Liouville's theorem, identity principle, maximum modulus principle, harmonic functions

1. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic doubly periodic function (it means that $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z \in \mathbb{C}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent vectors). Then f is constant.*

Sol.: A doubly periodic function is necessarily bounded, as it takes all its values on the compact set $P = \{z = a\omega_1 + b\omega_2 \ : \ a, b \in [0,1]\}$. Hence it is constant by Liouville theorem.

2. *Let $U \subset \mathbb{C}$ be an open neighbourhood of 0. Show that there are no (non-constant) holomorphic functions $f\colon U \to \mathbb{C}$, such that:*
   (a) $f(\frac{1}{n}) = (-1)^n \frac{1}{n^2}$,
   (b) $f(\frac{1}{n}) = \frac{1}{2^n}$,
   (c) $|f^{(n)}(0)| > n!\, n^n$.

Sol.: (a) $f(\frac{1}{n})$ coincides with $g(z) = z^2$, for n even, while it coincides with $h(z) = -z^2$, for n odd. Since the sets

$$\{\frac{1}{2k}\} \cup \{0\} \qquad \text{and} \qquad \{\frac{1}{2k+1}\} \cup \{0\}, \qquad k \in \mathbb{N}$$

are uniqueness sets for f, there is no holomorphic function satisfying $f(\frac{1}{n}) = (-1)^n \frac{1}{n^2}$.
(b) We are going to show that f is identically zero, by showing that all the derivatives of f at $z = 0$ are zero. We have

$$f(0) = \lim_{n\to+\infty} \frac{1}{2^n} = 0 \qquad \text{and} \quad f'(0) = \lim_{n\to+\infty} \frac{\frac{1}{2^n} - 0}{1/n} = 0.$$

Now we proceed by induction. We assume that $f^{(h)}(0) = 0$, for all $h \leq k$, and prove that $f^{(k+1)}(0)$. By our assumption

$$f(z) = a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \ldots = z^{k+1}(a_{k+1} + a_{k+2}z + \ldots).$$

Define $g(z) = \frac{f(z)}{z^{k+1}}$. Then $f^{(k+1)}(0)$ if and only if $g(0) = 0$. Indeed

$$g(0) = \lim_{n\to+\infty} \frac{f(1/n)}{\frac{1}{n^{k+1}}} = \lim_{n\to+\infty} \frac{n^{k+1}}{2^n} = 0.$$

(c) Let $\sum_{n\geq 0} a_n z^n$ be the series expansion of f around $z = 0$. Then $a_n = \frac{1}{n!} f^{(n)}(0)$ satisfies $|a_n| > n^n$. The radius of convergence R of such a series is given by

$$\frac{1}{R} = \limsup_{n\to+\infty} |a_n|^{1/n} = \limsup_{n\to+\infty} n = +\infty.$$

It follows that $R = 0$ and there is no holomorphic function as in (c).

3. *Let D be a domain in $\mathbb{C}$ and let $f\colon D \to \mathbb{C}$ be a holomorphic function, not identically zero. Prove that the set of zeros of f in D is at most countable (use: D is a countable union of compact sets).*

Sol.: The zeros of a holomorphic, not identically vanishing function are discrete. Hence finite in a compact set. As D is a countable union of compact sets, the set of zeros of f in D is at most countable (a countable union of finite sets is countable).

4. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open neighbourhood U of the closed unit disc $\overline{\Delta}$. Assume that f is not identically zero.*
   (a) *Show that f has at most finitely many zeros in Δ.*
   (b) *Determine the zeros of $f(z) = \sin(\frac{1}{1-z})$ on the disc Δ and compare the result with (a).*

Sol.: (a) Since $\overline{\Delta}$ is compact, f has at most finitely many zeros in $\overline{\Delta}$ and therefore in Δ (we used that a discrete subset of a compact set is finite).
(b) $f(z) = \sin(\frac{1}{1-z}) = 0$ if and only if $z = 1 - \frac{1}{k\pi}$, with $k \in \mathbb{Z}$. The zeros in Δ are the ones with $k \in \mathbb{Z}_{>0}$. They are infinitely many and they accumulate in $z = 1$, which is a singularity of f.
Note: in this case f is not holomorphic on an open neighbourhood of the closed unit disc $\overline{\Delta}$. Hence the arguments in (a) do not apply.

5. *Let $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Re} z < \frac{\pi}{2}\}$. Determine whether there exists a holomorphic function $f\colon S \to \mathbb{C}$ such that*
   (a) $\mathrm{Re} f(z) = x^2y + y^2x + \sin x \sinh y$;
   (b) $\mathrm{Re} f(z) = y^3 - 3x^2y + \cos x \cosh y + \sin x \sinh y$.
   Justify your answer: either exhibit one such function or explain why it cannot exist.

Sol.: On the convex set S a function is the real part of a holomorphic function if and only if it is harmonic.
The function in (a) is not harmonic: it is the sum of the harmonic function $\sin x \sinh y = Im(\cos z)$ and the function $f(x,y) = x^2y + y^2x$, whose Laplacian is given by $\Delta f = 2(x+y) \not\equiv 0$.
The function in (b) is harmonic: it is the sum of the harmonic functions $Re(\cos z) = \cos x \cosh y$, $Im(\cos z) = \sin x \sinh y = -Re(i \cos z)$ and the function $u(x,y) = y^3 - 3x^2y$ whose Laplacian is identically zero. Hence it is the real part of the holomorphic function

$$f(z) = u(x,y) + iv(x,y) + \cos z - i\cos z,$$

where $v(x,y)$ is a harmonic conjugate of $u(x,y)$. We determine v by the Cauchy-Riemann equations:

$$v_y = u_x = -6xy \quad \Rightarrow \quad v(x,y) = -3xy^2 + \phi(x);$$

$$v_x = -3y^2 + \phi'(x) = -u_y = -(3y^2 - 3x^2) \quad \Leftrightarrow \quad \phi'(x) = 3x^2 \quad \Leftrightarrow \quad \phi(x) = x^3 + c, \ c \in \mathbb{R}.$$

Conclusion: $v(x,y) = -3xy^2 + x^3 + c$ and

$$f = y^3 - 3x^2y + i(-3xy^2 + x^3) + \cos z - i\cos z + ic, \quad c \in \mathbb{R}.$$

6. (*Liouville's theorem for harmonic functions*). *Let $u\colon \mathbb{C} \to \mathbb{R}$ be harmonic and bounded either from above or from below.*
   (a) *Show that u is constant.*
   (b) *Verify that the real and the imaginary parts of the following holomorphic functions are not bounded:*

$$e^z, \qquad \sin z, \qquad \cos z, \qquad z^2.$$

Sol.: (a) The harmonic function u is the real part of a holomorphic function

$$f\colon \mathbb{C} \to \mathbb{C}, \quad f(z) = u(z) + iv(z),$$

where v is a harmonic conjugate of u. Suppose u is bounded by above. Then so is the absolute value of the holomorphic function $e^{f(z)} = e^{u(z)}e^{iv(z)}$. By Liouville's theorem, $e^{f(z)} \equiv c$ is constant. Then also $f(z) = \log(c)$ is constant.

Suppose now that u is bounded by below. Then we can apply the above argument to the harmonic function $-u$.

7. *Set $D = D(z_0, r)$. Let $f\colon D \to \mathbb{C}$ be a holomorphic function. Show that*

$$f(z_0) = \frac{1}{Area(D)} \int_D f(z)dxdy\,.$$

Sol.: We compute the integral in polar coordinates. By the substitutions $x = \rho\cos t$, $y = \rho\sin t$, $dxdy = \rho d\rho dt$, the integral $\int_D f(z)dxdy$ becomes

$$\int_0^r \int_0^{2\pi} f(z_0 + \rho e^{it})\rho d\rho dt = \int_0^r \rho \left(\int_0^{2\pi} f(z_0 + \rho e^{it})dt \right) d\rho.$$

By the mean value property, the above integral equals

$$= \int_0^r \rho 2\pi f(z_0)d\rho = 2\pi f(z_0)(\rho^2/2|_0^r) = f(z_0)\pi r^2,$$

from which we deduce

$$f(z_0) = \frac{1}{\pi r^2} \int\int f(x,y)dxdy = \frac{1}{area(D)} \int\int f(x,y)dxdy,$$

as claimed.

8. *Let D be a domain in $\mathbb{C}$ and let $f\colon D \to \mathbb{C}$ be a nonconstant holomorphic function. Show that the local minima of $|f|$ coincide with the zeros of f.*

Sol.: Let $z_0 \in D$ be a point of local minimum for $|f(z)|$: it means that there is an open neighbourhood U of z_0 in D with the property that $|f(z_0)| \le |f(z)|$, for all $z \in U$. Suppose that $f(z_0) \neq 0$. Then $g(z) = 1/f(z)$ is a holomorphic function of U and z_0 is a local maximum for $|g(z)|$. Then g is constant on U and so is f. As D is connected, by the identity principle f is constant on D. Contradiction.

9. *Let $f\colon U \to \mathbb{C}$ be a nonconstant holomorphic function defined on a neighbourhood of the unit disc Δ. Show that if $|f|$ is constant on the boundary of Δ, then f admits at least one zero in Δ.*

Sol.: The closure of the disc $\overline{\Delta}$ is a compact set. Hence $|f|$ has maximum and minimum on $\overline{\Delta}$. They are distinct because f is non constant. Since the maximum of $|f|$ is attained on the boundary of Δ, then the minimum is necessarily attained in the interior. By the previous exercise, such minimum is 0. Conclusion: f admits at least one zero in Δ.