Weak and local versions of measurability

Paolo Lipparini

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Introduction.

- Large cardinals admit many equivalent formulations in terms of topology, model theory, combinatorics, Boolean algebras, etc.
- A classical paper about the subject: Keisler Tarski 1964, Fund. Math. 53.
- Keisler and Tarski already pointed out that measurability is "more closely tied up with mathematical problems outside of general set theory and more readily handled by mathematical methods", while, say, weak compactness has a more "metamathematical" character.

- Basically, our main aim is to exploit in the possible clearest way the mathematical content of weak compactness (partially work in progress!).
- Moreover, we shall introduce some “local” versions of weak compactness.
- This is parallel to classical “local” versions of measurability introduced by Chang, Prikry, Silver in the 70’s of the last millennium.

Measurability and local forms.

- A cardinal $\mu > \omega$ is *measurable* if there is some μ-complete ultrafilter D over μ.
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Now suppose that D is *not* μ-complete, and consider a counterexample π with the minimum possible number of classes. Say, π has λ classes. Then:

- If X is a union of $< \lambda$ classes from π, then X is not in D.

(Otherwise, say $X \in D$ is a union of ν classes, $\nu < \lambda$, then we could get a counterexample to μ-completeness by a partition of cardinality $\leqslant \nu$, contradicting minimality)

Taking the contrapositive, we get:

- An ultrafilter D is μ-complete if and only if for every $\lambda < \mu$ and every partition of μ into λ classes, some union of $< \lambda$ such classes belongs to D.

This suggests the following classical definition.

- An ultrafilter D over μ is λ-*indecomposable* if for every partition of μ into λ classes, some union of $< \lambda$ such classes belongs to D.

With this terminology,

- D over μ is μ-complete if and only if it is λ-indecomposable for every infinite $\lambda < \mu$.

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- Hence λ-indecomposability can be thought of as a local version (at λ) of measurability.

- Again, the existence of a λ-indecomposable uniform ultrafilter over μ admits many equivalent formulations in terms of topology, model theory, extended model theory (works by X. Caicedo, C. Chang, J. Makowsky, S. Shelah and the author).

We want to use similar ideas in order to obtain local versions of weak compactness.
In order to do this, we first have to describe weak compactness in a way similar to the above description of measurability.

- A cardinal $\kappa > \omega$ is *weakly compact* if it is (strongly) inaccessible, and the infinitary language $\mathcal{L}_{\kappa,\omega}$ is (κ, κ)-compact.
- Recall that $\mathcal{L}_{\kappa,\omega}$ is like first order logic, except that we allow infinite disjunctions and conjunctions of $< \kappa$ sentences.
- Recall that a logic is (κ, κ)-*compact* if every κ-satisfiable set of sentences of cardinality κ is satisfiable.
- Here κ-*satisfiable* means that every subset of cardinality $< \kappa$ has a model.

Usually the assumption of inaccessibility is included in the definition of weak compactness. However:

- (Boos 1976, JSL 41) Without assuming inaccessibility, we get a different notion of weak compactness.
- More precisely, there is a model of set theory with a cardinal $\kappa > \omega$ such that $\mathcal{L}_{\kappa,\omega}$ is (κ,κ)-compact, but κ is not inaccessible.

- We shall mostly deal with cardinals κ such that $\mathcal{L}_{\kappa,\omega}$ is (κ,κ)-compact (no matter whether κ is inaccessible or not).
- We shall call such cardinals *weakly compact w.i.* (short for "without inaccessibility")

We now can provide a characterization of weak compactness w.i. in a fashion similar to the characterization of measurability. The following theorem is probably folklore.

Theorem

A cardinal $\kappa > \omega$ is weakly compact w.i. if and only if for every set P of partitions of κ into $< \kappa$ classes, with $|P| \leqslant \kappa$, there is some uniform ultrafilter D over κ such that for every $\pi \in P$, say with λ classes, no union of $< \lambda$ classes is in D.

In other words, maybe D is actually λ-decomposable (= not λ-indecomposable), but no partition in P can see this.

- Had we allowed P in the above theorem to be of cardinality 2^κ, we would have obtained measurability (since then we could choose P to consist of *all* the partitions of κ into $< \kappa$ classes).

Summing up:

- A cardinal $\kappa > \omega$ is weakly compact w.i. *(resp. measurable)* if and only if for every set P of partitions of κ into $< \kappa$ classes, with $|P| \leqslant \kappa$ *(resp.* $|P| \leqslant 2^\kappa$*)*, there is some uniform ultrafilter D over κ such that for every $\pi \in P$, say with λ classes, no union of $< \lambda$ classes is in D.

In case $2^\kappa > \kappa^+$ there are of course intermediate possibilities, that is, allowing P to be of cardinality ν, for some ν with $2^\kappa > \nu > \kappa$.

- In equivalent formulations (under a cardinality constraint) such intermediate possibilities have been recently studied by J. Schanker.
- He showed that such possibilities can actually occur, that is, there is a notion of weak measurability which is strictly between weak compactness and measurability.

Local forms of weak compactness.

- Now it is obvious how to get "local versions" of weak compactness.
- Just consider the statement of the above theorem, but restrict only to partitions with exactly λ-classes.

In general, we shall consider a principle which depends on

- A "starting" cardinal κ (the candidate to be a large cardinal).
- A cardinal ν (the maximum number of classes allowed)
- A set Λ of infinite cardinals $< \kappa$.

- When ν is large, we get notions related to measurability.
- When $\nu = \kappa$ we get notions related to weak compactness.
- There are intermediate possibilities.

On the other hand

- When Λ is the set of all infinite cardinals $< \kappa$ we get exactly measurability or weak compactness w.i.
- As we make Λ smaller, we get more and more local versions.

In details (κ, ν cardinals, Λ a class of cardinals):

- Consider all sets P of partitions of κ such that:
  - $|P| \leqslant \nu$ and
  - For every $\pi \in P$, the number of classes of π is a cardinal in Λ.

Say that a partition π with λ classes is a *decomposition* of an ultrafilter D if no union of $< \lambda$ classes belongs to D.
Then our principle reads:

- For every set P of partitions as in the above items, there is a uniform ultrafilter D such that no partition in P is a decomposition of D.

The above principle seems interesting in itself, since it incorporates and generalizes measurability, weak compactness, and the notion of a λ-indecomposable ultrafilter.

Is there any other use for this?

- We characterize exactly the (topological) compactness properties of products of cardinals, considered as topological spaces with the order topology.
- Various other applications to topology.
- There are clear model theoretical characterizations involving the realizability of certain types in linearly ordered models.

- An exact characterization of the compactness property of infinitary languages. (Say, we might ask whether $\mathcal{L}_{\omega_1,\omega}$ is (κ,κ)-compact; here $\omega_1 \neq \kappa$)
- Applications to other extensions of first order logic (not everything yet completely clear).
- Equivalent formulations in terms of filters and Boolean algebras (work in progress)
- What about equivalent formulations in terms of partition calculus? (recall that weak compactness does have a formulation in terms of partition calculus)

Thank you!