We discuss the geometry of Wigner-Yanase-Dyson information via the so-called Amari-Nagaoka embeddings in $L^p$-spaces of quantum trajectories.

1. Introduction

The Wigner-Yanase-Dyson information was introduced in 1963\textsuperscript{28}. Wigner and Yanase observed that “According to quantum mechanical theory, some observables can be measured much more easily than others: the observables which commute with the additive conserved quantities ... can be measured with microscopic apparatuses; those which do not commute with these quantities need for their measurements macroscopic systems. Hence the problem of defining a measure of our knowledge with respect to the
latter quantities arises ...”. After the discussion of the requirements such a measure should satisfy (convexity, ...) they proposed, tentatively, the following formula and called it skew information:

\[ I_\rho(A) := -\frac{1}{2} \text{Tr}([\rho^\frac{1}{2}, A]^2). \]

More generally they defined (following a suggestion by Dyson)

\[ I_\rho^\beta(A) := -\frac{1}{2} \text{Tr}([\rho^\beta, A] \cdot [\rho^{1-\beta}, A]), \quad \beta \in [0, 1]. \]

The latter is known as WYD-information. The skew information should be considered as a measure of information contained in a state \( \rho \) with respect to a conserved observable \( A \).

From that fundamental work WYD-information has found applications in a manifold of different fields. A possibly incomplete list should mention: i) strong subadditivity of entropy\(^{23,22} \); ii) homogeneity of the state space of factors (of type III\(_1\))\(^6 \); hypothesis testing \(^3 \) iii) measures for quantum entanglement \(^4,19 \); iv) uncertainty relations\(^{24,25,21,27,7,10,11,12,13} \).

Such a variety should be not surprising at the light of the result showing that WYD-information is just an example of monotone metric, namely it is a member of the vast family of quantum Fisher informations\(^9 \). On the other hand one can prove that, among the family of all the quantum Fisher informations, the geometry of WYD-information is rather special\(^{8,16} \).

In this paper we want to discuss the particular features of WYD-information emphasizing the relation with the embedding of quantum dynamics in \( L_p \)-spaces.

2. Preliminary notions of matrix analysis

Let \( M_n := M_n(\mathbb{C}) \) (resp. \( M_{n,sa} := M_n(\mathbb{C})_{sa} \)) be the set of all \( n \times n \) complex matrices (resp. all \( n \times n \) self-adjoint matrices). We shall denote general matrices by \( X, Y, ... \) while letters \( A, B, ... \) (or \( H \)) will be used for self-adjoint matrices. Let \( D_n \) be the set of strictly positive elements of \( M_n \) while \( D_n^1 \subset D_n \) is the set of density matrices namely

\[ D_n^1 = \{ \rho \in M_n | \text{Tr} \rho = 1, \rho > 0 \}. \]

The tangent space to \( D_n^1 \) at \( \rho \) is given by \( T_\rho D_n^1 \equiv \{ A \in M_{n,sa} : \text{Tr}(A) = 0 \} \), and can be decomposed as \( T_\rho D_n^1 = (T_\rho D_n^1)^c \oplus (T_\rho D_n^1)^o \), where \( (T_\rho D_n^1)^c := \{ A \in T_\rho D_n^1 : [A, \rho] = 0 \} \), and \( (T_\rho D_n^1)^o \) is the orthogonal complement of \( (T_\rho D_n^1)^c \), with respect to the Hilbert-Schmidt scalar product \( \langle A, B \rangle := \)}
\[ \langle A, B \rangle_{HS} := \text{Tr}(A^* B) \] (the Hilbert-Schmidt norm will be denoted by \(|| \cdot |||\)).

A typical element of \((T_{\rho} D_n)^o\) has the form \(A = i[\rho, H]\), where \(H\) is self-adjoint.

In what follows we shall need the following result (pag. 124 in \textsuperscript{2}).

**Proposition 2.1.** Let \(A \in M_{n,sa}\) be decomposed as

\[ A = A^c + i[q, H] \]

where \(q \in D_n\), \([A^c, q] = 0\) and \(H \in M_{n,sa}\). Suppose \(\varphi \in C^1(0, +\infty)\). Then

\[ (D_q \varphi)(A) = \varphi'(q)A^c + i[\varphi(q), H]. \]

### 3. Schrödinger equation and quantum dynamics

Let \(\rho(t)\) be a curve in \(D_n^1\) and let \(H \in M_{n,sa}\). We say that \(\rho(t)\) satisfy the Schrödinger equation w.r.t. \(H\) if

\[ \frac{d}{dt}\rho(t) = i[\rho(t), H]. \]

This equation is also known in the literature as the Landau-von Neumann equation.

The solution of the above evolution equation (please note that \(H\) is time independent) is given by

\[ \rho_H(t) := e^{-itH} \rho e^{itH}. \]  \hspace{1cm} (1)

Therefore the commutator \(i[\rho, H]\) appears as the tangent vector to the quantum trajectory (1) (at the initial point \(\rho = \rho_H(0)\)) generated by \(H\). Suppose we are considering two different evolutions determined, through the Schrödinger equation, by \(H\) and \(K\). If we want to quantify how “different” the trajectories \(\rho_H(t), \rho_K(t)\) are, then it would be natural to measure the “area” spanned by the tangent vectors \(i[\rho, H], i[\rho, K]\) (with respect to some scalar product\textsuperscript{10}).

### 4. \(L^p\)-embedding for states and trajectories

The functions

\[ \rho \rightarrow \frac{\rho^\beta}{\beta}, \quad \beta \in (0, 1) \]

are known as Amari-Nagaoka embeddings\textsuperscript{1,14}. They can be considered as an immersion of the state manifold into \(L^p\)-spheres.
Proposition 4.1. Let \( \rho(t) \) be a curve in \( D^1_n \), let \( H \in M_{n,sa} \) and let \( \beta \in (0,1) \). The following differential equations are equivalent

\[
\frac{d}{dt} \rho(t) = i[\rho(t), H],
\]

(1)

\[
\frac{d}{dt} (\rho(t)^\beta) = i[\rho(t)^\beta, H].
\]

(2)

Proof. Let \( \phi_\beta(\rho) := \rho^\beta \). By Proposition 2.1 we get

\[
\frac{d}{dt} (\rho(t)^\beta) = D_{\rho^\beta} \phi_\beta \circ \frac{d}{dt} \rho(t) = D_{\rho^\beta} (\rho(t), H] = (i[\rho^\beta(\rho(t), H]) = i[\rho(t)^\beta, H].
\]

Therefore, Equation (1) implies Equation (2). Analogously, again using Proposition 2.1, Equation (2) implies Equation (1) because we have

\[
\frac{d}{dt} (\rho(t)) = \frac{d}{dt} \left( (\rho(t)^\beta) \right) = D_{\rho^\beta} \phi_\beta^{-1} \circ \frac{d}{dt} (\rho(t)^\beta) = D_{\rho^\beta} \phi_\beta^{-1} \circ i[\rho(t)^\beta, H] = D_{(g(t))} \phi_\beta^{-1} \circ i[g(t), H] = i[\phi_\beta^{-1}(g(t)), H] = i[\rho(t), H].
\]

5. WYD-information by pairing of dual trajectories

The Wigner-Yanase-Dyson information is defined as

\[
I_\beta^\rho(H) := -\frac{1}{2} \text{Tr}([\rho^\beta, H] \cdot [\rho^{1-\beta}, H]), \quad \beta \in (0,1).
\]

Let us explain the link between \( L^p \)-embeddings and WYD-information. Let \( V, W \) be vector spaces over \( \mathbb{R} \) (or \( \mathbb{C} \)). One says that there is a duality pairing if there exists a separating bilinear form \( \langle \cdot, \cdot \rangle : V \times W \rightarrow \mathbb{R} (\mathbb{C}) \).

In the case of \( L^p \) spaces the pairing is given by the \( L^2 \) scalar product. In our case this is just the HS-scalar product.

Note that using the function \( \rho \rightarrow \rho^\beta \) we may look at dynamics as a curve on a \( L^\frac{1}{2} \)-sphere. The function \( \rho \rightarrow \rho^{1-\beta} \) does the same on the dual space \( (L^\frac{1}{2})^* = L^{\frac{2}{1-\beta}} \).

Proposition 5.1. If \( \rho(t) \) satisfies the Schrödinger equation w.r.t. \( H \) then

\[
\frac{d}{dt} \rho(t)^\beta, \frac{d}{dt} \rho(t)^{1-\beta} = 2 \cdot I^\beta_{\rho(t)}(H) \quad \beta \in (0,1).
\]
Proof. Apply Proposition 4.1 to obtain
\[
\langle \frac{d}{dt} (\rho(t)^\beta), \frac{d}{dt} (\rho(t)^{1-\beta}) \rangle = \langle i[\rho(t)^\beta, H], i[\rho(t)^{1-\beta}, H] \rangle = -\text{Tr}(\rho(t)^\beta, H) [\rho(t)^{1-\beta}, H] \rangle.
\]

In this way WYD-information appears as the “pairing” of the dual $L^p$-embeddings of the same quantum trajectory.

6. Quantum Fisher informations

In the commutative case a Markov morphism is a stochastic map $T : \mathbb{R}^n \to \mathbb{R}^k$. In the noncommutative case a Markov morphism is a completely positive and trace preserving operator $T : M_n \to M_k$. Let

\[
\mathcal{P}_n := \{\rho \in \mathbb{R}^n | \rho_i > 0\} \quad \mathcal{P}_1^1 := \{\rho \in \mathbb{R}^n | \sum \rho_i = 1, \rho_i > 0\}.
\]

In the commutative case a monotone metric is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{P}_n^1\}$, $n \in \mathbb{N}$, such that

\[
g_m^{T(p)}(TX, TX) \leq g_n^{\rho}(X, X)
\]

holds for every Markov morphism $T : \mathbb{R}^n \to \mathbb{R}^m$ and all $\rho \in \mathcal{P}_n^1$ and $X \in T_\rho \mathcal{P}_n^1$.

In perfect analogy, a monotone metric in the noncommutative case is a family of Riemannian metrics $g = \{g^n\}$ on $\{\mathcal{D}_n^1\}$, $n \in \mathbb{N}$, such that

\[
g_m^{T(p)}(TX, TX) \leq g_n^{\rho}(X, X)
\]

holds for every Markov morphism $T : M_n \to M_m$ and all $\rho \in \mathcal{D}_n^1$ and $X \in T_\rho \mathcal{D}_n^1$.

Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is called operator monotone if, for any $n \in \mathbb{N}$, any $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is said symmetric if $f(x) := x f(x^{-1})$. With such operator monotone functions $f$ one associates the so-called Chentsov–Moroztova functions

\[
c_f(x, y) := \frac{1}{y f(xy^{-1})} \quad \text{for} \quad x, y > 0.
\]

Define $L_\rho(A) := \rho A$, and $R_\rho(A) := A \rho$. Since $L_\rho$ and $R_\rho$ commute we may define $c(L_\rho, R_\rho)$ (this is just the inverse of the operator mean associated to $f$ by Kubo-Ando theory\(^{10}\)). Now we can state the fundamental theorems about monotone metrics. In what follows uniqueness and classification are stated up to scalars (for reference see \(^{26}\)).
Theorem 6.1. (Chentsov 1982) There exists a unique monotone metric on $P_n^1$ given by the Fisher information.

Theorem 6.2. (Petz 1996) There exists a bijective correspondence between monotone metrics on $D_n^1$ and symmetric operator monotone functions. For $\rho \in D_n^1$, this correspondence is given by the formula

$$g_f(A, B) := g_{f, \rho}(A, B) := \text{Tr}(A \cdot c_f(L_{\rho}, R_{\rho})(B)).$$

Because of these two theorems, the terms “Monotone Metrics” and “Quantum Fisher Informations” are used with the same meaning. Note that usually monotone metrics are normalized so that $[A, \rho] = 0$ implies $g_{f, \rho}(A, A) = \text{Tr}(\rho^{-1}A^2)$, that is equivalent to set $f(1) = 1$.

7. The WYD monotone metric

The following functions are symmetric, normalized and operator monotone (see $9,16$). Let

$$f_\beta(x) := \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(1 - x^\beta)} \quad \beta \in (0, 1).$$

Proposition 7.1. For the QFI associated to $f_\beta$ one has

$$g_{f_\beta}(i[\rho, H], i[\rho, K]) = -\frac{1}{\beta(1 - \beta)} \text{Tr}([\rho^\beta, H] \cdot [\rho^{1 - \beta}, K]) \quad \beta \in (0, 1).$$

One can find a proof in $9,16$. Because of the above Proposition, $g_\beta$ is known as WYD($\beta$) monotone metric.

Of course what we have seen about $L^p$-embedding of quantum dynamics applies to this example of quantum Fisher information. Indeed we can summarize everything into the following final result.

Proposition 7.2. Let $H, K$ be selfadjoint matrices and $\rho$ be a density matrix. Choose two curves $\rho(t), \sigma(t) \subset D_n^1$ such that

i) $\rho(t)$ satisfies the Schrödinger equation w.r.t. $H$;

ii) $\sigma(t)$ satisfies the Schrödinger equation w.r.t. $K$;

iii) $\rho = \rho(0) = \sigma(0)$.

One has

$$g_{f_\beta}(i[\rho, H], i[\rho, K]) = \left( \frac{d}{dt} \left( \frac{\rho(t)^\beta}{\beta} \right), \frac{d}{dt} \left( \frac{\sigma(t)^{1 - \beta}}{1 - \beta} \right) \right)|_{t=0} \quad \beta \in (0, 1).$$
Proof. From Proposition 7.1, one gets
\[ g_{f_{\beta}}(i[\rho, H], i[\rho, K]) = -\frac{1}{\beta(1-\beta)} \text{Tr}([\rho^{\beta}, H] \cdot [\rho^{1-\beta}, K]) \]
\[ = -\frac{1}{\beta(1-\beta)} \text{Tr}([\rho(t)^{\beta}, H] \cdot [\sigma(t)^{1-\beta}, K])|_{t=0} \]
\[ = \left( \frac{d}{dt} \left( \frac{\rho(t)^{\beta}}{\beta} \right) \cdot \frac{d}{dt} \left( \frac{\sigma(t)^{1-\beta}}{1-\beta} \right) \right)|_{t=0} \]

\[ \square \]

8. Conclusion
All the ingredients of the above construction make sense on a von Neumann algebra: WYD-information, quantum dynamics, \( L^p \)-spaces, Amari-Nagoka embeddings and so on\(^{20,14} \). Nevertheless we are not aware of any attempt to see geometry of WYD-information along the lines described in the present paper, in the infinite-dimensional context. We plan to address this problem in future work.

References