1. Automorphisms of Δ.
   (a) Show that every automorphism of the unit disc Δ extends injectively to a neighbourhood of its closure $\overline{\Delta}$ and admits at least a fixed point in $\overline{\Delta}$.
   (b) Show that if f has a fixed point in Δ, then it is necessarily unique.

2. Define $D(0,r) = \{z \in \mathbb{C} \ : \ |z| < r\}$. Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \partial D(0,1)$. Show that if f is nonconstant, then there exists an automorphism g of $D(0,1)$ such that $f \circ g(0) = 0$.

3. Let $f : \Delta \to \Delta$ be a holomorphic function with a zero of order m at 0. Show that for all $z \in \Delta$ one has $|f(z)| \leq |z|^m$.

4. Verify that the Cayley transform is a biholomorphism between the unit disc Δ and the upper half plane $\mathbb{H}^+$.

5. Show that the stripe $S_r := \{ z \in \mathbb{C} \ : \ 0 < \mathrm{Im\, z} < r \}$, with $r > 0$, is biholomorphic to $\mathbb{H}^+$ (and therefore to Δ). Determine an explicit biholomorphism $f\colon S_1 \to \Delta$.

6. Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the border $\partial\Delta$ of the unit disc Δ.

7. Determine whether there exists a nonconstant holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ whose image $f(\mathbb{C})$ has empty intersection with the real line $\mathbb{R}$.

Laurent series and isolated singularities, Riemann extension theorem, Casorati-Weierstrass theorem, automorphisms of the plane, residue theorem.

8. *Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions with $g(z) \neq 0$, for every $z \in \mathbb{C}$. Assume that $|f(z)| \leq |g(z)|$, for every $z \in \mathbb{C}$. Show that there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ (cf. Exercise 17 in sheet 0). Show that the assumption "$g(z) \neq 0$, for every $z \in \mathbb{C}$" can be replaced with "g not identically zero".*

9. *(a) Let f be a bounded holomorphic function defined on $\mathbb{C} \setminus \{0,\, i,\, 1+i\}$ or on $C \setminus \mathbb{Z}$. Show that f is constant.*
   (b) Is it true that f is constant when it is bounded on $\mathbb{C} \setminus (\{1/n \, : \, n \in \mathbb{N}\} \cup \{0\})$?

10. *Show that the functions*
$$\frac{\sin z}{z}, \qquad \frac{e^z - 1}{z}, \qquad \frac{\cosh z - 1}{z}$$
are entire, i.e. they extend holomorphically to $\mathbb{C}$.

11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.*
   (a) Show that if f has a zero of order n in z_0, then $1/f$ has a pole of order n in z_0.
   (b) Let z_0 be a singularity (removable, polar, essential) of f. Determine the corresponding type of singularity of $1/f$.

12. *Let z_0 be a pole of f. Recall that for every $M >> 0$ there exists $\varepsilon > 0$ such that $f(D^*(z_0, \epsilon)) \subset \{|z| > M\}$.*

Show that, given $\varepsilon > 0$, there exists $M >> 0$ such that $\{|z| > M\} \subset f(D^*(z_0, \epsilon))$. (Justify your answer and mention all the results you used.)

13. *Determine and classify all the singularities of the following functions:*

$$\tan z, \quad \frac{1}{z^3}\sum_{n=2}^{\infty} 2^n z^n, \quad ze^{1/z}e^{-1/z^2}, \quad \frac{\sin\frac{1}{z}}{z^4}, \quad \frac{\sin z}{z^4}, \quad \frac{1}{z^3} - \cos z.$$

14. *Let $g(z) := e^{1/z} - e^{2/z}$. Determine $g(\mathbb{C} \setminus \{0\})$.*

15. *Let $f(z) = \frac{z^2}{z^2+1}$. Determine the Laurent serie of f around $z_0 = i$.*

16. *Let f be a holomorphic function on $D^*(0, r)$, for $0 < r \leq +\infty$. Assume that $z_0 = 0$ is a pole of f of order m. Then the Laurent serie of f around z_0 is given by*

$$f(z) = \sum_{n \geq -m} a_n z^n, \qquad a_n = \frac{1}{(n+m)!} g^{(n+m)}(0), \quad \text{where } g(z) = z^m f(z).$$

17. *Let $f(z) = \frac{e^z}{(z-i)^3(z+2)^2}$.*
 (i) *Show that $z_1 = i$ is a pole of f of order 3.*
 (ii) *Determine all the coefficients of the principal part of the Laurent series of f around z_1.*
 (iii) *Show that $z_2 = -2$ is a pole of f of order 2.*
 (iv) *Determine all the coefficients of the principal part of the Laurent series of f around z_2.*

18. *Let $\gamma_2(\theta) = e^{i\theta}$, where $\theta \in [0, 2\pi]$, and $\gamma_1(\theta) = 3e^{i\theta}$, where $\theta \in [0, 2\pi]$. Compute*

$$(a) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2+5z}{(z-2)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2+5z}{(z-2)}dz;$$

$$(b) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^2-2}{z}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^2-2}{z}dz;$$

$$(c) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{z^3-3z-6}{z(z+2)(z+4)}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{z^3-3z-6}{z(z+2)(z+4)}dz.$$

19. *Compute the residues of the following functions at the singular points:*

$$e^{3/z^2}, \qquad \frac{z^3}{z-1}, \qquad \frac{z^3}{(z-1)^2}, \qquad \frac{z^3}{1-z^4}, \qquad \frac{z^5}{(z^2-1)^2}, \qquad \frac{\cos z}{1+z+z^2}, \qquad \frac{1}{\sin z}.$$

20. *Show that the function defined by $\sum_{n=1}^{\infty} \frac{1}{n!z^n}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Compute its integral on $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$.*

21. *Compute*

$$\int_\gamma \frac{e^z}{z^3}dz, \qquad \int_\gamma \frac{e^{1/z}}{z^3}dz,$$

when $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 6\pi]$ and when $\gamma(\theta) = e^{i\theta}$, for $\theta \in [0, 2\pi]$.

22. *Use the "sector method" to compute*

$$\int_0^{+\infty} \frac{1}{1+x^4}dx\,.$$

See Sarason, Example 3, p.126.