Complex integration

1. Compute
$$\int_\gamma \frac{e^z}{z}dz, \qquad \int_\gamma \frac{e^z}{(z-1/2)^2}dz,$$
where $\gamma = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$.

2. Compute
$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-1)(z-2i)}dz,$$
where $\gamma = \{4e^{i\theta},\ \theta \in [0, 2\pi]\}$.

3. Compute
$$\int_\gamma \frac{\cos z}{(z-\pi/2)^{10}}dz,$$
where $\gamma = \{2e^{i\theta},\ \theta \in [0, 2\pi]\}$.

4. Let $|a| < 1 < |b|$. For $n, m \in \mathbf{Z}$, compute
$$\frac{1}{2\pi i}\int_\gamma \frac{(z-b)^m}{(z-a)^n}dz,$$
where $\gamma = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$.

Cauchy's formulas, identity principle, Liouville's theorem.

5. Let $\Omega \subset \mathbf{C}$ be an open connected set. Let $f, g \colon \Omega \longrightarrow \mathbf{C}$ be holomorphic functions such that $f \cdot g \equiv 0$. Then either $f \equiv 0$ or $g \equiv 0$.

6. Let $f \colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function, such that $|f(z)| \leq |e^z|$, for all $z \in \mathbf{C}$. Prove that $f(z) = ce^z$, for some constant $c \in \mathbf{C}$.

7. Let $f \colon \mathbf{C} \longrightarrow \mathbf{C}$ be a holomorphic function. Assume that there exist positive constants M and c for which $|f(z)| \leq M(c + |z|^n)$, for all $z \in \mathbf{C} \setminus B(0, R)$, with $R > 0$.
   (a) Show that f is a polynomial of degree $\leq n$.
   (b) Assume $|f(z)| \leq M|z|^2$. Determine $f(0)$ and $f'(0)$.

8. Determine all the zeros of $\sin z$ and $\cos z$. Let $U = \mathbf{C} \setminus \{\pi/2 \pm k\pi,\ k \in \mathbf{Z}\}$. Let f be a holomorphic function on U, such that $f(\pi/n) = \tan(\pi/n)$, $n \geq 3$. Deduce that f is not holomorphic on all $\mathbf{C}$ and that it does not assume the value i.