimate anisotropic Banach space." *J. Stat. Phys.* 166.3-4, pp. 525–557. MR: 3607580. Zbl: 1383.37020 (cit. on p. 106).

— (2018). *Dynamical zeta functions and dynamical determinants for hyperbolic maps*. Vol. 68. Ergebnisse der Mathematik und ihrer Grenzgebiete. A functional approach. Springer, Cham, pp. xv+291. MR: 3837132. Zbl: 1405.37001 (cit. on pp. ii, 81).

V. Baladi and M. F. Demers (2020). "On the measure of maximal entropy for finite horizon Sinai billiard maps." *J. Amer. Math. Soc.* 33.2, pp. 381–449. Zbl: 1444.37020 (cit. on p. 102).

V. Baladi, M. F. Demers, and C. Liverani (2018). "Exponential decay of correlations for finite horizon Sinai billiard flows." *Invent. Math.* 211.1, pp. 39–177. MR: 3742756. Zbl: 1382.37037 (cit. on pp. 74, 96, 102, 106, 107, 113, 114, 118, 119, 124, 128, 135, 141, 147, 149, 151, 153, 154).

V. Baladi and S. Gouëzel (2010). "Banach spaces for piecewise cone-hyperbolic maps." *J. Mod. Dyn.* 4.1, pp. 91–137. MR: 2643889. Zbl: 1226.37008 (cit. on pp. 84, 106).

V. Baladi and C. Liverani (2012). "Exponential decay of correlations for piecewise cone hyperbolic contact flows." *Comm. Math. Phys.* 314.3, pp. 689–773. MR: 2964773. Zbl: 1386.37030 (cit. on pp. 106, 123, 128, 136).

V. Baladi and D. Smania (2012). "Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps." *Ann. Sci. Éc. Norm. Supér. (4)* 45.6, 861–926 (2013). MR: 3075107. Zbl: 1277.37045 (cit. on p. 44).

V. Baladi and M. Tsujii (2007). "Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms." *Ann. Inst. Fourier (Grenoble)* 57.1, pp. 127–154. MR: 2313087. Zbl: 1138.37011 (cit. on pp. 81, 106).

— (2008). "Dynamical determinants and spectrum for hyperbolic diffeomorphisms." In: *Geometric and probabilistic structures in dynamics*. Vol. 469. Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 29–68. MR: 2478465. Zbl: 1154.37320 (cit. on p. 81).

V. Baladi and L.-S. Young (1993). "On the spectra of randomly perturbed expanding maps." *Comm. Math. Phys.* 156.2. See Erratum in Baladi and Young (1994), pp. 355–385. MR: 1233850. Zbl: 0809.60101 (cit. on pp. 18, 40, 178, 222).

— (1994). "Erratum: "On the spectra of randomly perturbed expanding maps"." *Comm. Math. Phys.* 166.1. Erratum for Baladi and Young (1993), pp. 219–220. MR: 1309547. Zbl: 0814.60102 (cit. on pp. 18, 222).

P. Bálint, O. Butterley, and I. Melbourne (2019). "Polynomial decay of correlations for flows, including Lorentz gas examples." *Comm. Math. Phys.* 368.1, pp. 55–111. MR: 3946405. Zbl: 1433.37006 (cit. on p. 147).

P. Bálint and I. Melbourne (2008). "Decay of correlations and invariance principles for dispersing billiards with cusps, and related planar billiard flows." *J. Stat. Phys.* 133.3, pp. 435–447. MR: 2448631. Zbl: 1161.82016 (cit. on p. 147).

J.-B. Bardet, S. Gouëzel, and G. Keller (2007). "Limit theorems for coupled interval maps." *Stoch. Dyn.* 7.1, pp. 17–36. MR: 2293068. Zbl: 1114.37006 (cit. on pp. 175, 177).

G. Birkhoff (1957). "Extensions of Jentzsch's theorem." *Trans. Amer. Math. Soc.* 85, pp. 219–227. MR: MR0087058(19,296a). Zbl: 0079.13502 (cit. on p. 187).

— (1979). *Lattice theory*. third. Vol. 25. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, R.I., pp. vi+418. MR: 598630. Zbl: 0505.06001 (cit. on pp. 186, 189).

M. Blank (2001). "Perron-Frobenius spectrum for random maps and its approximation." *Mosc. Math. J.* 1.3, pp. 315–344, 470. MR: 1877596. Zbl: 0996.37005 (cit. on p. 54).

M. Blank, G. Keller, and C. Liverani (2002). "Ruelle-Perron-Frobenius spectrum for Anosov maps." *Nonlinearity* 15.6, pp. 1905–1973. MR: 1938476. Zbl: 1021.37015 (cit. on pp. ii, 81, 82, 85, 106).

C. Bonatti, L. J. Díaz, and M. Viana (2005). *Dynamics beyond uniform hyperbolicity*. Vol. 102. Encyclopaedia of Mathematical Sciences. A global geometric and probabilistic perspective, Mathematical Physics, III. Springer-Verlag, Berlin, pp. xviii+384. MR: 2105774. Zbl: 1060.37020 (cit. on p. ii).

R. Bowen (1970). "Markov partitions for Axiom A diffeomorphisms." *Amer. J. Math.* 92, pp. 725–747. MR: 277003. Zbl: 0208.25901 (cit. on p. ii).

— (2008). *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*. revised. Vol. 470. Lecture Notes in Mathematics. With a preface by David Ruelle, Edited by Jean-René Chazottes. Springer-Verlag, Berlin, pp. viii+75. MR: 2423393. Zbl: 1172.37001 (cit. on p. ii).

F. E. Browder (1960/61). "On the spectral theory of elliptic differential operators. I." *Math. Ann.* 142, pp. 22–130. MR: 209909. Zbl: 0104.07502 (cit. on p. 165).

O. Butterley (2016). "A note on operator semigroups associated to chaotic flows—corrigendum [MR3519417]." *Ergodic Theory Dynam. Systems* 36.5, pp. 1409–1410. MR: 3517543. Zbl: 1417.37100 (cit. on p. 129).

N. Chernov (1998). "Markov approximations and decay of correlations for Anosov flows." *Ann. of Math. (2)* 147.2, pp. 269–324. MR: 1626741. Zbl: 0911.58028 (cit. on p. 106).

— (2007). "A stretched exponential bound on time correlations for billiard flows." *J. Stat. Phys.* 127.1, pp. 21–50. MR: 2313061. Zbl: 1302.37026 (cit. on p. 106).

N. Chernov and R. Markarian (2006). *Chaotic billiards*. Vol. 127. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, pp. xii+316. MR: 2229799. Zbl: 1101.37001 (cit. on pp. 149, 152).

P. H. Collet, H. Epstein, and G. Gallavotti (1984). "Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties." *Comm. Math. Phys.* 95.1, pp. 61–112. MR: 757055. Zbl: 0585.58022 (cit. on p. 105).

P. H. Collet and S. Isola (1991). "On the essential spectrum of the transfer operator for expanding Markov maps." *Comm. Math. Phys.* 139.3, pp. 551–557. MR: 1121133. Zbl: 0754.58032 (cit. on p. 3).

I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai (1982). *Ergodic theory*. Vol. 245. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from the Russian by A. B. Sosinskiĭ. Springer-Verlag, New York, pp. x+486. MR: 832433. Zbl: 0493.28007 (cit. on p. ii).

G. Darbo (1955). "Punti uniti in trasformazioni a codominio non compatto." *Rend. Sem. Mat. Univ. Padova* 24, pp. 84–92. MR: 70164. Zbl: 0064.35704 (cit. on p. 167).

E. B. Davies (2007). *Linear operators and their spectra*. Vol. 106. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xii+451. MR: 2359869. Zbl: 1138.47001 (cit. on pp. 120, 124).

J. De Simoi and C. Liverani (2015). "The martingale approach after Varadhan and Dolgopyat." In: *Hyperbolic dynamics, fluctuations and large deviations*. Vol. 89. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, pp. 311–339. MR: 3309102. Zbl: 1356.37010 (cit. on p. 21).

— (2016). "Statistical properties of mostly contracting fast-slow partially hyperbolic systems." *Invent. Math.* 206.1, pp. 147–227. MR: 3556527. Zbl: 1385.37044 (cit. on p. ii).

— (2018). "Limit theorems for fast-slow partially hyperbolic systems." *Invent. Math.* 213.3, pp. 811–1016. MR: 3842060. Zbl: 1401.37029 (cit. on pp. 27, 39).

A. Dembo and O. Zeitouni (2010). *Large deviations techniques and applications*. Vol. 38. Stochastic Modelling and Applied Probability. Corrected reprint of the second (1998) edition. Springer-Verlag, Berlin, pp. xvi+396. MR: 2571413. Zbl: 1177.60035 (cit. on p. 38).

M. F. Demers and C. Liverani (2008). "Stability of statistical properties in two-dimensional piecewise hyperbolic maps." *Trans. Amer. Math. Soc.* 360.9, pp. 4777–4814. MR: 2403704. Zbl: 1153.37019 (cit. on pp. 96, 97, 102, 107).

M. F. Demers and H.-K. Zhang (2011). "Spectral analysis of the transfer operator for the Lorentz gas." *J. Mod. Dyn.* 5.4, pp. 665–709. MR: 2903754. Zbl: 1321.37034 (cit. on pp. 96, 102, 107, 113, 114, 118, 153).

— (2013). "A functional analytic approach to perturbations of the Lorentz gas." *Comm. Math. Phys.* 324.3, pp. 767–830. MR: 3123537. Zbl: 1385.37050 (cit. on pp. 96, 102, 107).

— (2014). "Spectral analysis of hyperbolic systems with singularities." *Nonlinearity* 27.3, pp. 379–433. MR: 3168259. Zbl: 1347.37070 (cit. on pp. 96, 102, 107, 113).

W. Doeblin and R. Fortet (1937). "Sur des chaînes à liaisons complètes." *Bull. Soc. Math. France* 65, pp. 132–148. MR: 1505076. Zbl: 0018.03303 (cit. on p. 113).

D. Dolgopyat (1998). "On decay of correlations in Anosov flows." *Ann. of Math. (2)* 147.2, pp. 357–390. MR: 1626749. Zbl: 0911.58029 (cit. on p. 106).

— (2000). "On dynamics of mostly contracting diffeomorphisms." *Comm. Math. Phys.* 213.1, pp. 181–201. MR: 1782146. Zbl: 0964.37020 (cit. on p. ii).

— (2004a). "Limit theorems for partially hyperbolic systems." *Trans. Amer. Math. Soc.* 356.4, pp. 1637–1689. MR: 2034323. Zbl: 1031.37031 (cit. on p. 64).

— (2004b). "On differentiability of SRB states for partially hyperbolic systems." *Invent. Math.* 155.2, pp. 389–449. MR: 2031432. Zbl: 1059.37021 (cit. on p. 64).

— (2005). "Averaging and invariant measures." *Mosc. Math. J.* 5.3, pp. 537–576, 742. MR: 2241812. Zbl: 1396.37040 (cit. on p. 21).

L. Dubois (2009). "Projective metrics and contraction principles for complex cones." *J. Lond. Math. Soc. (2)* 79.3, pp. 719–737. MR: 2506695 (cit. on p. 15).

N. Dunford and J. T. Schwartz (1988). *Linear operators. Part I*. Wiley Classics Library. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, pp. xiv+858. MR: 1009162. Zbl: 0635.47001 (cit. on p. 155).

D. E. Edmunds and W. D. Evans (2018). *Spectral theory and differential operators*. Oxford Mathematical Monographs. Second edition of [MR0929030]. Oxford University Press, Oxford, pp. xviii+589. MR: 3823299 (cit. on pp. 164, 165).

L. C. Evans and R. F. Gariepy (2015). *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, pp. xiv+299. MR: 3409135. Zbl: 1310.28001 (cit. on p. 204).

F. Faure, N. Roy, and J. Sjöstrand (2008). "Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances." *Open Math. J.* 1, pp. 35–81. MR: 2461513 (cit. on p. 81).

P. Ferrero and B. Schmitt (1988). "Produits aléatoires d'opérateurs matrices de transfert." *Probab. Theory Related Fields* 79.2, pp. 227–248. MR: 958289 (cit. on p. 15).

D. Fried (1986). "The zeta functions of Ruelle and Selberg. I." *Ann. Sci. École Norm. Sup. (4)* 19.4, pp. 491–517. MR: 875085. Zbl: 0609.58033 (cit. on p. ii).

S. Galatolo and I. Nisoli (2014). "An elementary approach to rigorous approximation of invariant measures." *SIAM J. Appl. Dyn. Syst.* 13.2, pp. 958–985. MR: 3216642. Zbl: 1348.37118 (cit. on p. 18).

S. Galatolo, I. Nisoli, and B. Saussol (2015). "An elementary way to rigorously estimate convergence to equilibrium and escape rates." *J. Comput. Dyn.* 2.1, pp. 51–64. MR: 3390401. Zbl: 1353.37157 (cit. on p. 18).

S. Gouëzel (2010). "Characterization of weak convergence of Birkhoff sums for Gibbs-Markov maps." *Israel J. Math.* 180, pp. 1–41. MR: 2735054 (cit. on pp. 44, 183).

S. Gouëzel and C. Liverani (2006). "Banach spaces adapted to Anosov systems." *Ergodic Theory Dynam. Systems* 26.1, pp. 189–217. MR: 2201945. Zbl: 1088.37010 (cit. on pp. iv, 44, 81, 85, 91, 92, 94, 101, 106, 114, 178, 183, 184).

— (2008). "Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties." *J. Differential Geom.* 79.3, pp. 433–477. MR: 2433929. Zbl: 1166.37010 (cit. on pp. 81, 101).

Y. Guivarc'h and J. Hardy (1988). "Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov." *Ann. Inst. H. Poincaré Probab. Statist.* 24.1, pp. 73–98. MR: 937957 (cit. on p. 21).

H. Hennion (1993). "Sur un théorème spectral et son application aux noyaux lipchitziens." *Proc. Amer. Math. Soc.* 118.2, pp. 627–634. MR: 1129880. Zbl: 0772.60049 (cit. on pp. 5, 121, 174).

H. Hennion and L. Hervé (2001). *Limit Theorem for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness*. Vol. 1766. Lecture Notes in Mathematics. Springer-Verlag, Berlin. MR: 1862393 (cit. on p. 174).

F. Hofbauer and G. Keller (1982). "Ergodic properties of invariant measures for piecewise monotonic transformations." *Math. Z.* 180.1, pp. 119–140. MR: 656227. Zbl: 0485.28016 (cit. on p. ii).

L. Hörmander (1990). *The analysis of linear partial differential operators. I*. Second. Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, pp. xii+440. MR: 1065993 (cit. on p. 87).

A. V. Isaev and S. G. Krantz (2000a). "Invariant distances and metrics in complex analysis." *Notices Amer. Math. Soc.* 47.5, pp. 546–553. MR: 1750665. Zbl: 1040.32012 (cit. on p. 186).

— (2000b). "Invariant distances and metrics in complex analysis." *Notices Amer. Math. Soc.* 47.5, pp. 546–553. MR: 1750665. Zbl: 1040.32012 (cit. on p. 15).

O. Jenkinson, M. Pollicott, and P. Vytnova (2018). "Rigorous computation of diffusion coefficients for expanding maps." *J. Stat. Phys.* 170.2, pp. 221–253. MR: 3744387. Zbl: 1390.37069 (cit. on p. 18).

L. V. Kantorovič and G. Š. Rubinšteĭn (1958). "On a space of completely additive functions." *Vestnik Leningrad. Univ.* 13.7, pp. 52–59. MR: 0102006 (cit. on p. 57).

F. Kasun and C. Liverani (2021). "Edgeworth expansion for weakly dependent random variables." *Ann. Inst. Henri Poincaré Probab. Stat.* 57.1, pp. 469–505. MR: 4255182 (cit. on p. 19).

T. Kato (1995). *Perturbation theory for linear operators*. Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, pp. xxii+619. MR: 1335452. Zbl: 0836.47009 (cit. on pp. 22, 50, 155).

A. Katok (1994). "Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems." *Ergodic Theory Dynam. Systems* 14.4. With the collaboration of Keith Burns, pp. 757–785. MR: 1304141. Zbl: 0816.58029 (cit. on p. 124).

A. Katok and B. Hasselblatt (1995). *Introduction to the modern theory of dynamical systems*. Vol. 54. Encyclopedia of Mathematics and its Applications. With a supplementary chapter by Katok and Leonardo Mendoza. Translation to Portuguese in Katok and Hasselblatt (2005). Cambridge University Press, Cambridge, pp. xviii+802. MR: 1326374. Zbl: 0878.58020 (cit. on pp. 29, 39, 83, 85, 214, 227).

— (2005). *A moderna teoria de sistemas dinâmicos*. Translation by Luis Barreira from the original in English Katok and Hasselblatt (1995). Fundação Calouste Gulbenkian, p. 839 (cit. on p. 227).

M. Keane, R. Murray, and L.-S. Young (1998). "Computing invariant measures for expanding circle maps." *Nonlinearity* 11.1, pp. 27–46. MR: 1492949. Zbl: 0903.58019 (cit. on p. 18).

G. Keller (1979). "Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan." *C. R. Acad. Sci. Paris Sér. A-B* 289.12, A625–A627. MR: 556443 (cit. on p. ii).

— (1982). "Stochastic stability in some chaotic dynamical systems." *Monatsh. Math.* 94.4, pp. 313–333. MR: 685377. Zbl: 0496.58010 (cit. on p. 42).

— (1984). "On the rate of convergence to equilibrium in one-dimensional systems." *Comm. Math. Phys.* 96, pp. 181–193. MR: 0768254. Zbl: 0576.58016 (cit. on p. 3).

G. Keller (1998). *Equilibrium states in ergodic theory*. Vol. 42. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, pp. x+178. MR: 1618769. Zbl: 0896.28006 (cit. on pp. 38, 39).

G. Keller and C. Liverani (1999). "Stability of the spectrum for transfer operators." *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 28.1, pp. 141–152. MR: 1679080. Zbl: 0956.37003 (cit. on pp. iv, 40, 41, 178, 179).

— (2006). "Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension." *Comm. Math. Phys.* 262.1, pp. 33–50. MR: 2200881. Zbl: 1107.37019 (cit. on p. 44).

— (2009a). "Map lattices coupled by collisions." *Comm. Math. Phys.* 291.2, pp. 591–597. MR: 2530174. Zbl: 1188.37044 (cit. on p. 44).

— (2009b). "Rare events, escape rates and quasistationarity: some exact formulae." *J. Stat. Phys.* 135.3, pp. 519–534. MR: 2535206. Zbl: 1179.37010 (cit. on pp. 44, 183).

G. Keller and H. H. Rugh (2004). "Eigenfunctions for smooth expanding circle maps." *Nonlinearity* 17.5, pp. 1723–1730. MR: 2086147. Zbl: 1066.37024 (cit. on p. 7).

Y. Kifer (1988). *Random perturbations of dynamical systems*. Vol. 16. Progress in Probability and Statistics. Birkhäuser Boston, Inc., Boston, MA, pp. vi+294. MR: 1015933. Zbl: 0659.58003 (cit. on pp. 40, 178).

A. Y. Kitaev (1999). "Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness." *Nonlinearity* 12.1, pp. 141–179. MR: 1668543. Zbl: 0917.58029 (cit. on pp. ii, 100).

B. O. Koopman (May 1931). "Hamiltonian Systems and Transformation in Hilbert Space." *Proc. Natl. Acad. Sci. USA* 17 (5), pp. 315–318. Zbl: 0002.05701 (cit. on p. ii).

A. Lasota and J. A. Yorke (1973). "On the existence of invariant measures for piecewise monotonic transformations." *Trans. Amer. Math. Soc.* 186, 481–488 (1974). MR: 335758. Zbl: 0298.28015 (cit. on pp. ii, 113).

E. H. Lieb and M. Loss (2001). *Analysis*. Second. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. xxii+346. MR: 1817225. Zbl: 0966.26002 (cit. on p. 34).

C. Liverani (1995a). "Decay of correlations." *Ann. of Math. (2)* 142.2, pp. 239–301. MR: 1343323. Zbl: 0871.58059 (cit. on pp. 15, 64, 102).

— (1995b). "Decay of correlations for piecewise expanding maps." *J. Statist. Phys.* 78.3-4, pp. 1111–1129. MR: 1315241. Zbl: 1080.37501 (cit. on pp. 15, 16).

— (1996). "Central limit theorem for deterministic systems." In: *International Conference on Dynamical Systems (Montevideo, 1995). A tribute to Ricardo Mañé*. Vol. 362. Pitman Res. Notes Math. Ser. Longman, Harlow, pp. 56–75. MR: 1460797. Zbl: 0871.58055 (cit. on p. 21).
— (2001). "Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study." *Nonlinearity* 14.3, pp. 463–490. MR: 1830903. Zbl: 0991.37002 (cit. on pp. 18, 43).
— (2003). "Invariant measures and their properties. A functional analytic point of view." In: *Dynamical systems. Part II*. Pubbl. Cent. Ric. Mat. Ennio Giorgi. Scuola Norm. Sup., Pisa, pp. 185–237. MR: 2071241. Zbl: 1066.37013 (cit. on pp. i, 178).
— (2004). "On contact Anosov flows." *Ann. of Math. (2)* 159.3, pp. 1275–1312. MR: 2113022. Zbl: 1067.37031 (cit. on pp. 105–107, 119, 136).
— (2013). "Multidimensional expanding maps with singularities: a pedestrian approach." *Ergodic Theory Dynam. Systems* 33.1, pp. 168–182. MR: 3009108. Zbl: 1268.37026 (cit. on p. 44).
C. Liverani and V. Maume-Deschamps (2003). "Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set." *Ann. Inst. H. Poincaré Probab. Statist.* 39.3, pp. 385–412. MR: 1978986. Zbl: 1021.37002 (cit. on p. 15).
C. Liverani, B. Saussol, and S. Vaienti (1998). "Conformal measure and decay of correlations for covering weighted systems." *Ergodic Theory Dynam. Systems* 18.6, pp. 1399–1420. MR: 1658635. Zbl: 0915.58061 (cit. on p. 190).
C. Liverani and M. P. Wojtkowski (1995). "Ergodicity in Hamiltonian systems." In: *Dynamics reported*. Vol. 4. Dynam. Report. Expositions Dynam. Systems (N.S.) Springer, Berlin, pp. 130–202. MR: 1346498. Zbl: 0824.58033 (cit. on p. 124).
A. N. Livšic (1971a). "Certain properties of the homology of Y-systems." *Mat. Zametki* 10. Translation to English in Livšic (1971b), pp. 555–564. MR: 293669 (cit. on pp. 24, 229).
— (1971b). "Homology properties of Y-systems." *Math. Notes* 10. Translation to English of the Russian original in Livšic (1971a), pp. 754–757 (cit. on p. 229).
— (1972a). "Cohomology of dynamical systems." *Math. U.S.S.R. Iz.* 6. Translation to English of the Russian original in Livšic (1972b), pp. 1278–1301 (cit. on p. 229).
— (1972b). "Cohomology of dynamical systems." *Izv. Akad. Nauk SSSR Ser. Mat.* 36. Translation to English in Livšic (1972a), pp. 1296–1320. MR: 0334287 (cit. on pp. 24, 229).

V. Maume-Deschamps (2001). "Projective metrics and mixing properties on towers." *Trans. Amer. Math. Soc.* 353.8, pp. 3371–3389. MR: 1828610 (cit. on p. ii).

I. Melbourne (2007). "Rapid decay of correlations for nonuniformly hyperbolic flows." *Trans. Amer. Math. Soc.* 359.5, pp. 2421–2441. MR: 2276628. Zbl: 1184.37024 (cit. on p. 106).

C. C. Moore (1987). "Exponential decay of correlation coefficients for geodesic flows." In: *Group representations, ergodic theory, operator algebras, and mathematical physics (Berkeley, Calif., 1984)*. Vol. 6. Math. Sci. Res. Inst. Publ. Springer, New York, pp. 163–181. MR: 880376. Zbl: 0625.58023 (cit. on p. 105).

S. V. Nagaev (1957). "Some limit theorems for stationary Markov chains." *Teor. Veroyatnost. i Primenen.* 2, pp. 389–416. MR: 0094846. Zbl: 0078.31804 (cit. on p. 21).

J. von Neumann (1932). "Proof of the quasi-ergodic hypothesis." *Proc. Natl. Acad. Sci. USA* 18, pp. 70–82. Zbl: 0004.31004 (cit. on p. ii).

R. D. Nussbaum (1969). "The fixed point index and asymptotic fixed point theorems for k-set-contractions." *Bull. Amer. Math. Soc.* 75, pp. 490–495. MR: 246285. Zbl: 0174.45402 (cit. on p. 167).

— (1970). "The radius of the essential spectrum." *Duke Math. J.* 37, pp. 473–478. MR: 264434. Zbl: 0216.41602 (cit. on p. 170).

— (1988). "Hilbert's projective metric and iterated nonlinear maps." *Mem. Amer. Math. Soc.* 75.391, pp. iv+137. MR: 961211. Zbl: 0666.47028 (cit. on pp. 186, 187).

Y. B. Pesin and Y. G. Sinai (1982). "Gibbs measures for partially hyperbolic attractors." *Ergodic Theory Dynam. Systems* 2.3-4, 417–438 (1983). MR: 721733. Zbl: 0519.58035 (cit. on p. 64).

M. Pollicott (1985). "On the rate of mixing of Axiom A flows." *Invent. Math.* 81.3, pp. 413–426. MR: 807065. Zbl: 0591.58025 (cit. on p. 105).

— (1992). "Exponential mixing for the geodesic flow on hyperbolic three-manifolds." *J. Statist. Phys.* 67.3-4, pp. 667–673. MR: 1171148. Zbl: 0892.58060 (cit. on p. 105).

M. Ratner (1987). "The rate of mixing for geodesic and horocycle flows." *Ergodic Theory Dynam. Systems* 7.2, pp. 267–288. MR: 896798. Zbl: 0623.22008 (cit. on p. 105).

M. Reed and B. Simon (1980). *Methods of modern mathematical physics. I.* Second. Functional analysis. Academic Press, Inc. [Harcourt Brace Jovanovich,

Publishers], New York, pp. xv+400. MR: 751959. Zbl: 0459.46001 (cit. on p. 155).

D. Ruelle (1976a). "A measure associated with axiom-A attractors." *Amer. J. Math.* 98.3, pp. 619–654. MR: 415683. Zbl: 0355.58010 (cit. on p. ii).

— (1976b). "Zeta-functions for expanding maps and Anosov flows." *Invent. Math.* 34.3, pp. 231–242. MR: 420720. Zbl: 0329.58014 (cit. on p. ii).

— (1978). *Thermodynamic formalism*. Vol. 5. Encyclopedia of Mathematics and its Applications. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota. Addison-Wesley Publishing Co., Reading, Mass., pp. xix+183. MR: 511655 (cit. on p. ii).

— (1983). "Flots qui ne mélangent pas exponentiellement." *C. R. Acad. Sci. Paris Sér. I Math.* 296.4, pp. 191–193. MR: 692974. Zbl: 0531.58040 (cit. on p. 105).

H. H. Rugh (1992). "The correlation spectrum for hyperbolic analytic maps." *Nonlinearity* 5.6, pp. 1237–1263. MR: 1192517. Zbl: 0768.58027 (cit. on p. ii).

— (1996). "Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems." *Ergodic Theory Dynam. Systems* 16.4, pp. 805–819. MR: 1406435 (cit. on p. ii).

— (2010). "Cones and gauges in complex spaces: spectral gaps and complex Perron-Frobenius theory." *Ann. of Math. (2)* 171.3, pp. 1707–1752. MR: 2680397 (cit. on p. 15).

O. M. Sarig (1999). "Thermodynamic formalism for countable Markov shifts." *Ergodic Theory Dynam. Systems* 19.6, pp. 1565–1593. MR: 1738951. Zbl: 0994.37005 (cit. on p. ii).

B. Saussol (2000). "Absolutely continuous invariant measures for multidimensional expanding maps." *Israel J. Math.* 116, pp. 223–248. MR: 1759406. Zbl: 0982.37003 (cit. on pp. 15, 44, 52).

Y. G. Sinai (1968). "Markov partitions and C-diffeomorphisms." *Funkcional. Anal. i Priložen* 2.1, pp. 64–89. MR: 0233038. Zbl: 0182.55003 (cit. on p. ii).

— (1970). "Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards." *Uspehi Mat. Nauk* 25.2 (152), pp. 141–192. MR: 0274721. Zbl: 0252.58005.

— (1972a). "Gibbs measure in ergodic theory." *Russian Math. Surveys* 27. Translation to English of the Russian original in Sinai (1972b), pp. 21–69. MR: 0399421. Zbl: 0255.28016 (cit. on p. 232).

Y. G. Sinai (1972b). "Gibbs measures in ergodic theory." *Uspehi Mat. Nauk* 27.4(166). Translation to English in Sinai (1972a), pp. 21–64. MR: 0399421. Zbl: 0246.28008 (cit. on pp. ii, 231).

M. Tsujii (2018). "Exponential mixing for generic volume-preserving Anosov flows in dimension three." *J. Math. Soc. Japan* 70.2, pp. 757–821. MR: 3787739. Zbl: 1393.37028 (cit. on p. 106).

M. Tsujii and Z. Zhang (2020). "Smooth mixing Anosov flows in dimension three are exponential mixing" (cit. on p. 106).

S. R. S. Varadhan (2001). *Probability theory*. Vol. 7. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, pp. viii+167. MR: 1852999. Zbl: 0980.60002 (cit. on pp. 20, 21, 25, 200).

— (2007). *Stochastic processes*. Vol. 16. Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, pp. x+126. MR: 2354349. Zbl: 1133.60004 (cit. on p. 21).

— (2016). *Large deviations*. Vol. 27. Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, pp. vii+104. MR: 3561097. Zbl: 1377.60006 (cit. on p. 38).

M. Viana (1997). *Stochastic dynamics of deterministic systems*. XXI Braz. Math. Colloq. Rio de Janeiro: IMPA (cit. on pp. iv, 40, 202).

F. Wolf (1959). "On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators." *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.* 21, pp. 142–147. MR: 0107751. Zbl: 0086.08203 (cit. on p. 165).

K. Yosida (1995). *Functional analysis*. Classics in Mathematics. Reprint of the sixth (1980) edition. Springer-Verlag, Berlin, pp. xii+501. MR: 1336382 (cit. on p. 157).

L.-S. Young (1998). "Statistical properties of dynamical systems with some hyperbolicity." *Ann. of Math. (2)* 147.3, pp. 585–650. MR: 1637655. Zbl: 0945.37009 (cit. on p. ii).

— (1999). "Recurrence times and rates of mixing." *Israel J. Math.* 110, pp. 153–188. MR: 1750438. Zbl: 0983.37005 (cit. on pp. ii, 63).

Index of Notation

Index

Títulos Publicados — 33º Colóquio Brasileiro de Matemática

Geometria Lipschitz das singularidades – *Lev Birbrair e Edvalter Sena*

Combinatória – *Fábio Botler, Maurício Collares, Taísa Martins, Walner Mendonça, Rob Morris e Guilherme Mota*

Códigos geométricos, uma introdução via corpos de funções algébricas – *Gilberto Brito de Almeida Filho e Saeed Tafazolian*

Topologia e geometria de 3-variedades, uma agradável introdução – *André Salles de Carvalho e Rafał Marian Siejakowski*

Ciência de dados: algoritmos e aplicações – *Luerbio Faria, Fabiano de Souza Oliveira, Paulo Eustáquio Duarte Pinto e Jayme Luiz Szwarcfiter*

Discovering Poncelet invariants in the plane – *Ronaldo A. Garcia e Dan S. Reznik*

Introdução à geometria e topologia dos sistemas dinâmicos em superfícies e além – *Víctor León e Bruno Scárdua*

Equações diferenciais e modelos epidemiológicos – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

Differential Equation Models in Epidemiology – *Marlon M. López-Flores, Dan Marchesin, Vítor Matos e Stephen Schecter*

A friendly invitation to Fourier analysis on polytopes – *Sinai Robins*

PI-álgebras: uma introdução à PI-teoria – *Rafael Bezerra dos Santos e Ana Cristina Vieira*

First steps into Model Order Reduction – *Alessandro Alla*

The Einstein Constraint Equations – *Rodrigo Avalos e Jorge H. Lira*

Dynamics of Circle Mappings – *Edson de Faria e Pablo Guarino*

Statistical model selection for stochastic systems with applications to Bioinformatics, Linguistics and Neurobiology – *Antonio Galves, Florencia Leonardi e Guilherme Ost*

Transfer Operators in Hyperbolic Dynamics - an introduction – *Mark F. Demers, Niloofar Kiamari e Carlangelo Liverani*

A course in Hodge Theory: Periods of Algebraic Cycles – *Hossein Movasati e Roberto Villaflor Loyola*

A dynamical system approach for Lane-Emden type problems – *Liliane Maia, Gabrielle Nornberg e Filomena Pacella*

Visualizing Thurston's geometries – *Tiago Novello, Vinícius da Silva e Luiz Velho*

Scaling problems, algorithms and applications to Computer Science and Statistics – *Rafael Oliveira e Akshay Ramachandran*

An introduction to Characteristic Classes – *Jean-Paul Brasselet*

impa

Instituto de
Matemática
Pura e Aplicada

ISBN 978-65-89124-26-9