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Congruence Identities Satisfied in n-Permutable Varieties.

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Sunto. – *In* [Lp1] *l'autore ha dimostrato che ogni varietà n-permutabile soddisfa ad identità reticolari non banali. Sorprendentemente, questo risultato è stato ottenuto coll'esclusivo ausilio di tecniche note da almeno un decennio: essenzialmente, alcune proprietà del commutatore. Scopo del presente lavoro è quello di ricavare, come conseguenze della n-permutabilità, identità più forti delle precedenti. Si indicano anche alcuni metodi di dimostrazione che non fanno uso della teoria del commutatore.*

In [Lp1] we showed that, for every $n \geqslant 2$, there is a non trivial identity (in the pure language of lattices) holding in every congruence lattice of algebras in n-permutable varieties, thus generalizing results obtained by A. Day and J. B. Nation [Jo, Lemma 3.10] and D. Hobby and R. McKenzie [HMK, Theorem 9.19].

The identity found in [Lp1] is rather weak (stating that certain interval sublattices are modular): the aim of this paper is to give much stronger identities; also, we analyse in more details the cases when n is small.

To this purpose, we prove and apply a very strong version of a theorem of W. Taylor asserting that every abelian algebra belonging to an n-permutable variety is affine; and, as in [Lp1], we use this theorem together with a rather rough commutator theory.

In [Lp2] a general commutator theory is provided, which applies to a broad class of varieties, and even to single algebras (see Definition 1.4); and some interesting and concrete results are obtained. However, for n-permutable varieties, it is likely that a much finer commutator theory can be developed: though this goes outside the scope of the present paper, it may happen that some of our results will serve in future as a basis for such a theory.

We assume the reader is familiar with the basic concepts of universal algebra: [BS] or [MKNT] can be used as accessible

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textbooks; see also [MK] and the introduction of [Sm] for a less technical presentation of the subject and of its aims. The books [Sm], [FMK], [Gu] develop commutator theory for permutable and modular varieties: knowledge of such theory is not necessary for a formal understanding of this paper, but is fundamental in order to grasp the entirety of motivations.

A large part of commutator theory for modular varieties has been developed in Darmstadt; curiously enough, the main ideas for proving the quoted result of [Lp1] came to us there, while attending the Symposium on Lattice Theory in honour of the 70th birthday of G. Birkhoff.

Most of the results of this paper have been announced in [Lp1].

If not otherwise specified, we use the notations of [FMK]. We recall only the basic notions: an *algebra* (algebraic structure) is just a set together with some operations; a class of algebras of the same type is a *variety* just in case it is closed under taking direct products, substructures and homomorphic images (a celebrated theorem by G. Birkhoff states that variety is the same as a class of algebras satisfying a given set of equationa). A *term* (or word, or derived operation) of an algebra $\boldsymbol{A}$ is just an operation obtained by composition from the basic operations of $\boldsymbol{A}$. A *congruence* on an algebra is just a compatible (admissible) equivalence relation, that is, the kernel of some homomorphism. Con $(\boldsymbol{A})$, the set of all congruences of the algebra $\boldsymbol{A}$, is naturally equipped with a *lattice* structure, with *meet*, denoted by juxtaposition, set theoretical intersection, and *join* denoted by $+$. Given two congruences α and β, we can form the relational product $\alpha \circ \beta$, which is a compatible relation on $\boldsymbol{A}$, but not necessarily a congruence. If $n \geqslant 2$, two congruences α and β are said to *n-permute* iff $\beta \circ \alpha \circ \beta \circ \alpha \ldots = \alpha \circ \beta \circ \alpha \circ \beta \ldots$ (n factors on each side): so that, in particular, $\alpha + \beta = \alpha \circ \beta \circ \alpha \circ \beta \ldots$ (n factors). *Permute* is the same as 2-permute. A variety V is (congruence) *n-permutable* iff every pair of congruences of every algebra in V n-permute.

■ denotes the end of a proof.

1. – Taylor's theorem revisited.

We now introduce some commutators which will play an essential role in what follows. Since their definitions may appear very technical, the reader is referred to [Lp1] for simple applications (more exactly: deep applications with relatively simple proofs: see in particular Remark *a*) after Theorem 1 of [Lp1]).

DEFINITION 1.1. – Let A be any algebra, and $\alpha,\beta \in \mathrm{Con}(A)$.
$M(\alpha,\beta)$ is the set of all *matrices* of the form:

$$\begin{vmatrix} t(\bar{a},\bar{b}) & t(\bar{a}.\bar{b}') \\ t(\bar{a}',\bar{b}) & t(\bar{a}',\bar{b}') \end{vmatrix}$$

where $\bar{a},\bar{a}' \in A^n$, $\bar{b},\bar{b}' \in A^m$, for some $n,m \geqslant 0$, t is any $(m+n)$-ary term operation of A, and $\overline{a}\alpha\overline{a'}$, $\overline{b}\beta\overline{b'}$.

The *commutator of* α *and* β *truncated at the n-th concatenation*, denoted $[\alpha,\beta|n]$, is defined as follows:

$$[\alpha,\beta|0] = 0_A,$$

$$[\alpha,\beta|n+1] = \mathrm{Cg}\left\{(z,w) \middle| \begin{vmatrix} x & y \\ z & w \end{vmatrix} \in M(\alpha,\beta), \text{ for some } x[\alpha,\beta|n]y\right\},$$

where Cg means «the congruence generated by».

By notational convenience, we also put $[\alpha,\beta|-1] = 0_A$.

The *commutator* $[\alpha,\beta]$ is $\bigvee_{n\in\omega}[\alpha,\beta|n]$. This last commutator was denoted by $C(\alpha,\beta)$ in [FMK]; as a consequence of Proposition 4.2 there, within modular varieties, $[\alpha,\beta] = [\alpha,\beta|1]$.

The *solvable* series are defined as follows:

$$\alpha^{(0|0)} = \alpha^{(0|-1)} = 0_A;$$

$$\alpha^{(0)} = \alpha^{(0|s)} = \alpha, \quad \text{for every } s \geqslant 1;$$

$$\alpha^{(m+1|s)} = [\alpha^{(m|s)}, \alpha^{(m|s)}|s];$$

$$\alpha^{(m+1)} = [\alpha^{(m)}, \alpha^{(m)}].$$

Now for the generalization of [Ta, Theorem 2] (see also the remark on p. 23 there).

THEOREM 1.2. – *Suppose that* V *is an* $(n+2)$*-permutable variety,* $s,t \geqslant 0$ *and* $m \geqslant 1$. *Then there are ternary terms* $p_1, p_2, \dots, p_{n-s-t+1}$ *such that, for every* $A \in V$, $\alpha \in \mathrm{Con}(A)$, $a,b \in A$, $a\alpha b$, *the following identities hold:*

a) *If* $s+t \leqslant n-2$:

$$(1,m,s) \qquad a \equiv p_1(a,b,b) \qquad (\mathrm{mod}\, \alpha^{(m|s)}),$$

$$(2,m,s) \qquad p_1(a,a,b) \equiv p_2(a,b,b) \qquad (\mathrm{mod}\, \alpha^{(m|s-1)}),$$

$$(3,i) \qquad p_i(a,a,b)=p_{i+1}(a,b,b) \qquad \textit{for } 2\leqslant i\leqslant n-s-t-1\,,$$

$$(4,m,t) \qquad p_{n-s-t}(a,a,b)\equiv p_{n-s-t+1}(a,b,b) \qquad (\mathrm{mod}\,\alpha^{(m|t-1)})\,,$$

$$(5,m,t) \qquad p_{n-s-t+1}(a,a,b)\equiv b \qquad (\mathrm{mod}\,\alpha^{(m|t)})\,.$$

b) *If* $s+t=n-1$:

$$a\equiv p_1(a,b,b) \qquad (\mathrm{mod}\,\alpha^{(m|s)})\,,$$

$$p_1(a,a,b)\equiv p_2(a,b,b) \qquad (\mathrm{mod}\,\alpha^{(m|\sup\{s-1,t-1\})})\,,$$

$$p_2(a,a,b)\equiv b \qquad (\mathrm{mod}\,\alpha^{(m|t)})\,.$$

c) *If* $s+t=n$:

$$a\equiv p(a,b,b) \qquad (\mathrm{mod}\,\alpha^{(m|\sup\{s,t-1\})})\,,$$

$$p(a,a,b)\equiv b \qquad (\mathrm{mod}\,\alpha^{(m|\sup\{s-1,t\})})\,.$$

PROOF. – The case $s=t=0$, m arbitrary is the Hagemann Mitschke Mal'cev characterization of $(n+2)$-permutability. The general case is proved by triple induction on m, s, t: induction on s is a variant of the procedure of [Ta, Theorem 2]; induction on t is the symmetric argument, and induction on m is a broad generalization of the methods in [Gu, p. 64].

Now for details: the basis of the induction ($s=t=0$, m arbitrary) is just a restatement of the main result of [HM], since $\alpha^{(m|0)}=0$, for every m.

From terms satisfying the theorem for given s, t, m, we shall construct (in many steps) other terms satisfying the theorem for $s+1$, t, m. A completely symmetric argument gives the step from t to $t+1$, so that the result follows from simultaneous induction on s and t (by checking that we get exactly parts *b*) and *c*) when starting with 3 or 2 terms).

So, let us start with terms $p_1,p_2,\dots,p_{n-s-t+1}$ satisfying $(1,m,s)$, $(2,m,s)$, $(3,i)$ (for $2\leqslant i\leqslant n-s-t-1$), $(4,m,t)$, $(5,m,t)$. Define:

$$p_1'(xyz)=p_2(p_1(xyx),p_1(yyx),p_1(zyx))\,,$$

$$p_i'=p_{i+1}\,, \qquad (1<i\leqslant n-s-t)\,.$$

The arguments of [Ta, Theorem 2] apply verbatim to show that $p_1',p_2',\dots,p_{n-s-t}'$ satisfy $(2,m,s+1)$ and $(1,1,s+1)$, as well as, trivially, $(3,i)$ (for $2\leqslant i\leqslant n-s-t$), $(4,m,t)$, $(5,m,t)$: just replace $=$ with $\equiv$, and use the fact that $\alpha^{(m|s)}\geqslant\alpha^{(m|s-1)}$ and that $\alpha^{(1|s)}\geqslant\alpha^{(m|s)}$.

CLAIM. – If for some $r \geq 1$ there are terms $q_1, q_2, \ldots q_{n-s-t}$ satisfying $(1, r, s+1)$, $(2, m, s+1)$, $(3, i)$ (for $2 \leq i \leq n-s-t$), $(4, m, t)$, $(5, m, t)$, then there are terms $q_1', q_2', \ldots, q_{n-s-t}'$ which in addition satisfy $(1, r+1, s+1)$.

Using the claim, we obtain terms satisfying $(1, m, s+1)$, $(2, m, s+1)$, $(3, i)$ (for $2 \leq i \leq n-s-t$), $(4, m, t)$, $(5, m, t)$, by induction on $r \leq m$: the basis of the induction are $p_1', p_2', \ldots$; and the claim is the induction step.

So, it remains to prove the claim. Define:

$$q_i'(xyz) = q_i(x, q_i(xyy), q_i(xyz)), \qquad (\text{for } 1 \leq i \leq n-s-t).$$

If $a\alpha b$, then $a\alpha^{(r|s+1)} q_1(a, b, b)$, by $(1, r, s+1)$, hence, again by $(1, r, s+1)$:

$$a \equiv q_1(a, q_1(abb), q_1(abb)) = q_1'(abb),$$

where the congruence is modulo $(\alpha^{(r|s+1)})^{(r|s+1)} \leq \alpha^{(r+1|s+1)}$, so that $(1, r+1, s+1)$ is proved.

Now compute:

$$q_1'(aab) = q_1(a, q_1(aaa), q_1(aab)) = q_1(a, a, q_1(aab)) \equiv$$

$$q_1(a, a, q_2(abb)) \equiv q_2(a, q_2(abb), q_2(abb)) = q_2'(abb),$$

where $\equiv$ is congruence modulo $\alpha^{(m|s)}$; and $q_1(a, a, a) = a$ and $q_1(a, a, b)\,\alpha a \alpha q_2(a, b, b)$, since all terms we have constructed are idempotent (that is, they satisfy $q(x, x, x) = x$ identically): this is because we started with the Hagemann Mitschke terms, which all are idempotent.

That $q_1', q_2', \ldots, q_{n-s-t}'$ satisfy $(3, i)$ (for $2 \leq i \leq n-s-t$), $(4, m, t)$, $(5, m, t)$ is proved exactly in the same way as $(2, m, s+1)$. ■

REMARK 1.3. – Notice that the hypothesis of $(n+2)$-permutability is used only in the basis of the induction (the Hagemann Mitschke terms). Hence, if an algebra possesses $n-s-t+1$ terms satisfying the conclusion of Theorem 1.2 for given m, s, t, then it has terms satisfying Theorem 1.2 for every $s' \geq s$ and $t' \geq t$ and the given m, provided $n-s'-t'+1 \geq 1$.

A similar remark applies to most of the results of this paper. In particular, it has applications to the following notion introduced in [Lp2]:

DEFINITION 1.4. – If A is an algebra, a ternary term d is said to be a *weak difference term* iff for every $\alpha \in \mathrm{Con}(A)$, $a\alpha b \in A$:

$$d(b,b,a) \equiv a \equiv d(a,b,b), \qquad (\mathrm{mod}\,[\alpha,\alpha]).$$

The proof of the claim in 1.2 gives (in the simplest particular case of just one term):

PROPOSITION 1.5. – *If d is a weak difference term for the algebra A, then for every k there is a ternary term d_k such that $d_k(a,a,b)\,\alpha^{(k)}\,b\,\alpha^{(k)}\,d_k(b,a,a)$, for every $a\alpha b \in A \in V$.* ■

Algebras having weak difference terms include all algebras in modular and n-permutable varieties, in locally finite varieties omitting type 1 [HMK, Theorem 9.6(6)], in neutral varieties (those satisfying $[\alpha,\alpha]=\alpha$ identically); as well as all algebras with a commutative semigroup operation satisfying $x^n = x$ (for some fixed $n \geq 2$) [Lp2].

The class of varieties all whose algebras have a weak difference term can be proved to be a Mal'cev class, and the proof also shows that, for every such variety, there is an n such that, in Definition 1.4, $[\alpha,\alpha]$ may be replaced by $[\alpha,\alpha|n]$.

Algebras satisfying Definition 1.4 are studied in [Lp2], where we develop a general commutator theory for those algebras: applications include congruence identities (in the languge with $\circ$ added) and many permutability results (similar to, e.g., Corollary 2.6).

Also, if M_3 is a sublattice of the congruence lattice of an algebra with a weak difference term, then this sublattice is abelian (in the sense that $[\alpha,\alpha] \leq \beta$, where α and β are the largest and smallest elements of the sublattice); on the contrary, every sublattice isomorphic to N_5 is not abelian (even not solvable). This implies that there are lattices (e.g. M_3 «with an N_5 inside») which cannot be sublattices of congruence lattices of algebras having a weak difference term.

Moreover, if A is such an algebra, and $\alpha,\beta \in \mathrm{Con}(A)$, then, for every m, $(\alpha+\beta)^{(m)} \leq \alpha$ iff $\beta^{(m)} \leq \alpha\beta$: that is, projective quotients are either both abelian (solvable) or both non abelian (non solvable).

2. – Congruence identities.

In this section we use Theorem 1.2 in order to obtain some identities satisfied by congruences of algebras in n-permutable varieties. In general, our identities involve the various commutators and the composition operation $\circ$, in addition to the lattice operations $+$ and $\cdot$. However, at the end of this section we shall show how to obtain, in some particular cases, identities in the pure language of lattices (as we did in [Lp1]).

Some of our identities give permutability results: this is obtained by refining techniques used by H. P. Gumm and others (see [Gu, chapter 8]) in the simpler case of modular varieties.

We begin with a very important corollary of Theorem 1.2. If $\boldsymbol{A}$ is an algebra and $\Theta \subseteq A \times A$, let $\bar{\Theta}$ denote the least reflexive compatible relation on A containing Θ; that is, the subalgebra of $\boldsymbol{A} \times \boldsymbol{A}$ generated by $\Theta \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$.

COROLLARY 2.1. – *Suppose that, for given n, s, t, and m, an algebra $\boldsymbol{A}$ has ternary terms $p_1, p_2, \ldots, p_{n-s-t+1}$ sayisfying the conclusion of Theorem 1.2. (in particular, this holds if $\boldsymbol{A}$ belongs to some $(n+2)$-permutable variety).*

If $\Theta_1, \Theta_2, \ldots, \Theta_{n-s-t}$ are reflexive compatible relations on $\boldsymbol{A}$, and α, β, γ, $\delta \in \mathrm{Con}(\boldsymbol{A})$, then:

a) if $s+t \leqslant n-2$:

$$\alpha \circ \Theta_1 \circ \beta \circ \Theta_2 \circ \Theta_3 \ldots \Theta_{n-s-t-1} \circ \gamma \circ \Theta_{n-s-t} \circ \delta \subseteq$$
$$\alpha^{(m|s)} \circ \overline{(\alpha \circ \Theta_1) \cup (\Theta_1 \circ \beta)} \circ \beta^{(m|s-1)} \circ \overline{\beta \cup \Theta_2} \circ \overline{\Theta_2 \cup \Theta_3} \circ$$
$$\overline{\Theta_3 \cup \Theta_4} \circ \ldots \circ \overline{\Theta_{n-s-t-2} \cup \Theta_{n-s-t-1}} \circ \overline{\Theta_{n-s-t-1} \cup \gamma} \circ$$
$$\gamma^{(m|t-1)} \circ \overline{(\gamma \circ \Theta_{n-s-t}) \cup (\Theta_{n-s-t} \circ \delta)} \circ \delta^{(m|t)}$$

(it will be useful to keep in mind that there are $n-s-t+4$ factors on the left side of $\subseteq$, and $n-s-t+5$ factors on the right: expressions within a parenthesis or grouped below a bar are counted just as one factor!);

b) if $s+t = n-1$:

$$\alpha \circ \Theta \circ \beta \circ \Psi \circ \gamma \subseteq \alpha^{(m|s)} \circ \overline{(\alpha \circ \Theta) \cup (\Theta \circ \beta)}$$
$$\circ \beta^{(m|\sup\{s-1,t-1\})} \circ \overline{(\beta \circ \Psi) \cup (\Psi \circ \gamma)} \circ \gamma^{(m|t)};$$

c) if $s+t=n$:

$$\alpha \circ \Theta \circ \beta \subseteq \alpha^{(m|\sup\{s,t-1\})} \circ \overline{(\alpha \circ \Theta) \cup (\Theta \circ \beta)} \circ \beta^{(m|\sup\{s-1,t\})}.$$

PROOF. – We shall prove c) and a) leaving b) to the reader. For c), suppose $a \alpha b \Theta c \beta d$.

Then

$$a\,\alpha^{(m|\sup\{s,t-1\})}\,p_1(abb)\,\overline{(\alpha\circ\Theta)\cup(\Theta\circ\beta)}\,p_1(ccd)\,\beta^{(m|\sup\{s-1,t\})}\,d.$$

For a), suppose

$$a_1\,\alpha a_2\,\Theta_1\,a_3\beta a_4\,\Theta_2\,a_5\,\Theta_3\,\ldots$$
$$a_{n-s-t+1}\,\Theta_{n-s-t-1}\,a_{n-s-t+2}\,\gamma a_{n-s-t+3}\,\Theta_{n-s-t}\,a_{n-s-t+4}\,\delta a_{n-s-t+5}.$$

Then

$$a_1\,\alpha^{(m|s)}\,p_1(a_1\,a_2\,a_2)\,\overline{(\alpha\circ\Theta_1)\cup(\Theta_1\circ\beta)}\,p_1(a_3\,a_3\,a_4)$$
$$\beta^{(m|s-1)}\,p_2(a_3\,a_4\,a_4)\,\overline{\beta\cup\Theta_2}\,p_2(a_4\,a_4\,a_5)=p_3(a_4\,a_5\,a_5)\,\overline{\Theta_2\cup\Theta_3}$$
$$p_3(a_5\,a_5\,a_6)=p_4(a_5\,a_6\,a_6)\,\overline{\Theta_3\cup\Theta_4}\,p_4(a_6\,a_6\,a_7)\ldots$$
$$\ldots p_{n-s-t-1}(a_{n-s-t}\,a_{n-s-t+1}\,a_{n-s-t+1})\,\overline{\Theta_{n-s-t-2}\cup\Theta_{n-s-t-1}}$$
$$p_{n-s-t-1}(a_{n-s-t+1}\,a_{n-s-t+1}\,a_{n-s-t+2})=$$
$$p_{n-s-t}(a_{n-s-t+1}\,a_{n-s-t+2}\,a_{n-s-t+2})\,\overline{\Theta_{n-s-t-1}\cup\gamma}$$
$$p_{n-s-t}(a_{n-s-t+2}\,a_{n-s-t+2}\,a_{n-s-t+3})\,\gamma^{(m|t-1)}$$
$$p_{n-s-t+1}(a_{n-s-t+2}\,a_{n-s-t+3}\,a_{n-s-t+3})\,\overline{(\gamma\circ\Theta_{n-s-t})\cup(\Theta_{n-s-t}\circ\delta)}$$
$$p_{n-s-t+1}(a_{n-s-t+4}\,a_{n-s-t+4}\,a_{n-s-t+5})\,\delta^{(m|t)}\,a_{n-s-t+5}.\qquad\blacksquare$$

REMARK 2.2. – In the case $s=t=0$, a), b), c) above give back the conditions for $(n+2)$-permutability, For example, putting $\Theta=\Psi=0$ and $\alpha=\gamma$, b) becomes $\alpha\circ\beta\circ\alpha\subseteq\overline{\alpha\cup\beta}\circ\beta\circ\overline{\beta\cup\alpha}\subseteq\beta\circ\alpha\circ\alpha\circ\beta=\beta\circ\alpha\circ\beta$.

This, together with the symmetric form, is exactly 3-permutability (n here is $s+t-1=1$).

From now on, let $\alpha^*=\alpha$ if n is an odd integer, and $\alpha^*=\beta$ is n is even. Similarly, let $\beta^*=\beta$ if n is odd, and $\beta^*=\alpha$ if n is even. [ ] denotes *integer part*.

COROLLARY 2.3. – *Suppose that V is $(n+2)$-permutable, $A\in V$, $\alpha,\beta\in\mathrm{Con}(A)$ and $m\geq 1$. Then:*

(i) $\alpha+\beta=\alpha^{(m|1)}\circ\beta\circ\alpha\circ\ldots\circ\alpha^*=\alpha^*\circ\beta^*\circ\ldots\circ\beta\circ\alpha^{(m|1)}$ $(n+2$ *factors on each side*);

(ii) *if $n\geq 2$ then* $\alpha\circ\beta\circ\ldots\circ\beta^*\circ\alpha^*$ (n *factors*) $\subseteq\alpha^{(m|1)}\circ\beta\circ\alpha\circ\ldots\circ\alpha^*\circ\beta^*\circ\alpha^{(m|1)}$ ($n+2$ *factors*);

(iii) *if* $3 \leq r \leq n+2$, *r is odd and β permutes with* $\alpha^{(m|[(n-r+5)/2])}$ *then α and β $(r+1)$-permute.*

PROOF. – (i) For simplicity suppose, say, n odd. $\alpha+\beta = \alpha\circ\beta\circ\ldots\circ\beta\circ\alpha = \beta\circ\alpha\circ\ldots\circ\alpha\circ\beta$ ($n+2$ factors on each side) is the definition of $(n+2)$-permutability. Then, applying Corollary 2.1 with $s=1$ and $t=0$, we get:

$$\alpha\circ\beta\circ\ldots\circ\alpha\circ\beta\ (n+1\ \text{factors}) =$$

$$\alpha\circ 0\circ\beta\circ\alpha\circ\ldots\circ\alpha\circ\beta\circ\alpha\circ 0\circ\beta\ (n+3\ \text{factors}) \subseteq$$

$$\alpha^{(m|1)}\circ\overline{\alpha\cup\beta}\circ 0\circ\overline{\beta\cup\alpha}\circ\overline{\alpha\cup\beta}\circ\ldots\circ\overline{\beta\cup\alpha}\circ 0\circ\overline{\alpha\cup\beta}\circ 0\ (n+4\ \text{factors})\subseteq$$

$$\alpha^{(m|1)}\circ(\beta\circ\alpha)\circ(\alpha\circ\beta)\circ\ldots\circ(\beta\circ\alpha)\ (n+1\ \text{factors})\subseteq$$

$$\alpha^{(m|1)}\circ\beta\circ\alpha\circ\ldots\circ\beta\circ\alpha\ (n+2\ \text{factors}).$$

But now

$$\alpha+\beta = \alpha\circ\beta\circ\ldots\circ\beta\circ\alpha\ (n+2\ \text{factors}) =$$

$$(\alpha\circ\beta\circ\ldots\circ\beta)\circ\alpha \subseteq (\alpha^{(m|1)}\circ\beta\circ\ldots\circ\alpha)\circ\alpha =$$

$$\alpha^{(m|1)}\circ\beta\circ\ldots\circ\alpha(n+2\ \text{factors})\subseteq\alpha\circ\beta\circ\ldots\circ\alpha\ (n+2\ \text{factors}) = \alpha+\beta.$$

This proves the first equality. The other equality follows by symmetry; and the case n even is entirely similar.

(ii) is similar and easier, taking $s=t=1$.

(iii) let $s=[(n-r+4)/2]$ and $t=[(n-r+5)/2]$ (thus, in any case, $s+t=n-r+4$, that is, $r=n-s-t+4$). If $r>5$ then, by Corollary 2.1 (a):

$$\alpha\circ\beta\circ\alpha\circ\ldots\circ\alpha\circ\beta\circ\alpha\ (r\ \text{factors})\subseteq\alpha^{(m|s)}\circ\overline{(\alpha\circ\beta)\cup(\beta\circ\alpha)}\circ$$

$$\alpha^{(m|s-1)}\circ\overline{\alpha\cup\beta}\circ\ldots\circ\overline{\alpha\cup\beta}\circ\alpha^{(m|t-1)}\circ\overline{(\alpha\circ\beta)\cup(\beta\circ\alpha)}\circ$$

$$\alpha^{(m|t)}\ (r+1\ \text{factors})\subseteq\alpha^{(m|t)}\circ(\beta\circ\alpha\circ\beta)\circ\alpha^{(m|t)}\circ(\beta\circ\alpha)\circ$$

$$(\alpha\circ\beta)\circ\ldots\circ(\beta\circ\alpha)\circ(\alpha\circ\beta)\circ\alpha^{(m|t)}\circ(\beta\circ\alpha\circ\beta)\circ$$

$$\alpha^{(m|t)}\ (r+1\ \text{factors}) = \alpha^{(m|t)}\circ\beta\circ\alpha\circ\beta\circ\alpha^{(m|t)}\circ\beta\circ\alpha\circ\beta\ldots\circ$$

$$\alpha\circ\beta\circ\alpha^{(m|t)}\circ\beta\circ\alpha\circ\beta\circ\alpha^{(m|t)}\ (r+6\ \text{factors}) = \beta\circ\alpha^{(m|t)}\circ\alpha\circ$$

$$\alpha^{(m|t)}\circ\beta\circ\beta\circ\alpha\circ\ldots\circ\beta\circ\beta\circ\alpha^{(m|t)}\circ\alpha\circ\alpha^{(m|t)}\circ\beta,$$

since β permutes with $\alpha^{(m|t)}$; but this last expression is equal to $\beta \circ \alpha \circ \beta \circ \ldots \circ \alpha \circ \beta$ (r factors), and then the inequality implies (at least) that α and β $(r+1)$-permute.

The cases $r = 3$ and $r = 5$ use 2.1, parts *c*) and *b*). ■

THEOREM 2.4. – *Suppose that V is $(n+2)$-permutable, $A \in V$, $\alpha, \beta \in \mathrm{Con}(A)$ and $m \geq 1$, and let $s = [(n+1)/2]$ and $t = [n/2]$. Then:*

$$\alpha + \beta = \underbrace{\alpha^{(m|s)} \circ \beta^{(m|s)} \circ \ldots}_{n \text{ factors}} \circ \alpha \circ \beta \circ \underbrace{\alpha^{(m|t)} \circ \beta^{(m|t)} \circ \ldots}_{n \text{ factors}}.$$

PROOF. – Notice that, in any case, $s + t = n$.

One inclusion is trivial. For the other one, we shall prove by induction on r that, for every $\alpha, \beta \in \mathrm{Con}(A)$:

$$(*) \qquad \alpha \circ \beta \circ \ldots \ (r+2 \text{ factors}) =$$

$$\alpha^{(m|s)} \circ \underbrace{\overline{\alpha^{(m|s)} \cup \beta^{(m|s)}} \circ \ldots}_{r \text{ factors}} \circ \overline{\alpha \cup \beta} \circ \underbrace{\overline{\alpha^{(m|t)} \cup \beta^{(m|t)}} \circ \ldots}_{r \text{ factors}} \circ \alpha^{*(m|t)},$$

where $\alpha^* = \alpha$ if r is odd, and $\alpha^* = \beta$ otherwise.

By Corollary 2.1 (*c*) we have

$$\alpha \circ \beta = \alpha \circ 0 \circ \beta \subseteq \alpha^{(m|s)} \circ \overline{\alpha \cup \beta} \circ \beta^{(m|t)};$$

that is, the basis of the induction.

Suppose now that $(*)$ holds for given r, say r even. Then:

$$\underbrace{\alpha \circ \beta \circ \ldots \circ \beta \circ \alpha}_{r+3 \text{ factors}} = \alpha \circ \underbrace{(\beta \circ \ldots \circ \beta)}_{r+1 \text{ factors}} \circ \alpha \subseteq$$

$$\alpha^{(m|s)} \circ \overline{\underbrace{(\alpha \circ \beta \circ \ldots \circ \beta)}_{r+2 \text{ factors}} \cup \underbrace{(\beta \circ \alpha \circ \ldots \circ \alpha)}_{r+2 \text{ factors}}} \circ \alpha^{(m|t)},$$

by Corollary 2.1 (*c*).

But the inductive hypothesis $(*)$, together with its symmetric, implies that

$$\underbrace{(\alpha \circ \beta \circ \ldots \circ \beta)}_{r+2 \text{ factors}} \cup \underbrace{(\beta \circ \alpha \circ \ldots \circ \alpha)}_{r+2 \text{ factors}} \subseteq (\alpha^{(m|s)} \cup \beta^{(m|s)}) \circ \underbrace{\overline{\alpha^{(m|s)} \cup \beta^{(m|s)}} \circ \ldots}_{r \text{ factors}} \circ \overline{\alpha \cup \beta} \circ$$

$$\underbrace{\overline{\alpha^{(m|t)} \cup \beta^{(m|t)}} \circ \ldots}_{r \text{ factors}} \circ (\alpha^{(m|t)} \cup \beta^{(m|t)});$$

and the desired inclusion follows by taking closure under $\overline{}$. ■

The same proof applies almost unchanged to show [Lp2]:

THEOREM 2.5. – *If A has a weak difference term, $m \geqslant 1$ and $\alpha, \beta \in$ Con(A) then:*

$$\alpha+\beta=(\alpha^{(m)}+\beta^{(m)}) \circ \alpha \circ \beta \circ (\alpha^{(m)}+\beta^{(m)}) .$$

Of course, also the following corollary of Theorem 2.4 admits a version for algebras possessing a weak difference term [Lp2]:

COROLLARY 2.6. – *Suppose that V is an $(n+2)$-permutable variety, $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, $m \geqslant 1$ and $s=[(n+1)/2]$ and $t=[n/2]$. Then:*

(i) If β permutes with $\alpha^{(m|s)}$ then

$$\alpha+\beta=\beta \circ \alpha \circ \beta=\beta \circ \alpha \circ \beta^{(m|t)} .$$

(ii) If $\alpha^{(m|s)}$ and $\beta^{(m|s)}$ permute then

$$\alpha+\beta=\beta^{(m|s)} \circ \alpha \circ \beta \circ \alpha^{(m|s)} .$$

(iii) In particular, if (i) or (ii) holds then α and β 4-permute.

(iv) If α permutes with $\beta^{(m|s)}$ and β permutes with $\alpha^{(m|t)}$ then α and β permute.

(v) $\quad \gamma(\alpha+\beta) \leqslant \alpha(\beta+\gamma+\alpha^{(m|s)})+\beta(\alpha+\gamma+\beta^{(m|s)})$.

PROOF. – (i) By Theorem 2.4,

$$\begin{aligned}
\alpha+\beta=(\alpha^{(m|s)}+\beta^{(m|s)}) \circ \alpha \circ \beta \circ (\alpha^{(m|t)}+\beta^{(m|t)}) \subseteq \\
(\alpha^{(m|s)}+\beta) \circ \alpha \circ \beta \circ (\alpha^{(m|s)}+\beta) \subseteq \beta \circ \alpha^{(m|s)} \circ \alpha \circ \beta \circ \alpha^{(m|s)} \circ \beta= \\
\beta \circ \alpha \circ \alpha^{(m|s)} \circ \beta \circ \beta=\beta \circ \alpha \circ \beta=\beta \circ (\alpha \circ \beta) \subseteq \\
\beta \circ (\alpha^{(m|s)} \circ \beta \circ \alpha \circ \beta^{(m|t)})=\beta \circ \alpha \circ \beta^{(m|t)} \subseteq \alpha+\beta ,
\end{aligned}$$

where we used Corollary 2.1 (c).

(ii) and (iv) are completely similar; (iii) is immediate.

(v) Suppose $a\gamma(\alpha+\beta)b$. Then $a\alpha^{(m|s)}+\beta^{(m|s)}c\alpha d\beta e\alpha^{(m|t)}+\beta^{(m|t)}b$, for some $c, d, e \in A$, by Theorem 2.4. But now $c\alpha^{(m|s)}+\beta^{(m|s)}a\gamma b\alpha^{(m|t)}+\beta^{(m|t)}e\beta d$, so that c and d are congruent modulo $\alpha(\beta+\gamma+\alpha^{(m|s)})$; similarly, d and e are congruent modulo $\beta(\alpha+\gamma+$

$\beta^{(m|s)}$), so that a and b are congruent modulo $\alpha^{(m|s)} + \beta^{(m|s)} + \alpha(\beta + \gamma + \alpha^{(m|s)}) + \beta(\alpha + \gamma + \beta^{(m|s)}) = \alpha(\beta + \gamma + \alpha^{(m|s)}) + \beta(\alpha + \gamma + \beta^{(m|s)})$.

Notice that we proved more, that is, $\gamma(\alpha + \beta) \subseteq (\alpha^{(m|s)} + \beta^{(m|s)}) \circ \alpha(\beta + \gamma + \alpha^{(m|s)}) \circ \beta(\alpha + \gamma + \beta^{(m|s)}) \circ (\alpha^{(m|t)} + \beta^{(m|t)})$. ■

Given α, β, γ congruences, define recursively: $\beta_0 = \gamma_0 = 0$; $\beta_{n+1} = \beta + \alpha\gamma_n$; $\gamma_{n+1} = \gamma + \alpha\beta_n$.

We are now ready to give a congruence identity involving only $+$ and $\cdot$.

COROLLARY 2.7. – *Suppose that V is $(n+2)$-permutable. Then the following identity holds in every congruence lattice of every algebra in V:*

$$\delta(\alpha(\beta + \gamma) + \alpha'(\beta' + \gamma')) \leq \alpha(\beta + \gamma)(\delta + \alpha'(\beta' + \gamma') + \alpha\beta_r\gamma_r) + \alpha'(\beta' + \gamma')(\delta + \alpha(\beta + \gamma) + \alpha'\beta'_r\gamma'_r),$$

where $r = (n+2)[(n+1)/2]$.

PROOF. – By [Lp1, Lemma 1], if two congruences m-permute, then $[\beta + \gamma, \alpha | s] \leq \alpha\beta_{ms}\gamma_{ms}$. The corollary is now immediate from Corollary 2.6 (v), since $[\alpha(\beta + \gamma), \alpha(\beta + \gamma) | s] \leq [\beta + \gamma, \alpha | s]$. ■

It is questionable whether the subscript r in Corollary 2.7 is the best possible. Indeed, [Lp1, Lemma 1] holds for every algebra, but it is conceivable that better results hold for $(n+2)$-permutable varieties. Indeed, in 3-permutable varieties $[\beta + \gamma, \alpha | s] = [\beta + \gamma, \alpha] \leq [\beta, \alpha] + [\gamma, \alpha] \leq \beta\alpha + \gamma\alpha$, by the commutator theory for modular varieties, and since 3-permutability implies modularity.

As promised, we shall study now some particular cases when n is small, improving, in these cases, the identities we have obtained.

The cases $n = 0$ and $n = 1$ correspond to permutability and 3-permutability, which both imply modularity. In those cases Theorem 2.4 is the best possible, and, for $n = 1$ gives an expression of $\alpha + \beta$ found by H. P. Gumm for modular varieties.

If $n = 0$, the identity in Corollary 2.7 becomes equivalent to modularity, since all congruences with subscript r become 0; so we have an almost optimal result. If $n = 1$, however, this identity is weaker than modularity, suggesting, again, that Corollary 2.7 can still be improved.

The following proposition holds for 4-permutable and 5-permutable varieties:

PROPOSITION 2.8. – *The following identity holds in every 5-permutable variety*:

$$\gamma(\alpha+\beta) \leqslant \alpha(\beta+\gamma+\alpha^{(m|1)}) + \beta(\alpha+\gamma+\beta^{(m|1)}).$$

REMARK. – This improves on Corollary 2.6 (v), since there $s=2$, as $n=3$. Hence, in Corollary 2.7 we can take $r=5$ instead of $r=10$.

PROOF. – By Corollary 2.1 (b):

$$\alpha\circ\beta\circ\alpha = \alpha\circ 0\circ\beta\circ 0\circ\alpha \subseteq \alpha^{(m|1)}\circ\beta\circ\alpha\circ\beta^{(m|0)}\circ\alpha\circ\beta\circ\alpha^{(m|1)};$$

hence

$$\alpha+\beta = \beta\circ\alpha\circ\beta\circ\alpha\circ\beta = \beta\circ\alpha^{(m|1)}\circ\beta\circ\alpha\circ\beta\circ\alpha^{(m|1)}\circ\beta;$$

and, as in the proof of Corollary 2.6 (v):

$$\gamma(\alpha+\beta)\subseteq\beta+\alpha(\beta+\gamma+\alpha^{(m|1)}).$$

Using this identity twice, we get:

$$\begin{aligned}\gamma(\alpha+\beta) &\leqslant \gamma(\beta+\alpha(\beta+\gamma+\alpha^{(m|1)})) \leqslant \\ &\alpha(\beta+\gamma+\alpha^{(m|1)}) + \beta(\alpha(\beta+\gamma+\alpha^{(m|1)})+\gamma+\beta^{(m|1)}) \leqslant \\ &\alpha(\beta+\gamma+\alpha^{(m|1)}) + \beta(\alpha+\gamma+\beta^{(m|1)}). \quad \blacksquare\end{aligned}$$

PROBLEM. – Is it always possible to have $s=[n/2]$ in Corollary 2.6 (v)? We can do this, but adding some commutators of γ:

THEOREM 2.9. – *Suppose that V is $n+2$ permutable, $A\in V$, $\alpha,\beta,\gamma\in \mathrm{Con}(A)$ and $m\geqslant 1$, and let $s=[(n+1)/2]$ and $t=[n/2]$. Then*:

$$\gamma(\alpha+\beta)\subseteq\gamma^{(m|s)}\circ(\alpha^{(m|t)}+\beta^{(m|t)})\circ\alpha\circ\beta\circ(\alpha^{(m|t)}+\beta^{(m|t)}).$$

PROOF. – Let $\delta=\alpha^{(m|t)}+\beta^{(m|t)}$. We shall prove by induction on $r\geqslant 2$ that:

$$\gamma\cap(\delta\circ\underbrace{\alpha\circ\beta\circ\ldots\circ\alpha^{*}}_{r\text{ factors}}\circ\delta)\subseteq\gamma^{(m|s)}\circ\delta\circ\alpha\circ\beta\circ\delta.$$

The case $r = 2$ is trivial. So, suppose the inclusion true for r. If $a\gamma \cap (\delta \circ \underbrace{\alpha \circ \beta \circ \ldots \circ \beta^*}_{r+1 \text{ factors}} \circ \delta)\, b$, then $a\delta \circ \underbrace{\alpha \circ \beta \circ \ldots \circ \alpha^*}_{r \text{ factors}} c\beta^* d\delta b$, for some $c, d \in A$. By Theorem 1.2:

$$a\gamma^{(m|s)} p(a,b,b)\, \delta \circ \underbrace{\alpha \circ \beta \circ \ldots \circ \alpha^*}_{r \text{ factors}} p(c,c,d)\, \beta^{*(m|t)} d\delta b\,;$$

so that $p(a,b,b)\, \delta \circ \underbrace{\alpha \circ \beta \circ \ldots \circ \alpha^*}_{r \text{ factors}} \circ \delta b$.

Since $p(a,b,b)\, \gamma b$, the inductive hypothesis implies that $p(a,b,b)\, \gamma^{(m|s)} \circ \delta \circ \alpha \circ \beta \circ \delta b$; so that also $a\gamma^{(m|s)} \circ \delta \circ \alpha \circ \beta \circ \delta b$. ■

PROBLEMS. – If n is even, do n-permutable and $n+1$-permutable varieties satisfy the same identities (in the language of lattices)?

Gumm showed that, in some sense, modularity is permutability «composed» with distributivity. What is «n-permutability composed with distributivity»?

REMARK. – Of course, in principle, from the Hagemann Mitschke terms for n-permutability it must be possible to construct terms giving the Mal'cev conditions for the various identities we have found (the situation is entirely similar to [LTT]).

However, in the present case the situation seems much more complicated, and even just writing down the Mal'cev conditions could be very difficult.

3. – Further results. (Added january 1994).

Meanwhile, we have found a very simple and short proof of the result that n-permutable varieties satisfy non trivial lattice identities. This new proof does not use commutator theory and gives identities whose strength is not comparable with previously obtained identities. Actually, the proof shows:

THEOREM 3.1. – *Suppose that V is $(n+2)$-permutable, $A \in V$ $\alpha, \beta, \gamma \in \operatorname{Con} A$. Then $\{\delta \in \operatorname{Con} A \mid \alpha(\beta(\alpha+\gamma) + \gamma(\alpha+\beta)) \leqslant \delta \leqslant \alpha(\beta+\gamma)\}$ satisfies all the congruence identities (even when composition is allowed) holding in every n-permutable variety.*

A full proof will be presented elsewhere. However, we shall exemplify our methods in the particular case of 5-permutable varieties.

THEOREM 3.2. – *Suppose that V is 5-permutable, $\boldsymbol{A} \in V$, $\alpha, \beta, \gamma, \delta, \varepsilon \in \operatorname{Con} \boldsymbol{A}$, and let $\alpha' = \alpha(\beta + \gamma\alpha)$. Then:*

(i) *if t is a ternary term satisfying $x = t(x,y,y)$, for every $x, y \in \boldsymbol{A}$, and $a\alpha' b$, then $a \gamma\alpha \circ \alpha\beta \circ \gamma\alpha t(a,a,b)$.*

(ii) $\alpha' \circ \delta \circ \alpha' \subseteq \gamma\alpha \circ \alpha\beta \circ \gamma\alpha \circ \delta \circ \alpha' \circ \delta \circ \gamma\alpha \circ \alpha\beta \circ \gamma\alpha$,

(iii) $\varepsilon(\alpha(\beta + \gamma\alpha) + \delta) \leqslant \delta + \alpha(\varepsilon + \delta + \gamma\alpha + \alpha\beta)$,

(iv) $\varepsilon(\alpha(\beta + \gamma\alpha) + \alpha^*(\beta^* + \gamma^* \alpha^*))$
$$\leqslant \alpha(\varepsilon + \alpha^* + \gamma\alpha + \alpha\beta) + \alpha^*(\varepsilon + \alpha + \gamma^*\alpha^* + \alpha^*\beta^*),$$

(v) $\varepsilon(\alpha(\beta + \gamma) + \delta) \leqslant \delta + \alpha(\varepsilon + \delta + \alpha(\beta(\alpha + \gamma) + \gamma(\alpha + \beta)))$.

PROOF. – (i) Since $\boldsymbol{A}$ is 5-permutable, $\alpha' = \alpha \cap (\gamma\alpha \circ \beta \circ \gamma\alpha \circ \beta \circ \gamma\alpha) = \gamma\alpha \circ (\alpha \cap (\beta \circ \gamma\alpha \circ \beta)) \circ \gamma\alpha$, whence $a \gamma\alpha c \beta d \gamma\alpha e \beta f \gamma\alpha b$, and $c \alpha f$, for some $c, d, e, f \in \boldsymbol{A}$. It follows that $a \gamma\alpha c = t(c,e,e) \gamma\alpha t(c,d,e) \beta t(c,c,f) \cdot \gamma\alpha t(a,a,b)$, and $t(c,d,e) \alpha t(c,e,e) = c = t(c,c,c) \alpha t(c,c,f)$, from which (i) follows.

(ii) If $a \alpha' b \delta c \alpha' d$, then, using the Hagemann-Mitschke terms, by (i) and its symmetric version:

$$a \gamma\alpha \circ \alpha\beta \circ \gamma\alpha p_1(a,a,b) = p_2(a,b,b) \delta p_2(a,b,c) \alpha' p_2(b,b,d) =$$
$$p_3(b,d,d) \alpha' p_3(b,c,d) \delta p_3(c,c,d) = p_4(c,d,d) \gamma\alpha \circ \alpha\beta \circ \gamma\alpha d.$$

(iii) By 5-permutability, and using (ii):

$$\alpha' + \delta = \delta \circ (\alpha' \circ \delta \circ \alpha') \circ \delta \subseteq (\delta + \gamma\alpha + \alpha\beta) \circ \alpha' \circ (\delta + \gamma\alpha + \alpha\beta),$$

from which (iii) is immediate.

The proof that (iii) implies (iv) is similar to the last step in the proof of Proposition 2.8.

A proof of (v) seems considerably more complex, and shall be given elsewhere. ■

The following theorem furnishes some more results on $n + 2$-permutable varieties:

THEOREM 3.3. – *Suppose that V is $(n+2)$-permutable, $t = [n/2]$, $u = [(n-1)/2]$, and $\boldsymbol{A} \in V$, $\alpha, \beta \in \operatorname{Con} \boldsymbol{A}$. Then:*

(i) *If* $\alpha \leqslant \beta$ *then* $[\alpha,\beta] \leqslant [\beta,\alpha|t+1] + \beta^{(m|t-1)}$.

(ii) *In particular,* $[\alpha,\alpha] \leqslant [\alpha,\alpha|t+1]$.

(iii) $\alpha+\beta = (\alpha+\beta^{(m|t)}) \circ (\beta+\alpha^{(m|t-1)}) \circ (\alpha+\beta^{(m|u)})$.

(iv) $\gamma(\alpha+\beta) \leqslant \alpha + (\beta+\alpha^{(m|t-1)})(\alpha+\beta^{(m|t)}+\gamma)$; *or, more generally.*

(v) $\gamma(\alpha+\beta) \leqslant (\alpha+(\beta+\alpha^{(m|t-1)})^{(m|t-1)})(\gamma+\beta+\alpha^{(m|t)}) + (\beta+\alpha^{(m|t-1)})(\alpha+\beta^{(m|t)}+\gamma)$.

DEFINITION 3.4. – If A is any algebra, and α, β, γ are reflexive compatible relations on A, let $M(\alpha,\beta)$, $[\alpha,\beta|n]$, $[\alpha,\beta]$ be defined as in Definition 1.1. Moreover, let

$$K(\alpha,\beta;\gamma) = \left\{(z,w) \,\middle|\, \begin{vmatrix} x & y \\ z & w \end{vmatrix} \in M(\alpha,\beta), \text{ for some } x\gamma y\right\}.$$

Thus, $[\alpha,\beta|n+1]$ is the congruence generated by $K(\alpha,\beta;[\alpha,\beta|n])$, and $[\alpha,\beta]$ is the least congruence γ such that $K(\alpha,\beta;\gamma) \leqslant \gamma$. Sometimes, the relation $K(\alpha,\beta;\gamma) \leqslant \gamma$ is denoted by $C(\alpha,\beta;\gamma)$; in words, α *centralizes* β *modulo* γ.

REMARK 3.5. – Most of our results depend on the fact that we can sometimes replace $[\alpha,\beta]$ with $[\alpha,\beta|n]$, for some n. Similarly, we can refine some results using smaller relations defined in terms of K (in what follows α is always supposed to be a congruence).

For example, in Theorem 1.2, in the case $s=1$, we can get $aK(\alpha,\alpha;0)\,p_1(a,b,b)$.

In the case $s=2$, we can get

$$aK(\alpha,\alpha;0) \circ K((\alpha,\alpha;K(\alpha,\alpha;0))\,p_1(a,b,b).$$

These improvements do not seem to be of much use, due to the fact that generally $K(\alpha,\beta;\gamma)$ is not a congruence.

The following examples dash any hope to improve Theorem 1.2 in more substantial ways:

a) In a 4-permutable variety there do not necessarily exist ternary terms p_1 and p_2 satisfying

$$a = p_1(a,b,b),$$

$$p_1(a,a,b)[\alpha,\alpha]\,p_2(a,b,b),$$

$$p_2(a,a,b) = b,$$

whenever $a\alpha b$.

Indeed, this would imply $\alpha \circ \beta \circ \alpha \subseteq \beta \circ \alpha \circ [\beta, \beta] \circ \alpha \circ \beta$, which fails, for example, in the algebra constructed in [Lp2, Example 4.6].

b) In a 4-permutable variety there does not necessarily exist a ternary term p satisfying

$$a[\alpha, \alpha] p(a, b, b),$$

$$p(a, a, b) = b,$$

whenever $a \alpha b$.

Again, this would imply $\alpha \circ \beta \subseteq \beta \circ \alpha \circ [\beta, \beta]$, which fails as well in the mentioned example.

c) There are a 4-permutable variety $\boldsymbol{V}$, an algebra $A \in \boldsymbol{V}$ and $\alpha, \beta \in \operatorname{Con} \boldsymbol{A}$ such that $[\alpha, \beta | 1] < [\alpha, \beta | 2]$.

For example, add to the algebra constructed in [Lp2, Example 4.6] a unary operation f defined by: $f((0,0)) = f((1,0)) = f((1,1)) = (0,0)$, $f((0,1)) = (0,1)$. Then, γ and β still remain congruences, and $\beta = [1, \gamma | 1] \neq [1, \gamma | 2] = \gamma$.

REMARK. – 3.6. – In general, «commutator identities» do not give rise to Mal'cev classes: for example, the class of varieties satisfying $[\alpha, \alpha] = 0$ is not a weak Mal'cev class, since every variety with only unary operations satisfies $[\alpha, \alpha] = 0$.

However, we have the following generalization of [Jo, Theorem 2.16], whose proof is essentially the same:

THEOREM 3.7. – *Theorem* 2.16 *in* [Jo] *holds even if we let* r *contain the ternary operation* $K(\alpha, \beta; \gamma)$ *and the binary operations* $[\alpha, \beta]$, $[\alpha, \beta | n]$.

In some cases we can even get strong Mal'cev conditions. Let $K_{rs}(\alpha, \beta; \gamma)$ be defined as $K(\alpha, \beta; \gamma)$ (Definition 3.4), with the difference that we allow only matrices in which $\bar{a}$, $\bar{a}'$ are r-tuples, $\bar{b}$, $\bar{b}'$ are s-tuples and, of course, t is $r+s$-ary. Then classical arguments give:

THEOREM 3.8. – *If* p, q *are terms constructed using* $\cdot$ *and* $\circ$, *and* q *is allowed to contain* $K_{rs}(\alpha, \beta; \gamma)$, *then the class of varieties* V *for which* $V \models_{\mathrm{Con}} p \leqslant q$ *is a strong Mal'cev class.*

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