ADDENDUM TO "EXISTENTIALLY COMPLETE CLOSURE ALGEBRAS"

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(14 maggio 1998)

SOMMARIO. Si mostra, usando i metodi di [2], che la teoria universale delle algebre di chiusura è decidibile.

A *Closure algebra* is a Boolean algebra together with an operator K satisfying the formal properties of closure in a topological space. McKinsey and Tarsky [1] announced that the (first-order) theory of closure algebras is undecidable. In contrast, the proof of [2, Theorem 1] shows:

Theorem. *The set of universal consequences of the theory of closure algebras is decidable.*

We now show how the arguments in [2] can be used to prove the theorem.

A universal sentence ψ is a consequence of the theory of closure algebras iff there is no closure algebra which is a model of the existential sentence $\neg\psi$, that is, iff no extension of the two-element closure algebra is a model of $\neg\psi$.

In [2] we dealt with a more general problem, that is, we considered an arbitrary closure algebra $\mathcal{A}$ and an existential formula φ with constants from A, and we found necessary and sufficient conditions for the existence of an extension of $\mathcal{A}$ satisfying φ (we supposed disjunctions were factored out, but this is no loss of generality). First, we showed how to reduce the existential formula φ to an equivalent form which involved only a polynomial P with certain coefficients a_{st} (defined in terms of the constants from A); then we constructed increasing sequences $w_{st}(m,\ddot{a})$, such that $w_{st}(0,\ddot{a})=a_{st}$, and such that an extension as above exists iff for all m the $w_{st}(m,\ddot{a})$'s satisfy a certain property.

By our first remark here, we need to consider the particular case when $\mathcal{A}$ is the two-element closure algebra; in this case each a_{st} and each $w_{st}(m,\ddot{a})$ is either 0 or 1; since each $w_{st}(m,\ddot{a})$ is increasing, as a function of m, it is eventually constant; moreover, the indexes s and t vary on a finite set, so that it is enough to consider a finite number of m's (actually, just one). The procedure described in [2] is effective, and, by the preceding period, in this particular case, terminates in a finite number of steps, so that we have an effective way to decide whether a universal sentence is a consequence of the theory of closure algebras.

REMARKS. (i) A different proof can be found in [1]. We suspect that the algorithm furnished by the present proof is more efficient, but we have not checked this.

(ii) The constructions in [2] show that if a universal sentence fails in some closure algebra then it fails in a closure algebra whose underlying Boolean algebra is atomic and complete; that is, the sentence fails in a topological space (an atomic and complete Boolean algebra is isomorphic to the set of subsets of a set; we get a topological space by taking, as closed subsets, those x's satisfying Kx=x).

(iii) The same arguments show that set of universal consequences of the theory V_α is decidable, too (cf. [2, Remark 2].

We discovered the above arguments shortly after [2] had been printed, but only now we have written down the details.

References.

[1] McKinsey, A.Tarski, The algebra of topology, Ann.Math. (2) 45, 141-191 (1944).
[2] P. Lipparini, Existentially complete closure algebras, Bollettino UMI, Serie VI, Vol.I-D, N.1 (1982).

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