Weakly coupled lattices of 1D piecewise expanding maps: Limit theorems and phase transition

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Abstract:
Since Kaneko [1] introduced coupled map lattices around 1984, many authors investigated numerically systems (on the lattice $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$) of the type

$$x_i(t+1) = \tau(x_i(t)) + \frac{\epsilon}{2d} \sum_{j \in \mathcal{N}(i)} \tau(x_j(t)) \tag{1}$$

where τ is a chaotic 1D map, e.g. a logistic or a tent map on $[0,1]$.

The first rigorous mathematical result on the statistical properties of coupled map lattices in the limit of infinite system size (i.e. $\Lambda = \mathbb{Z}^d$) was published by Bunimovich and Sinai [2] in 1988. It dealt with piecewise expanding local maps τ on $[0,1]$ with onto branches and with a very special class of couplings which ensures that the symbolic dynamics of the system is a full $d+1$-dimensional shift over finitely many symbols. Despite numerous efforts and beautiful results by many authors [3] it was not possible so far to treat systems with the simple diffusive coupling (1) and a mixing piecewise expanding map (e.g. a tent map or a β-transformation) as a local map τ. Here I will report about some progress on these systems that was achieved between 2004 and 2007 in various cooperations with C. Liverani, J.-B. Bardet and S. Gouëzel. All results assume that $\inf |\tau'| > 2$.

For systems as above (and more general variants of them) we prove:

Theorem 1 ([6] with reference to [4, 5]) *For sufficiently small coupling strength $\epsilon > 0$, the dynamical system on $\Omega := [0,1]^{\mathbb{Z}^d}$ given by (1) has a* **unique invariant probability measure** *μ_ϵ in the class of all measures which have densities h_{Λ_0} on finite "boxes" $\Lambda_0 \subset \Lambda$ and for which the variation of h_{Λ_0} grows subexponentially with the diameter of Λ_0. With this measure the system has* **exponentially decaying correlations** *both in time and in space, and μ_ϵ is the* **SRB measure** *of the system in the following sense:*
- *μ_ϵ is stable under small perturbations of the dynamics by smooth noise, and*
- *for each continuous observable $\psi : X \to \mathbb{R}$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} \psi(x(t)) = \int_X \psi \, d\mu_\epsilon$$

exists for almost every initial configuration $x(0)$ – "almost every" with respect to the infinite Lebesgue product measure on X.

The key ingredient of the proof is a **new decoupling procedure** for the coupled system. A refinement of this procedure yields:

Theorem 2 ([7]) *If $\psi : \Omega \to \mathbb{R}^\Lambda$ is a local Lipschitz observable, then the partial sum process $(\sum_{t=1}^n \psi(x(t)))_{n=1,2,3,\dots}$ defined on the probability space (Ω, μ_ϵ) satisfies a central limit theorem and also a local limit theorem.*

Finally I plan to describe the construction of a piecewise linear Markov map τ giving rise to the coupled dynamics (on $\Omega = \mathbb{R}^{\mathbb{Z}^2}$)

$$x_i(t+1) = \tau(x_i(t)) + \epsilon \cdot \left(\frac{1}{2} \left(\tau(x_{i+\binom{0}{1}}(t) + \tau_{i+\binom{1}{0}}(t) \right) - \tau(x_i(t) \right) \tag{2}$$

which shows a phase transition in the coupling strength ϵ:

Theorem 3 ([8])

a) *On a finite periodic lattice $\Lambda = (\mathbb{Z}/\mathbb{Z})^2$ holds: For all $\epsilon \in [0, \frac{1}{4}]$ the system has a unique invariant density with respect to which the system is exponentially mixing in time.*

b) *On the infinite lattice $\Lambda = \mathbb{Z}^2$ holds: There are $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \leq \frac{1}{4}$ such that*

   i) *For $0 \leq \epsilon \leq \epsilon_1$ the systemn has a unique SRB measure μ_ϵ as described in Theorem 1.*

   ii) *For $\epsilon_2 \leq \epsilon \leq \epsilon_3$ the system has at least two invariant measures with densities on finite boxes as described in Theorem 1.*

The proof relates the dynamics of the system to those of a stochastic cellular automaton of Toom's type [9].

Tentative schedule:
1. lecture: Introduction of the model, main results, relevant spaces of functions and measures
2. lecture: Main ideas in the proofs of Theorems 1 and 2
3. lecture: Main idea in the proof of Theorem 3.

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