Difference terms and commutators.

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Paolo Lipparini

After J.D.H. Smith's monograph [Sm], and subsequent generalizations (see [Gu, FMK]) the commutator has become a major tool for a refined analysis of congruence modular varieties.

Outside modular varieties commutator theory cannot be applied as it stands, since the commutator may be defined in several ways, which turn out to be non equivalent, in general; nonetheless, some applications have been found, under rather mild assumptions [HMK, Lp1-4].

Recently, it has been discovered that a substantial part of the modular commutator theory follows already from the existence of a difference term: see [Ke], parts of [Lp4], [Lp5] (mainly in connection with [Ki]) and, especially, [Lp6]; recall that every algebra in a congruence modular variety has a difference term, but there are non-modular varieties still having a difference term.

Actually, in [Lp4-6] we develop commutator theory for single algebras with a difference term, and use it to provide improvements and simplified proofs of many results obtained for modular varieties. For example, in [Lp6] we show that if **A** has a difference term and $\delta=(\alpha_0+\beta_0)(\alpha_1+\beta_1)(\alpha_2+\beta_2)$, then for every n: $\delta=(\alpha_0 \circ \beta_0)(\alpha_1 \circ \beta_1)(\alpha_2 \circ \beta_2) \circ \delta^{(n)}$ ($\delta^{(n)}$ denotes the solvable series). The above identity is used to solve a problem in [Ts] (condition C there is equivalent to congruence modularity), and to improve the main result from [FJ] to: if **A** has a difference term and Con **A** is modular then Con **A** is arguesian.

The present paper is a complement to [Lp4-6]: we show that if a single algebra has a difference term, then many definitions of the commutator coincide. In particular, we strengthen a result from [Ki]: there he showed, among other things, that in every congruence modular variety the usual commutator (defined by the Term Condition) coincides with the commutator defined using the Two Terms Condition. Here we furnish a very direct proof of this result, actually using only a difference term, and indeed we get a stronger conclusion: the TC-commutator coincides with a ω-commutator, whose definition involves n terms.

The conditions defining the ω-commutator are specializations of the m-implications introduced in [Qu]; taking all m-implications into account leads to the definition of the linear commutator (see [KQ], [KP, p.167]): we also show that there exists a finite algebra in which the linear and the ω commutators are distinct.

A preliminary version of this work has been circulating since April 1994 (the result in the Proposition has been obtained in July 1994). K.Kearnes has provided many suggestions which have been used in order to improve the manuscript.

We are going to define a commutator $[\alpha,\beta]_\omega$ in such a way that an algebra **A** satisfies $[1,1]_\omega=0$ iff in $\mathbf{A}\times\mathbf{A}$ the congruence θ generated by $\{(a,a) | a\in A\}$ is a common complement of the kernels π_1 , π_2 of the two canonical projections; that is, θ, π_1 and π_2 generate a lattice isomorphic to M_3 (actually, $\theta\pi_1=0$ is enough to imply $[1,1]_\omega=0$).

$[\alpha,\beta]_\omega$ will be the union of an increasing chain of commutators $[\alpha,\beta]_n$ $(n\geq 1)$; the definition of each $[\alpha,\beta]_n$ involves n terms.

For n=1 we get the "classical" commutator $[\alpha,\beta]=[\alpha,\beta]_1$ of, say, [FMK, Definition 3.1(3)] (that is, the commutator defined by the specialization of the Term Condition; the Term Condition then becomes $[1,1]=0$).

For n=2 we get the Two Terms commutator defined in [Ki] (actually, [Ki] defined the α-β two terms condition which, in the present terminology, corresponds to $[\alpha,\beta]_2=0$).

The most general form of "conditions" involving n terms are the "n-implications" of [Qu] (but here we deal with terms, rather than operations); it is trivial to see that every 1-implication is equivalent to the Term Condition; and that a 2-implication either reduces to the Term Condition or is equivalent to the two terms condition.

For $n\geq 3$ there are many different kinds of n-implications: each one can be described by a pair of permutations (on the set $\{1,\dots,n\}$) ; in defining $[\alpha,\beta]_n$ we select those n-implications which are described by two identical cycles; thus we sometimes call $[\alpha,\beta]_n$ and $[\alpha,\beta]_\omega$ the (n-terms) <u>cycle</u> commutator.

The proof of [Qu, Theorem 6] shows that, for every n, there exists a finite groupoid in which $0=[1,1]_n<[1,1]_{n+1}$; so that, in general, $[\alpha,\beta]_n$ grows larger as n grows larger (since we are dealing with terms, rather than basic operations, the proof of [Qu, Theorem 6] has to be slightly modified: the groupoid has to satisfy $x(yz)=x(y'z')$ and $(xy)z=(x'y')z$; but this can be easily obtained: it goes exactly as in the proof of the Proposition below*).

Moreover, the proof of the quoted theorem from [Qu] can be adapted to build a finite groupoid G satisfying $0=[1,1]_\omega<[1,1]_L$ ($[-,-]_L$ denotes the linear commutator [Qu, KQ, KP]); in other words, G is not quasi affine, that is, does not satisfy the Additive Term Condition [Qu]; actually, a 3-implication fails in G (see the proof of the Proposition below).

(*) THIS IMPROVES QUACKENBUSH'S RESULT IN THE SENSE THAT THE GROUPOID CAN BE CHOOSEN TO BELONG TO A LOCALLY FINITE VARIETY.

In conclusion, we get that $[\alpha,\beta]=[\alpha,\beta]_1\leq\ldots\leq[\alpha,\beta]_n\leq[\alpha,\beta]_{n+1}\leq\ldots\leq[\alpha,\beta]_\omega\leq[\alpha,\beta]_L$, and that each inequality may be strict.

The situation changes if we assume the existence of a weak difference term: in the case $\alpha=\beta$ all the above commutators coincide, as shown in [Lp3, Remark 3.3(a)]. On the other side, [Lp4, Remark 4.9] shows that there is an algebra with a weak difference term (even, belonging to a 4-permutable variety) and with two congruences α, β for which $[\alpha,\beta]<[\alpha,\beta]_2$.

We show in the present paper that if there is a difference term then $[\alpha,\beta]=[\alpha,\beta]_n=[\alpha,\beta]_\omega$, for every n, α and β. Meanwhile K. Kearnes and A. Szendrei improved this to $[\alpha,\beta]=[\alpha,\beta]_L$: actually, they used a much weaker hypothesis (it holds in the particular case of congruence modular varieties). Previously, E.Kiss and R. Quackenbush [KQ] had proved that in modular varieties $[\alpha,\beta]=[\alpha,\beta]_L$ when α and β permute).

Now for the relevant definitions.

Let **A** be an algebra, and α, β, γ be congruences on **A** (actually, the definitions make sense and have some interest even when α,β and γ are relations on A).

$M(\alpha,\beta)$ denotes the set of all matrices of the form:

$$\begin{vmatrix} t(\overline{a}, \overline{b}) & t(\overline{a}, \overline{b}') \\ t(\overline{a}',\overline{b}) & t(\overline{a}',\overline{b}') \end{vmatrix}$$

where $\overline{a},\overline{a}'\in A^n$, $\overline{b},\overline{b}'\in A^m$, for some n, m ≥ 0 , t is any m+n-ary term operation of **A**, and $\overline{a}\alpha\overline{a}'$, $\overline{b}\beta\overline{b}'$.

$K_n(\alpha,\beta;\gamma)$ denotes the set of all couples (c,d) such that there are matrices

$$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \quad \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \quad \begin{vmatrix} a_n & b_n \\ c_n & d_n \end{vmatrix}$$

in $M(\alpha,\beta)$ such that $b_n\gamma a_1$, $b_i\gamma a_{i+1}$ $(1\leq i<n)$, $d_i\gamma c_{i+1}$ $(1\leq i<n)$, and $c=c_1$, $d=d_n$.

$K_\omega(\alpha,\beta;\gamma)$ is $\bigcup_{n\in\omega} K_n(\alpha,\beta;\gamma)$.

For $n\leq\omega$, the <u>n-cycle</u> <u>commutator</u> $[\alpha,\beta]_n$ is the least congruence γ such that $K_n(\alpha,\beta;\gamma)\leq\gamma$. Thus, the <u>cycle</u> commutator $[\alpha,\beta]_\omega$ is $\bigcup_{n\in\omega}[\alpha,\beta]_n$.

Notice that (mainly for n=1) some authors denote the relation $K_n(\alpha,\beta;\gamma)\leq\gamma$ by $C_n(\alpha,\beta;\gamma)$, and say that α <u>centralizes</u> β <u>modulo</u> γ (in the sense of the n-Cycle Condition).

REMARKS. (a) It is well-known that an algebra **A** satisfies $[1,1]=0$ iff $\{(a,a)|a\in A\}$ is a block of some congruence θ on $\mathbf{A}\times\mathbf{A}$; similarly, $[1,1]_\omega=0$ means that θ and the (kernels of the) two projections generate a lattice isomorphic to M_3 . More generally, it is immediate from (the symmetric version of) [FMK, Lemma 4.8] (which is proved without using modularity) that $[\alpha,\beta]_\omega=0$ iff in the algebra $A(\alpha)$ $\Delta_{\alpha,\beta}\cdot\pi_1=0$, iff $\Delta_{\alpha,\beta}\cdot\pi_2=0$ (see [FMK, Definition 4.7] for notation; π_i is the kernel of the restriction of the i-th projection).

(b) As a consequence of (a) and of [FMK, Theorem 4.9(i)$\Leftrightarrow$(iv)], in every modular variety $[\alpha,\beta]=[\alpha,\beta]_\omega$.

Indeed, many properties of the modular commutator follow just from the fact that $[\alpha,\beta]=[\alpha,\beta]_\omega$, as implicit in [FMK, chapter IV], and explicitly pointed out to us by K.Kearnes (cf. also [KMK]).

(c) Notice also that the proof of [FMK, Theorem 4.10] shows that $C^{(V)}(\alpha,\beta)\leq[\alpha,\beta]_\omega$ holds in <u>every</u> variety. (and hence also $C^{(V)}(\alpha,\beta)\leq[\alpha,\beta]_\omega\cdot[\beta,\alpha]_\omega$, since $C^{(V)}$ is commutative). Actually, the proof shows that $C^{(V)}(\alpha,\beta)\leq K_\omega(\alpha,\beta;0)$ in every variety.

(d) Also Condition (iii) in [HH, Theorem 1.4] can be used to define a commutator operation $C^*(\alpha,\beta)$. The proof of [FMK, Theorem 4.11] shows that in every variety $C^*(\alpha,\beta)\leq[\alpha,\beta]_\omega$.

(e) There is a useful bound for $[\alpha+\beta,\gamma]$ (see [Lp1]); also left semidistributivity of $[\alpha,\beta]$ has very important consequences.

In this respect, $[\alpha,\beta]_\omega$ seems to be not so well behaved. However, suppose (for the rest of this remark) $\alpha+\alpha'=\alpha \circ \alpha'$: then we can compute $K_n(\alpha+\alpha',\beta;\gamma)\leq K_n(\alpha',\beta;K_n(\alpha,\beta;(\alpha+\gamma)(\alpha'+\gamma))$; and, in particular, $K_n(\alpha+\alpha',\beta;\gamma)\leq(\beta+\gamma)(\alpha'+(\alpha+\gamma)(\beta+(\alpha+\gamma)(\alpha'+\gamma)))$. This identity can be iterated as in [Lp1, Lemma 1(ii)] in order to give a bound (not depending on $n\leq\omega$) for $[\alpha+\alpha',\beta]_n$: define $[\alpha,\beta|0]_n=0$ and $[\alpha,\beta|m+1]_n=CgK_n(\alpha,\beta;[\alpha,\beta|m]_n)$ and observe that $[\alpha,\beta]_n=\bigcup_{m\in\omega} [\alpha,\beta|m]_n$).

Also a weak form of semidistributivity holds: if $[\alpha,\beta]_n=[\alpha',\beta]_n\geq\alpha\alpha'$ then $[\alpha+\alpha',\beta]_n=[\alpha,\beta]_n$.

Without the supposition $\alpha+\alpha'=\alpha \circ \alpha'$ we still can get a bound for $[\alpha\circ\alpha',\beta]_n$ (as mentioned before, the definition of $[\alpha,\beta]_n$ can be given for α a relation, rather than a congruence)

(f) Trivially, $[\alpha,\beta]_2=[\beta,\alpha]_2$. Probably $n=2$ is the only n for which $[-,-]_n$ is commutative (in [KP, Example 5.17] it is shown that $[-,-]$ is not necessarily commutative).

If **A** is an algebra, a ternary term t is a <u>difference term</u> for **A** iff $a=t(a,b,b)$ and $t(a,a,b)[\alpha,\alpha]b$, for every $\alpha\in$Con **A** and $a,b\in A$ such that $a\alpha b$.

The following theorem generalizes [Ki, Proposition 3.10], and gives another proof which does not make use of quaternary terms:

THEOREM. If **A** has a difference term, then $[\alpha,\beta]=[\alpha,\beta]_\omega$, for every $\alpha,\beta\in$Con **A**.

Let us first exemplify the method by proving that $[\alpha,\beta]=0$ implies $[\alpha,\beta]_2=0$. Indeed, if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \begin{vmatrix} b & a \\ d & e \end{vmatrix}$$

belong to $M(\alpha,\beta)$, then also

$$\begin{vmatrix} t(a,b,b) & t(a,a,a) \\ t(c,d,d) & t(c,c,e) \end{vmatrix} = \begin{vmatrix} a & a \\ c & t(c,c,e) \end{vmatrix}$$

belongs to $M(\alpha,\beta)$, so that $[\alpha,\beta]=0$ implies $c=t(c,c,e)$. But $c\alpha a\alpha e$ and $c\beta d\beta e$, whence $c\alpha\beta e$ and $t(c,c,e)[\alpha\beta,\alpha\beta]e$, since t is a difference term. In conclusion, $t(c,c,e)=e$, since $[\alpha\beta,\alpha\beta]\leq[\alpha,\beta]$, so that $c=e$, what we had to show.

<u>Proof of the Theorem</u>. Clearly, $[\alpha,\beta]\leq[\alpha,\beta]_\omega$. For the converse, we prove by induction on n that $C_n(\alpha,\beta;[\alpha,\beta])$ holds, for every n.

So, suppose that $\gamma=[\alpha,\beta]$, and that c,d are as in the definition of $K_n(\alpha,\beta;\gamma)$.

If $n=1$, then clearly $c[\alpha,\beta]d$.

Otherwise,

$$\begin{vmatrix} t(a_1, b_1, a_i) & t(a_1, b_1, b_i) \\ t(c_1, d_1, c_i) & t(c_1, d_1, d_i) \end{vmatrix} \quad (2\leq i<n)$$

and

$$\begin{vmatrix} t(a_1, b_1, a_n) & t(a_1, a_1, b_n) \\ t(c_1, d_1, c_n) & t(c_1, c_1, d_n) \end{vmatrix}$$

are $n-1$ matrices satisfying the conditions in the definition of $K_{n-1}(\alpha,\beta;\gamma)$.

Indeed, clearly $t(a_1, b_1, b_i)\gamma t(a_1, b_1, a_{i+1})$, and $t(c_1, d_1, d_i)\gamma t(c_1, d_1, c_{i+1})$, for $2\leq i<n$. Moreover, $t(a_1, a_1, b_n)\gamma t(a_1, a_1, a_1)=a_1=t(a_1, b_1, b_1)\gamma$ $t(a_1, b_1, a_2)$.

Whence, by the inductive hypothesis, $t(c_1, c_1, d_n)\gamma$ $t(c_1, d_1, c_2)$. We have that $t(c_1, d_1, c_2)\gamma$ $t(c_1, d_1, d_1)=c_1$; moreover, $c_1\alpha a_1\gamma b_n\alpha d_n$, and $c_1\beta d_1\gamma c_2$ $\ldots \gamma c_n\beta d_n$, so that $c_1\alpha\beta d_n$, since $\gamma\leq\alpha\beta$, and hence $t(c_1, c_1, d_n)\gamma d_n$, since $[\alpha\beta,\alpha\beta]\leq[\alpha,\beta]$ and t is a difference term. In conclusion, $c_1\gamma d_n$, what we had to show. ⊔

PROPOSITION. There exists a finite algebra (indeed, a groupoid G with 50 elements) which satisfies $[1,1]_\omega$ but which is not quasi affine. (BELONGING TO A LOCALLY FINITE VARIETY

Proof. The proof is very similar to the proof of [Qu, Theorem 6].

The multiplication table of the groupoid G is:

	f_1	f_2	f_3	f_4	f_5	f_6		...	
f_1	a_1	a_2	*	*	*	*	c_1	...	c_1
f_2	a_3	a_4	*	*	*	*	c_2	...	c_2
f_3	*	*	a_5	a_1	*	*	c_3	...	c_3
f_4	*	*	a_4	a_6	*	*	c_4	...	c_4
f_5	*	*	*	*	a_2	a_5	c_5	...	c_5
f_6	*	*	*	*	a_6	a_7	c_6	...	c_6
	d_1	d_2	d_3	d_4	d_5	d_6	e	...	e
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱	⋮
	d_1	d_2	d_3	d_4	d_5	d_6	e	...	e

where each * represents a different element from a set $\{ b_1 \dots b_{24} \}$, and a_1 , ..., f_6 are all distinct.

The disposition of the a_i's implies that G is not quasi affine, because of Quackenbush's characterization [Qu, Theorem 5]: a 3-implication fails in G, since $a_3 \neq a_7$.

In comparison with [Qu, Theorem 6], here the bordering of the multiplication table seems to be relevant: the elements f_1 , ..., f_6 have been added so that G satisfies $x(yz)=x(y'z')$ and $(xy)z=(x'y')z$, for every $x,x',y,\dots$ (AND SO G GENERATES A LOCALLY FINITE VARIETY); in particular, all matrices in M(1,1) are either trivial (that is, two pairs of entries are equal), or have the form:

$$\begin{vmatrix} xy & xz \\ wy & wz \end{vmatrix} \text{ or } \begin{vmatrix} xy & zy \\ xw & zw \end{vmatrix}.$$

By looking at all possible pairs of matrices in $M(1,1)$ having a column in common, the problem of showing $[1,1]_\omega=0$ reduces to showing $[1,1]_2=0$, and this can be easily verified. □

PROBLEMS. (a) Can we generalize the Proposition showing that there exists a variety satisfying $[1,1]_\omega=0$ which is not quasi affine?

(b) More generally, is there a variety V in which every algebra satisfies (*), but V has an algebra which does not satisfy (**)? (where (*) and (**) can be anyone of $[1,1]_W=0$, $[1,1]_n=0$, $[1,1]_\omega=0$, $[1,1]_L=0$; $[-,-]_W$ denotes the weak commutator implicitly defined in [Ke1, §5]).

(c) If $[\alpha,\beta]=[\alpha,\beta]_2$ holds in V, does $[\alpha,\beta]=[\alpha,\beta]_L$ hold in V, too? (similar problems can be posed for other commutators, as in (b)).

(d) Solve the above problems at least for locally finite varieties.

(e) Are the following equivalent? (i) $[1,1]_\omega=0$ in **A**; (ii) In **A**×**A** there exists a common complement δ of the kernels of the two projections. (iii) Same as (ii) and, moreover, $\pi_i \circ \delta \circ \pi_i^{-1}$ $(i=1,2)$.

By Remark (a) above, (i)⇒(iii); and (iii)⇒(ii) trivially. By Remark (e), if (ii) holds then (in **A**×**A**) $[1,\delta]_\omega=0$, and also $[1,1]=0$, but this does not seem enough to imply (i).

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Paolo Lipparini
Dipartimento di matematica
Università di Cagliari.

Current address:
Dipartimento di matematica
Università di Tor Vergata
I-00133 ROME
ITALY