If every subalgebra of $\mathbf{A}^4$ is congruence modular then $\mathbf{A}$ satisfies the tolerance identity $\Gamma^* \cap \Theta^* = (\Gamma \cap (\Theta \circ \Theta))^*$

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Abstract. It has been recently discovered that congruence modular varieties satisfy a very useful tolerance identity, called TIP.

We show that it is enough to suppose that all subalgebras of $\mathbf{A}^4$ are congruence modular in order to obtain that $\mathbf{A}$ satisfies a slightly weaker tolerance identity.

1. Introduction

It has been recently discovered that algebras in congruence modular varieties satisfy the following identity, called Tolerance Intersection Property:

$$\text{(TIP)} \qquad \Gamma^* \cap \Theta^* = (\Gamma \cap \Theta)^*$$

for every pair of tolerances Γ and Θ of $\mathbf{A}$, and where * denotes transitive closure.

The discovery is surprising in at least two respects: first, the result has a really elementary, simple and short proof, but had passed unnoticed in about 30 years of deep analysis of congruence modular varieties, starting with A. Day [5] characterization in terms of a Mal'cev condition, and culminating, perhaps, with commutator theory (see [7], [8]). Actually, TIP can be obtained simply as a corollary of the 1969 Day's result!

The second surprising fact is that, though really simple, TIP has many important consequences, which came completely unexpected, and that TIP can be used in order to obtain new deep results, and very simpler proofs of older ones. G. Czédli, E. Horváth, K. Kearnes, S. Radeleczki (in alphabetical order) are among the workers who contributed either to the discovery of TIP, or to the development of its applications, or both. We are not in the position to give explicit credits.

For a proof that congruence modular varieties satisfy TIP, and for some applications, see [1], [2], [3], [4] (there might be other published or unpublished material).

In this paper we address the following problems:

Problem 1.1. If $\mathbf{A}$ is an algebra such that, for every n, every subalgebra of $\mathbf{A}^n$ is congruence modular, does $\mathbf{A}$ satisfy TIP?

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Is there some (fixed) n such that if every subalgebra of $\mathbf{A}^n$ is congruence modular then $\mathbf{A}$ satisfies TIP?

In particular, is this true for $n = 2$?

Though we have not solved the above problems yet, we present here a partial solution. We show that if all subalgebras of $\mathbf{A}^4$ are congruence modular then $\mathbf{A}$ satisfies the following tolerance identity:

$$(\mathrm{wTIP}_2) \qquad \Gamma^* \cap \Theta^* = (\Gamma \cap (\Theta \circ \Theta))^*$$

In Section 4 we state some further problems. Actually, those problems have been the main motivation leading to this paper.

2. Notations

$\mathbf{A}$, $\mathbf{B}$... denote *algebras*; α, β are *congruences*; Γ, Θ are *tolerances*, that is, reflexive and symmetric compatible relations.

According to notational convenience we shall write $a \alpha b$ to mean $(a, b) \in \alpha$, and similarly for tolerances. We shall use chains of the above notation: for example, $a \Theta b \alpha c \Gamma d$ means $(a, b) \in \Theta$, $(b, c) \in \alpha$ and $(c, d) \in \Gamma$.

For sake of brevity, we sometimes denote intersection by juxtaposition; in particular $\alpha\beta$ denotes the meet of the congruences α and β.

R^* denotes the transitive closure of the binary relation R; in particular, Θ^* is the smallest congruence which contains Θ. $\Theta^n = \Theta \circ \Theta \circ \dots \Theta$ (n occurrences of Θ).

3. The proof

Theorem 3.1. *Suppose that every subalgebra of $\mathbf{A}^4$ generated by 4 elements is congruence modular. Then $\mathbf{A}$ satisfies*

$$(\mathrm{wTIP}_2) \qquad \Gamma^* \cap \Theta^* = (\Gamma \cap (\Theta \circ \Theta))^*$$

for all tolerances Θ and Γ of $\mathbf{A}$.

Lemma 3.2. *If an algebra $\mathbf{A}$ satisfies*

$$(\Gamma \circ \Gamma)(\Theta \circ \Theta) \subseteq (\Gamma(\Theta \circ \Theta))^*$$

for all tolerances Θ and Γ of $\mathbf{A}$, then $\mathbf{A}$ satisfies

$$\Gamma^* \Theta^* = (\Gamma(\Theta \circ \Theta))^*$$

for all tolerances Θ and Γ of $\mathbf{A}$.

Proof. By the hypothesis,

$$\Gamma^{2^{n+1}}(\Theta \circ \Theta) = (\Gamma^{2^n} \circ \Gamma^{2^n})(\Theta \circ \Theta) \subseteq (\Gamma^{2^n}(\Theta \circ \Theta))^*$$

for all integers $n \geq 0$, since Γ^{2^n} is a tolerance.

Since $X^{**} = X^*$ for all subsets X of $\mathbf{A}$, we get by induction on $n \geq 1$:

$$\Gamma^{2^n}(\Theta \circ \Theta) \subseteq ((\Gamma \circ \Gamma)(\Theta \circ \Theta))^*$$

for all integers $n \geq 1$.

By applying the above identity twice (the second time with the role of Θ and Γ exchanged) we get

$$\Gamma^{2^n}\Theta^{2^m} \subseteq \Gamma^{2^n}(\Theta^{2^m} \circ \Theta^{2^m}) \subseteq ((\Gamma \circ \Gamma)(\Theta^{2^m} \circ \Theta^{2^m}))^* \subseteq ((\Gamma \circ \Gamma)(\Theta^{2^{m+1}}))^* \subseteq \\ ((\Gamma \circ \Gamma)(\Theta \circ \Theta))^* \subseteq (\Gamma(\Theta \circ \Theta))^*$$

for all positive integers m and n, where the last inclusion follows by applying the hypothesis again.

Now let $(a, b) \in \Gamma^*\Theta^*$. Then $(a, b) \in \Gamma^h\Theta^k$ for some integers h and k. For sufficiently large n and m, $h \leq 2^n$ and $k \leq 2^m$. Thus

$$\Gamma^h\Theta^k \subseteq \Gamma^{2^n}\Theta^{2^m} \subseteq (\Gamma(\Theta \circ \Theta))^*$$

that is, $(a, b) \in (\Gamma(\Theta \circ \Theta))^*$.

This implies that $\Gamma^*\Theta^* \subseteq (\Gamma(\Theta \circ \Theta))^*$.

The reverse inclusion is trivial. □

Proof of the Theorem. In view of the above Lemma, it is enough to show that

$$(\Gamma \circ \Gamma)(\Theta \circ \Theta) \subseteq (\Gamma(\Theta \circ \Theta))^*$$

So, let $a, b, c, d \in \mathbf{A}$ be such that $a\Gamma b\Gamma c$ and $a\Theta d\Theta c$. We need to show that $(a, c) \in (\Gamma(\Theta \circ \Theta))^*$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^4$ generated by the four elements

$$(daad) \quad (dabb) \quad (dcbb) \quad (dccd)$$

where we omit commas in order to improve readibility. Notice that if $(xyzw) \in \mathbf{B}$ then $x\Theta y$, $y\Gamma z$ and $z\Theta w$, since all generators satisfy the above conditions, and Θ, Γ are compatible, reflexive and symmetric.

Let us denote $(0_\mathbf{A} \times 0_\mathbf{A} \times 1_\mathbf{A} \times 1_\mathbf{A})|_\mathbf{B}$ by $(0011)_\mathbf{B}$, and the same for similar congruences of $\mathbf{B}$. We have

$$\begin{array}{ccc} (daad) & (0011)_\mathbf{B} & (dabb) \\ (0110)_\mathbf{B} & & (0100)_\mathbf{B} \\ (dccd) & (0011)_\mathbf{B} & (dcbb) \end{array}$$

that is,

$$((daad), (dccd)) \in (0110)_\mathbf{B} \cap ((0011)_\mathbf{B} + (0100)_\mathbf{B}) = \\ ((0110)_\mathbf{B} \cap (0011)_\mathbf{B}) + (0100)_\mathbf{B} = (0010)_\mathbf{B} + (0100)_\mathbf{B}$$

since $\mathbf{B}$ is congruence modular, and $(0110)_\mathbf{B} \supseteq (0100)_\mathbf{B}$.

This means that there is some n, and there are quadruples $(x_iy_iz_iw_i) \in \mathbf{B}$ such that

$$(daad) = (x_0 y_0 z_0 w_0)$$

$$(x_i y_i z_i w_i)(0010)_{\mathbf{B}}(x_{i+1} y_{i+1} z_{i+1} w_{i+1}) \qquad \text{for } i \text{ even}$$

$$(x_i y_i z_i w_i)(0100)_{\mathbf{B}}(x_{i+1} y_{i+1} z_{i+1} w_{i+1}) \qquad \text{for } i \text{ odd}$$

$$(x_n y_n z_n w_n) = (dccd)$$

This implies:

$$d = x_0 = x_1 = x_2 \cdots = x_n = d$$

$$d = w_0 = w_1 = w_2 \cdots = w_n = d$$

$$a = y_0 \quad y_n = c \quad z_n = c$$

$$z_i = z_{i+1} \qquad \text{for } i \text{ even}$$

$$y_i = y_{i+1} \qquad \text{for } i \text{ odd}$$

Since $(x_i y_i z_i w_i) \in \mathbf{B}$, by the remark made after the definition of $\mathbf{B}$, $y_i \Gamma z_i$ $d = x_i \Theta y_i$ and $z_i \Theta w_i = d$.

Hence, for all i, $y_i \Theta d \Theta z_i$, thus $y_i \Gamma(\Theta \circ \Theta) z_i$, and, since Γ and Θ are symmetric, $z_i \Gamma(\Theta \circ \Theta) y_i$.

Thus, the sequence

$$a = y_0, \quad z_0 = z_1 \quad y_1 = y_2 \quad \ldots \quad y_n = z_n = c$$

witnesses that $(a, c) \in (\Gamma(\Theta \circ \Theta))^*$. □

4. Further problems

R. Freese and B. Jónsson [6] showed that every congruence modular variety is in fact congruence arguesian. In [3] it has been shown, among other things, how to use TIP in order to give a short proof of R. Freese and B. Jónsson's result.

Indeed, [3] shows that every algebra satisfying TIP satisfies not only the arguesian identity, but also the higher arguesian identities introduced by M. Haiman [9]. Since every algebra in a congruence modular variety satisfies TIP, we get a new proof of R. Freese and B. Jónsson's result. More: we get

Theorem 4.1. [3] *Every congruence modular variety satisfies all Haiman's higher arguesian identities.*

However, the proof given in [3] does not entirely subsume the results in [6], since [6] actually proved the following local version.

Theorem 4.2. [6] *If every subalgebra of* $\mathbf{A}^2$ *is congruence modular then* $\mathbf{A}$ *is arguesian.*

The results obtained in the present note arose in the effort of getting a proof of Theorem 4.2 along the lines of [3]. Hopefully, one should also be able to generalize Theorem 4.2.

Problem 4.3. If every subalgebra of $\mathbf{A}^2$ is congruence modular does $\mathbf{A}$ satisfy Haiman's higher arguesian identities?

Since, as we mentioned, every algebra satisfying TIP satisfies Haiman's higher arguesian identities, an obvious way to use the methods from [3] would be to show that if every subalgebra of $\mathbf{A}^2$ is congruence modular, then $\mathbf{A}$ satisfies TIP; however, this is still an open problem. Towards this direction, we could only get Theorem 3.1 (and Theorem 4.6 below).

Let us mention that, as far as we know, it is even possible that wTIP$_2$ and TIP are actually equivalent. In fact, they are equivalent within a variety: a variety satisfies wTIP$_2$ if and only if it satisfies TIP, if and only if it is congruence modular [11].

Problem 4.4. Find (if it exists) an example of an algebra satisfying wTIP$_2$ but not TIP.

As far as we know, it is possible that the following has an affirmative answer.

Problem 4.5. If $\mathbf{A}$ satisfies wTIP$_2$, does $\mathbf{A}$ satisfies Haiman's higher arguesian identities?

In view of Theorem 3.1, an affirmative solution of Problem 4.5 would imply that Problem 4.3 has an affirmative solution, provided the exponent 2 is changed to 4.

Of course, another problem is whether the hypothesis that every subalgebra of $\mathbf{A}^4$ is congruence modular in Theorem 3.1 could be weakened to "every subalgebra of $\mathbf{A}^2$ is congruence modular".

We have a partial result towards this direction [10], [11].

Theorem 4.6. *Suppose that every subalgebra of* $\mathbf{A}^2$ *is congruence modular. Then* $\mathbf{A}$ *satisfies*

$$\text{(wTIP)} \qquad \Gamma^* \cap \Theta^* = (\Gamma \cap \Theta^*)^*$$

for all tolerances Θ *and* Γ *of* $\mathbf{A}$.

What makes wTIP particularly interesting is that it is equivalent (even for single algebras) to H.-P. Gumm's Shifting Principle, as defined in 3.1 on p. 14 in [8]. In particular, wTIP implies congruence modularity (there is a short direct proof along the lines of the proof given in [3] of Lemma 1 D there). Throughout a variety, wTIP is equivalent to modularity.

Of course, we could ask the problems of the present section replacing wTIP$_2$ by wTIP. Moreover, we could ask Problem 4.3 with the exponent 2 replaced by some larger exponent n.

Until (some of) the above problems are settled, the definition of $wTIP_2$, and the proof that it holds under the hypothesis of Theorem 3.1 appear to be the relevant contributions of the present note.

A final comment: of course, it is always interesting to prove some local version of a global result (a positive answer to Problem 4.3 would give a local version of Theorem 4.1). However, the main point is that the proof of a local version could shed more light on the global version, too. The original proof [6] of Theorem 4.2 uses geometrical ideas (a lattice-theoretical version of the well known result that projective geometries of dimension ≥ 3 are arguesian). On the other side, [8] develops a theory of algebras in congruence modular varieties deeply inspired by geometrical instruments. However, no direct connection between [6] and [8] is known. In the terminology of [8] "no transition between the projective geometry of algebras (as manifested in their congruence lattices) and the affine geometry (Kongruenzklassengeometrie) is known".

The main interest in Problem 4.3 is that a proof could give hints on how to obtain a correlation between the projective geometries and the affine geometries of algebras.

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