# Advanced topics on Algorithms

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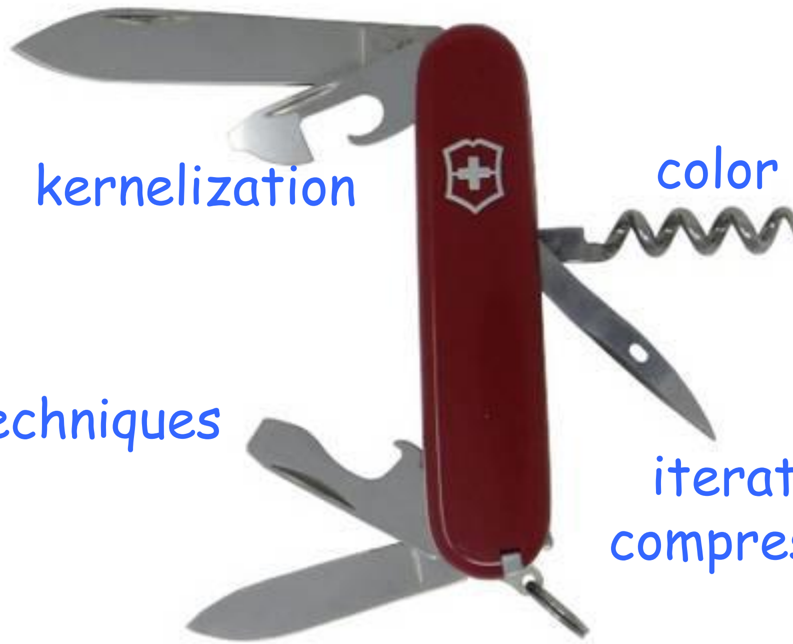
# Parameterized algorithms
# Episode IV

# Toolbox (to show a problem is FPT)



# Lower bounds

tools and theory
of the parameterized intractability

## What kind of negative results we can prove?

- Can we show that a problem (e.g., k-Clique) is not FPT?
- Can we show that a problem (e.g., k-Vertex Cover) does not have an algorithm running in time $2^{o(k)}n^{O(1)}$?

obs: we have to assume P≠NP
(if P=NP, k-Clique can be solved in polynomial time, and hence is FPT)

→ conditional lower bounds

idea: develop a theory that provides evidence that a parameterized problem is hard (e.g., not FPT)

# Parameterized complexity

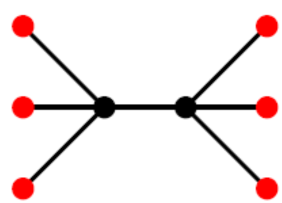
To build a complexity theory for parameterized problems, we need two ingredients:

- An appropriate notion of reduction
- An appropriate (hardness) hypothesis

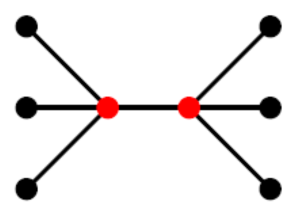
obs: Polynomial-time reductions are not good for our purposes

Example: G has an Independent Set of size k iff has a Vertex Cover of size n-k

IS problem



Vertex Cover problem



Complexity: NP-complete | NP-complete

no $n^{o(k)}$-time algorithm is known

a $O(2^k n^{O(1)})$ algorithm exists

# Parameterized reduction

**Parameterized reduction** from problem P to problem Q: a function $\phi$ mapping an instance $(x,k)$ of P into an instance $(x',k') = \phi(x,k)$ of Q, such that

- $(x,k)$ is a YES-instance of P iff $(x',k')$ is a YES-instance of Q;
- $(x',k')$ can be computed in time $f(k)n^{O(1)}$;
- $k' \leq g(k)$ for some function g.

Note: if Q is FPT then P is also FPT.

Equivalently: if P is not FPT then Q is not FPT.

Non-example: from Independent Set to Vertex Cover

$$(G,k) \Rightarrow (G,n-k)$$

Example: from Independent Set to Clique

$$(G,k) \Rightarrow (\overline{G},k)$$

# Multicolored Clique

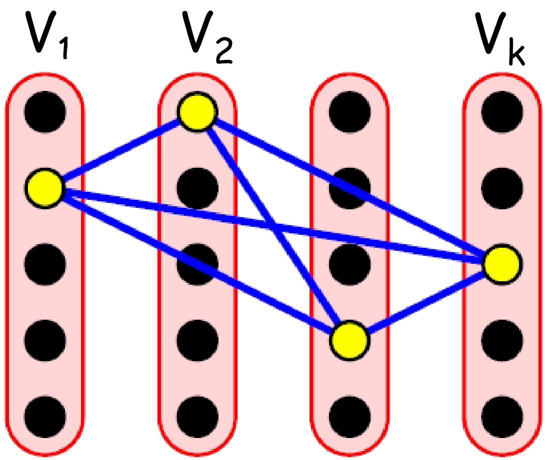
Input:

- a graph $G=(V,E)$, vertices are colored with k colors
- a nonnegative integer k

question:

is there a clique of size k containing one vertex for each color

parameter: k

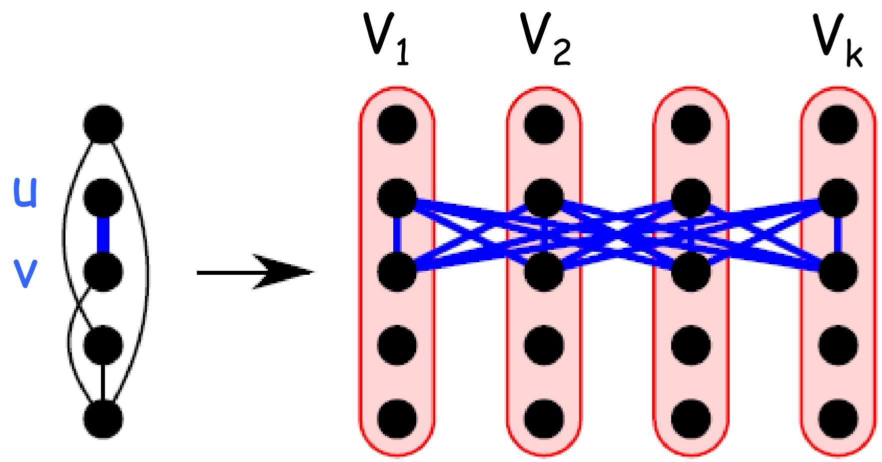


## Theorem

There is a parameterized reduction form Clique to Multicolored Clique.

## proof

- for each vertex v of G, G' has k vertices $v_1,\ldots v_k$, one for each color
- if u and v are adjacent in G, connect all copies of u with all copies of v




G has a k-clique ⟺ G' has a multicolored k-clique

Similarly: reduction from k-Clique to multicolored k-Independent Set

# k-Dominating Set

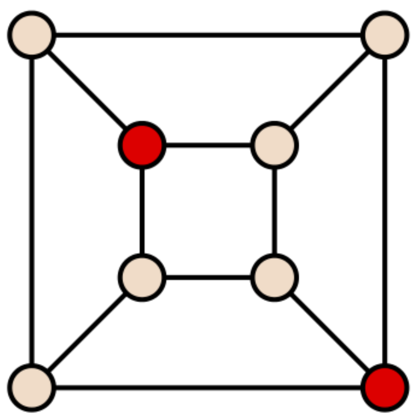
Input:

- a graph G=(V,E)
- a nonnegative integer k

question:

is there a set U of vertices of size |U|≤k such that each v∈V\U is adjacent to a vertex u∈U

parameter: k

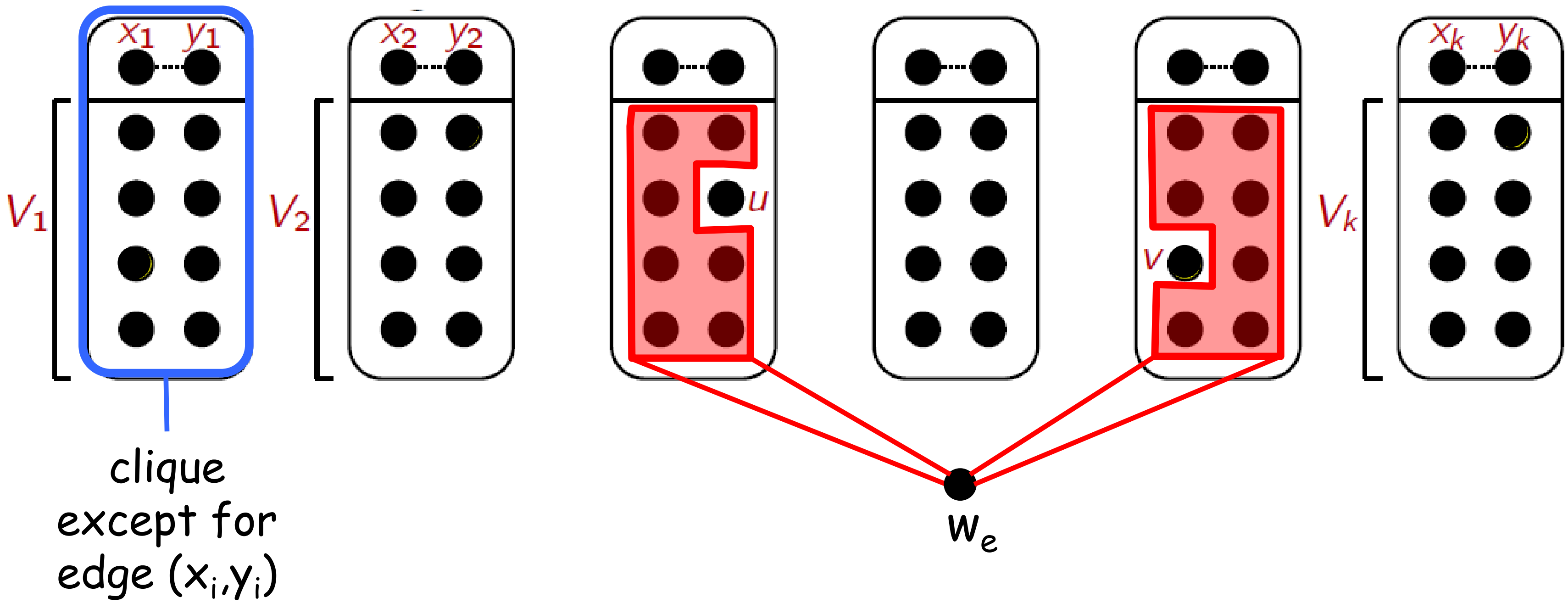


**Theorem**

There is a parameterized reduction form Multicolored Independent Set to Dominating Set.

**proof**

- $G'$ has all vertices of $G$ plus vertices $x_i$, $y_i$, for each color $i$
- for each edge $(u,v)$ in $G$ with $u \in V_i$ and $v \in V_j$, add a vertex $w_e$ to $G'$ adjacent to every vertex of $(V_i \cup V_j) \setminus \{u,v\}$




Claim: a k-DS must choose a vertex from each $V_i$ and such vertices must form and independent set in $G$.

**Theorem**

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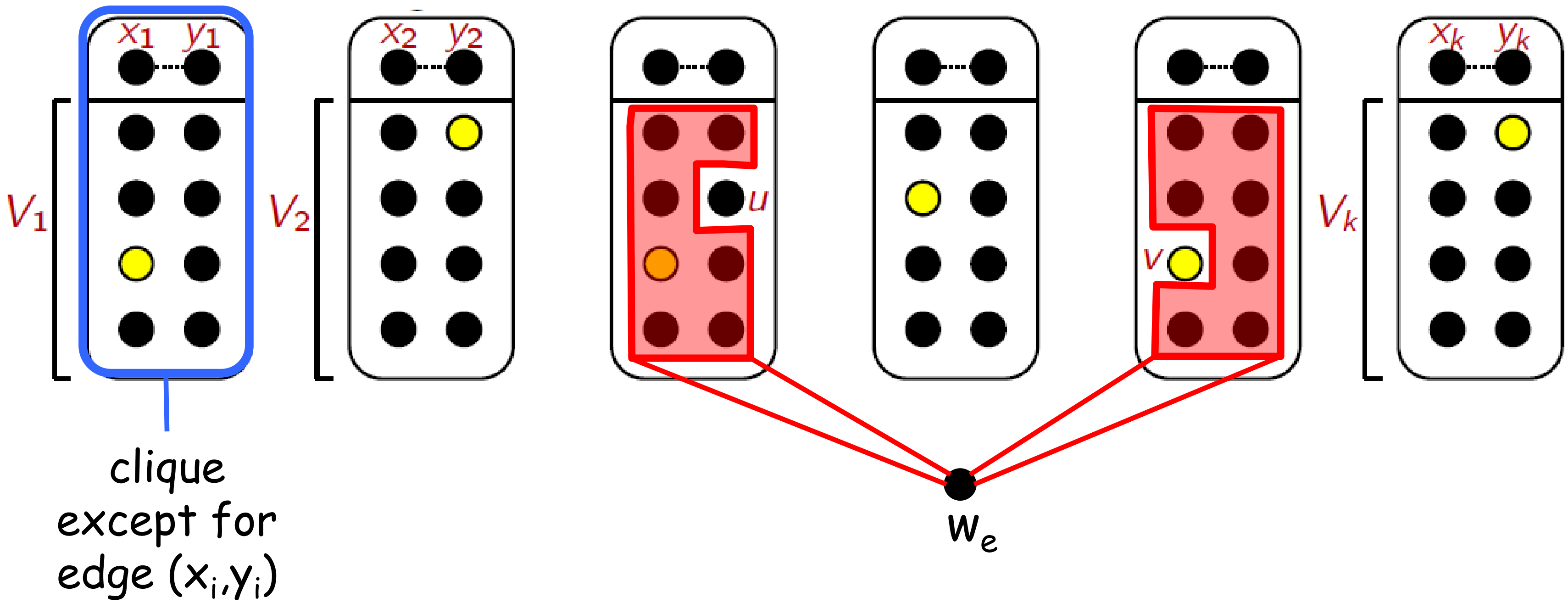



Claim: a k-DS must choose a vertex from each $V_i$ and such vertices must form and independent set in $G$.

# Hard problems

Hundreds of parameterized problems are known to be at least as hard as Clique:

- Independent Set
- Dominating Set (even in bipartite graphs)
- Set Cover
- Hitting Set
- Connected Dominating Set
- Partial Vertex Cover (parameterized by the size of the cover)
- ...

We believe that none of these problems are FPT

## Basic Hypotesis

It seems we have to assume something stronger that P≠NP

Let's choose a basic hypothesis:

**Engineers' Hypothesis**

k-Clique cannot be solved in time $f(k)\ n^{O(1)}$.

⇕

**Theorists' Hypothesis**

k-Step Halting Problem (is there a path of a give Nondeterministic Turing Machine that stops in k steps?) cannot be solved in time $f(k)\ n^{O(1)}$.

⇑

**Exponential Time Hypothesis (ETH)**

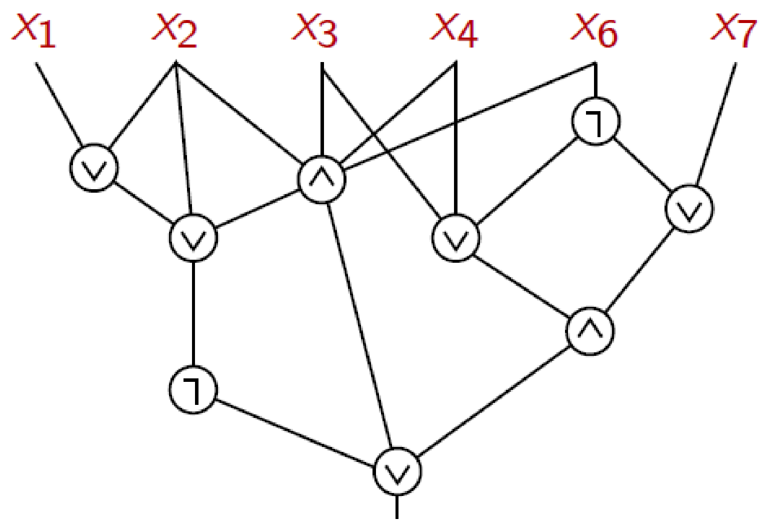
n-variable 3-SAT cannot be solved in time $2^{o(n)}$.

which hypothesis is most plausible?

# Some observations

- k-Clique and k-Step Halting problem can be reduced to each other
  - Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- k-Clique and k-Step Halting problem can be reduced to k-Dominating Set
- Is there a parameterized reduction from k-Dominating Set to k-Clique?
- Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
  - Independent Set is W[1]-complete
  - Dominating Set is W[2]-complete
- Does not matter if we only care about whether a problem is FPT or not!

a Boolean circuit consists of input gates, negation gates, AND gates, OR gates, and a single output gate

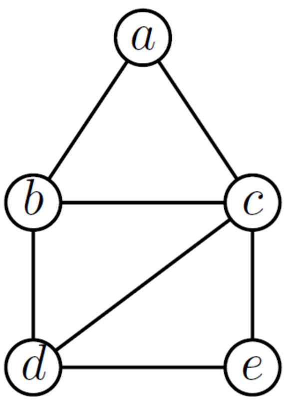


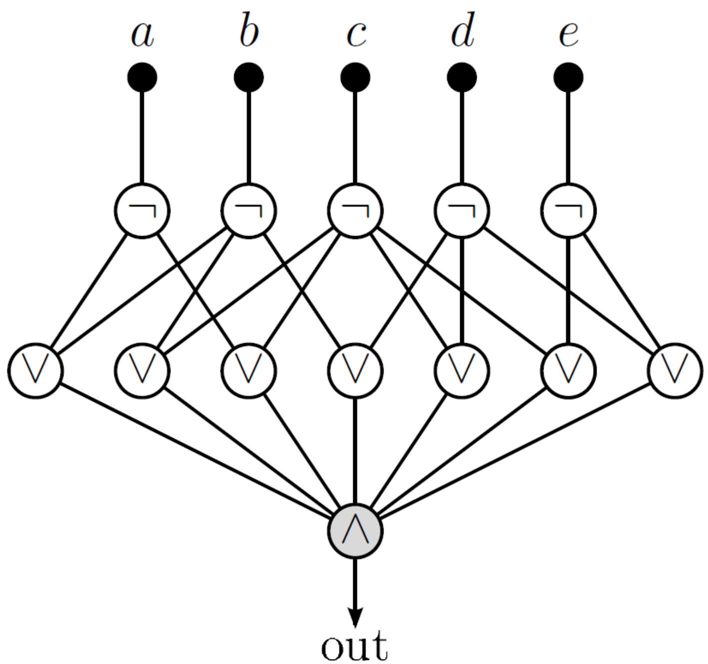

**Circuit Satisfiability**
Given a Boolean circuit $C$, decide if there is an assignment on the inputs of $C$ making the output true

weight of an assignment: number of true variables

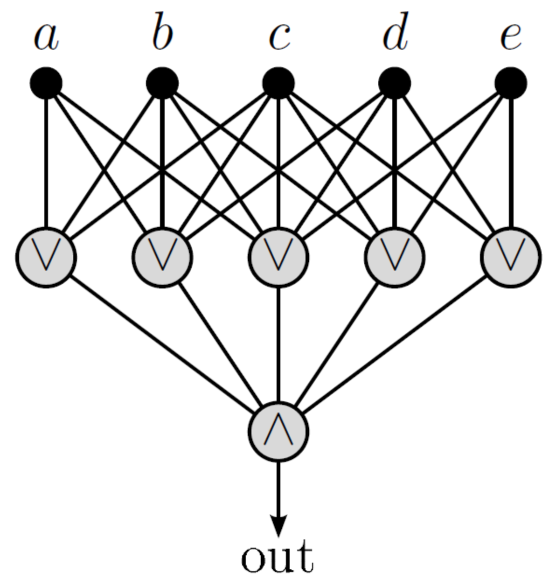
**Weighted Circuit Satisfiability**
Given a Boolean circuit $C$ and an integer $k$, decide if there is an assignment of weight $k$ making the output true

Both k-Independent Set and k-Dominating Set can be reduced to Weighted Circuit Satisfiability




k-Independent Set

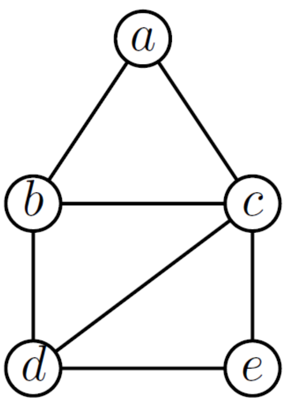



k-Dominating Set

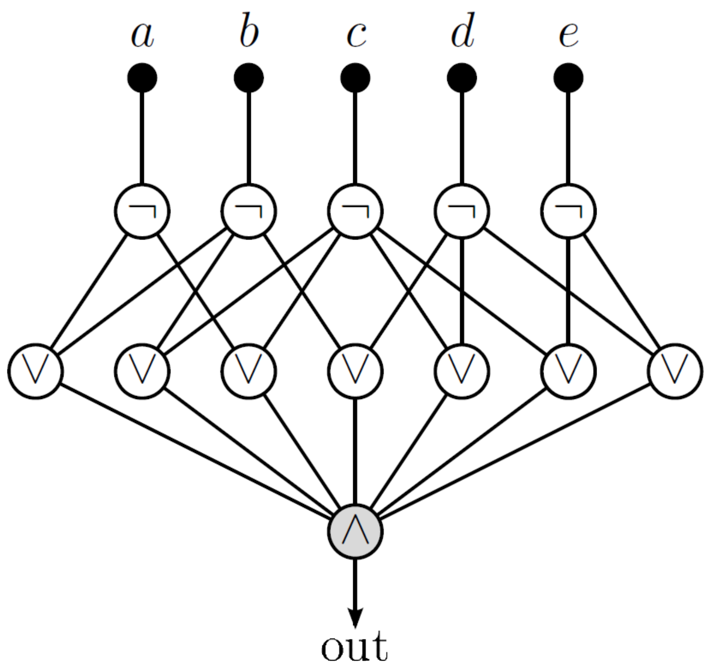
idea: DS is harder than IS because we need a more complicated circuit depth of a circuit: the maximum length of an input-output path

a gate is large if it has more than 2 inputs

weft of a circuit: the maximum number of large gates in an input-output path

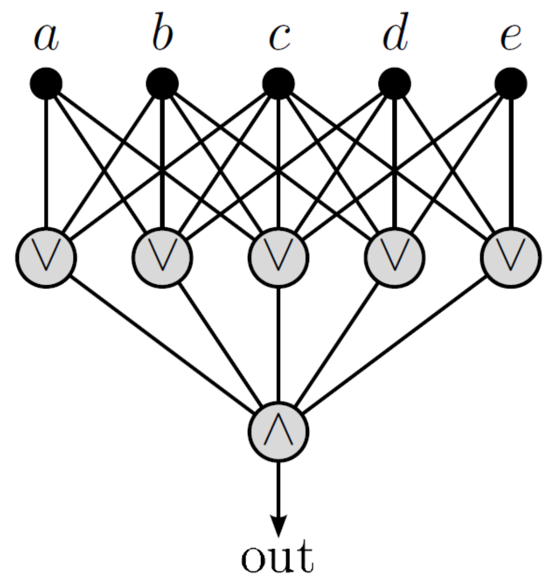



depth: 3 weft: 1




k-Independent Set

depth: 2 weft: 2




k-Dominating Set

## The W-hierarchy

Let C[t; d] be the set of all circuits having weft at most t and depth at most d

**Definition**
A problem P is in the class W[t] if there is a constant d and a parameterized reduction from P to Weighted Circuit Satisfiability of C[t;d]

Independent Set is in W[1] and Dominating Set is in W[2]

fact: Independent Set is W[1]-complete

fact: Dominating Set is W[2]-complete

a problem is complete for a given class if every other problem in the class can be reduced to it

⇒ a reduction from DS to IS would imply W[1]=W[2]

# ETH and some cool consequences

**Exponential Time Hypothesis (ETH)**

There is no $2^{o(n)}$-time algorithm for n-variable 3-SAT

Note: current best algorithm is $1.30704^n$ [Hertli 2011].

Note: an n-variable 3-SAT formula can have $\Omega(n^3)$ clauses.

**Sparsification lemma [Impagliazzo, Paturi, Zane 2001]**

There is no $2^{o(n)}$-time algorithm for n-variable 3-SAT

$$\Updownarrow$$

There is no $2^{o(m)}$-time algorithm for m-clause 3-SAT

# Transferring lower bounds: an example

**Exponential Time Hypothesis (ETH)**
There is no $2^{o(m)}$-time algorithm for m-clause 3-SAT

The textbook reduction from 3-SAT to 3-Coloring:

| 3-SAT formula F | → | a graph G |
|---|---|---|
| n variables | | O(n+m) vertices |
| m clauses | | O(n+m) edges |

## Transferring lower bounds: an example

**Exponential Time Hypothesis (ETH)**

There is no $2^{o(m)}$-time algorithm for m-clause 3-SAT

The textbook reduction from 3-SAT to 3-Coloring:

| 3-SAT formula F<br>n variables<br>m clauses | ⇒ | a graph G<br>O(m) vertices<br>O(m) edges |
| --- | --- | --- |

**Corollary**

Assuming ETH, there is no $2^{o(n)}$-time algorithm for 3-coloring on an n-vertex graph

# Transferring lower bounds

There are many similar reductions from 3-SAT to other graph problems.

**Consequence:**

Assuming ETH, there is no $2^{o(n)}$-time algorithm on an n-vertex graph for:

- Independent Set
- Clique
- Dominating Set
- Vertex Cover
- Longest Path
- ...

# Transferring lower bounds

There are many similar reductions from 3-SAT to other graph problems.

Consequence on the $f(k)$ game:

Consequence:

Assuming ETH, there is no $2^{o(k)}n^{O(1)}$ time algorithm on an n-vertex graph for:

- k-Independent Set
- k-Clique
- k-Dominating Set
- k-Vertex Cover
- k-Path
- ...

(k-Vertex Cover and k-Path:) roughly tights since they can be solved in time $2^{O(k)}n^{O(1)}$

**Engineers' Hypothesis**

k-Clique cannot be solved in time $f(k)\ n^{O(1)}$.

**Theorists' Hypothesis**

k-Step Halting Problem (is there a path of a give Nondeterministic Turing Machine that stops in k steps?) cannot be solved in time $f(k)\ n^{O(1)}$.

**Exponential Time Hypothesis (ETH)**

n-variable 3-SAT cannot be solved in time $2^{o(n)}$.

Assuming ETH we can prove that k-Clique is not FPT.

Indeed, we can prove a much stronger and interesting result:

**Theorem [Chen et al. 2004]**

k-Clique cannot be solved in time $f(k)\ n^{o(k)}$ for any computable function f proof

assume you can find a k-clique on a graph H in time $f(k)\ |V(H)|^{k/s(k)}$,

s(k): (positive) nondecreasing unbounded function

we show you can find a 3-coloring of G in $2^{o(n)}$ time (contradicting ETH)

technical assumption:

$f(k) \geq \max\{k, k^{k/s(1)}\}$ otherwise set $f'(k) = \max\{f(k), k, k^{k/s(1)}\}$

partition the n vertices of G into k groups of at most $\lceil n/k \rceil$ vertices each

build H as follows:

- each vertex corresponds to a proper 3-coloring of one of the groups
- two vertices of H are connected iff the corresponding colorings are compatible

$|V(H)| \leq k\ 3^{\lceil n/k \rceil}$

Claim: there is a k-clique in H iff G admits a proper 3-coloring

now, we suitably choose k...

for a given n, let k be the largest integer such that $f(k) \le n$

$k := g(n)$ nondecreasing unbounded function on n (satisfying $g(n) \le n$)

time to compute a 3-coloring of $G$:

$$f(k)\,|V(H)|^{k/s(k)} \le f(k)\left(k\,3^{\lceil n/k \rceil}\right)^{k/s(k)} \quad \text{using } f(k) \le n$$

$$\le n\left(k\,3^{\lceil n/k \rceil}\right)^{k/s(k)} \quad \text{using } k=g(n) \le n$$

$$\le n\left(k\,3^{2n/k}\right)^{k/s(k)} \quad \text{using } s(k) \text{ nondecreasing}$$

$$\le n\; k^{k/s(1)}\; 3^{2n/s(k)} \quad \text{using } k^{k/s(1)} \le f(k) \le n$$

$$\le n^2\, 3^{2n/s(g(n))} \quad \text{function } s(g(n)) \text{ is nondecreasing \& unbounded}$$

$$= 2^{o(n)}$$

# Strong ETH

# k-Dominating Set

Input:

- a graph $G=(V,E)$
- a nonnegative integer k

question:

is there a set U of vertices of size $|U| \leq k$ such that each $v \in V \setminus U$ is adjacent to a vertex $u \in U$

parameter: k

naive: $n^{k+1}$

$n^{k/10}$ ?

smarter: $n^{k+o(1)}$

$n^{k-1}$ ?

assuming ETH: no $f(k)\, n^{o(k)}$

Exponential Time Hypothesis (ETH)

There is no $2^{o(n)}$-time algorithm for n-variable 3-SAT

Note: current best algorithm is $1.30704^n$ [Hertli 2011].

Strong ETH (SETH)

There is no $(2-\varepsilon)^n$-time algorithm for CNF-SAT

for any fixed k, a $n^{k-0.01}$ time algorithm for k-DS would violate SETH

assuming SETH:

- no $n^{2.99}$ time algorithm for 3-DS
- no $n^{3.99}$ time algorithm for 4-DS
- no $n^{4.99}$ time algorithm for 5-DS
- ...