Complex integration.

1. *Compute*
$$\int_\gamma \frac{e^z}{z}dz, \qquad \int_\gamma \frac{e^z}{(z-1/2)^2}dz,$$
where $\gamma = \{e^{i\theta},\ \theta \in [0,2\pi]\}$.

Sol.: The function e^z is holomorphic on $\mathbf{C}$ and coincides with all its derivatives. Hence by Cauchy integral formulas we have
$$\int_\gamma \frac{e^z}{z}dz = 2\pi i\, e^0 = 2\pi i$$
and
$$\int_\gamma \frac{e^z}{(z-1/2)^2}dz = 2\pi i (e^z)'_{|z=1/2} = 2\pi i\, e^{1/2}.$$

2. *Compute*
$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-1)(z-2i)}dz,$$
where $\gamma = \{4e^{i\theta},\ \theta \in [0,2\pi]\}$.

Sol.: Both $z=1$ and $z=2i$ are inside the circle of center 0 and radius 4. Write
$$\frac{1}{(z-1)(z-2i)} = \frac{a}{(z-1)} + \frac{b}{(z-2i)}, \qquad a = (1+2i)/5, \quad b = -(1+2i)/5.$$
Then
$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-1)(z-2i)}dz = \frac{1}{2\pi i}\int_\gamma \frac{a}{(z-1)} + \frac{1}{2\pi i}\int_\gamma \frac{b}{(z-2i)} = a+b = 0,$$
by direct computation or by applying Cauchy integral formula to the constant functions $f(z)=a$ and $g(z)=b$.

3. *Compute*
$$\int_\gamma \frac{\cos z}{(z-\pi/2)^{10}}dz,$$
where $\gamma = \{2e^{i\theta},\ \theta \in [0,2\pi]\}$.

Sol.: The point $\pi/2$ is inside the circle of center 0 and radius 2 and the function $\cos z$ is holomorphic on $\mathbf{C}$. Hence
$$\int_\gamma \frac{\cos z}{(z-\pi/2)^{10}}dz = 2\pi i \frac{1}{9!}\cos^{(9)}(\pi/2),$$
where $\cos^{(9)}(\pi/2) = -\sin(\pi/2) = -1$.

4. *Let $|a| < 1 < |b|$. For $n, m \in \mathbf{Z}$, compute*

$$\frac{1}{2\pi i}\int_\gamma \frac{(z-b)^m}{(z-a)^n}dz,$$

where $\gamma = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$.

Sol.: Observe that $(z-b)^m$ is holomorphic inside the unit disc, for every $m \in \mathbf{Z}$. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{(z-b)^m}{(z-a)^n}dz = 0,$$

for all $m \in \mathbf{Z}$ and $n \leq 0$. Now let $m \in \mathbf{Z}$ and $n > 0$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{(z-b)^m}{(z-a)^n}dz = cf^{(n-1)}(a),$$

where $f(z) = (z-b)^m$ and $c \in \mathbf{C}$ is some constant...to be computed.

Chauchy's formulas, identity principle, Liouville theorem.

5. *Let $\Omega \subset \mathbf{C}$ be an open connected set. Let $f, g\colon \Omega \longrightarrow \mathbf{C}$ be holomorphic functions such that $f \cdot g \equiv 0$. Then either $f \equiv 0$ or $g \equiv 0$.*

Sol.: Suppose for example that $f \not\equiv 0$. Then there exist a point $z_0 \in \Omega$ and an open neighbourhood U_{z_0} of z_0 in Ω such that $f(z) \neq 0$, for all $z \in U_{z_0}$. But then $g(z) = 0$, for all $z \in U_{z_0}$ and, by the identity principle, $g \equiv 0$ on Ω.

6. *Let $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function, such that $|f(z)| \leq |e^z|$, for all $z \in \mathbf{C}$. Prove that $f(z) = ce^z$, for some constant $c \in \mathbf{C}$.*

Sol.: Consider the function $g(z) := f(z)/e^z$. Since e^z is holomorphic and $\neq 0$ for all $z \in \mathbf{C}$, the function g is well defined and holomorphic on $\mathbf{C}$. Moreover $|g(z)| \leq 1$, for all $z \in \mathbf{C}$. Hence $g \equiv c$, for some constant $c \in \mathbf{C}$ and $f(z) = ce^z$, as stated.

7. *Let $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function. Assume that there exist positive constants M and c for which $|f(z)| \leq M(c + |z|^n)$, for all $z \in \mathbf{C} \setminus B(0, R)$, with $R > 0$.*
   (a) Show that f is a polynomial of degree $\leq n$.
   (b) Assume $|f(z)| \leq M|z|^2$. Determine $f(0)$ and $f'(0)$.

Sol.: (a) As f is entire, it admits a series expansion around 0,

$$f(z) = \sum_{k\geq 0} a_k z^k, \qquad a_k = \frac{1}{k!}f^{(k)}(0) = \frac{1}{2\pi i}\int_{|z|=R}\frac{f(\xi)}{\xi^{k+1}}d\xi, \quad (\forall R > 0),$$

converging on all $\mathbf{C}$. We need to show that $a_k = 0$, for all $k > n$.
We have

$$|a_k| \leq \frac{1}{2\pi}\,2\pi R\,\frac{\sup_{|z|=R}|f(z)|}{R^{k+1}} \leq \frac{M(c+R^n)}{R^k},$$

which tends to 0, for $k > n$, when $R \to \infty$.
(b) It is clear that $f(0) = 0$, since $\lim_{|z|\to 0} |f(z)| = 0$.
Moreover $\frac{|f(z)-f(0)|}{|z|} \leq M|z|$. Hence $|f'(0)| = \lim_{|z|\to 0} \frac{|f(z)-f(0)|}{|z|} \leq M|z| = 0$.

8. *Determine all the zeros of* $\sin z$ *and* $\cos z$. *Let* $U = \mathbf{C} \setminus \{\pi/2 \pm k\pi,\ k \in \mathbf{Z}\}$. *Let* f *be a holomorphic function on* U, *such that* $f(\pi/n) = \tan(\pi/n)$, $n \geq 3$. *Deduce that* f *is not holomorphic on all* $\mathbf{C}$ *and that it does not assume the value* i.

Sol.: Let $z = x + iy \in \mathbf{C}$. One has

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \qquad \cos z = \cos x \cosh y + i \sin x \sinh y.$$

Hence

$$\sin z = 0 \quad \Leftrightarrow \quad \begin{cases} \sin x = 0 \\ \sinh y = 0 \end{cases} \quad \Leftrightarrow \quad z = k\pi,\ k \in \mathbf{Z};$$

$$\cos z = 0 \quad \Leftrightarrow \quad \begin{cases} \cos x = 0 \\ \sinh y = 0 \end{cases} \quad \Leftrightarrow \quad z = \frac{\pi}{2} + k\pi,\ k \in \mathbf{Z}.$$

The functions $f(z)$ and $\tan(z)$ are holomorphic on the same open connected set U, containing 0. Since $\{\frac{\pi}{n}\}_{n\in\mathbf{Z}_{\geq 3}}$ is a set with an accumulation point in 0, it is a "uniqueness set" for f. It follows that $f(z) = \tan(z)$, for all $z \in U$. In particular f is unbounded near the points $\{\frac{\pi}{n}\}_{n\in\mathbf{Z}_{\geq 3}}$ and cannot be holomorphic on all of $\mathbf{C}$. Now assume that f takes the value i for some $z_0 \in U$. Then

$$f(z_0) = \tan(z_0) = i \quad \Leftrightarrow \quad \sin(z_0)/\cos(z_0) = i \quad \Leftrightarrow \quad \begin{cases} \sin x_0(\cosh y_0 - \sinh y_0) = 0 \\ \cos x_0(\cosh y_0 + \sinh y_0) = 0. \end{cases}$$

Since $\cos x \neq 0$ on U and $\sinh y + \cosh y = e^y \neq 0$, we have a contradiction.