Statistical Properties of Hyperbolic Billiards

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Contents

A Foreword

These are partial notes for the lecture on billiards held at Budapest in August 2024. The lectures are intertwined with the ones given by Mark Demers, but Demers' lectures are more technical. The purpose here is to provide a basic understanding for people with essentially no prior knowledge of the subject. First, I will discuss how to establish hyperbolicity, and then I will discuss the statistical properties. I will put the emphasis on some mathematical techniques useful to tackle such problems (e.g. standard pairs, dynamical functional spaces and transfer operators, strictly invariant cones, and Hilber metric). The notes are both more and less extensive than the lectures. I apologize for that, but writing notes is a rather time-consuming activity for a slow person like me. In addition, since I have written them in a hurry, they may contain mistakes. So read at your own risk, and apologies again.

Chapter 1

Hyperbolicity

Here, we discuss how to establish hyperbolicity for symplectic maps and flows. The ideas put forward can also be used for more general systems, but sympletcity provides an extra structure that allows to develop a much richer theory. Since Billiards are Hamiltonian systems, and hence give rise to symplectic flows and maps, this theory is relevant for Billiards.

1.1 Hamiltonian flows and Symplectic structure

Given the matrix $2d \times 2d$ defined by

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

Hamilton's equations can be written as[1]

$$\dot{x} = J\nabla H(x) \tag{1.1.1}$$

where $x = (q, p)$. Note that $J^2 = -\mathbb{1}$ e $J^T = -J$.[2] The matrix J plays a fundamental role in the Hamiltonian structure. In particular, one can define the bilinear form on $\mathbb{R}^{2d}$

$$\omega(v, w) := \langle v, Jw \rangle. \tag{1.1.2}$$

The form ω is called the *symplectic form.* A matrix A with the property $\omega(Av, Aw) = \omega(v, w)$, for every $v, w \in \mathbb{R}^{2d}$, is called *symplectic.* A transformation $F \in \mathcal{C}^1(\mathbb{R}^{2d}, \mathbb{R}^{2d})$ such that $DF(x)$ is symplectic for every $x \in \mathbb{R}^{2d}$ is said to be *symplectic transformation.*

[1] The gradient of a function $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is given by the vector $\nabla f := (\partial_{x_i} f)$.

[2] Note the similarity with the imaginary number i, where the transpose takes the place of the complex conjugation; this is no accident!

Lemma 1.1.1 *For each Hamiltonian H the Hamiltonian flow ϕ_t is a symplectic transformation.*

PROOF. Let $\Xi(x,t) = D\phi_t$, then

$$\dot{\Xi}(x,t) = JD^2H \circ \phi_t(x) \cdot \Xi(x,t)$$

hence, for each $v, w \in \mathbb{R}^{2d}$,

$$\frac{d}{dt}\omega(\Xi v, \Xi w) = \omega(\dot{\Xi} v, \Xi w) + \omega(\Xi v, \dot{\Xi} w) = \langle JD^2H\Xi v, J\Xi w\rangle - \langle \Xi v, D^2H\Xi w\rangle = 0,$$

where we used the fact that D^2H is a symmetric matrix.[3] □

Lemma 1.1.2 *The set of symplectic matrices form a group (called $Sp(2d,\mathbb{R})$). Furthermore, if $L \in Sp(2d,\mathbb{R})$, then $L^T \in Sp(2d,\mathbb{R})$.*

PROOF. First note that a matrix is symplectic if and only if $L^TJL = J$. Then it is trivial to verify that $\mathbb{1} \in Sp(2d,\mathbb{R})$. Furthermore, if $L, B \in Sp(2d,\mathbb{R})$, then

$$(LB)^TJLB = B^TL^TJLB = J,$$

therefore $LB \in Sp(2d,\mathbb{R})$. Moreover, $L[-JL^TJ] = \mathbb{1}$ shows that L is invertible and $L^{-1} = -JL^TJ$, furthermore

$$(L^{-1})^TJL^{-1} = (-JL^TJ)^TJL^{-1} = JLL^{-1} = J.$$

Hence $L^{-1} \in Sp(2d,\mathbb{R})$. Finally, if $L \in Sp(2d,\mathbb{R})$, then $L^{-1}J(L^T)^{-1} = J$ which impleis $(L^T)^{-1} \in Sp(2d,\mathbb{R})$ and $L^T \in Sp(2d,\mathbb{R})$. □

Next, we provide a useful decomposition.

Lemma 1.1.3 *If $L := \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \in Sp(2d,\mathbb{R})$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are $d \times d$ matrices, and $\det(\boldsymbol{a}) \neq 0$, then there exist symmetric $d \times d$ matrices R, P such that*

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix}, \tag{1.1.3}$$

[3]Obviously we are assuming that $H \in \mathcal{C}^2$ and symmetry follows from Schwartz's Lemma.

PROOF. A direct computation shows that $L \in Sp(2d, \mathbb{R})$ if and only if

$$\boldsymbol{c}^T \boldsymbol{a} = (\boldsymbol{a}^T \boldsymbol{c})^T = \boldsymbol{a}^T \boldsymbol{c}\,; \quad \boldsymbol{d}^T \boldsymbol{b} = (\boldsymbol{b}^T \boldsymbol{d})^T = \boldsymbol{b}^T \boldsymbol{d}\,; \quad \boldsymbol{a}^T \boldsymbol{d} - \boldsymbol{c}^T \boldsymbol{b} = \mathbb{1}. \quad (1.1.4)$$

Since $\boldsymbol{a}$ is invertible, we can write

$$L = \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}, \quad (1.1.5)$$

where $R = \boldsymbol{a}^{-1}\boldsymbol{b}$, $P = \boldsymbol{a}^T \boldsymbol{c}$ and $H = \boldsymbol{a}^T \boldsymbol{d}$. Condition (1.1.4) implies that $\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix}$ is symplectic. Then, by Lemma 1.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & R \\ P & H \end{pmatrix}$ must be symplectic. Accordingly, (1.1.4) implies

$$P^T = P\,; \quad H = \mathbb{1} + P^T R = \mathbb{1} + PR.$$

On the other hand, by Lemma 1.1.2, also the matrix $\begin{pmatrix} \mathbb{1} & P \\ bR^T & H^T \end{pmatrix}$ is symplectic, hence (1.1.4) implies

$$R^T = R$$

from which the Lemma follows. □

Note that $L^T J L = J$ implies $\det(L)^2 = 1$. In fact, since the symplectic group is connected, the above decomposition implies that $\det(L) = 1$ by continuity.

1.2 Symplectic Poincarè sections and time one maps

Let $\tau : \mathbb{R}^{2d} \to \mathbb{R}_+$ be a piecewise differentiable function and define the map $f(x) = \phi_{\tau(x)}(x)$. Where f is differentiable, we have

$$D_x f = D_x \phi_\tau + J\nabla H(\phi_\tau(x)) \otimes \nabla\tau.$$

We restrict the map f to a constant energy surface $M_E = \{x \in \mathbb{R}^{2d} : H(x) = E\}$. Then, for $v \in TM_E$ we have $\langle \nabla H, v\rangle = 0$. It follows that, for $v, w \in TM_E$,

$$\begin{aligned}\omega(Dfv, Dfw) =& \langle D\phi_\tau v + J\nabla H(\phi_\tau(x))\langle\nabla\tau, v\rangle, J(D\phi_\tau w + J\nabla H(\phi_\tau(x))\langle\nabla\tau, w\rangle)\rangle \\ =& \omega(v, w) + \langle \nabla H(\phi_\tau(x)), D\phi_\tau w\rangle\langle\nabla\tau, v\rangle \\ &- \langle D\phi_\tau v, \nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, w\rangle \\ &+ \langle \nabla H(\phi_\tau(x)), J\nabla H(\phi_\tau(x))\rangle\langle\nabla\tau, v\rangle\langle\nabla\tau, w\rangle = \omega(v, w).\end{aligned}$$

It is then natural to introduce the equivalence relation $v \sim w$ is $v - w = \lambda J\nabla H$ for some $\lambda \in \mathbb{R}$. Let $\mathbb{V}_x = T_x M_E/\sim$ be the vector space formed by the equivalence classes. Note that

$$\begin{aligned} D_x f(v + \lambda J\nabla H(x)) &= D_x f v + \lambda D_x \phi_\tau J\nabla H(x) + \lambda J\nabla H(\phi_\tau(x))\langle \nabla\tau, J\nabla H(x)\rangle \\ &= D_x f v + \lambda J\nabla H(f(x))\left[1 + \langle \nabla\tau, J\nabla H(x)\rangle\right]. \end{aligned}$$

Hence, the action of Df from $T_x M_E$ to $T_{f(x)} M_E$ quotients naturally in an action between $\mathbb{V}_x$ and $\mathbb{V}_{f(x)}$. On the other hand, for $v \in \mathbb{V}_x$ we have

$$\omega(J\nabla H, v) = \langle \nabla H, v\rangle = 0.$$

Thus $\omega(v + \lambda J\nabla H, w + \mu J\nabla H) = \omega(v, w)$, that is we can quotient ω as well on $\mathbb{V}_x$. It follows that ω induces canonically a symplectic form, which we still call ω, on each $\mathbb{V}_x$. By the above discussion the d dimensional spaces $W_1^+ = \{(v, 0) \ : \ v \in \mathbb{R}^d\}$ and $W_2^+ = \{(0, v) \ : \ v \in \mathbb{R}^d\}$ quotient to $d-1$ dimensional spaces W_i in each $\mathbb{V}_x$, moreover $\omega(w, w') = 0$ for each $w, w' \in W_1$ or $w, w' \in W_2$ (such subspaces, as we will see briefly, are called Lagrangian). Next, one can check that it is possible to choose basis $\{e_i\}$ in W_1 and $\{f_i\}$ in W_2 such that $\omega(e_i, f_j) = \delta_{ij}$. Then we can write any vector $a \in \mathbb{V}_x$ as $a = \sum_{i=1}^{d-1} \xi_i e_i + \sum_{i=1}^{d-1} \eta_i f_i$ and

$$\omega(a, a') = \sum_{i,j} \xi_i \eta'_j \omega(e_i, f_j) + \eta_i \xi'_j \omega(f_i, e_j) = \sum_i \xi_i \eta'_i - \xi' \eta_i = \langle (\xi, \eta), J(\xi', \eta').\rangle$$

That is, in such coordinates, the symplectic form has the standard form (1.1.2). We can thus identify all the spaces $\mathbb{V}_x$ and, in such coordinates, $Df|_{\mathbb{V}}$ is symplectic.

By choosing $\tau \equiv 1$, the map ϕ_1 can be seen as a $2d-2$ symplectic map. Moreover, if Σ is a Poincarè section for the flow, then we can choose τ to be the first return time and since $\mathbb{V}$ is naturally isomorphic to $T\Sigma$, again we have that the Poincarè map $f(x) = \phi_{\tau(x)}(x)$ is symplectic.

1.3 Hyperbolicity and how to establish it

Since we will discuss *hyperbolic billiards*, we must say exactly what we mean and how to see if a billiard is hyperbolic

First of all, recall Oseledec [35] (see [45] for a nice introduction and [24] for a generalization and more recent bibliography). We content ourselves with the following version.

Theorem 1.3.1 (Wojtkowski [43]) *Let (X, μ) be a probability space and $f : X \to X$ a measure-preserving transformation. Let $L : X \to GL(n, \mathbb{R})$ be a measurable mapping to $n \times n$ matrices such that $\log_+ \|L(\cdot)\| \in L^1(X, \mu)$. Then for μ-almost all $x \in X$ there are subspaces $\{0\} = V_x^0 \subset V_x^1 \subset \cdots \subset V_x^n = \mathbb{R}^n$ and numbers $\lambda_1(x) \le \cdots \le \lambda^n(x)$ such that, for all $i \in \{1, \ldots, n\}$.*

$$\lim_{k\to\infty} \frac{1}{k} \ln \|L(f^{k-1}(x)) \cdots L(f(x))L(x)v\| = \lambda_i(x)$$

if $v \in V_x^i \setminus V_x^{i-1}$.

1.4 Two dimensions

We are interested in the case $L(x) = D_x\phi_1$, where ϕ_t is the billiard flow. Of course, the flow will have a zero Lyapunov exponent (the flow direction).

Definition 1 *A Billiard is hyperbolic if the only zero Lyapunov exponent is the one associated with the flow direction. Equivalently, a Billiard is hyperbolic if the Poincarè map has no zero Lyapunov exponent.*

The problem is to have a tool to establish hyperbolicity. The following theorem provides a very efficient tool.

Theorem 1.4.1 (Wojtkowski [43]) *Let X be a Riemannian manifold, possibly with boundaries, $\{\mathcal{C}(x) \subset \mathcal{T}_x X \ : \ x \in X\}$ a family of closed cones in the tangent space. Let $f : X \to X$ and $L : X \to SL(n, \mathbb{R})$ as in Theorem 1.3.1. If for μ almost $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $L(f^{n(x)-1}) \cdots L(x)\mathcal{C}(x) \subset \text{int}(\mathcal{C}(f^{n(x)}(x)))$, then the maximal Lyapunov exponent is strictly positive.*

The above theorem suffices for planar billiards, where there are two Lyapunov exponents λ_i and, by volume conservation $\lambda_1 = -\lambda_2$. For higher dimensional billiard, it does not control all the Lyapunov exponents. To achieve this, we have to use more heavily the fact that the Billiards flows are Hamiltonian, and hence symplectic. In addition, while a two-dimensional cone is simply a sector, a higher-dimensional cone can have many different shapes, and it is not obvious what is a natural cone shape.

1.5 Higher dimensions: the symplectic structure

Given a symplectic form ω, which is left invariant by map $f : M \to M$, we have a symplectic flow. If $\mathcal{T}M = \mathbb{R}^{2d}$, then a d-dimensional subspace $V \subset \mathbb{R}^{2d}$ is called *Lagrangian* if $\omega|_V \equiv 0$. Given two transversal Lagrangian subspaces V_1, V_2, we can write uniquely $v \in \mathbb{R}^{2d}$ ad $v = v_1 + v_2$, with $v_i \in V_i$. we can then define the quadratic function

$$Q(v) = \omega(v_1, v_2).$$

This allows us to define special cones with remarkable properties:

$$\mathcal{C} = \{v \in \mathbb{R}^{2n} \ : \ Q(v) > 0\}. \tag{1.5.6}$$

Accordingly, if we specify a field of transversal Lagrangian subspace, we have the quadratic functions Q_x and the cone field $\mathcal{C}_x$.

Obviously, if $Q_{f(x)}(d_x f v) \geq Q_x(v)$, then $d_x f \mathcal{C}_x \subset \mathcal{C}_{f(x)}$, hence we have cone invarince. Such maps are called *monotone*.

If $Q_{f(x)}(d_x f v) > Q_x(v)$ for all $v \neq 0$, then $d_x f(\overline{\mathcal{C}_x} \setminus \{0\}) \subset \mathcal{C}_{f(x)}$, such maps are called *strictly monotone*.

Lemma 1.5.1 ([32], Sections 6) *A map is monotone if and only if the cone field is invariant. The same is true for strict monotonicity.*

Theorem 1.5.2 ([32] Sections 5, 6, or [31]) *If a map is eventually strictly monotone, then all its Lyapunov exponents are non-zero.*

This is proven exactly as Theorem 1.6.1, so we refer to the proof of the latter.
The above also has a continuous version: a Hamiltonian flow in a $2d+2$ dimensional manifold, is determined by a Hamiltonian

1.5.1 Lagrangian subspaces

By a symplectic change of variables, we can assume that the space is $\mathbb{R}^{2d}$, the vectors are written as (ξ, η), $\xi, \eta \in \mathbb{R}^d$ and the symplecit form is given by

$$\omega((\xi, \eta), (\eta', \eta') = \langle \xi, \eta' \rangle - \langle \eta, \xi' \rangle.$$

Then, $A \in GL(2d, \mathbb{R})$ is sympletc if and only if $\omega(Av, Aw) = \omega(v, w)$ for all $v, w \in \mathbb{R}^{2d}$. That is if

$$A^T J A = J$$
$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

To introduce an appropriate higher dimensional formalism, it is convenient to discuss briefly *Lagrangian subspaces*.

Definition 2 *A d-dimensional subspace $\mathbb{V}$ of $\mathbb{R}^{2d}$ is Lagrangian iff*

$$\omega(v, w) = 0$$

for all $v, w \in \mathbb{V}$.

Lemma 1.5.3 *For each $d \times d$ matrix U, the space $\mathbb{V} = \{(v, Uv) \ : \ v \in \mathbb{R}^d\}$ is Lagrangian iff U is symmetric.*

PROOF. Clearly $\mathbb{V}$ is d-dimensional. To conclude, it suffices to compute

$$\omega((v, Uv), (w, Uw)) = \langle v, Uw\rangle - \langle w, Uv\rangle$$

which is zero only if U is symmetric. □

Let $V_1, V_2 \in \mathbb{R}^{2d}$ two transversal Lagrangian subspaces, then, for each $v \in \mathbb{R}^{2d}$ we can write uniquely $v = v_1 + v_2$ with $v_i \in V_i$. We then write

$$Q(v) := \omega(v_1, v_2)$$

By a symplectic change of variable, we can always reduce the general case to the case $V_1 = \{(v_1, 0) \ : \ v_1 \in \mathbb{R}^d\}$ $V_1 = \{(0, v_2) \ : \ v_2 \in \mathbb{R}^d\}$. In this case

$$Q((v_1, v_2)) = \langle v_1, v_2\rangle.$$

We say that a symplectic matrix L is monotone if $Q(Lv) \geq Q(v)$ for each $v \in \mathbb{R}^{2d}$, and we say that a symplectic matrix L is strictly monotone if $Q(Lv) > Q(v)$ for each $v \in \mathbb{R}^{2d} \setminus \{0\}$.

Lemma 1.5.4 *A Lagrangian space $\mathbb{V}$ belongs to $\mathcal{C} \cup \{0\}$ iff it is of the form (v, Uv), with U strictly positive.*[4]

PROOF. If $\pi_i(v_1, v_2) = v_i$, then $\pi_1 : \mathbb{V} \to \mathbb{R}^n$ is injective. If not, there exists $(v_1, v_2) \in \mathbb{V} \setminus \{0\}$ such that $v_1 = 0$. But then $Q((v_1, v_2)) = 0$ contrary to the hypothesis. We can then define $U := \pi_2 \circ \pi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{V} = \{(v, Uv) \ : \ v \in \mathbb{R}^d\}$. Then, by Lemma 1.5.3 U must be symmetric. Finally, for $v \neq 0$,

$$0 < Q((v, Uv)) = \langle v, Uv\rangle$$

hence U is strictly positive. The opposit implication is trivial. □

[4]Recall the defintion of $\mathcal{C}$ in (1.5.6).

Lemma 1.5.5 *A symplectic matrix* $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ *is strictly monotone if and only if* $\det \boldsymbol{a} \neq 0$ *and the matrices* R, P *in the factorization* (1.1.3) *are strictly positive.*

Proof. Indeed, if $\det \boldsymbol{a} = 0$, then there exists $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\boldsymbol{a}\xi = 0$, but then

$$Q(L(\xi, 0)) = \langle \boldsymbol{c}\xi, \boldsymbol{a}\xi \rangle = 0 = Q((\xi, 0))$$

contrary to the hypothesis. We can then apply Lemma 1.1.3 to write

$$\begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & (\boldsymbol{a}^{-1})^T \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & R \\ 0 & \mathbb{1} \end{pmatrix} (v_1, v_2) = (\boldsymbol{a}(v_1 + Rv_2), (\boldsymbol{a}^{-1})^T (Pv_1 + (\mathbb{1} + PR)v_2)).$$

Thus,

$$Q(L(v_1, v_2)) = \langle v_1 + Rv_2, Pv_1 + (\mathbb{1} + PR)v_2 \rangle \tag{1.5.7}$$

If $v_2 = 0$, then we have

$$0 < Q(L(v_1, 0)) = \langle v_1, Pv_1 \rangle$$

hence P is a strictly positive matrix. On the other and, for each $\mu > 0$ and $\|v\| = 1$, we have that

$$\mu < Q(L(v, \mu v)) = \langle v + \mu Rv, Pv + \mu(\mathbb{1} + PR)v \rangle$$

We can then chose v to be an eigenvector of R, so $Rv = \lambda v$. Then we obtain

$$\mu < \langle (1 + \mu\lambda)v, Pv + \mu(\mathbb{1} + \lambda P)v \rangle = (1 + \lambda)\mu + (1 + \lambda\mu)^2 \langle v, Pv \rangle$$

that is

$$\lambda\mu + (1 + \lambda\mu)^2 \langle v, Pv \rangle > 0$$

It follows that it must be $\lambda \geq 0$ otherwise we can choose $\mu = -\lambda^{-1}$ and obtain the contradiction $-1 > 0$. On the other hand, if $\lambda = 0$, then

$$0 < Q(L(0, v)) = \langle 0, v \rangle = 0$$

which is also impossible. On the other hand, if $\det(\boldsymbol{a}) \neq 0$ and the matrices P, R are strictly positive, then

$$Q(L(v_1, v_2)) = \langle v_1, v_2 \rangle + \langle v_2, Rv_2 \rangle + \langle v_1 + Rv_2, P(v_1 + Rv_2) \rangle > Q((v_1, v_2)).$$

□

The above implies that if L is strictly monotone, then $LV_i \subset \mathcal{C}\cup\{0\}$. There is a useful partial converse of this fact.[5]

Lemma 1.5.6 *If $LV_i \subset \mathcal{C}\cup\{0\}$ and, for all $v \in \mathbb{R}^d$, $\omega(L(0, a^T v), (0, v)) \geq 0$, then L is strictly monotone.*

PROOF. If $LV_i \subset \mathcal{C}$, that is, for all vectors in $V_1 \cup V_2$, $Q(Lv) > 0$, then L is strictly monotone. First of all, note that

$$0 < Q(L(v, 0)) = \langle Q((\boldsymbol{a}v, \boldsymbol{c}v)) = \langle v, \boldsymbol{c}^T \boldsymbol{a}v\rangle.$$

Since (1.1.4) implies that $\boldsymbol{c}^T \boldsymbol{a}$ is a symmetric matrix, it follows that $\boldsymbol{c}^T \boldsymbol{a}$ is strictly positive, hence $\det(\boldsymbol{a}) \neq 0$. We can then use the decomposition (1.1.3) which yields the expression (1.5.7) which implies

$$0 < Q(L(v, 0)) = \langle v, Pv\rangle$$

which implies that P is a strictly positive matrix. This implies that

$$Q\left(\begin{pmatrix} \mathbb{1} & 0 \\ P & \mathbb{1} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = Q((v_1, Pv_1 + v_2)) = Q((v_1, v_2)) + \langle v_1, Pv_1\rangle \geq Q((v_1, v_2)).$$

On the other hand

$$0 < Q(L(0, v)) = \langle Rv, (\mathbb{1} + PR)v\rangle = \langle v, (R + RPR)v\rangle,$$

that is $R + RPR$ is strictly positive matrix. Since R is symmetric it has d eigenvectors, let w, $\|w\| = 1$, and eigenvector and λ the corresponding eigenvalue, then

$$0 < \langle w, (R + RPR)w\rangle = \lambda + \lambda^2 \langle w, Pw\rangle$$

which implies $\lambda \neq 0$. Finally, setting $w = \boldsymbol{a}^T v$,

$$0 \leq \omega(L(0, w), (0, (\boldsymbol{a}^T)^{-1} w)) = \langle Rw, w\rangle$$

implies that R is positive and hence strictly positive. The Lemma follows then from Lemma 1.5.5. □

[5] Note that [32, Proposition 8.4] is false as the example $L = \begin{pmatrix} \mathbb{1} & 0 \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & -2\mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix}$ shows.

To measure precisely how much the quadric form increases, it is convenient to introduce the cones

$$\mathcal{C} = \{v \in \mathbb{R}^{2d} \ : \ Q(v) > 0\} \ ; \quad \overline{\mathcal{C}} = \{v \in \mathbb{R}^{2d} \ : \ Q(v) \geq 0\}.$$

Accordingly, if a symplectic map L is monotone, then $L\mathcal{C} \subset \mathcal{C}$. Let us define

$$\sigma(L) = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}}.$$

Lemma 1.5.7 *If a symplectic matrix* $L = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$ *is strictly monotone, then the eaigenvalues of* $\boldsymbol{c}^T\boldsymbol{b}$ *are all strictly positive and, calling* t *the minimal such eigenvalue, we have*

$$\sigma(L) \geq \sqrt{t} + \sqrt{1+t}.$$

Proof. We use the decomposition (1.1.3) and note that the matrix

$$\mathcal{R} = \begin{pmatrix} R^{-\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{pmatrix}$$

is a Q-isometry, that is $Q(\mathcal{R}v) = Q(v)$ for all $v \in \mathbb{R}^{2d}$. In particular, this implies that $\mathcal{R}\mathcal{C} = \mathcal{C}$. Hence, setting $\mathcal{L} = \begin{pmatrix} \mathbb{1} & R \\ P & \mathbb{1} + PR \end{pmatrix}$,

$$\inf_{v \in \mathcal{C}} \sqrt{\frac{Q(Lv)}{Q(v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{L}v)}{Q(v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}(\mathcal{R}v))}{Q(\mathcal{R}v)}} = \inf_{v \in \mathcal{C}} \sqrt{\frac{Q(\mathcal{R}\mathcal{L}\mathcal{R}^{-1}v)}{Q(v)}}.$$

Note that, setting $T = R^{\frac{1}{2}} P R^{\frac{1}{2}}$,

$$\mathcal{R}\mathcal{L}\mathcal{R}^{-1} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ T & \mathbb{1} + T \end{pmatrix} =: \mathcal{T}$$

Note that T is a strictly positive matrix; hence, calling t_i its eigenvalues and w_i the associated eigenvector, we have $t_i > 0$. In addition, we have

$$PR(R^{-\frac{1}{2}} w_i) = R^{-\frac{1}{2}} T w_i = t_i R^{-\frac{1}{2}} w_i.$$

That is, the eigenvalues of T are also the eigenvalues of $PR = \boldsymbol{c}^T \boldsymbol{a} \boldsymbol{a}^{-1} \boldsymbol{b} = \boldsymbol{c}^T \boldsymbol{b}$, where we have used (1.1.5) and the fact that $P^T = P$. To conclude, we note that, setting $v = (v-1, v_2)$ and calling t the minimal eigenvalue ot T,

$$\begin{aligned}
\frac{Q(\mathcal{T}v)}{Q(v)} &= \frac{\langle v_1, v_2\rangle + \langle v_2, v_2\rangle + \langle (v_1+v_2), T(v_1+v_2)\rangle}{\langle v_1, v_2\rangle} \geq 1 + \frac{\|v_2\|^2 + t\|v_1+v_2\|^2}{\langle v_1, v_2\rangle} \\
&= 1 + \frac{(1+t)\|v_1\|^2 + 2t\langle v_1, v_2\rangle + t\|v_2\|^2}{\langle v_1, v_2\rangle} \\
&= 1 + \frac{2t\langle v_1, v_2\rangle + (1+t)^{\frac{1}{2}} t^{\frac{1}{2}} \left[(1+t)^{\frac{1}{2}} t^{-\frac{1}{2}} \|v_1\|^2 + (1+t)^{-\frac{1}{2}} t^{\frac{1}{2}} \|v_2\|^2\right]}{\langle v_1, v_2\rangle} \\
&\geq 1 + \frac{2\left[t + (1+t)^{\frac{1}{2}} t^{\frac{1}{2}}\right] \langle v_1, v_2\rangle}{\langle v_1, v_2\rangle} = \left[\sqrt{t} + \sqrt{1+t}\right]^2.
\end{aligned}$$

□

1.6 Higher dimensions: hyeprbolicity

We say that a Hamiltonian flow (M, ϕ_t) is hyperbolic on a constant energy surface M_E if, when restricted to such a surface, all his Lyponov exponents, but one (the one in the flow direction), are non-zero. For simplicity, we restrict to the case $M \subset \mathbb{R}^{2d}$, but the result holds for general symplectic manifolds. Also, we require that M_E is compact. Let μ be the Liouville measure normalized so that $\mu(M) = 1$. The goal of this section is to prove the following theorem:

Theorem 1.6.1 ([43], or see [32] Sections 5, 6, or [31]) *If a flow on M_E is* eventually strictly monotone*, then all its Lyapunov exponents, apart from the one in the flow direction, are non-zero.*

By the results of section 1.2, we can restrict ourselves to a discrete-time analysis. We will consider the time one map $f = \phi_1$ with the differential acting on the quotient space there described; the study of the Poincarè map being similar. For $x \in M$, le $s(x) = \min\{k \ : \ Df^k$ is strictly monotone$\}$.

By *eventually strictly monotone*, we mean that, for almost all $x \in M$, $D_x f$ is monotone and $s(x) < \infty$.

Proof of Theorem 1.6.1. Let $A_m = \{x \in M \ : \ s(x) = m\}$. For such

m we define the return time to A_m: we define the first return time as

$$n_m(x) = \begin{cases} 0 & \text{if } x \notin A_m \\ \min\{k \in \mathbb{N} \setminus \{0\} \ : \ f^k(x) \in A_m\} & \text{otherwise.} \end{cases}$$

Lemma 1.6.2 (Kac' s theorem) *For each $m \in \mathbb{N}$, $n_m \in L^1$.*

PROOF. If $\mu(A_m) = 0$, the statement is trivial. We can then limit ourselves to the case $\mu(A_m) > 0$. Let $A_{m,k} = \{x \in A_m \ : \ n_m(x) = k\}$. Note that f, being the time one map of a flow, is invertible, so f^{-1} is measurable and preserves the measure. Moreover, if $x \in f^j(A_{m,k}) \cap f^l(A_{m,k'})$ for some $j < l \leq k'$ and $j \leq k$, then, setting $y = f^{-l}(x) \in A_{m,k'}$ and $w = f^{-j}(x) \in A_{m,k}$ we have $f^j(w) = f^l(y)$, that is $f^{l-j}(y) = w \in A_m$ which contradicts the fact that $y \in A_{m,k'}$ since $l - j < l \leq k'$. It follows that $f^j(A_{m,k}) \cap f^l(A_{m,k'}) = \emptyset$ for all $j \neq l \leq k$. Then

$$\int_{A_m} n_m(s)\mu(dx) = \sum_{k=1}^{\infty} k\mu(A_{m,k}) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \mu(f^j(A_{m,k})) = \mu(\cup_{j=0}^{\infty} f^j(A_m)) \leq 1.$$

The statement follows since, by definition, $\int_M n_m(s)\mu(dx) = \int_{A_m} n_m(s)\mu(dx)$. □

If f is eventually strictly monotone, then $\sum_{m=1}^{\infty} \mu(A_m) = 1$. Let m so that $\mu(A_m) > 0$. Then, defining the return map $F(x) = f^{n_m(x)}(x)$, for each $n \in \mathbb{N}$ we can define $k(x) = \max\{k \in \mathbb{N} \ : \ \sum_{j=0}^{k} n_m(F^j(x)) \leq n\}$. Then

$$\begin{aligned} \sigma(Df^n) &\geq \sigma(D_{F^{k(x)-1}(x)} f^{n_m(F^{k(x)-1}(x))} \cdots D_x f^{n_m(x)}) \\ &\geq \prod_{j=0}^{k(x)-1} \sigma(D_{F^j(x)} f^{n_m(F^j(x))}). \end{aligned}$$

Also, note that, by definition, it must be $n_m(s) \geq m$. So, by Lemma 1.5.7, we have, for each $y \in A_m$, $\sigma(D_y f^{n_m(y)}) \geq \sqrt{t}(x) + \sqrt{1 + t(x)} =: e^{\alpha(x)}$ where $\alpha(x) > 0$. Accordingly,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \ln \sigma(D_x f^n) &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \ln \sigma(D_{F^j(x)} f^{n_m(F^j(x))}) \\ &\geq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{k(x)-1} \alpha(F^j(x)) \\ &\geq \frac{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \alpha(F^j(x))}{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} n_m(F^j(x))}. \end{aligned}$$

By Birkhoff's ergodic theorem, the limits exist almost surely and are L^1 functions. Hence, the limit can be zero on a positive measure set only if the numerator is. Also, the points for which the numerator is zero form an invariant set $B \subset A_m$. But if $\mu(B) > 0$, then we can restrict the above argument to B and we obtain, for almost al $x \in B$, the contradiction

$$0 = \int_B \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \alpha(F^j(x)) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_B \alpha(F^j(x)) = \int_B \alpha(x) > 0.$$

The above implies that for each $v \in \mathcal{C}$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \|Df^n v\| = \lim_{n\to\infty} \frac{1}{2n} \ln \|Df^n v\|^2 \geq \lim_{n\to\infty} \frac{1}{2n} \ln Q(Df^n v)\\ \geq \lim_{n\to\infty} \frac{1}{2n} \ln \sigma(Df^n) > 0.\end{aligned}$$

Since the Lagrangian space $\mathbb{V} = \{(w,w)\} \subset \mathcal{C} \cup \{0\}$, we have a d-dimensional subspace with strictly poistive Lyapunov exponents. On the other hand, for each n we can set $\mathcal{C}_n(x) = D_{f^{-n}(x)} f^n \overline{\mathcal{C}}$. Since $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ there exists a Lagrangian subsapce $(w, Uw) \subset \cap_{n=0}^{\infty} \mathcal{C}_n$. Let, $\|w\| = 1$ and let $\xi_n = \|D_x f^{-n}(w,Uw)\|^{-1} D_x f^{-n}(w,Uw)$ so

$$(w, Uw) = \|D_x f^{-n}(w,Uw)\| D_{f^{-n}(x)} f^n \xi_n \in \mathcal{C}(x).$$

and

$$\begin{aligned}\|(w,Uw)\| = \|D_x f^{-n}(w,Uw)\| \|D_{f^{-n}(x)} f^n \xi_n\|\\ \|D_x f^{-n}(w,Uw)\| = \|D_x f^{-n}(w,Uw)\| \|\xi_n\| = \frac{\|(w,Uw)\|}{\|D_{f^{-n}(x)} f^n \xi_n\|}.\end{aligned}$$

Then, arguing as before,

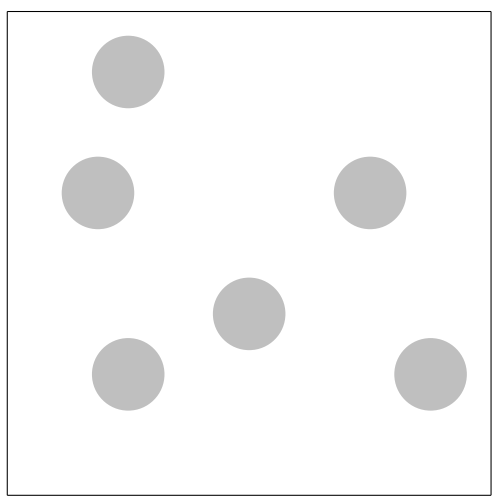
$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \|D_x f^{-n}(w,Uw)\| &= \lim_{n\to\infty} \frac{1}{n} \ln \frac{\|(w,Uw)\|}{\|D_{f^{-n}(x)} f^n (w,w)\|}\\ &= -\lim_{n\to\infty} \frac{1}{n} \ln \|D_{f^{-n}(x)} f^n (w,w)\|\\ &\leq -\lim_{n\to\infty} \frac{1}{2n} \ln \sigma(D_{f^{-n}(x)} f^n)\\ &\leq -\frac{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \alpha(F^{-j}(x))}{\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} n_m(F^{-j}(x))} < 0\end{aligned}$$

since the forward and backward ergodic averages are equal almost surely. We have thus d Lyapunov exponents strictly negative proving the theorem. □

Chapter 2

Billiards

The study of billiards has a double parallel history. On the one hand, starting at least with G. Birkhoff, they are seen as simple examples of dynamical systems and a tool to understand issues of integrability (billiard in an ellipse, polygonal billiards) and tool to understand strongly irregular motion (Sinai and Bunimovich Billiards). We here will concentrate on the second class of models. The genesis of the study of the latter type of billiards goes back at least to Boltzmann who proposed to study the properties of a gas imagining that it consists of balls colliding elastically.



A two dimensional gas of particles in a box

The (seemingly ridiculous) simplest case is a gas of two particles in two dimensions. For simplicity, let us consider two particles of radius $r < \frac{1}{2}$ in a torus of size one. Let $x_1, x_2 \in \mathbb{T}^2$ be the coordinate of the center of the disks, the velocity changes at collision according to the law

$$\begin{cases} v_1^+ = v_1^- - \langle n, v_2^- - v_1^- \rangle n \\ v_2^+ = v_2^- + \langle n, v_2^- - v_1^- \rangle n \end{cases} \tag{2.0.1}$$

where n is a unit vector in the direction $x_2 - x_1$.[1]

Here, there are three integrals of motion: the energy $E = \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2)$ and the total momentum $P = v_1 + v_2$. Thus, if we want to obtain an ergodic system, we have to reduce the system. We will then consider that phase spaces

$$X_{E,P} = \left\{ (x_1, x_2, v_1, v_2) \in \mathbb{T}^4 \times \mathbb{R}^4 \;\middle|\; \frac{1}{2}(\|v_1\|^2 + \|v_2\|^2) = E;\; v_1 + v_2 = P \right\}.$$

Since, in the velocity space, the previous conditions correspond to the intersection between the surface of a four-dimensional sphere (S^3) and a two-dimensional linear space, the velocity vectors $(v_1 + v_2)$ are contained in a one-dimensional circle. Thus, topologically, $X_{E,P} = \mathbb{T}^4 \times S^1$.[2] It is then natural to choose an angle θ as coordinate on S^1, moreover, since

$$2E = \|v_1\|^2 + \|v_2\|^2 = \frac{1}{2}\|v_1 - v_2\|^2 + \frac{1}{2}\|P\|^2,$$

it is hard to resist setting $v_2 - v_1 = v(\theta)$.[3] Hence,

$$\begin{cases} v_1 = \frac{1}{2}(P - v(\theta)) \\ v_2 = \frac{1}{2}(P + v(\theta)). \end{cases}$$

The free motion is then given by

$$\begin{cases} x_1(t) = x_1(0) + \frac{1}{2}(P - v(\theta))t \\ x_2(t) = x_2(0) + \frac{1}{2}(P + v(\theta))t. \end{cases}$$

Accordingly,

$$\begin{cases} x_1(t) + x_2(t) = x_1(0) + x_2(0) + Pt \mod 1 \\ x_2(t) - x_1(t) = x_2(0) - x_1(0) + v(\theta)t \mod 1. \end{cases}$$

[1] To be precise $x_2 - x_1$ has no meaning since $\mathbb{T}^2$ it is not a linear space. Yet, at collision, the distance between the two disks is $2r$, so the global structure of $\mathbb{T}^2$ is irrelevant, and we can safely confuse it with a piece of $\mathbb{R}^2$.

[2] Of course, we are considering only the cases $E \neq 0$.

[3] As usual $v(\theta) = (\sin\theta, \cos\theta)$.

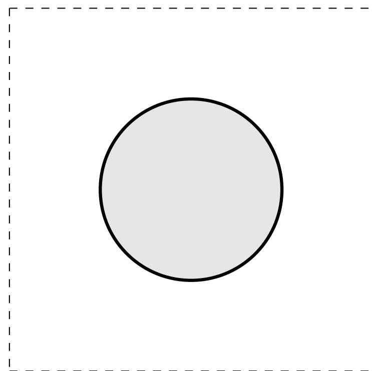


Figure 2.1: Sinai Billiard with infinite horizon

It is then clear the need to introduce the two new variables $Q = x_1 + x_2$ and $\xi = x_2 - x_1$. The variable Q performs a translation on the torus, such a motions are completely understood, and we can then disregard it. The only relevant motion is the one in the variables (ξ, θ). The reduced phase space is then $\mathcal{B} \times S^1$ where $\mathcal{B} = \mathbb{T}^2 \backslash \{\|\xi\| \leq 2r\}$, that is, the torus minus a disk of radius $2r$. The domain $\mathcal{B}$ is represented in the next Figure and, apart from the different choices of the fundamental domain, it corresponds exactly to the simplest Sinai billiard. The free motion corresponds to the free motion of a point as well, while at collision, from (2.0.1), we have

$$v(\theta^+) = v(\theta^-) - 2\Big\langle \frac{\xi}{2r}, v(\theta^-)\Big\rangle v(\theta^-)$$

that is exactly the elastic reflection from the disk!

It is then natural to consider the general problem of a particle moving in a region with reflecting boundary conditions. Let $\mathcal{B} \subset \mathbb{R}^d$ (or $\mathcal{B} \subset \mathbb{T}^d$) be the region and suppose that the boundary $\partial\mathcal{B}$ is made of finitely many smooth manifolds. Calling $(x, v) \in \mathcal{B} \times \mathbb{R}^d$ the position and the velocity, respectively, the motion inside $\mathcal{B}$ is described by a free flow

$$\phi_t(x, v) = (x + vt, v), \tag{2.0.2}$$

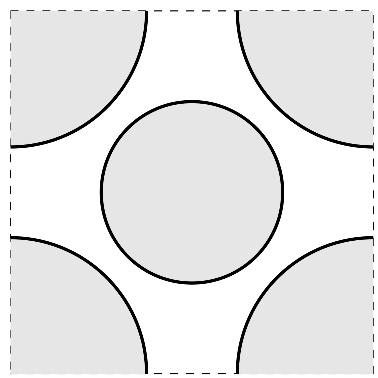
When $x \in \partial\mathcal{B}$, a collision takes place. If $n \in \mathbb{R}^d$, $\|n\| = 1$, is the normal to $\partial\mathcal{B}$ at x, then, calling v_- and v_+ the velocities before and after collision, respectively, the elastic collision is described by

$$v_+ = v_- - 2\langle v_-, n\rangle n.$$

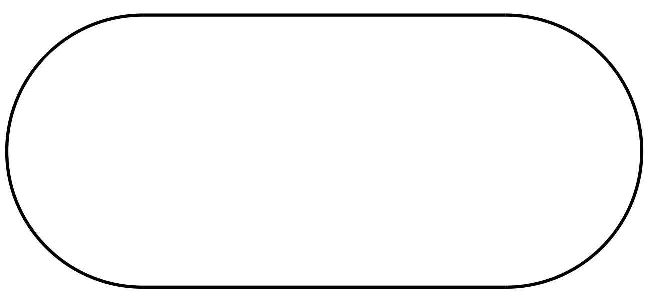
Remark 2.0.1 *Here, I will provide a few ideas on billiards and hyperbolicity. This should allow the reader to be able to easily takle a more complete account of the theory (in particular [9]).*

2.1 Some Billiard tables

In the two-dimensional case, there are many possible billiards tables that have been studied. The two most famous are the Sinai Billiard and the Bunimovich Stadium.



Sinai Billiard with finite horizon



Bunimovich stadium

Further interesting billiard tables can be found in [44, 7, 9] and references therein.

2.2 Collision map and Jacobi fields

To compute, in general, the collision map, it is helpful to introduce appropriate coordinates in $\mathcal{T}X$. A very interesting choice is constituted by the *Jacobi fields*.[4] Let X_- be the set of configurations just before collision. For each $(x, v) \in X \backslash X_-$ there exists $\delta > 0$ such that

$$\phi_t(x, v) = (x + vt, v) \quad 0 \leq t \leq \delta.$$

Let us consider the curve in $\mathcal{X}$

$$\xi(\varepsilon) = (x(\varepsilon), v(\varepsilon)),$$

with $\xi(0) = (x, v)$ and $\|v(\varepsilon)\| = 1$.

For each t such that $\phi_t(\xi(0)) \notin X_-$, let

$$\xi(\varepsilon, t) = (x(\varepsilon, t), v(\varepsilon, t)) = \phi_t(\xi(\varepsilon)).$$

The Jacobi field $J(t)$ is defined by

$$J(t) \equiv \left.\frac{\partial x}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Note that, since $x(0, t) \notin X_-$, for $s < \delta$

$$\xi(\varepsilon, t + s) = \xi(\varepsilon, t) + (v(\varepsilon, t)s, 0),$$

so

$$J'(t) = \frac{dJ(t)}{dt} = \left.\frac{\partial v(\varepsilon, t)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

That is, $(J(t), J'(t)) = d\phi_t \xi'(0)$.

At each point $\xi = (x, v) \in X$ we choose the following base for $\mathcal{T}_\xi X$:[5]

$$\eta_0 = (v, 0); \quad \eta_1 = (v^\perp, 0); \quad \eta_2 = (0, v^\perp);$$

[4] The Jacobi Fields are a widely used instrument in Riemannian geometry (see [18]) and have an important rôle in the study of Geodetic flows, although we will not insist on this aspect at present. Here, they appear in a very simple form.

[5] Here $v^\perp = Jv$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

where $\|v^{\perp}\| = 1$, $\langle v, v^{\perp}\rangle = 0$.

The vector η_0 corresponds to a family of trajectories along the the flow direction and it is clearly invariant; η_1 to a family of parallel trajectories and η_2 to a family of trajectories just after focusing. It is very useful the following graphic representation. We represent a tangent vector by drawing a curve that it is tangent to it. A curve in $\mathcal{T}X$ is given by a base curve that describes the variation of the x coordinate equipped with a direction at each point (specified by an arrow) which show how varies the velocity.

A direct check shows that each vector η perpendicular to the flow direction will stay so i.e.

$$\langle d\phi_t\eta, (v_t, 0)\rangle = \langle d\phi_t\eta, d\phi_t(v, 0)\rangle = \langle \eta, (v, 0)\rangle = 0.$$

So the free flow is described by

$$d\phi^t\eta_0 = \eta_0; \quad d\phi^t\eta_1 = \eta_1; \quad d\phi^t\eta_2 = \eta_2 + t\eta_1,$$

that is, in the above coordinates

$$d\phi^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}. \tag{2.2.3}$$

Let us see now what happens at a collision.

Let $x_0 \in \partial\mathcal{B}$ be the collision point and let $\xi_c = (x_0, v)$ be the configuration at the collision. We want to compute $R_\varepsilon := d_{\phi^{-\varepsilon}\xi_c}\phi^{2\varepsilon}$, that is the tangent map from just before to just after the collision. Clearly $R_\varepsilon\eta_0 = \eta_0$. If $\gamma(s)$ is the curve associated to η_1 at the point $\phi^{-\varepsilon}\xi_c$,

$$d\phi^{2\varepsilon}\gamma(s) = \left(v_+^{\perp}\left[s + \varepsilon\frac{2s}{r\sin\varphi}\right], \frac{2s}{r\sin\varphi}\right) + \mathcal{O}(s^2)$$

where r is the radius of the osculating circle (that is the circle tangent to the boundary up to second order) which is the inverse of the curvature $K(x_0)$ of the boundary at the collision point.

The above equation means that

$$J(\varepsilon) = (1 + \frac{2\varepsilon K(x_0)}{\sin\varphi})v_+^{\perp}.$$

Accordingly, calling $R = \lim_{\varepsilon\to 0} R_\varepsilon$ the collision map, we have

$$R\eta_1 = \eta_1 + \frac{2K}{\sin\varphi}\eta_2; \quad R\eta_2 = \eta_2.$$

Hence,

$$DR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2K}{\sin\varphi} \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2.4}$$

The above computations provide the following formula for the derivative of the Poincaré section from the boundary of the obstacle, just after collision, to the boundary of the obstacle just after the next collision

$$DT = \begin{pmatrix} 1 & \frac{2K}{\sin\varphi} \\ \tau & 1 + \frac{2\tau K}{\sin\varphi} \end{pmatrix}, \tag{2.2.5}$$

where τ is the flying time between the two collisions and φ the collision angle. Formula (2.2.5) is sometimes called *Benettin formula* (e.g., [26]).

2.3 Hyperbolicity of Sinai Billiard

As an example let us consider the Sinai Billiard depicted in Figure 2.1. Note that the system cannot be uniformly hyperbolic since there are trajectories that never hit the obstacle, and hence have clearly zero Lyapunov exponents. We define a cone family in the plane perpendicular to the flow direction $(v, 0)$, that is in the plane η_1, η_2, this plane is naturally isomorphic to the tangent space of $\mathcal{M}$ (just project along the flow direction) in each non-tangent point.

In the case in which no collision takes place, we have seen that the parallel family η_1 stays parallel, while the most divergent family (the vector η_2) becomes less divergent (a linear combination of η_1 and η_2 with positive coefficients). This means that the first quadrant (in the η_i coordinates goes into itself but the η_1 side stays put). Let us study what happens at a reflection. Any divergent family of trajectories will be divergent after the collision, and in particular, the parallel family will be strictly divergent. To be more precise the η_2 family will go into itself from just before to just after the collision, while the parallel one will be strictly divergent. Again the cone goes strictly inside itself but one side (the η_2 one this time) stays put. Nevertheless, the combination of free motion and reflection clearly sends the cone strictly inside itself.

Note that if a trajectory has a velocity with components with irrational ratios, then the flow without the obstacle is ergodic. This means that it is impossible that the trajectory does not hit the obstacle. Since the set of trajectories with velocities having components with rational ratios are of zero measure, it follows that almost all trajectories experience a collision. Hence, the billiard cocycle is eventually strictly monotone, and Wojtkowski's theorem applies. Accordingly, all the Lyapunov exponents are different from zero almost everywhere for the dynamical system $(\mathcal{M}, T, m)$.

2.4 Hyperbolicity of Bunimovich stadium

The naïve understanding of the previous example is that the obstacle acts as a defocusing mirror and thus makes the trajectories diverge, whereby creating instability. This idea was already present in Krylov work [29] and was considered the natural mechanism producing hyperbolicity. With this point of view in mind it seems that a table with convex boundaries (in which parallel trajectories are focused after reflections) is unlikely to yield hyperbolic behavior. This impression can be only confirmed by the presence of caustics in smooth convex billiards [30]. It came then as a surprise the discovery by Bunimovich that perturbations of the circle[6] could be hyperbolic.

The main intuition behind it is that, although the trajectories after reflection maybe focusing, after some time, they focus and then become divergent, so if there is enough time between two consecutive collisions, we can have divergent families going into divergent families, again (provided we look at the right place). Another equivalent point of view is that the instability is measured not just by the change in position but also, by the change in velocity, from this point of view, a very strong convergence is not so different from a strong divergence.

To find a new invariant family of cones, let us consider first a circular billiard. The collision angle is a conserved quantity of the motion. It is then natural to consider, at each point in phase space, the tangent vector η_3 associated to a family of trajectories that, at the next collision with one of the two half circles, will have the same collision angle.

We have defined η_3 in geometrical terms, clearly its expression in terms of η_1 and η_2 changes from point to point. Yet, there are special points (the middle of the cord between two consecutive collision with the same half-circle) in which η_3 coincides with η_2 (this is seen immediately by geometric considerations).

Clearly, in a sequence of collisions with the same circumference the vector η_3 is invariant. Also, from the above considerations follows that before

[6] Clearly non smooth perturbations such as the stadium, otherwise the KAM theorem would apply, see [22].

collisions η_3 corresponds to a diverging family, while immediately after a collision it corresponds to a convergent family.

What happens to the parallel family η_1? Since divergent families becomes convergent it is obvious that the parallel family, after reflection, becomes even more convergent. Hence, it will focus before the middle point to the next collision (the point where the family η_3 focuses).

The previous considerations suggest to consider the cone $\mathcal{C}(x, v) = \{\xi \in \mathcal{TM} \mid \alpha\eta_1 + \beta\eta_3 \text{ with } \alpha\beta \geq 0\}$.

Hence, for a trajectory that collides only with a half circle the cone just defined is invariant but not strictly invariant. Since this would be true also for a billiard inside a circle it is clearly not sufficient (the billiard inside a circle has zero Lyapunov exponents, since, as we have already remarked, the motion is integrable).

Let us go back to the Bunimovich stadium. Clearly, it will behave as a circular billiard for trajectories colliding only with a half circle. So we need to see what happens if a trajectory goes from one circumference to the other (which will happen with probability one). In this case, the infinitesimal motion is the same that would happen if the straight line would be not present. In fact, if we reflect the billiard table along the straight lines we can imagine that the motion proceeds in a straight line.

Hence the family η_3 will first focus and than diverge for a longer time (and so get closer to the parallel family) than would happen if the collision would be in the same circle. This is exactly what we need to get strict invariance of the cone family.

In conclusion the cone family is strictly invariant each time that the trajectory goes from one half circle to the other. Since this happens almost surely, again we have proven hyperbolicity of the system.

It is interesting to notice that the cone family coincide with the one used in the Sinai Billiard (divergent trajectories) if one looks at it at the right point: the point laying in the intersection between the trajectory before collision and the circle of radius $r/2$ (if r is the radius of the half-circles forming the table) tangent to the the boundary at the next collision with a half-circle (but nowhere else).[7]

[7] If the last collision was with a flat wall, then the point is obtained by reflecting the billiard so the trajectory backward looks straight, determining the point and then reflecting back to find the real point on the trajectory.

2.5 Hard spheres

For hard balls or radius $\frac{1}{2}$, and mass one, in dimension d, the flow is given by $\phi_t(q,p) = q + tp$ if no collision occurs. If the ball i collides with the ball j, then let p_i^-, p_j^- and p_i^+, p_j^+ be the velocities just before and after the collision, respectively. Note that for the balls to collide is must be that before the collision

$$0 > \frac{d}{dt}\|q_i - q_j\|^2 = \langle q_i - q_j, p_i - p_j\rangle.$$

Thus, at collision, $\langle q_i - q_j, p_i - p_j\rangle \le 0$. Let $n = q_i - q_j$, then

$$\begin{aligned} p_i^+ &= p_i^- - \langle n, p_i^- - p_j^-\rangle n \\ p_j^+ &= p_j^- + \langle n, p_i^- - p_j^-\rangle n. \end{aligned} \tag{2.5.6}$$

Let us $d_{(q,p)}\phi_t(\delta q, \delta p)$ across a collision. If τ is the collision time of the trajectory then $\|q_i(\tau) - q_j(\tau)\| = 1$. If we consider the trajectories $\phi_t((q,p) + s(\delta q, \delta p))$, then the collision time $\tau(s)$ satisfies

$$\langle q_i(\tau) - q_j(\tau), \delta q_i - \delta q_j\rangle + \langle q_i(\tau) - q_j(\tau), p_i(\tau) - p_j(\tau)\rangle \tau'(0) = 0.$$

If the collision is non tangent (i.e. $\langle n, p_i(\tau) - p_j(\tau)\rangle \neq 0$), then,

$$\tau'(0) = -\frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle}.$$

To compute $d\phi_t$ is then convenient to shift along the flow direction by τ so all the trajectories $(q,p) + s(\delta q, \delta p)$ collide simultaneously. Let us call $(\tilde{\delta} q, \tilde{\delta} p)$, the shifted tangent vector. For such a tangent vector, we have that (2.5.6) yields

$$\begin{aligned} \tilde{\delta} q_i^+ =& \tilde{\delta} q_i^- \\ \tilde{\delta} q_j^+ =& \tilde{\delta} q_j^- \\ \tilde{\delta} p_i^+ =& \tilde{\delta} p_i^- - \langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, p_i^- - p_j^-\rangle n - \langle n, p_i^- - p_j^-\rangle (\tilde{\delta} q_i^- - \tilde{\delta} q_j^-) \\ & - \langle n, \tilde{\delta} p_i^- - \tilde{\delta} p_j^-\rangle n \\ \tilde{\delta} p_j^+ =& \tilde{\delta} p_j^- + \langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, p_i^- - p_j^-\rangle n + \langle n, p_i^- - p_j^-\rangle (\tilde{\delta} q_i^- - \tilde{\delta} q_j^-) \\ & + \langle n, \tilde{\delta} p_i^- - \tilde{\delta} p_j^-\rangle n. \end{aligned} \tag{2.5.7}$$

And the derivative is then obtained shifting back along the flow direction. Note that, by construction $\langle \tilde{\delta} q_i^- - \tilde{\delta} q_j^-, n\rangle = 0$.

To apply Theorem 1.6.1 we have thus to construct the quadratic form Q. We choose the lagrangian spaces $\mathbb{V}_1 = \{\delta q = 0\}$ and $\mathbb{V}_2 = \{\delta p = 0\}$. The energy is only kinetic energy, then the vectors tangent to the constant energy are $\langle p, \delta p\rangle = 0$. This yields the form $Q(\delta q, \delta p) = \langle \delta q, \delta p\rangle$. The vector field is $(p, 0)$, and $Q(\delta q + \alpha p, \delta p) = Q(\delta q, \delta p)$, so Q is well defined on the quotient and we can restrict ourselves to the vectors $\{(\delta q, \delta p) \,:\, \langle p, \delta p\rangle = \langle p, \delta q\rangle = 0\}$. Note that

$$Q((\delta q + t\delta p, \delta p)) = Q(\delta q, \delta p) + t\|\delta p\|^2 \geq 0.$$

and if just a collision takes place in the interval $[0, t]$, then

$$Q((\delta q + tp, \delta p)) = Q(\tilde{\delta} q, \tilde{\delta} p) = Q(\delta q, \delta p) - \langle n, p_i^- - p_j^-\rangle \|\tilde{\delta} q_i^- - \tilde{\delta} q_j^-\|^2 \geq 0.$$

The invariance of the cone follows.

Note that we have strict invariance if $\delta p \neq 0$. If $\delta p = 0$, then we have the strict invariance if $\tilde{\delta} q_i^- \neq \tilde{\delta} q_j^-$. This fails only if

$$\delta q_i^- - \frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle} p_i = \delta q_j^- - \frac{\langle n, \delta q_i - \delta q_j\rangle}{\langle n, p_i(\tau) - p_j(\tau)\rangle} p_j,$$

i.e. there exists $z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \delta q_i &= z + \lambda p_i \\ \delta q_j &= z + \lambda p_j. \end{aligned} \tag{2.5.8}$$

To see how to use the above facts, it is convenient to introduce a bit of notation.

2.6 Collision graphs

First of all I will introduce a *collision graph* to describe pictorially the relevant features of a trajectory, it will be a directed graph (the direction being given by time). The graph starts with n roots (each one representing one ball), from each root starts an edge (representing the path of a ball). A collision is represented by a vertex in the graph (I will idicate it pictorially by a star not to confuse it with edges that crosses on due to the two dimensional reprentation). If the collision involves k balls, then the vertex will have degree $2k$ with k entering edges–representing the incoming particles–and k exiting edges–representing the outcoming particles. Note that typically each vertex will have degree four, yet in the following we will generalize the meaning of a vertex and vertex of higher degree will play an important role.

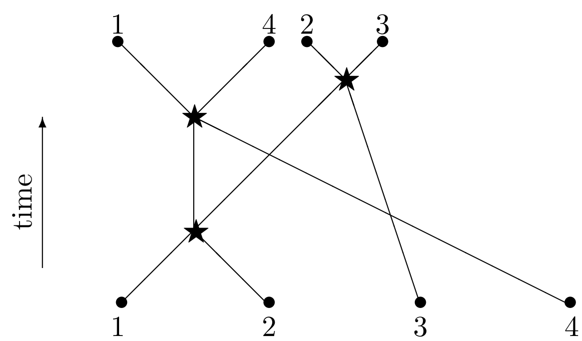


Figure 2.2: A simple collision graph (the stars are the collisions)

See figure 2.2 for the case of four balls in which number one collides with two, then two with four, and finally two with three.[8]

Next, let us call $\mathcal{G}$ a collision graph and let $V(\mathcal{G})$ be the collection of its vertexes, $\tilde{B}(\mathcal{G})$ the collection of its edges and $B(\mathcal{G})$ the collection of edges that connect starred veteces. In addition for each edge $b \in B(\mathcal{G})$ let $\nu(b), \nu_+(b)$ be the two vertices joined by the edge.[9]

To follow the history of a vector of type $(\delta q, 0)$ that stubbornly refuses to enter strictly in the cone it is convenient to specify at each vertex the values (λ_ν, z_ν) appearing in the associated equation (2.5.8). Of course, to recover the tangent vectors from the $\{(\lambda_\nu, z_\nu)\}_{\nu \in V(\mathcal{G})}$, it is necessary to specify the velocities. To this end we specify for each edge the velocity $v(b)$ of the particle associated to such a line. We can then decorate a graph with the above informations and we obtain a full description of the history of a tangent vector that keeps being not increased by the dynamics in the trajectory piece described by the graph (of course provided such a vector exists at all).

Now consider a edge $b \in B(\mathcal{G})$, if it represents the trajectory of the particle j between the collision corresponding to the the vertex $\nu(b)$ and the one corresponding to the vertex $\nu_+(b)$, then the corresponding component of the tangent vector at such times can be written both as $\delta q_j = z(\nu(b)) + \lambda(\nu(b))v(b)$ and $\delta q_j = z(\nu_+(b)) + \lambda(\nu_+(b))v(b)$. Accordingly, the following

[8] The rule for tracing the graph is that the order of the balls is not changed at collision, so the line on the left represents the particles entering the collision vertex from the left. Remark that the collision graph is only a symbolic device and does not respect the geometry of the actual collisions, so the ordering of the balls is only a device to tell them apart and has no relation with the actual geometry of the associated configuration. Keeping this in mind, in figure 2.2 the final disposition of the balls is: one, four, two, three.

[9] By convention $\nu(b)$ corresponds to the lower collision and $\nu_+(b)$ to the upper.

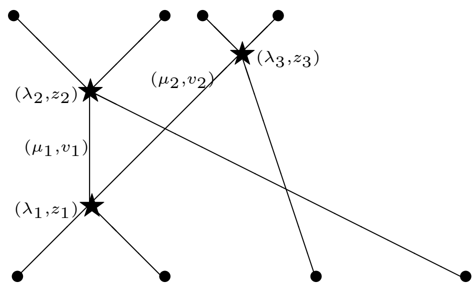


Figure 2.3: A decorated collision graph

compatibility condition must be satisfied:

$$z(\nu(b)) - z(\nu_+(b)) = [\lambda(\nu_+(b)) - \lambda(\nu(b))]v(b). \tag{2.6.9}$$

It is then natural to define another decoration, this time associated to edges that connect two collision vertexes,

$$\mu(b) := \lambda(\nu_+(b)) - \lambda(\nu(b)). \tag{2.6.10}$$

By decorated collision graph, we will mean a graph with $(\lambda(\nu), z(\nu))$ attached to each vertex and $\mu(b), v(b)$ to each edge connecting two collisions, with a mild abuse of notations we will call such decorated graph $\mathcal{G}$ as well, see figure 2.3.[10]

2.7 Cycles

As time progress the graph will grow more complex, in particular it may develop *cycles*. By a cycle I mean a connected path of edges that leave a vertex and go back to it, e.g. the thick edges in the graph of figure 2.4.

Once a cycle is formed a remarkable compatibility condition can be derived. In fact, let $C \subset \mathcal{G}$ be a cycle, let us run it counterclockwise and define, for each edge $b \in C$, $\varepsilon_C(b) = 1$ if the edge is run from bottom to top and $\varepsilon_C(b) = -1$ if it is run from top to bottom. We have, by definition (2.6.10), $\sum_{b\in B(C)} \varepsilon_C(b)\mu(b) = 0$. In addition, we can sum equation (2.6.9) for each

[10] Note that the above description is quite redundant due to (2.6.9), yet we will see in the following that such a description is quite convenient.

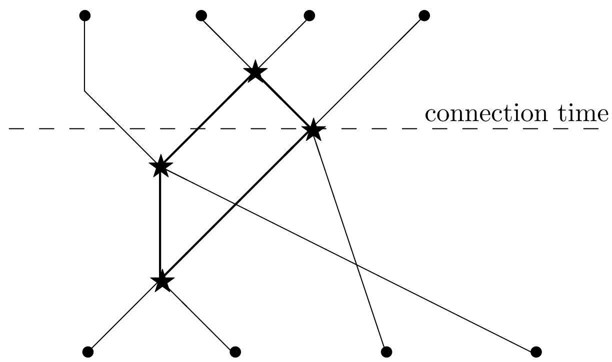


Figure 2.4: A cycle

edge in the cycle and obtain

$$\begin{aligned} &\sum_{b\in B(C)} \mu(b)\varepsilon_C(b)v(b) = 0 \\ &\sum_{b\in B(C)} \varepsilon_C(b)\mu(b) = 0. \end{aligned} \tag{2.7.11}$$

The above formula is essentially the *closed path formula* introduced by Simanyi in [37]. Such a formula expresses a compatibility condition that puts a clear restriction on the possible existence of the decorated collision graph, and hence of the corresponding nonincreasing vector. Studying the combinatorics of such collisions, it is possible to establish the hyperbolicity, and ergodicity, of a gas of n particles. This has been done in a series of papers of the *Hungarian team* [28, 37, 38, 39, 40].

2.7.1 Exemples: 2 ball in $d \geq 2$

First of all recall that that there are always zero Lyapunov exponents connected with the flow direction and the momenta conservation. To avoid that we consider only the situation in which the center of mass is at rest. This implies that $\sum_i \delta q_i = 0$.

Next, notice that the situation in which the two balls never collide is of zero measure: if there is no collision, the two balls just perform a translation on the torus. One can then see it as a translation on $\mathbb{T}^{2d}$, which is ergodic if the velocities have entries that are not rational among them, which is a zero-measure condition. Hence, the ball will collide with probability one.

Once the two ball collide for the second time we have a close cycle made of just two bonds $\{b_1, b_2\}$. In this case (2.7.11) implies

$$\mu(b_1)(v(b_1) - v(b_2)) = 0,$$

which has solutions either $\mu(b_1) = 0$ or $v(b_1) = v(b_2)$. The latter condition is a zero measure one, hence with probability one $\mu(b_1) = 0$. But then the second of (2.7.11) implies $\mu(b_2) = 0$. It follows that $\delta q = (z, z)$ which implies $z = 0$, hence eventual strict monotonicity and hence hyperbolicity.

2.7.2 Exemples: 3 ball in $d \geq 2$

Again we can assume $\sum_i \delta q_i = 0$. Also, we would like to know that the situation in which a ball does not collide with the other two has zero measure. This is a bit more complex and needs the mixing of the two ball systems. Let us assume it and proceed.

After the first collision (say of particles $1, 2$), we wait until a collision involving particle 3, say a collision with particle 1. The next collision will close a cycle. If the cycle involves particles $1, 3$, then the previous discussion implies $\mu(b_1) = \mu(b_3) = 0$. This situation will persist until there is a collision with particle 2, say with particle 3. At that point (2.7.11) implies $\mu(b_2) = 0$. It follows that, almost surely $\delta q = (z, z, z)$ and hence $z = 0$.

The case in which the cycle involves all the particles remains to be analyzed, say particle 3 collides with particle 2. In this case (2.7.11) implies

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) + \mu(b_3)\varepsilon_C(b_3)(v(b_3) - v(b_2)) = 0$$

where b_1 is the edge associated to the particle 1 before the collisione with 3, b_3 is the edge associated to the partile 3 after the collision with 1 and b_2 the edge associated to the particle 2 after the collision with 1. On the other had, calling b_3^- the edge associated to 3 before the collision with 1, by (2.5.6) we have

$$v(b_3) = v(b_3^-) - \langle n, v(b_3^-) - v(b_1)\rangle n.$$

Note that $R := \langle n, v(b_3^-) - v(b_1)\rangle = 0$ is a codimension one condition hence it happens on a zero measure set. Accordingly, just before the collision of 1 and 3 we have

$$\mu(b_1)\varepsilon_C(b_1)(v(b_1) - v(b_2)) - \mu(b_3)\varepsilon_C(b_3)(v(b_1) - v(b_3^-) - Rn) = 0.$$

Again, for $v(b_1) - v(b_2)$ and $v(b_1) - v(b_3^-) - Rn$ to be linearly dependent is a codimension one condition. It follows $\mu(b_1) = \mu(b_3) = 0$ and then (2.7.11) implies $\mu(b_2) = 0$. So $\delta q_i = 0$, thus the form is eventually strictly increasing.

2.8 Geometry of foliations and ergodicity (very few words)

Once we know that the system is hyperbolic, we can try to take advantage of hyperbolicity: the first step is to construct stable and unstable manifolds. The strategy is the usual one: e.g., to construct the unstable manifold at x, consider the trajectory $f^{-n}(x)$ (for simplicity, we consider the Poincarè map). If the trajectory does not meet a discontinuity, then we can consider a manifold W, with tangent space in the unstable cone, centered at $f^{-n}(x)$ and push it forward with the dynamics. In this way, we obtain a sequence of manifolds $W_n = f^n(W)$ that we expect to converge to a limit object. Yet, one has to take into account that the manifold can be cut by singularities, and this could be a serious problem.

In the uniformly hyperbolic case, the analysis is especially simple: since the manifold W expands exponentially ($|W_n| \geq e^{\lambda n}|W|$), we have that the manifolds are cut at a distance shorter than δ only if the distance of $f^{-n}(x)$ from the singularities is less than $\delta e^{-\lambda n}$. This means that the manifold is cut short only if $f^{-n}(x)$ belongs to a neighborhood $\mathcal{S}_n$ of measure $\delta e^{-\lambda n}$. But since the measure is preserved, we have

$$\text{Leb}\left(\cup_{n=0}^{\infty} f^n(\mathcal{S}_n)\right) \leq \sum_{n=0}^{\infty} e^{-\lambda n}\delta \leq C\delta.$$

It follows that there exists a set of measure $1 - C\delta$ in which the unstable manifold has a length larger than δ.

Implementing the above basic idea can be technically challenging, especially since the formula (2.2.5) shows that the derivative blows up near tangencies. Yet, it can be done, for details, see [26, 9]. A technical tool used to deal with the blow-up of the differential at tangent collisions is the introduction, by Sinai, of homogeneity strips. See [9] for details.

The above construction provides a stable foliation, yet the foliation has very poor regularity properties, and this makes it very hard to use it; in general, it is only measurable. Luckily, the holonomy is absolutely continuous. Moreover, it turns out that it can be approximated by a foliation with much better properties that can be conveniently used, see [2, Section 6] for details.

The next step is to prove ergodicity. Once we have an absolutely continuous foliation, you can try to copy Hopf's argument. Such an argument is based on the observation that the ergodic averages of continuous functions are constant along stable and unstable manifolds. This was achieved by Sinai [41]. But see [31] for a more general version. In addition, [31] discusses a piecewise linear example in which the technical difficulties are reduced to a bare minimum, and hence Sinai's argument can be easily understood. The idea is to prove local ergodicity, and then a global argument can be employed to prove ergodicity. The same argument proves that all the powers of the Poincare maps are ergodic, which implies mixing.

It remains the problem of flows. Since the flow can be seen as a suspension over the Poicnarè map, the ergodicity of the flow follows from the ergodicity of the map. Not so for mixing: think of a suspension with a constant ceiling. Mixing for the flows follows from the contact structure. Forgetting for one second the discontinuities, the fact that the flow is contact implies that is we do a cycle stable, unstable, stable, unstable, we move in the flow direction, see Figure 2.5. Indeed,

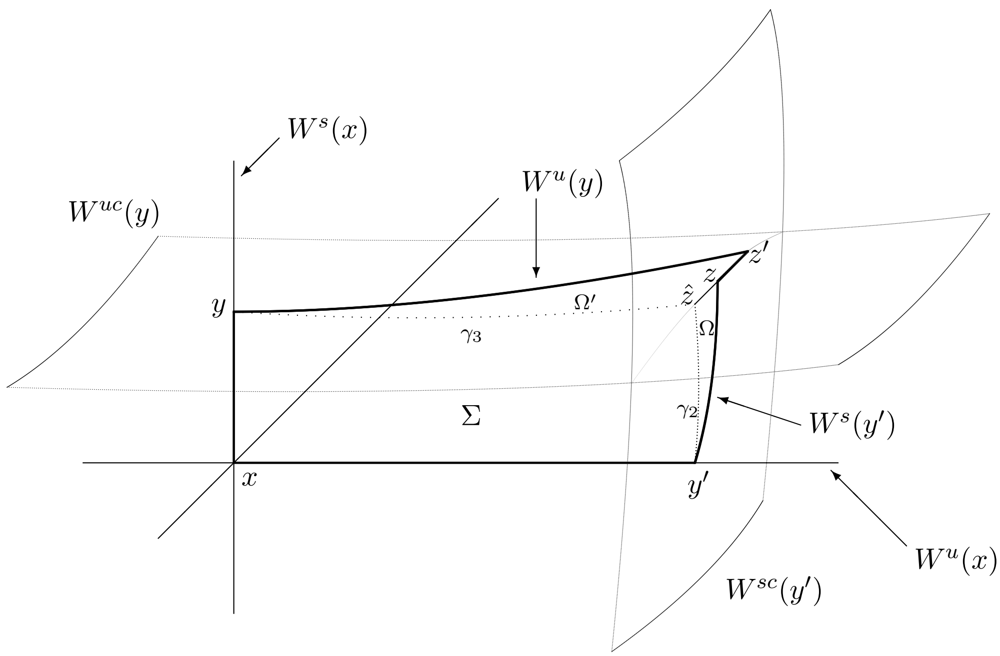


Figure 2.5: Definition of the *temporal function* $\Delta(y, y')$ and related quantities

let α be the contact form, then if v is a strong unstable or a strong stable vector, then $\alpha(v) = 0$, while $\alpha((p, 0)) = 1$, where $(p, 0)$ is the flow direction, it follows that if the cycle in bold in figure 2.5, call it γ, has sides of length δ, then

$$\delta^2 = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = \int_\gamma \alpha$$

which equals exactly the displacement in the flow direction, which is then non-zero. It follows that the stable and unstable foliations are not *jointly integrable*, and this property shows that the flow cannot be reduced to a constant flow suspension by a change of coordinates (since, in such a case, the foliations would indeed be jointly integrable). This suffices to prove the mixing of the flow, eventhough the argument is a bit more technical than this.

Chapter 3

Statistical Properties

3.1 The problem and a brief overview

Given a topological Dynamical System we would like first to characterize the invariant measures in order to have a clearer picture of which measurable Dynamical Systems can be associated with them. This is still at the qualitative level. In addition, we would like to have tools to actually compute such invariant measures with a given precision, and this is a first quantitative issue.

Next, we would like to study in-depth statistical properties for some measures that we deem interesting. The type of questions we would like to address are

If we make repeated finite time and precision measurements, what do we observe?

Remember that a measurement is represented by the evaluation of a function. The fact that the measurement has a finite precision corresponds to the fact that the function has some uniform regularity (otherwise, we could identify the point with an arbitrary precision). The fact that the measure is made for finite time means that we are able only to measure finite time averages. In other words, we would like to understand the behavior of

$$\sum_{k=0}^{N-1} f \circ T^k$$

for large but finite N.

We will see that to achieve this, it is necessary, first and foremost, an estimate of the speed of mixing. In the case of two-dimensional hyperbolic billiards, Bunimovich and Sinai first achieved this [5] for the Poincarè map, while the result for the flow is due to Demers, Baladi Liverani [11], almost forty years later (not for lack of trying). For higher dimensional billiards, the problem is still open.

Several techniques have been developed to study the speed of decay of correlations, the main one are

1. coding the system via Markov Partitions (Bunimovich and Sinai [5])
2. coding the systems via towers (Lai-Sang Young [46, 47])
3. standard pairs and coupling (Lai-Sang Young [47], Dolgopyat [19])
4. operator renewal theory (Sarig [36])
5. Functional spaces adapted to the transfer operator (Blank, Keller, Liverani [4]; Liverani, Gouezel [25]; Baladi, Tsujii [1]; Demers, Liverani [12]; Demers, Zhang [16])
6. Hibert metric (Ferrero, Schmitt [23], Liverani [33]; Demers, Liverani [17])
7. Random perturbations (Liverani, Saussol, Vaienti [34])

The most powerful techniques are probably (5, 6), but they can work only if the decay of correlations is exponential. For polynomial decay of correlations (2, 4) or even the rougher (7) are the way to go. While (3) is unquestionably the more versatile technique.

For an introduction to (3,5,6) see [13].

To conclude, let me recap part of the state of the art, giving a, idiosyncratic, list of results.

The ergodicity of various billiard tables was established in many papers, e.g., [44, 7]. Ergodicity results also exist for billiards in which the particle is subject to a soft potential, rather than a hard core one, e.g. [27, 21]. The ergodicity of a gas of hard spheres was established, building on a rather long string of papers, in [40]. The statistical properties of billiards with finite and infinite horizon can be found in [8, 20] where the standard pair technology is put to work. The functional analytic approach has been developed in [14, 16]; such an approach also allows establishing how the statistical properties depend on the billiard shape [15]. In addition, the functional approach has proven instrumental in the proof of exponential mixing for two dimensional uniformly hyperbolic billiard flow [2]. Many limit theorems have been obtained for billiard systems for which mixing properties have been established. Notable results are the polynomial decay of correlations in the Bunimovich stadium [3] and the monumental study of one massive particle interacting with a light one in a box [10].

All the previous papers deal with isolated systems, if the system changes in time (e.g. a time-dependent billiard table), then the simple study of the spectral properties of the transfer operator does not suffice; one has to deal with the product of different operators. This can be done using perturbation theory if the change in time is very slow [42]. However, if the change in time is more violent, perturbation theory fails, and a new approach is needed. This has been recently achieved in [17] using Hilbert metrics on invariant cones of densities.

Even though the above list of results is very partial, I hope it gives an idea of the breadth of the field and of the many directions along which the research is developing.

Given a Dynamical System, it is, in general, very hard to study its ergodic properties, especially if the goal is to have a *quantitative* understanding. To make clear what is meant by a *quantitative understanding* and which type of obstacles may prevent it I refere the reader to the first chapters of the book [13], available online.

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