Automorphisms of Δ

1. Let Δ be the unit disc.
(a) Show that every automorphism of Δ extends injectively to a neighbourhood of its closure $\overline{\Delta}$ and admits at least a fixed point in $\overline{\Delta}$.
(b) Show that if f has a fixed point in Δ, then it is necessarily unique.

Sol.: Let $f(z) = e^{i\theta}\frac{z-z_0}{1-z\bar{z}_0}$ be an automorphism of the disc, where $\theta \in \mathbb{R}$ and $z_0 \in \Delta$. Since the rotation $z \mapsto e^{i\theta}z$ is defined and injective on all $\mathbb{C}$, it is sufficient to consider $g(z) = \frac{z-z_0}{1-z\bar{z}_0}$. The map g is well defined provided that $1 - z\bar{z}_0 \neq 0$, in particular for all z with $|z| < 1/|z_0|$. Note that $1/|z_0| > 1$, meaning that g is well defined on an open neighbourhood of $\overline{\Delta}$. To prove injectivity, we solve

$$w = \frac{z-z_0}{1-z\bar{z}_0} \quad \Leftrightarrow \quad z = \frac{w+z_0}{1+w\bar{z}_0}.$$

To see that there exists at least a fixed point in $\overline{\Delta}$, we consider the equation

$$e^{i\theta}\frac{z-z_0}{1-z\bar{z}_0} = z \quad \Leftrightarrow \quad z^2 + z\,\frac{e^{i\theta}-1}{\bar{z}_0} - e^{i\theta}\frac{z_0}{\bar{z}_0} = 0.$$

The equation shows that there are at most 2 fixed points. Since the modulus of their product is $|e^{i\theta}\frac{z_0}{\bar{z}_0}| = 1$, then either both fixed points are on $\partial\Delta$ or one is inside and the other is outside Δ.

2. *Define $D(0,r) = \{z \in \mathbb{C} \ : \ |z| < r\}$. Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \partial D(0,1)$. Show that if f is nonconstant, then there exists an automorphism g of $D(0,1)$ such that $f \circ g(0) = 0$.*

Sol.: By the maximum modulus principle, $|f(z)| < 1$, for all $z \in D(0,1)$. In other words, f maps the unit disc into itself. Since $|f|$ is constant on the boundary of the disc, then there exists $z_0 \in D(0,1)$ such that $f(z_0) = 0$ (cf. Exercise 9). Since the automorphism group of the disc acts transitively on it, then there exists an automorphism g such that $g(0) = z_0$. It follows that $f(g(0)) = f(z_0) = 0$.

3. *Let $f : \Delta \to \Delta$ be a holomorphic function with a zero of order m at 0. Show that for all $z \in \Delta$ one has $|f(z)| \leq |z|^m$.*

Sol.: This exercise is a variation of the Schwarz Lemma. The power expansion of f around 0 is $f(z) = a_m z^m + a_{m+1}z^{m+1} + \dots$. Consider the holomorphic function

$$g(z) := \begin{cases} \frac{f(z)}{z^m} = a_m + a_{m+1}z + \dots, & z \neq 0 \\ f^{(m)}(0), & z = 0. \end{cases}$$

For $0 < r < 1$, one has

$$|g(z)| \leq 1/r, \quad \forall z \ : \ |z| = r,$$

and by the maximum principle the above inequality holds for all z with $|z| \leq r$. By letting $r \to 1$, we obtain $|g(z)| \leq 1$, for all $z \in \Delta$. Equivalently

$$|f(z)| \leq |z|^m,$$

as claimed.

4. *Verify that the Cayley transform*
$$C(z) := \; i\,\frac{1+z}{1-z}$$
is a biholomorphism between the unit disc Δ *and the upper half plane* $\mathbb{H}^+$.

Sol.: The map C is holomorphic on Δ. Since
$$Im\left(i\,\frac{1+z}{1-z}\right) = \frac{1-|z|^2}{|1-z^2|} > 0, \quad \forall z \in \Delta,$$
the map C takes the disc onto the upperhalfplane. Finally C is injective, as it has inverse
$$C^{-1}\colon \mathbb{H} \to \Delta, \quad w \mapsto \frac{w-i}{w+i}.$$

5. *Show that the strip* $S_r := \{\, z \in \mathbb{C} \;:\; 0 < \mathrm{Im}\, z < r\,\}$, *with* $r > 0$, *is biholomorphic to* $\mathbb{H}$ *(and therefore to* Δ*). Determine an explicit biholomorphism* $f\colon S_1 \to \Delta$.

Sol.: The exponential map e^z determines a biholomorphism between the strip S_π and the upper-halplane $\mathbb{H}$. It is holomorphicn and injective
$$e^z = e^w,\ z, w \in S_\pi \quad \Leftrightarrow \quad e^{z-w} = 1 \quad \Leftrightarrow \quad z - w \in \mathbb{Z}2\pi i \quad \Leftrightarrow \quad z = w.$$
It is surjective: horizontal lines in the strip S_π are mapped into halflines from the origin in $\mathbb{H}$.
Any strip S_r, with $r > 0$, is biholomorphic to S_π via the map $z \mapsto \frac{\pi}{r}z$.
An explicit biholomorphism $f\colon S_1 \to \Delta$ is given by
$$z \mapsto C^{-1}(\exp(\pi z)),$$
where $C^{-1}\colon \mathbb{H} \to \Delta$ is the inverse Cayley transform of Execise 13.
Note: The strip S_r is a proper convex subset of $\mathbb{C}$. In particular it is simply connected. We'll see later in the course that any such set is biholomorphic to the unit disc.

6. *Determine whether there exists a nonconstant holomorphic function* $f\colon \mathbb{C} \to \mathbb{C}$ *whose image* $f(\mathbb{C})$ *has empty intersection with the border* $\partial\Delta$ *of the unit disc* Δ.

Sol.: If $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc Δ, then either $f(\mathbb{C}) \subset \Delta$ or $f(\mathbb{C}) \subset \mathbb{C} \setminus \Delta$. In the first case f is bounded, and therefore constant. In the second case $f(\mathbb{C})$ is contained in a domain D biholomorphic to the unit disc, i.e. there exists a biholomorphism $g\colon D \to \Delta$ such that $g \circ f\colon \mathbb{C} \to \Delta$ is a bounded holomorphic function. By Liouville's theorem, $g \circ f$ is constant and therefore also f is constant.

7. *Determine whether there exists a nonconstant holomorphic function* $f\colon \mathbb{C} \to \mathbb{C}$ *whose image* $f(\mathbb{C})$ *has empty intersection with the real line* $\mathbb{R}$.

Sol.: If $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$, then it either contained in the upperhalf plane or in the lower half plane in $\mathbb{C}$. In both halfplanes are biholomorphic to the unit disc. Then the same argument as in the previous exercise shows that there are no nonconstant holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$.

Laurent series and isolated singularities, Riemann extension theorem, Casorati-Weierstrass theorem, automorphisms of the plane, residue theorem

8. *Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions with $g(z) \neq 0$, for every $z \in \mathbb{C}$. Assume that $|f(z)| \leq |g(z)|$, for every $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ (cf. Exercise 17 in sheet 0). Show that the assumption "$g(z) \neq 0$, for every $z \in \mathbb{C}$" can be replaced with "g not identically zero".*

Sol.: Under the assumption that g never vanishes, f/g is a bounded holomorphic function. Hence it is constant by Liouville's theorem, i.e. $f/g = c$.

Without such assumption, zeros of g are isolated singularities of f/g. However f/g is bounded near such singularities. By the Riemann extension theorem, they are removable singularities and the statement follows from Liouville's theorem.

9. *(a) Let f be a bounded holomorphic function defined on $\mathbb{C} \setminus \{0, i, 1+i\}$ or on $C \setminus \mathbb{Z}$. Show that f is constant.*
   (b) Is it true that f is constant when it is bounded on $\mathbb{C} \setminus (\{1/n : n \in \mathbb{N}\} \cup \{0\})$?

Sol.: (a) The boundedness assumption implies that all singularities of f are removable. In other words, f is holomorphic and bounded on all $\mathbb{C}$. Therefore it is constant.
(b) By the boundedness assumption, all isolated singularities $\{1/n\}_{n\in\mathbb{N}}$ are removable. Hence f is holomorphic and bounded at least on $\mathbb{C} \setminus \{0\}$. On the other hand, if 0 were singular, then f could not be bounded in any neighbourhood of 0. Conclusion: f is holomorphic and bounded on all $\mathbb{C}$, and therefore constant.

10. *Show that the functions*

$$\tfrac{\sin z}{z}, \qquad \tfrac{e^z - 1}{z}, \qquad \tfrac{\cosh z - 1}{z}$$

are entire, i.e. they extend holomorphically to $\mathbb{C}$.

Sol.: The above functions are holomorphic on all $\mathbb{C}$, except possibly for $z = 0$. By expanding the numerators around $z = 0$ we find

$$\frac{\sin z}{z} = \frac{z - z^3/3! + z^5/5! - \dots}{z} = 1 - z^2/3! + z^4/5! - \dots$$

$$\frac{e^z - 1}{z} = \frac{1 + z + z^2/2! + z^3/3! + \dots - 1}{z} = 1 + z/2! + z^2/3! + \dots$$

$$\frac{\cosh z - 1}{z} = \frac{1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \dots - 1}{z} = \frac{1}{2}z + \frac{1}{24}z^3 + \dots,$$

which show that $z = 0$ is indeed a removable singularity, i.e. the three functions are entire.

11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.*
   (a) Show that if f has a zero of order n in z_0, then $1/f$ has a pole of order n in z_0.
   (b) Let z_0 be a singularity (removable, polar, essential) of f. Determine the corresponding type of singularity of $1/f$.

Sol.: (a) Write $f(z) = (z - z_0)^n u(z)$, where $u(z) = a_n + a_{n+1}(z - z_0) + \dots$ is a holomorphic function with $u(z_0) \neq 0$. Then

$$1/f(z) = \frac{1}{(z - z_0)^n} \frac{1}{u(z)}.$$

Since $1/u(z)$ is holomorphic around z_0, then z_0 is a pole of $1/f$ of order n.
(b) Let z_0 be a removable singularity for f and let $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$ be the Taylor expansion of f around z_0. If $f(z_0) \neq 0$, then $1/f$ is holomorphic around z_0; if z_0 is a zero of f of order n, then z_0 is a pole of order n of $1/f$.
Let z_0 be a pole of f or order m and let

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} \dots = \frac{1}{(z - z_0)^m}(a_{-m} + a_{-m+1}(z - z_0) + \dots)$$

be the Laurent expansion of f around z_0, where $g(z) = a_{-m} + a_{-m+1}(z - z_0) + \dots$ is a holomorphic function with $g(z_0) \neq 0$. Now it is clear that z_0 is a zero of $1/f = \frac{(z-z_0)^m}{g(z)}$ of order n.
(c) Let z_0 be an essential singularity for f. Then the image of any disc $D(z_0, r) \setminus \{z_0\}$ is dense in $\mathbb{C}$ and the same is true for $\frac{1}{f}$. Hence z_0 is an essential singularity for $1/f$ as well.

12. *Let z_0 be a pole of f. Recall that for every $M >> 0$ there exists $\varepsilon > 0$ such that $f(D^*(z_0, \epsilon)) \subset \{|z| > M\}$.*
Show that, given $\varepsilon > 0$, there exists $M >> 0$ such that $\{|z| > M\} \subset f(D^*(z_0, \epsilon))$.
(Justify your answer and mention all the results you used.)

Sol.: If z_0 is a pole, then $\lim_{z \to z_0} |f(z)| = +\infty$. Equivalently, for every $M >> 0$ there exists $\varepsilon > 0$ such that $f(D^*(z_0, \epsilon)) \subset \{|z| > M\}$.

View the map f valued in $¶^1(\mathbb{C})$, with $f(z_0) = \infty$. One easily cheks that f is a non constant holomorphic map to $¶^1(\mathbb{C})$, viewed as a Riemann surface. In particular f is open and its image contains a neighbourhood of ∞. Therefore, given $\varepsilon > 0$, there exists $M >> 0$ such that $\{|z| > M\} \subset f(D^*(z_0, \epsilon))$.

13. *Determine and classify all the singularities of the following functions:*

$$\tan z, \quad \frac{1}{z^3}\sum_{n=2}^{\infty} 2^n z^n, \quad z e^{1/z} e^{-1/z^2}, \quad \frac{\sin\frac{1}{z}}{z^4}, \quad \frac{\sin z}{z^4}, \quad \frac{1}{z^3} - \cos z.$$

Sol.: (a) The function $f(z) = \tan z = \sin z / \cos z$ is defined and holomorphic on $\mathbb{C} \setminus \{\pi/2 + k\pi,\ k \in \mathbb{Z}\}$, where $\cos z \neq 0$, and it is periodic of period π. So it is sufficient to determine the type singularity at $\pi/2$. For this note that $\sin z$ is non-zero close to $\pi/2$ while $\cos z$ has a zero of order one in $\pi/2$. It follows that all points $\{\pi/2 + k\pi,\ k \in \mathbb{Z}\}$ are simple poles for $\tan z$ (cf. ex. 4).
(b) The function

$$\frac{1}{z^3}\sum_{n=2}^{\infty} 2^n z^n = \sum_{n=2}^{\infty} 2^n z^{n-3} = 2^2\frac{1}{z} + 2^3 + 2^4 z + \dots$$

has a pole of order 1 at $z = 0$.
(c) The function

$$z e^{1/z} e^{-1/z^2} = z e^{1/z - 1/z^2} = z(1 + (\frac{1}{z} - \frac{1}{z^2}) + \frac{1}{2}(\frac{1}{z} - \frac{1}{z^2})^2 + \dots)$$

has an essential singularity at $z = 0$.
(d) The function

$$\frac{\sin(1/z)}{z^4} = \frac{1}{z^4}(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \dots)$$

has an essential singularity at $z = 0$.
(e) The function

$$\frac{\sin z}{z^4} = \frac{1}{z^4}(z - \frac{1}{3!}z^3 + \dots)$$

has a pole of order 3 at $z = 0$. Alternatively, one can observe that

$$\lim_{z\to 0} z^3 \frac{\sin z}{z^4} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$$

(f) Since $\cos z$ is holomorphic on $\mathbb{C}$, the function

$$\frac{1}{z^3} - \cos z$$

has a pole of order 3 at $z = 0$.

14. *Let* $g(z) := e^{1/z} - e^{2/z}$. *Determine* $g(\mathbb{C} \setminus \{0\})$.

Sol.: The Laurent series expansion of g around $z = 0$ is given by

$$g(z) = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \dots - (1 + \frac{2}{z} + \frac{1}{2!}\frac{2^2}{z^2} + \frac{1}{3!}\frac{2^3}{z^3} + \dots$$

$$= -\frac{1}{z} - \frac{1}{2!}\frac{2^2-1}{z^2} - \frac{1}{3!}\frac{2^3-1}{z^3} + \dots$$

One sees that $z = 0$ is an essential singularity for g. Hence $g(\mathbb{C} \setminus \{0\})$ is dense in $\mathbb{C}$, by Casorati-Weierstrass theorem. Also, the equation $w = e^{1/z}(1 - e^{1/z})$ can be solved for every $w \in \mathbb{C}$. In order to see this, set $\eta := e^{1/z}$. Then such equation reads as $\eta^2 - \eta + w = 0$. For $w \neq 0$ fixed, one obtains two different roots in the punctured complex plane $\mathbb{C}^*$, which is the image of the map $z \to e^{\frac{1}{z}}$. For $w = 0$, one root is $\eta = 1$ and, indeed, for $\frac{1}{z} = 2\pi i k$, with $k \neq 0$, one obtains $e^{1/z} - e^{2/z} = 0$. Therefore $g(\mathbb{C} \setminus \{0\}) = \mathbb{C}$.

15. *Let* $f(z) = \frac{z^2}{z^2+1}$. *Determine the Laurent serie of* f *around* $z_0 = i$.

Sol.: The function $f(z) = \frac{z^2}{(z+i)(z-i)}$ is holomorphic on $\mathbb{C}$, except for $\pm i$ where it has simple poles. The Laurent serie of f around $z_0 = i$ is given by

$$f(z) = \frac{1}{(z-i)} \sum_{n\geq 0} a_n (z-i)^n,$$

where $\sum_{n\geq 0} a_n (z-i)^n$ is the Taylor series expansion of $g(z) = \frac{z^2}{(z+i)}$, which is holomorphic around $z_0 = i$. The coefficients of such series are

$$a_0 = g(i) = i/2, \quad a_1 = g'(i) = 3/4, \dots, \qquad a_n = \frac{1}{n!} g^{(n)}(i).$$

16. *Let f be a holomorphic function on $D^*(0,r)$, for $0 < r \leq +\infty$. Assume that $z_0 = 0$ is a pole of f of order m. Then the Laurent serie of f around z_0 is given by*

$$f(z) = \textstyle\sum_{n \geq -m} a_n z^n, \qquad a_n = \frac{1}{(n+m)!} g^{(n+m)}(0), \quad \text{where } g(z) = z^m f(z).$$

Sol.: The Laurent series expansion of f around $z = 0$ is given by

$$f(z) = \frac{a_{-m}}{z^m} + \frac{a_{-(m-1)}}{z^{m-1}} + \ldots = \frac{1}{z^m}(a_{-m} + a_{-(m-1)}z + \ldots),$$

and $g(z) = z^m f(z) = b_0 + b_1 z + \ldots$ is a holomorphic function with $g(z) \neq 0$. The coefficients of the series expansion of g are

$$b_k = a_{-m+k} = \frac{1}{k!} g^{(k)}(0).$$

If $n = -m + k$, then

$$a_n = \frac{1}{(n+m)!} g^{(n+m)}(0).$$

17. *Let $f(z) = \frac{e^z}{(z-i)^3(z+2)^2}$.*
   (i) Show that $z_1 = i$ is a pole of f of order 3.
   (ii) Determine all the coefficients of the principal part of the Laurent series of f around z_1.
   (iii) Show that $z_2 = -2$ is a pole of f of order 2.
   (iv) Determine all the coefficients of the principal part of the Laurent series of f around z_2.

Sol.: (i) The limit $\lim_{z \to i}(z-i)^3 \cdot f(z) = \frac{e^i}{(i+2)^2}$ is finite. Hence $z_1 = i$ is a pole of f of order 3.

(ii) The Laurent series expansion of f around $z_1 = i$ is given by

$$\frac{1}{(z-i)^3} \sum_{n \geq 0} a_n (z-i)^n,$$

where $\sum_{n \geq 0} a_n (z-i)^n$ is the Taylor series expansion of the holomorphic function $g(z) = \frac{e^z}{(z+2)^2}$ around z_1. The principal part of the Laurent series of f around z_1 is

$$a_{-3} \frac{1}{(z-i)^3} + a_{-2} \frac{1}{(z-i)^2} + a_{-1} \frac{1}{(z-i)}.$$

The coefficients can be computed as in Exercise 9

$$a_{-3} = g(i) = \frac{e^i}{(i+2)^2}, \qquad a_{-2} = g'(i) = \frac{e^i(1+i)}{(i+2)^2} \qquad a_{-1} = \frac{1}{2!} g''(i) = \ldots.$$

(iii) The limit $\lim_{z \to i}(z+2)^2 \cdot f(z) = \frac{e^{-2}}{(-2-i)^3}$ is finite. Hence $z_2 = -2$ is a pole of f of order 2.

(iv) The Laurent series expansion of f around $z_2 = -2$ is given by

$$\frac{1}{(z+2)^2} \sum_{n \geq 0} a_n (z+2)^n,$$

where $\sum_{n\geq 0} a_n(z+2)^n$ is the Taylor series expansion of the holomorphic function $g(z) = \frac{e^z}{(z+2)^3}$ around z_2. The principal part of the Laurent series of f around z_2 is

$$a_{-2}\frac{1}{(z+2)^2} + a_{-1}\frac{1}{(z+2)}.$$

The coefficients can be computed as in Exercise 9

$$a_{-2} = g(-2), \qquad a_{-1} = g'(-2).$$

18. *Let $\gamma_2(\theta) = e^{i\theta}$, where $\theta \in [0, 2\pi]$, and $\gamma_1(\theta) = 3e^{i\theta}$, where $\theta \in [0, 2\pi]$. Compute*

$$(a) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2+5z}{(z-2)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2+5z}{(z-2)}dz;$$

$$(b) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2-2}{z}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2-2}{z}dz;$$

$$(c) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^3-3z-6}{z(z+2)(z+4)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^3-3z-6}{z(z+2)(z+4)}dz.$$

Sol.: The curves γ_1 and γ_2 are circles centered in the origin, oriented counterclockwise, of radius 3 and radius 1, respectively. (a) The function $f(z) = \frac{z^2+5z}{(z-2)}$ is holomorphic on $\mathbb{C} \setminus \{2\}$ and has a pole of order 1 in $z = 2$ with residue $Res_f(2) = \lim_{z\to 2}(z-2)f(z) = 14$. The pole is inside γ_1 and outside γ_2.
Hence the integral in (a) is equal to $2\pi i Res_f(2) = 28\pi i$.
(b) The function $f(z) = \frac{z^2-2}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a pole of order 1 in $z = 0$ with residue $Res_f(0) = \lim_{z\to 0}(z)f(z) = -2$. The pole is inside γ_1 and inside γ_2.
Hence the integral in (b) is equal to 0.
(c) The function $f(z) = \frac{z^3-3z-6}{z(z+2)(z+4)}$ has simple poles in $z = 0$, inside both circles, $z = -2$ inside γ_1 and outside Γ_2, and $z = -4$, outside both circles.
Hence the integral in (c) is given by $2\pi i(Res_f(0) + Res_f(-2) - Res_f(0)) = 2\pi i Res_f(-2) = 4\pi i$.

19. *Compute the residues of the following functions at the singular points:*

$$e^{3/z^2}, \qquad \frac{z^3}{z-1}, \qquad \frac{z^3}{(z-1)^2}, \qquad \frac{z^3}{1-z^4}, \qquad \frac{z^5}{(z^2-1)^2}, \qquad \frac{\cos z}{1+z+z^2}, \qquad \frac{1}{\sin z}.$$

Sol.:

$$f(z) = e^{3/z^2} = 1 + \frac{3}{z^2} + \frac{1}{2}(\frac{3}{z^2})^2 + \ldots, \qquad Res_f(0) = 0$$

$$f(z) = \frac{z^3}{z-1}, \qquad Res_f(1) = \lim_{z\to 1}(z-1)f(z) = 1;$$

$$f(z) = \frac{z^3}{(z-1)^2} = \frac{1}{(z-1)^2} + \frac{2}{(z-1)} + \ldots, \qquad Res_f(1) = 2;$$

$$f(z) = \frac{z^3}{1-z^4} = \frac{z^3}{(z-1)(z+1)(z-i)(z+i)} \qquad Res_f(1) = \lim_{z\to 1}(z-1)f(z) = \frac{1}{4}, \quad etc\ldots$$

$$f(z) = \frac{z^5}{(z^2-1)^2} = \frac{z^5}{(z+1)^2(z-1)^2} = \frac{1}{(z+1)^2}\left(-\frac{1}{4} + (z+1) + \ldots\right) \quad Res_f(-1) = 1;$$

$$f(z) = \frac{z^5}{(z^2-1)^2} = \frac{1}{(z-1)^2}\left(\frac{1}{4} + (z-1) + \ldots\right) \quad Res_f(1) = 1;$$

The points $\alpha = -1/2 + i\sqrt{3}/2$ and $\beta = -1/2 - i\sqrt{3}/2$ are simple poles for f

$$f(z) = \frac{\cos z}{1+z+z^2} = \frac{\cos z}{(z-\alpha)(z-\beta)}.$$

Then

$$Res_f(\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \frac{\cos\alpha}{\alpha-\beta}, \qquad Res_f(\beta) = \lim_{z\to\beta}(z-\beta)f(z) = \frac{\cos\beta}{\beta-\alpha}.$$

$$f(z) = \frac{1}{\sin z} \qquad Res_f(0) = Res_f(k\pi) = 1, \quad \forall k \in \mathbb{Z}.$$

20. *Show that the function defined by* $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ *is holomorphic on* $\mathbb{C}\setminus\{0\}$. *Compute its integral on* $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 2\pi]$.

Sol.: The function defined by $\sum_{n=1}^{\infty} \frac{1}{n! z^n}$ is just $e^{1/z}$, which is holomorphic on $\mathbb{C}\setminus\{0\}$. It ha an essential singularity at $z = 0$ with residue 1. Hence the intehgral on f on γ is equal to $2\pi i$.

21. *Compute*

$$\int_\gamma \frac{e^z}{z^3}dz, \qquad \int_\gamma \frac{e^{1/z}}{z^3}dz,$$

when $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 6\pi]$ *and when* $\gamma(\theta) = e^{i\theta}$, *for* $\theta \in [0, 2\pi]$.

Sol.: The curve $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$, is the unit circle, oriented counterclockwise, covered 1 time. Then

$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \ldots, \qquad \int_\gamma f(z)dz = 2\pi i Res_f(0) = 2\pi i\frac{1}{2} = \pi i,$$

$$f(z) = \frac{e^{1/z}}{z^3} = \frac{1}{z^3} + \frac{1}{z^4} + \ldots, \qquad \int_\gamma f(z)dz = 2\pi i Res_f(0) = 0.$$

The curve $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 6\pi]$, is the unit circle, oriented counterclockwise, covered 3 times. Then

$$f(z) = \frac{e^z}{z^3}, \qquad \int_\gamma f(z)dz = 3\cdot 2\pi i Res_f(0) = 3\pi i,$$

$$f(z) = \frac{e^{1/z}}{z^3}, \qquad \int_\gamma f(z)dz = 3\cdot 2\pi i Res_f(0) = 0.$$

22. *Use the "sector method" to compute*

$$\int_0^{+\infty} \frac{1}{1+x^4}dx\,.$$

Sol.: See Sarason, Example 3, p.126.