A REMARK ON THE N-INVARIANT GEOMETRY OF BOUNDED HOMOGENEOUS DOMAINS

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Abstract. Let $\mathbf{D}$ be a bounded homogeneous domain in $\mathbb{C}^n$. In this note we give a characterization of the Stein domains in $\mathbf{D}$ which are invariant under a maximal unipotent subgroup N of $Aut(\mathbf{D})$. We also exhibit an N-invariant potential of the Bergman metric of $\mathbf{D}$, expressed in a Lie theoretical fashion. These results extend the ones previously obtained in the symmetric case.

1. Introduction

By the results of Gindikin, Pijatetcki-Shapiro and Vinberg (see [GPSV68], [PS69]), every bounded domain $\mathbf{D}$ in $\mathbb{C}^n$ admits a realization as a Siegel domain. Such a realization relies on the existence of a simply transitive real split solvable group S of holomorphic automorphisms of $\mathbf{D}$. In the symmetric case, the group $G = Aut(\mathbf{D})$ is semisimple and $S = AN$, where A and N are the abelian and the unipotent subgroups arising from an Iwasawa decomposition of G.

In [GeIa23], the N-invariant Stein domains in irreducible symmetric Siegel domains were characterized. The goal of this note is to prove a similar characterization for N-invariant Stein domains in arbitrary irreducible Siegel domains, which form a much wider class of domains containing the symmetric ones as special cases.

As in the symmetric case, to an N-invariant domain D in $\mathbf{D}$ we associate an r-dimensional tube domain in $\mathbb{H}^r$, the product of r copies of the upper half plane $\mathbb{H}$ in $\mathbb{C}$ (here r is the rank of $\mathbf{D}$). Then we prove that D is Stein if and only if the base of the associated tube is convex and satisfies an additional geometric condition (see Theorem 3.4). In the symmetric case, such condition only depends on whether $\mathbf{D}$ is of tube type or of non-tube type, while in the general case it depends on the specific root decomposition of the normal J-algebra $\mathfrak{s} = Lie(S)$ of $\mathbf{D}$.

The univalence of holomorphically separable, N-equivariant, Riemann domains over $\mathbf{D}$ continues to hold true in this more general context, yielding a precise

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description of the envelope of holomorphy of an arbitrary N-invariant domain in $\mathbf{D}$ (see Corollary 3.5).

Finally, we exhibit an N-invariant potential of the Bergman metric of $\mathbf{D}$, expressed in a Lie theoretical fashion and obtained via an explicit N-moment map with respect to the Bergman Kähler structure of $\mathbf{D}$ (see Proposition 4.2).

2. Preliminaries

Every bounded domain $\mathbf{D}$ in $\mathbb{C}^n$ admits a real split solvable group S of holomorphic automorphisms acting simply transitively on $\mathbf{D}$. The Lie algebra $\mathfrak{s}$ of S has the structure of a *normal J-algebra*, with the complex structure J inherited from $\mathbf{D}$ and the linear form $-f_0 \in \mathfrak{s}^*$ inducing the Bergman metric (cf. [Kos55]). This means in particular that $\omega(X,Y) := -f_0([X,Y])$ is a non-degenerate skew-symmetric J-invariant bilinear form on $\mathfrak{s}$ and $\langle X,Y\rangle := -f_0([JX,Y])$ is a J-invariant positive definite inner product on $\mathfrak{s}$.

The normal J-algebra of a bounded domain. For the structure of normal J-algebras, we mainly refer to [RoVe73], Sect. 5, A. Further details and comments can be found in [GeIa23]. Denote by $\mathfrak{n} := [\mathfrak{s},\mathfrak{s}]$ the nilradical of $\mathfrak{s}$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$, with respect to the inner product $\langle\cdot,\cdot\rangle$. Then $\mathfrak{a}$ is an abelian subalgebra, whose dimension r is by definition the rank of D. The adjoint action of $\mathfrak{a}$ on $\mathfrak{s}$ is symmetric with respect to $\langle\cdot,\cdot\rangle$ and decomposes $\mathfrak{s}$ into the orthogonal direct sum of root spaces $\mathfrak{s}^\alpha = \{X \in \mathfrak{s} \mid [H,X] = \alpha(H)X,\ \forall H \in \mathfrak{a}\}$. There exist $e_1,\dots,e_r \in \mathfrak{s}^*$ such that the roots α are of the form

$$e_j - e_l, \quad e_j + e_l, \quad 1 \leqslant j < l \leqslant r, \qquad 2e_j, \quad e_j, \quad 1 \leqslant j \leqslant r.$$

In the non-symmetric case, not all possibilities need occur. Here the roots are normalized so that, in the symmetric case, they coincide with the restricted roots. The complex structure J permutes the root spaces as follows

$$J\mathfrak{a} = \bigoplus_j \mathfrak{s}^{2e_j}, \qquad J\mathfrak{s}^{e_j-e_l} = \mathfrak{s}^{e_j+e_l}, \qquad J\mathfrak{s}^{e_j} = \mathfrak{s}^{e_j}.$$

Let $H_1,\dots,H_r$ be the basis of $\mathfrak{a}$ dual to $e_1,\dots,e_r \in \mathfrak{a}^*$. As $\dim \mathfrak{s}^{2e_j} = 1$, for $j = 1,\dots,r$, one can fix generators $E^j \in \mathfrak{s}^{2e_j}$ such that the pairs $\{H_j,\ E^j\}$ satisfy

$$[H_j, E^l] = \delta_{jl} 2E^l, \quad JE^j = \tfrac{1}{2}H_j, \qquad \text{for } j,l = 1,\dots,r.$$

For $j = 1,\dots,r$, the real split solvable subalgebras generated by $\{H_j,\ E^j\}$ pairwise commute and are isomorphic to the $\mathfrak{a}\oplus\mathfrak{n}$-component of an Iwasawa decomposition of $\mathfrak{sl}(2,\mathbb{R})$.

Set $H_0 := \frac{1}{2}\sum_j H_j \in \mathfrak{a}$. The adjoint action of H_0 decomposes $\mathfrak{s}$ and $\mathfrak{n}$ as

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_{1/2} \oplus \mathfrak{s}_1, \qquad \mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_{1/2} \oplus \mathfrak{n}_1\,,$$

where $\mathfrak{n}_j = \mathfrak{n} \cap \mathfrak{s}_j$ and

$$\mathfrak{s}_0 = \mathfrak{a} \oplus \bigoplus_{1 \leqslant j < l \leqslant r} \mathfrak{s}^{e_j - e_l}, \quad \mathfrak{s}_{1/2} = \oplus_{1 \leqslant j \leqslant r} \mathfrak{s}^{e_j}, \quad \mathfrak{s}_1 = \oplus_{1 \leqslant j \leqslant r} \mathfrak{s}^{2e_j} \oplus \bigoplus_{1 \leqslant j < l \leqslant r} \mathfrak{s}^{e_j + e_l}.$$

If $\mathfrak{s}_{1/2} = \{0\}$, then the domain $\mathbf{D}$ is of *tube type*, otherwise it is of *non-tube type.*

Set $E_0 := \sum E^j$. The complex structure on $\mathfrak{s}_0$ is given by $JX = [E_0, X]$, for all $X \in \mathfrak{s}_0$. The orbit

$$V := Ad_{\exp \mathfrak{s}_0} E_0$$

is a sharp convex cone in $\mathfrak{s}_1$ and

$$F\colon \mathfrak{s}_{1/2} \times \mathfrak{s}_{1/2} \to \mathfrak{s}_1^{\mathbb{C}}, \qquad F(W, W') := \tfrac{1}{4}([JW', W] - i[W', W]),$$

is a V-valued Hermitian form, i.e. it is sesquilinear and $F(W, W) \in \overline{V}$ (the closure of V), for all $W \in \mathfrak{s}_{1/2}$. The group S acts on $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ by affine transformations, given by

$$s \cdot (Z, W) = (Ad_{\exp \gamma} Z + \xi + 2iF(Ad_{\exp \gamma} W, \zeta) + iF(\zeta, \zeta), Ad_{\exp \gamma} W + \zeta), \quad (1)$$

where $s = \exp \zeta \exp \xi \, \exp \gamma$, with $\zeta \in \mathfrak{s}_{1/2}$, $\xi \in \mathfrak{s}_1$, $\gamma \in \mathfrak{s}_0$. If we fix the base point $p_0 := (iE_0, 0) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$, then the map

$$\mathcal{L}:\ S \mapsto D(V, F), \qquad s \mapsto s \cdot p_0 \quad (2)$$

defines a biholomorphism between $\mathbf{D} \cong S$ and the Siegel domain

$$D(V, F) = \{(Z, W) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2} \mid Im(Z) - F(W, W) \in V\}$$

(cf. [RoVe73], Lem. 5.2, p. 330). Denote by

$$(E^1)^*, \ldots, (E^r)^*$$

the elements in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$, with the property that $(E^j)^*(E^l) = \delta_{jl}$ and $(E^j)^*(X) = 0$, for all $X \in \mathfrak{s}^\alpha$, with $\alpha \notin \{2e_1, \ldots, 2e_r\}$.

Lemma 2.1. (*a*) *The form* $-f_0\colon \mathfrak{s} \to \mathbb{R}$ *is given by* $-f_0 = \sum_k c_k (E^k)^*$, *for some* $c_k \in \mathbb{R}^{>0}$.

(*b*) *Let* $X \in \mathfrak{s}^{e_j - e_l} \backslash \{0\}$. *Then* $[JX, X] = sE^j$, *for some* $s \in \mathbb{R}^{>0}$. *Let* $X, Y \in \mathfrak{s}^{e_j - e_l} \backslash \{0\}$, *satisfying* $\langle X, Y \rangle = 0$. *Then* $[JX, Y] = 0$.

(*c*) *Let* $X \in \mathfrak{s}^{e_j} \backslash \{0\}$. *Then* $[JX, X] = tE^j$, *for some* $t \in \mathbb{R}^{>0}$. *Let* $X, Y \in \mathfrak{s}^{e_j} \backslash \{0\}$, *satisfying* $\langle X, Y \rangle = 0$. *Then* $[JX, Y] = 0$.

Proof. The proof of statement (a) is contained in [RoVe73]. For the sake of completeness we recall the main arguments. Let f_0 also denote the $\mathbb{C}$-linear extension of f_0 to $\mathfrak{s}^{\mathbb{C}}$. From the integrability of J one has that $f_0([X + iJX, Y + iJY]) = 0$, for all $X, Y \in \mathfrak{s}$. This implies that $f_0([H, X]) = f_0(J[H, X]) = 0$, for all $H \in \mathfrak{a}$ and $X \in \mathfrak{q} := \mathfrak{s}_{1/2} \oplus \bigoplus_{j<l} \mathfrak{s}^{e_j - e_l}$. Since $[\mathfrak{a}, \mathfrak{q}] = \mathfrak{q}$ and $J\mathfrak{q} = \mathfrak{s}_{1/2} \oplus \bigoplus_{j<l} \mathfrak{s}^{e_j + e_l}$, the form f_0 identically vanishes on $\mathfrak{q}$ and $-f_0 = \sum_j c_j (E^j)^*$, for

some $c_j \in \mathbb{R}$. The identity $c_j = -f_0(E^j) = -\frac{1}{2}f_0([H_j, E^j]) = -f_0([JE^j, E^j]) = \langle E^j, E^j\rangle > 0$ concludes the proof.
(b) Let $X \in \mathfrak{s}^{e_j - e_l}\backslash\{0\}$. Then $JX = [E^l, X] \in \mathfrak{s}^{e_j+e_l}$. Since $\mathfrak{s}^{2e_j}$ is one-dimensional, $[JX, X] = sE^j$, for some $s \in \mathbb{R}$. By applying $-f_0$ to both terms, one obtains $-f_0([JX, X]) = \langle X, X\rangle = c_j s > 0$. Since $c_j > 0$, also $s > 0$. For the second part of the statement, write $[JX, Y] = sE^j$, for some $s \in \mathbb{R}$. Then from

$$0 = \langle X, Y\rangle = -f_0([JX, Y]) = -\sum_k c_k(E^k)^*(sE^j) = c_j s,$$

one obtains $s = 0$ and therefore $[JX, Y] = 0$, as desired.
As $\mathfrak{s}^{e_j}$ is J-invariant, statement (c) follows in a similar way. □

Remark 2.2. *The forms $\sum_j c_j(E^j)^*$, where the c_j's vary in $\mathbb{R}^{>0}$ for $j = 1, \ldots, r$, determine all S-homogeneous Kähler metrics on $D(V, F)$* (cf. [Do85], Thm. 1, p. 304)*. By* [DA79], Thm. 4*, one such metric is Kähler-Einstein if and only if the quantity $\frac{1}{c_j}(1 + \frac{1}{4}\dim \mathfrak{s}^{e_j} + \frac{1}{2}\sum_{j<l}\mathfrak{s}^{e_j+e_l})$ is a constant independent of $j = 1, \ldots, r$.*

N-invariant domains in $D(V, F)$ and tube domains in $\mathbb{H}^r$. In $S = NA$, consider the unipotent abelian subgroup $R := \exp J\mathfrak{a}$, isomorphic to $\mathbb{R}^r$. The R-invariant set

$$R\exp(\mathfrak{a}) \cdot p_0$$

is an r-dimensional closed complex submanifold of $D(V, F)$, intersecting all N-orbits in $D(V, F)$. Define the positive octant in $J\mathfrak{a}$

$$J\mathfrak{a}^+ := \{\textstyle\sum y_k E^k \ : \ y_k > 0, \text{ for } k = 1, \ldots, r\}.$$

Then the map $\mathcal{L}$ defined in (1) and (2) restricts to a biholomorphism

$$R\exp(\mathfrak{a}) \to J\mathfrak{a} \oplus iJ\mathfrak{a}^+,$$

given by

$$\exp(\textstyle\sum_j e_j E^j)\exp(\sum_k h_k H_k) \mapsto \sum_j e_j E^j + iAd_{\exp(\sum_k h_k H_k)}E_0. \tag{3}$$

In particular $\mathcal{L}|_{\exp(\mathfrak{a})}$ defines a diffeomorphism $L\colon \mathfrak{a} \to J\mathfrak{a}^+$ given by

$$\sum_k h_k H_k \mapsto Ad_{\exp(\sum_k h_k H_k)}E_0 = \sum_j e^{2h_j}E^j. \tag{4}$$

Write an N-invariant domain in a rank-r homogeneous Siegel domain $D(V, F)$ as $D = N\exp\mathcal{D}\cdot p_0$, for some domain $\mathcal{D} \subset \mathfrak{a}$. Then, as in the symmetric case (see [GeIa23], Sect. 3), one can associate to D an r-dimensional tube domain.

Definition 2.3. *The r-dimensional tube domain associated to an N-invariant domain D in $D(V, F)$ is the image of the set $R\exp(\mathcal{D})$ under $\mathcal{L}$, namely*

$$D \cap (J\mathfrak{a} \oplus iJ\mathfrak{a}^+) = J\mathfrak{a} + i\Omega, \quad \textit{where } \Omega := L(\mathcal{D}).$$

3. N-invariant Stein domains in a homogeneous Siegel domain

Let $D(V,F)$ be a bounded domain. In this section we give a characterization of the N-invariant Stein domains D in $D(V,F)$ in terms of the associated tube domain. If D is Stein then such tube domain is Stein and its base Ω is an open convex set in $J\mathfrak{a}^+$. On the other hand, we will see that Ω must satisfy some further geometric conditions which depend on the specific root decomposition of the normal J-algebra of $D(V,F)$.

Let D be an N-invariant domain in $D(V,F)$. Then

$$D = \{(Z,W) \in D(V,F) \mid Im(Z) - F(W,W) \in \mathbf{\Omega}\},$$

where $\mathbf{\Omega}$ is the $Ad_{\exp \mathfrak{n}_0}$-invariant open subset in V determined by

$$i\mathbf{\Omega} := D \cap iV.$$

By (2), (3) and (4), the base of the associated tube is

$$\Omega = \mathbf{\Omega} \cap J\mathfrak{a}^+.$$

Note that, since $Ad_A E_0 = J\mathfrak{a}^+$, the set $i\Omega$ is a slice both for the $Ad_{\exp \mathfrak{n}_0}$-action on $i\mathbf{\Omega}$ and for the N-action on D.

For $D(V,F)$ irreducible, define a cone in $J\mathfrak{a}^+$ as follows

$$C := \begin{cases} \mathcal{C}_t, \text{in the tube case} \\ \mathcal{C}_{nt}, \text{in the non-tube case} \end{cases}$$

where $\mathcal{C}_t := cone\{E^j\}_j$, with $j \in \{1,\ldots,r-1\}$ such that $\mathfrak{s}^{e_j-e_l} \neq \{0\}$ for some $l > j$, and $\mathcal{C}_{nt} = cone\{E^j\}_j$, with $j \in \{1,\ldots,r\}$ such that either $\mathfrak{s}^{e_j-e_l} \neq \{0\}$ for some $l > j$, or $\mathfrak{s}^{e_j} \neq \{0\}$. (Here, given non-zero vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$, we set $cone\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} := \{\sum_j t_j\mathbf{v}_j,\ t_j > 0\}$).

In the reducible case, the normal J-algebra $\mathfrak{s}$ and all its related objects decompose accordingly. In particular, the cone decomposes as $C = C^{(1)} \times \ldots \times C^{(m)}$, where $C^{(i)}$ is the cone associated to the i^{th} irreducible component of $D(V,F)$

Example 3.1. *(a) If $D(V,F)$ is irreducible symmetric, then $\mathcal{C}_t = cone\{E^1,\ldots,E^{r-1}\}$ and $\mathcal{C}_{nt} = cone\{E^1,\ldots,E^r\}$ (see (9) in* [GeIa23]*).*
(b) Let $D(V)$ be the tube domain over the 5-dimensional Vinberg cone

$$D(V) = \left\{ \begin{pmatrix} z_{11} & 0 & z_{13} \\ 0 & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{pmatrix} \mid z_{ij} = x_{ij} + iy_{ij} \in \mathbb{C},\ \begin{cases} y_{11}y_{33} - y_{13}^2 > 0 \\ y_{22}y_{33} - y_{23}^2 > 0 \end{cases} \text{and} \quad y_{33} > 0 \right\}.$$

Then

$$\mathfrak{s} = \mathfrak{a} \oplus J\mathfrak{a} \oplus \mathfrak{s}^{e_1 \pm e_3} \oplus \mathfrak{s}^{e_2 \pm e_3}, \qquad \dim \mathfrak{a} = 3,\ \dim \mathfrak{s}^{e_j \pm e_l} = 1$$

and $\mathcal{C}_t = cone\{E^1, E^2\}$.

(b) Let $D(V,F)$ be the 4-dimensional non-symmetric domain

$$D(V,F)=\left\{\left(\begin{pmatrix} z_{11} & z_{12}\\ z_{12} & z_{22}\end{pmatrix},w\right)\ \mid\ z_{ij}=x_{ij}+iy_{ij},\ w\in\mathbb{C},\ \begin{cases}(y_{11}-|w|^2)y_{22}-y_{12}^2>0\\ y_{22}>0\end{cases}\right\}.$$

Then

$$\mathfrak{s}=\mathfrak{a}\oplus J\mathfrak{a}\oplus\mathfrak{s}^{e_1\pm e_2}\oplus\mathfrak{s}^{e_1},\qquad \dim\mathfrak{a}=2,\ \dim s^{e_1\pm e_2}=1,\ \dim\mathfrak{s}^{e_1}=2$$

and $\mathcal{C}_{nt}=cone\{E^1\}$.

Definition 3.2. *A domain $\Omega\subset J\mathfrak{a}^+$ is C-invariant if $E\in\Omega$ implies $E+C\subset\Omega$ or, equivalently, if $E\in\Omega$ implies $E+\overline{C}\subset\Omega$.*

Denote by

$$p\colon i\mathfrak{s}_1\to iJ\mathfrak{a}$$

the projection onto $iJ\mathfrak{a}$, parallel to $i(\bigoplus\mathfrak{s}^{e_j+e_l})$ and by

$$\widetilde{p}\colon\mathfrak{s}_1^{\mathbb{C}}\oplus\mathfrak{s}_{1/2}\to iJ\mathfrak{a}$$

the projection onto $iJ\mathfrak{a}$ parallel to $\mathfrak{s}_1\oplus i(\bigoplus\mathfrak{s}^{e_j+e_l})\oplus\mathfrak{s}_{1/2}$.

For simplicity, the next lemma is formulated in the irreducible case, and holds on each irreducible component.

Lemma 3.3. *The following statements hold true.*

(i) *Assume $\mathfrak{s}^{e_j-e_l}\neq\{0\}$, for some $l>j$, and let $X\in\mathfrak{s}^{e_j-e_l}$ be a non-zero element. Then $[[E^l,X],X]=sE^j$, for some $s\in\mathbb{R}^{>0}$.*

(ii) *Let $E=\sum y_kE^k\in J\mathfrak{a}^+$. Then $p(iAd_{\exp\mathfrak{n}_0}E)=i(E+\overline{\mathcal{C}}_t)$.*

(iii) *Let $E\in J\mathfrak{a}^+$. Then $\widetilde{p}(N\cdot(iE,0))=i(E+\overline{\mathcal{C}}_{nt})$.*

Proof. (i) Since $[[E^l,X],X]=[JX,X]$, then the statement follows from Lemma 2.1 (b).
(ii) Fix $1\leqslant j\leqslant r-1$ and define $\mathcal{L}_j:=\oplus_{l>j}\mathfrak{s}^{e_j-e_l}$. In each root space $\mathfrak{s}^{e_j-e_l}\neq\{0\}$ in $\mathcal{L}_j$, there exists an orthogonal basis $\{E^p_{jl}\}_p$ such that for $X=\sum_{l>j,\,p}x^p_{jl}E^p_{jl}\in\mathcal{L}_j$, one has

$$(E^j)^*(Ad_{\exp X}E)=y_j(1+\textstyle\sum_{l>j,\,p}(x^p_{jl})^2)\qquad\text{and}\qquad(E^r)^*(Ad_{\exp X}E)=y_r$$

(cf. Lemma 2.1, (b), (c)). Moreover, from a discussion similar to the one in [RoVe73], p. 363, one obtains

$$(E^j)^*(Ad_{\exp\mathfrak{n}_0}E)=(E^j)^*Ad_{\exp\mathcal{L}_j}E$$

([RoVe73], Theorem 4.10, formula (4.13)). Hence $p(iAd_{\exp\mathfrak{n}_0}E)=i(E+\overline{\mathcal{C}}_t)$, as claimed.
(iii) The N-orbit of the point $(iE,0)\in\mathfrak{s}_1^{\mathbb{C}}\oplus\mathfrak{s}_{1/2}$ is given by

$$\{(\xi+i(Ad_{\exp\mathfrak{n}_0}E+F(\zeta,\zeta)),\zeta)\ :\ \xi\in\mathfrak{s}_1,\ \zeta\in\mathfrak{s}_{1/2}\}.\qquad(5)$$

By (5) and Lemma 3.3 (ii), one has

$$\widetilde{p}(N\cdot(iE,0)) = i(E+\overline{\mathcal{C}}_t) + \{\tilde{p}(iF(\zeta,\zeta)) \ : \ \zeta\in\mathfrak{s}_{1/2}\}\,.$$

If $\mathfrak{s}^{e_j} \neq \{0\}$ and $\zeta \neq 0$ in $\mathfrak{s}^{e_j}$, then by Lemma 2.1(c) the element $F(\zeta,\zeta) = \frac{1}{4}[J\zeta,\zeta]$ is a positive multiple of E^j. Therefore $\tilde{p}(N\cdot(iE,0)) = i(E+\overline{\mathcal{C}}_{nt})$, as claimed. □

Theorem 3.4. *Let $D(V,F)$ be a homogeneous Siegel domain of rank r. Let D be an N-invariant domain in $D(V,F)$ and let Ω be the base of the associated tube domain. Then D is Stein if and only if Ω is convex and C-invariant.*

Proof of Theorem 3.4. We first prove that D Stein implies Ω convex and C-invariant. Then we show that Ω convex and C-invariant implies D convex and therefore Stein (cf. [Gun90], Vol.1, Thm.10, p. 67). In particular, if D is Stein, then it is necessarily convex. An essential fact is that the N-action on D is affine and every affine map commutes with taking convex hulls.

The tube case. An N-invariant domain D in a tube domain $D(V)$ is itself a tube domain with base the $Ad_{\exp\mathfrak{n}_0}$-invariant set $\mathbf{\Omega}$. Since D is Stein if and only if its base is convex, all we have to show is that Ω convex and C-invariant is equivalent to $\mathbf{\Omega}$ being convex.

If $\mathbf{\Omega}$ is convex, then Ω is clearly convex. In order to prove that Ω is C_t-invariant, let $E=\sum_k y_kE^k\in\Omega$, with $y_k>0$, for $k=1,\dots,r$. If the root space $\mathfrak{s}^{e_j-e_l}\neq\{0\}$, let X be a non-zero element therein. Since ad_X is 2-step nilpotent, for every $t\in\mathbb{R}$

$$Ad_{\exp tX}E = E + ty_l[X,E^l] + \tfrac{1}{2}t^2y_l[X,[X,E^l]]$$

is an element of $\mathbf{\Omega}$. As $\mathbf{\Omega}$ is convex, by replacing t with $-t$, one finds that also the midpoint $E+\frac{1}{2}t^2y_l[X,[X,E^l]]$ lies in $\mathbf{\Omega}$. This says that $E+\lambda E^j$ lies in $\mathbf{\Omega}$, for all $\lambda\geqslant 0$. The same argument applied to all $j\in\{1,\dots,r-1\}$ for which $\mathfrak{s}^{e_j-e_l}\neq\{0\}$, for some $l>j$, and the convexity of Ω imply that $\Omega+\mathcal{C}_t\subset\Omega$, as desired.

Conversely, assume that Ω is convex and C-invariant. We are going to prove that $conv(\mathbf{\Omega})\subset\mathbf{\Omega}$. Since $\mathbf{\Omega}=Ad_{\exp\mathfrak{n}_0}\Omega$, from Lemma 3.3 (ii) and the C-invariance of Ω, one has

$$p(i\mathbf{\Omega}) = p(iAd_{\exp\mathfrak{n}_0}\Omega) = i(\Omega+\overline{\mathcal{C}}_t)\subset i\Omega.$$

From the above inclusion and the convexity of Ω, one has

$$conv(i\mathbf{\Omega})\cap iJ\mathfrak{a}\subset p(conv(i\mathbf{\Omega})) = conv(p(i\mathbf{\Omega}))\subset i\Omega.$$

Finally, from the $Ad_{\exp\mathfrak{n}_0}$-invariance of $conv(i\mathbf{\Omega})$ it follows that

$$conv(i\mathbf{\Omega}) = Ad_{\exp\mathfrak{n}_0}(conv(i\mathbf{\Omega})\cap iJ\mathfrak{a})\subset Ad_{\exp\mathfrak{n}_0}i\Omega = i\mathbf{\Omega}.$$

This completes the proof of the theorem in the tube case.

The non-tube case. Let D be an N-invariant domain in a Siegel domain $D(V,F)$. Denote by $conv(D)$ the convex hull of D in $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$, which is N-invariant as well.

If D is Stein, then $D \cap (\mathfrak{s}_1^{\mathbb{C}} \times \{0\}) = \{(Z,0) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2} \mid Im(Z) \in \mathbf{\Omega}\}$ is biholomorphic to a Stein tube domain in $\mathfrak{s}_1^{\mathbb{C}}$, invariant under $\exp(\mathfrak{n}_0 \oplus \mathfrak{n}_1)$. Hence, by Theorem 3.4 in the tube case, the set Ω is convex and $\Omega + \overline{\mathcal{C}}_t \subset \Omega$. The fact that $\Omega + \overline{\mathcal{C}}_{nt} \subset \Omega$ follows from (5) and the fact that $F(\zeta,\zeta)$ is an arbitrary positive multiple of E^j, when ζ varies in $\mathfrak{s}^{e_j} \backslash \{0\}$.

Conversely, assume that Ω is convex and C-invariant. By Lemma 3.3 (iii), one has

$$\widetilde{p}(D) = \widetilde{p}(N \cdot i\Omega) = i(\Omega + \overline{\mathcal{C}}_{nt}) \subset i\Omega.$$

Moreover,

$$conv(D) \cap iJ\mathfrak{a} \subset \widetilde{p}(conv(D)) = conv(\widetilde{p}(D)) \subset i\Omega.$$

By the N-invariance of $conv(D)$, one obtains

$$conv(D) = N \cdot (conv(D) \cap iJ\mathfrak{a}) \subset N \cdot i\Omega = D.$$

Hence D is convex and therefore Stein (cf. [Gun90], Vol.1, Thm.10, p. 67). This concludes the proof of the theorem. □

We conclude this section by observing that holomorphically separable, N-equivariant, Riemann domains over a bounded domain $\mathbf{D}$ are univalent: the same proof as in the symmetric case works in the more general case (see [GeIa23], Prop. 3.7).

Corollary 3.5. *The envelope of holomorphy $\widehat{D}$ of an N-invariant domain D in $\mathbf{D}$ is the smallest Stein domain in $\mathbf{D}$ containing D. Namely, $\widehat{D}$ is the N-invariant domain such that the base $\widehat{\Omega}$ of the associated tube is the convex C-invariant hull of Ω.*

4. An N-invariant potential of the Bergman metric

Let $D(V,F)$ be a Siegel domain and let $(\mathfrak{s}, J, -f_0)$ be the associated normal J-algebra, where $-f_0 \in \mathfrak{s}^*$ is the form inducing the Bergman metric g on $D(V,F)$. In this section we exhibit an N-invariant potential of g, expressed in a Lie theoretical fashion. In order to do this we determine an explicit formula for the N-moment map associated to g.

For $X \in \mathfrak{s}$ denote by $\widetilde{X}$ the vector field on $D(V,F)$ induced by the left S-acion. Its value at $z = s \cdot p_0$ is given by $\widetilde{X}_z = \frac{d}{dt}\big|_{t=0} \exp tX \cdot z$. If $z = a \cdot p_0$, with $a = \exp H$ and $H \in \mathfrak{a}$, and $X \in \mathfrak{s}^\alpha$, then $\widetilde{X}_z = e^{-\alpha(H)} a_* X$.

Lemma 4.1. *(a) The map* $\mu_S\colon D(V,F)\to\mathfrak{s}^*$, *defined by*

$$\mu_S(z)(X) := -f_0(Ad_{s^{-1}}X), \qquad z = s\cdot p_0,\ X\in\mathfrak{s},$$

is an S-moment map with respect to g.
(b) The map $\mu_N\colon D(V,F)\to\mathfrak{n}^*$, *defined by*

$$\mu_N(z)(X) := -(Ad_a^*f_0)(Ad_{n^{-1}}X), \qquad z = na\cdot p_0,\ X\in\mathfrak{n}$$

is an N-moment map with respect to g.

Proof. (a) By definition, the map μ_S is S-equivariant and satisfies $\mu_S(p_0) = -f_0$. We identify $D(V,F)$ with the group S by the map (1), and prove that

$$d\mu_S^X(s)(Z) = \omega_s(\widetilde{X}_s, Z), \qquad Z\in T_sS,\ X\in\mathfrak{s}. \tag{6}$$

Let $W\in T_eS\cong\mathfrak{s}$. Then

$$d\mu_S^X(W) = \tfrac{d}{dt}\big|_{t=0}\mu_S^X(\exp tW) = \tfrac{d}{dt}\big|_{t=0} - f_0(Ad_{\exp(-tW)}X) = -f_0(\tfrac{d}{dt}\big|_{t=0}e^{ad_{-tW}}X)$$

$$= -f_0(-[W,X]) = -f_0([X,W]) = \omega(X,W).$$

Now take $s\in S$ and let $s_*W\in T_sS\cong s_*\mathfrak{s}$. On the left hand side of (6), we find

$$(d\mu_S^X)(s_*W) = \tfrac{d}{dt}\big|_{t=0}\,\mu_S^X(s\exp tW) = \tfrac{d}{dt}\big|_{t=0} - f_0(Ad_{\exp -tW}Ad_{s^{-1}}X)$$
$$= -f_0(-[W,Ad_{s^{-1}}X]) = -f_0([Ad_{s^{-1}}X,W]).$$

Since also the right hand side of (6) is given by

$$\omega_s(\frac{d}{dt}\big|_{t=0}\exp tX\cdot s, s_*W) = \omega_s(s_*Ad_{s^{-1}}X, s_*W) = \omega(Ad_{s^{-1}}X,W) = -f_0([Ad_{s^{-1}}X,W]),$$

the proof of (a) is complete.

(b) The restriction of μ_S to $\mathfrak{n}$ defines an N-moment map μ_N on $D(V,F)$. Since μ_N is N-equivariant, it is uniquely determined by $\mu_N(a\cdot p_0)(X) = -Ad_a^*f_0(X)$, for $X\in\mathfrak{n}$. It follows that $\mu_N(z)(X) = -(Ad_a^*f_0)(Ad_{n^{-1}}X)$, as claimed. □

The moment map μ_S defined in Lemma 4.1 is an embedding of $D(V,F)$ in $\mathfrak{s}^*$ as the coadjoint orbit of $-f_0$, with trivial isotropy subgroup. The image $\mu_S(D(V,F))$ is the convex domain $\mathfrak{s}_0^* + \mathfrak{s}_{1/2}^* + V^*$ in $\mathfrak{s}^*$, where $V^* := \{\phi\in\mathfrak{s}_1^* \mid \phi(X)>0,\ \forall X\in\overline{V}\backslash\{0\}\}$ is the dual cone of V in $\mathfrak{s}_1$ (cf. [RoVe73], Lem. 3.5, p. 350). Similarly, the image $\mu_N(D(V,F))$ is the convex domain $\mathfrak{n}_0^* + \mathfrak{s}_{1/2}^* + V^*$ in $\mathfrak{n}^*$.

Convexity properties of the moment map have been studied in several settings (see [HiNePl94], and references therein). Here we show that the image under μ_N of a Stein N-invariant domain in $D(V,F)$ is not necessarily convex.

Let $D(V,F)$ be a Siegel domain, and let $D = N \cdot i\Omega$ be an N-invariant Stein domain therein. One has

$$(J\mathfrak{a})^* \cap \mu_N(D(V,F)) = (J\mathfrak{a})^* \cap V^* = (J\mathfrak{a}^+)^*$$

and one can easily verify that μ_N maps $A \cdot p_0 = iJ\mathfrak{a}^+$ bijectively onto$(J\mathfrak{a}^+)^*$.

Therefore $\mu_N(i\Omega) = (J\mathfrak{a}^+)^* \cap \mu_N(D)$. Consequently, if $\mu_N(i\Omega)$ is not convex, then $\mu_N(D)$ is not convex either.

Example 4.2. Let $\mathcal{P} = \{Z = X + iY \mid Z^t = Z,\, Y \gg 0\}$ be the Siegel upper half-plane of rank 2. Then

$$A \cdot p_0 = \left\{ i \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \mid y_1,\, y_2 > 0 \right\}$$

and

$$\mu_S\left(i \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \right) = -3(\tfrac{1}{y_1}(E^1)^* + \tfrac{1}{y_2}(E^2)^*).$$

Let $D := N \cdot i\Omega$ be the Stein N-invariant domain in $\mathcal{P}$ associated to the convex, C_t-invariant domain

$$\Omega := \left\{ i \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \mid y_1 y_2 > 1 \right\}.$$

The image

$$\mu_N(i\Omega) = \{\eta_1(E^1)^* + \eta_2(E^2)^* \mid \eta_1,\, \eta_2 < 0,\, \eta_1\eta_2 < 9\}$$

is clearly not convex. Therefore $\mu_N(D)$ is not convex either.

As the domain $\Omega \subset J\mathfrak{a}^+$ is convex and also $\mathcal{C}_{nt}$-invariant, a similar construction actually provides examples of N-invariant Stein domains with non-convex moment image in all rank-2 symmetric Siegel domains, both of tube type and of non tube type.

Proposition 4.3. *The N-invariant function $\rho\colon D(V,F) \to \mathbb{R}$, given by*

$$\rho(na \cdot p_0) := 2 \textstyle\sum_k c_k h_k,$$

where $a = \exp H$, for $H = \sum_k h_k H_k \in \mathfrak{a}$, and $-f_0 = \sum_k c_k (E^k)^*$, is a potential for the Kähler metric induced by $-f_0$.

Proof. As in the previous Lemma, we identify $D(V,F)$ with S. In order to check that $-dd^c\rho = \omega$, we need to show that $d^c\rho(\widetilde{X}_s) = \mu_N^X(s)$, for all $s \in S$. By the N-invariance of ρ and of J one has

$$d^c\rho(\widetilde{X}_{na}) = d^c\rho(\widetilde{\mathrm{Ad}_{n^{-1}}X_a}),$$

for every $na \in S$. Then, as μ_N is N-equivariant, it is enough to show that

$$d^c\rho(\widetilde{X}_a) = \mu_N^X(a), \tag{7}$$

for all $a \in A$ and $X \in \mathfrak{n}$. If $X = E^j$, then

$$d^c\rho((\widetilde{E^j})_a) = e^{-2e_j(H)}d\rho(a_*JE^j) = \tfrac{1}{2}e^{-2h_j}\tfrac{d}{ds}\big|_{s=0}\rho(\exp(H+sH_j))$$

$$= e^{-2h_j}c_j = -f_0(Ad_{a^{-1}}E^j) = \mu_N^{E^j}(a).$$

If $X \in \mathfrak{s}^\alpha$, with $0 \neq \alpha \notin \{2e_1, \ldots, 2e_r\}$, then $JX \in \mathfrak{s}^\beta$, with $0 \neq \beta \notin \{2e_1, \ldots, 2e_r\}$. By the N-invariance of ρ, one obtains

$$d^c\rho(\widetilde{X}_a) = e^{-\alpha(H)}d\rho(a_*JX) = e^{-\alpha(H)+\beta(H)}\tfrac{d}{ds}\big|_{s=0}\rho(\exp(sJX)a) = 0.$$

Since

$$\mu_N^X(a) = -f_0(Ad_{a^{-1}}X) = -e^{-\alpha(H)}f_0(X) = 0,$$

equation (7) holds true and the proposition follows. □

Remark. *The above computation produces an N-invariant potential and an associated N-moment map, for any S-invariant Kähler metric on $\mathbf{D}$ induced by an element $\sum_j d_j(E^j)^* \in \mathfrak{s}^*$, with $d_j \in \mathbb{R}^{>0}$, for $j = 1, \ldots, r$ (cf. Rem. 2.2).*

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