Advanced topics on Algorithms

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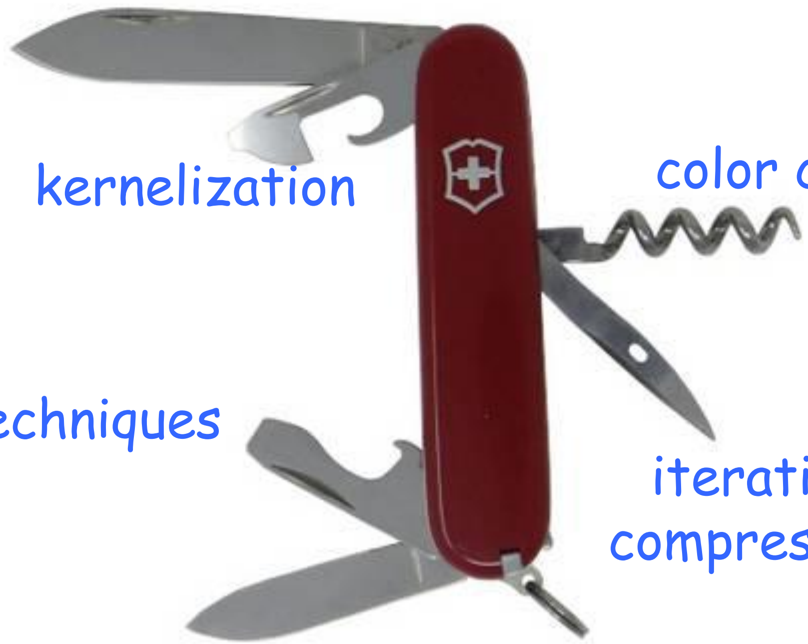
Parameterized algorithms
Episode II

Toolbox (to show a problem is FPT)

bounded-search trees

kernelization

color coding



algebraic techniques

iterative compression

treewidth

color coding

k-Path

Input:

- a graph $G=(V,E)$
- a nonnegative integer k

question:

is there a simple path of k vertices

parameter: k

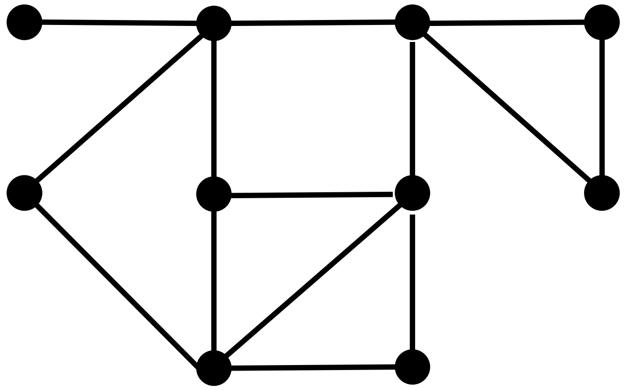
obs: NP-hard since it contains the Hamiltonian path as special case

Theorem [Alon, Yuster, Zwick 1994]
k-Path can be solved in time $2^{O(k)} n^{O(1)}$.

previous best algorithms had running time $k^{O(k)} n^{O(1)}$.

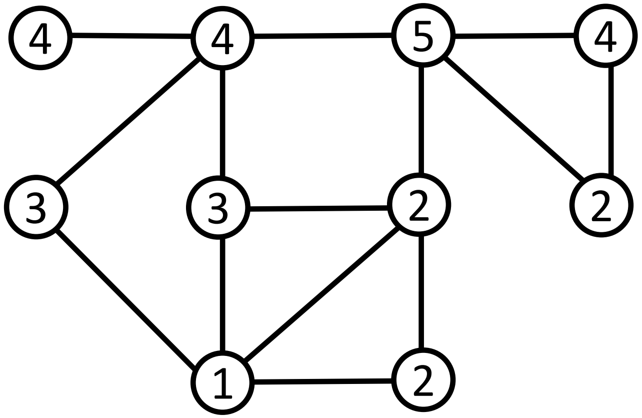
color coding

- assign colors from $\{1,\ldots,k\}$ to vertices $V(G)$ uniformly and independently at random.



color coding

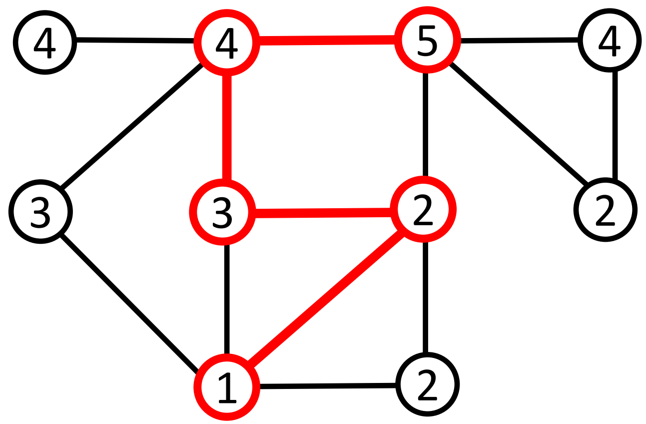
- assign colors from {1,...,k} to vertices V(G) uniformly and independently at random.



- check if there is a path colored 1-2-...-k and output YES or NO

color coding

- assign colors from {1,...,k} to vertices V(G) uniformly and independently at random.



- check if there is a path colored 1-2-...-k and output YES or NO

obs1: if there is no k-path: no path colored 1-2-...-k exists

⇒ NO

obs2: if there is a k-path: there is some probability that this path is colored 1-2-...-k

probability of success: k^{-k}.

⇒ YES with probability k^{-k}.

boosting the probability of success: independent repetitions

Useful fact

If the probability of success is at least p, then the probability that the algorithm does not say "YES" after $1/p$ repetitions is at most

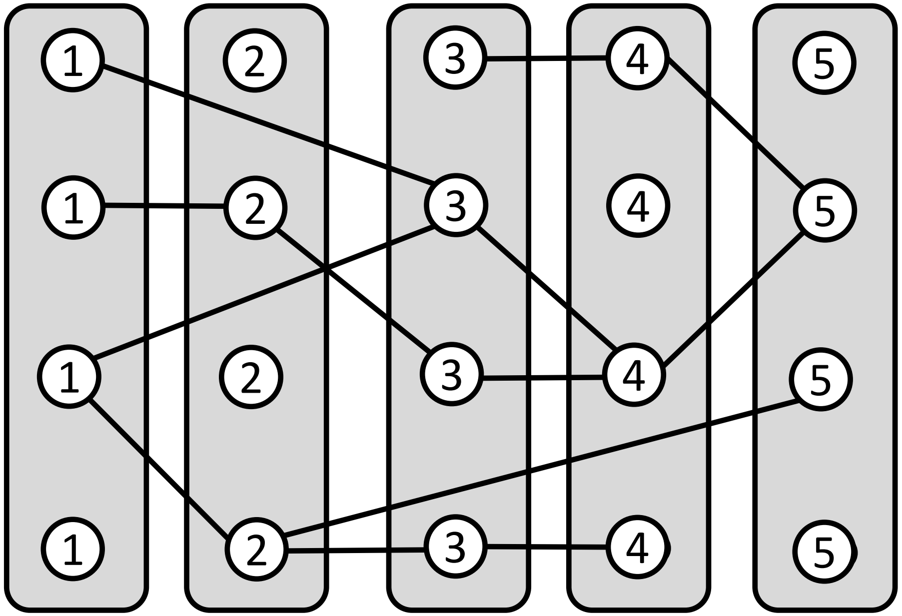
$$(1-p)^{1/p} \leq (e^{-p})^{1/p} = 1/e \approx 0.38$$

- Thus, if $p \geq k^{-k}$ then error probability is at most $1/e$ after k^k repetitions

- repeating the whole algorithm a constant number of times can make the error probability an arbitrary small constant

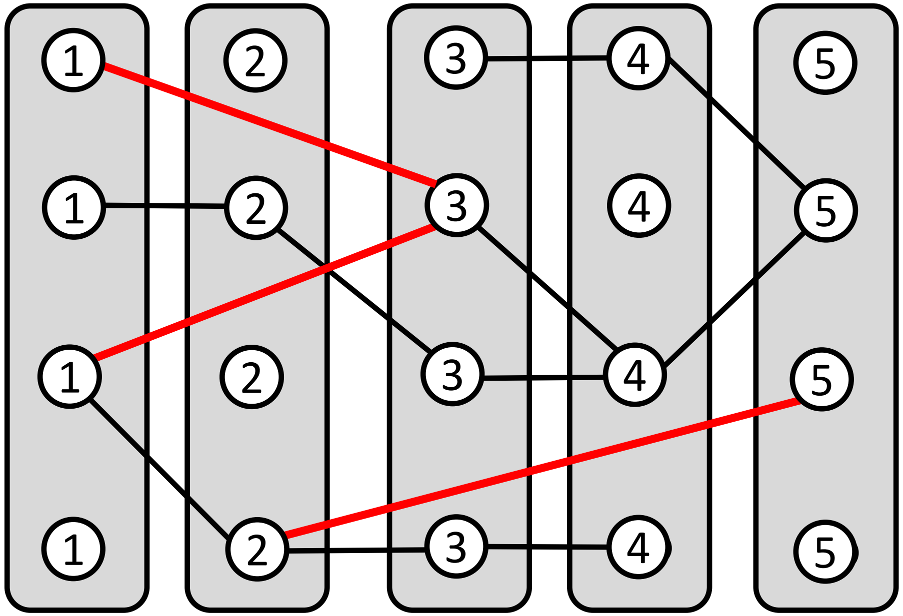
  example: trying $100\ k^k$ random colorings, the probability of a wrong answer is at most $(1/e)^{100}$.

Finding a path colored 1-2-...-k



- edges connecting nonadjacent color classes are removed
- the remaining edges are directed towards the larger class
- all we need to check if there is a directed path from class 1 to class k

Finding a path colored 1-2-...-k



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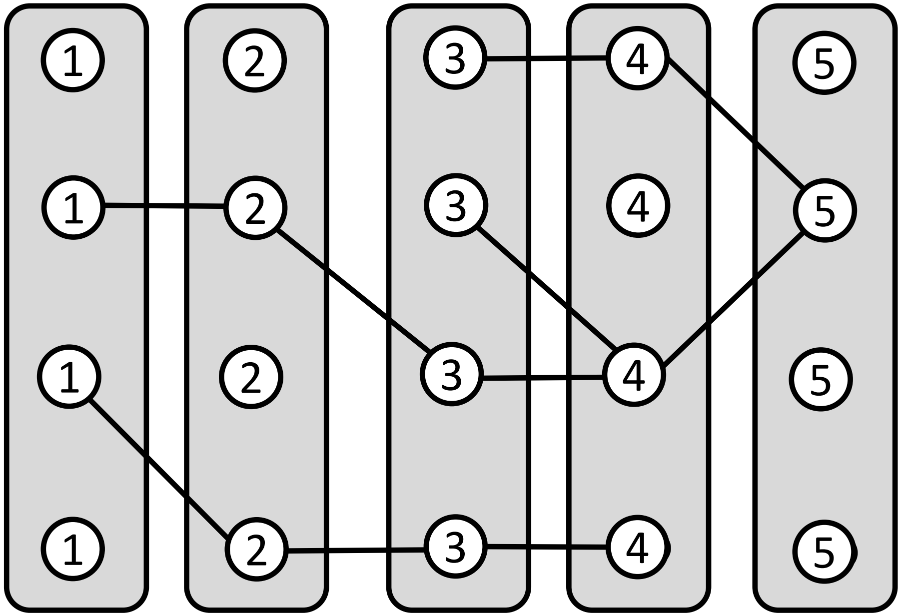

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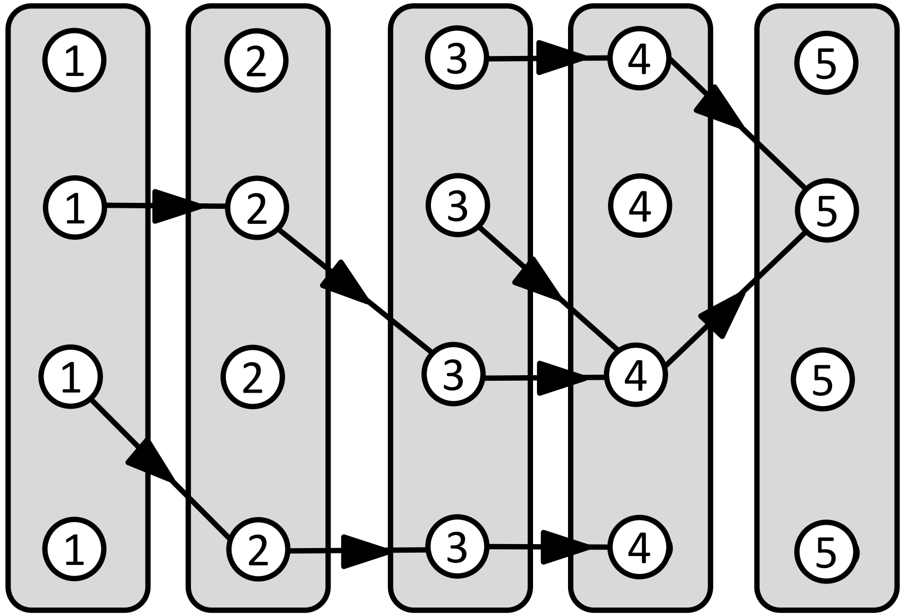

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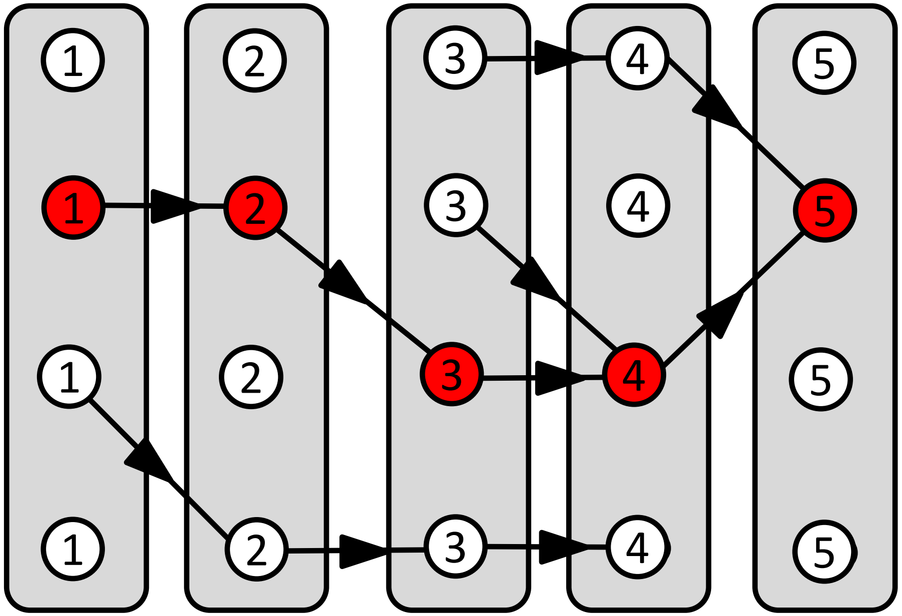
Finding a path colored 1-2-...-k



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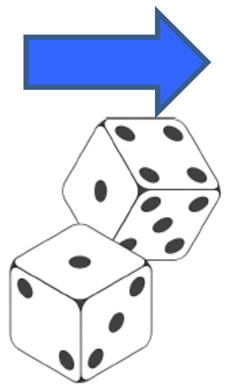
color coding

k-Path

color coding
success probability
k^{-k}

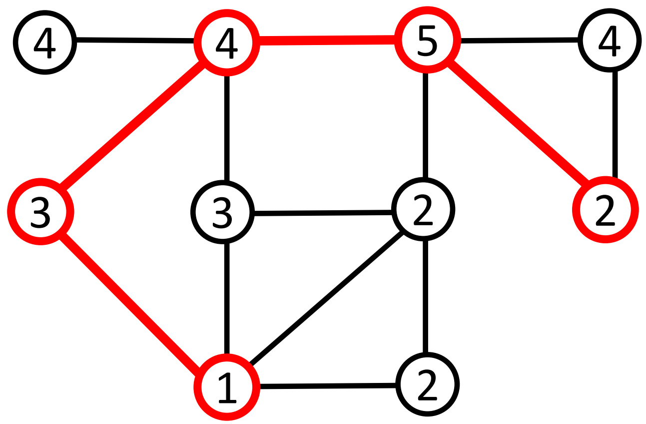


finding a
1-…-k
colored path

polynomial time
solvable

improved color coding

- assign colors from {1,...,k} to vertices V(G) uniformly and independently at random.



- check if there is a colorful path (each color appears exactly once) and output YES or NO

obs1: if there is no k-path: no colorful path exists

➡ NO

obs2: if there is a k-path: there is some probability that it is colorful

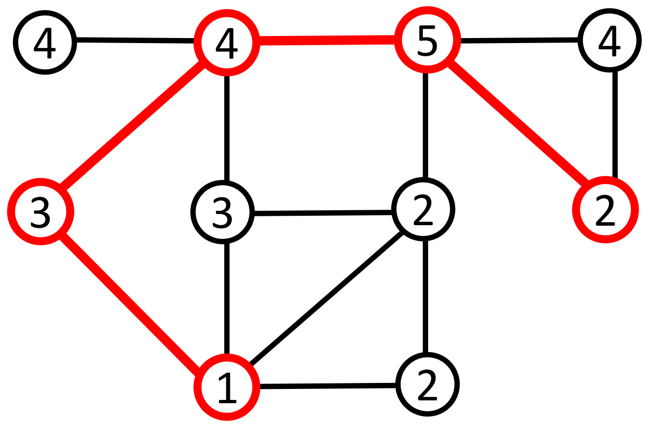
probability of success:

$$\frac{k!\, k^{n-k}}{k^n} = \frac{k!}{k^k} \geq \frac{(k/e)^k}{k^k} = e^{-k}$$

➡ YES with probability e^{-k}.

improved color coding

- assign colors from {1,...,k} to vertices V(G) uniformly and independently at random.



- repeating the algorithm 100 e^k times, the probability of a wrong answer is at most $(1/e)^{100}$.

how to find a colorful path?

- try all permutations: $k!\, n^{O(1)}$ time
- dynamic programming: $2^k\, n^{O(1)}$ time

finding a colorful path

subproblems:

$v \in V$, non-empty subset $S \subseteq \{1,\ldots,k\}$

Path(v,S): is there a path P ending at v such that each color of S appears in P exactly once and no other color appears in P?

obs1: There is a colorful path iff Path(v,{1,...,k})=TRUE for some v

obs2: # of subproblems $2^k n$

finding a colorful path

subproblems:

$v \in V$, non-empty subset $S \subseteq \{1,\ldots,k\}$

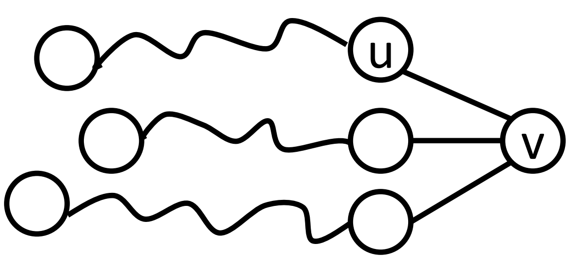
Path(v,S): is there a path P ending at v such that each color of S appears in P exactly once and no other color appears in P?

|S|=1 (base case)

Path(v,S)=TRUE iff S={col(v)}

|S|>1

$$\text{Path}(v,S) = \begin{cases} \displaystyle\operatorname*{OR}_{(u,v)\in E} \text{Path}(u, S-\{col(v)\}) & \text{if } col(v)\in S \\ \text{FALSE} & \text{otherwise} \end{cases}$$



color coding

k-Path

color coding
success probability
e^{-k}

finding a colorful path

solvable in
$2^k n^{O(1)}$ time

Theorem

There is a randomized algorithm for k-Path that runs in time $(e2)^k n^{O(1)}$ that either reports a failure or find a path of k vertices. Moreover, the algorithm finds a solution of a YES-instance with constant probability.

Kernelization

a 2k-vertex kernel for VC
based on linear programming

an Integer Linear Programming (ILP) formulation of VC

minimize $\sum_{v \in V} x_v$

subject to $x_u + x_v \geq 1$ $\quad e=(u,v) \in E$

$x_v \in \{0,1\}$ $\quad v \in V$

relax with
$x_v \geq 0$ & $x_v \leq 1$

redundant

LP-relaxation

minimize $\sum_{v \in V} x_v$

subject to $x_u + x_v \geq 1$ $\quad e=(u,v) \in E$

$x_v \geq 0$ $\quad v \in V$

a feasible solution is
a fractional VC

OPT_f: cost of the min fractional VC

$$OPT_f \leq OPT$$

Let $\mathbf{x}$ be an optimal fractional solution.

$$V_0 = \{ v \in V : x_v < \tfrac{1}{2} \}$$

$$V_{0.5} = \{ v \in V : x_v = \tfrac{1}{2} \}$$

$$V_1 = \{ v \in V : x_v > \tfrac{1}{2} \}$$

Theorem (Nemhauser-Trotter)

There is a minimum vertex cover S of G such that

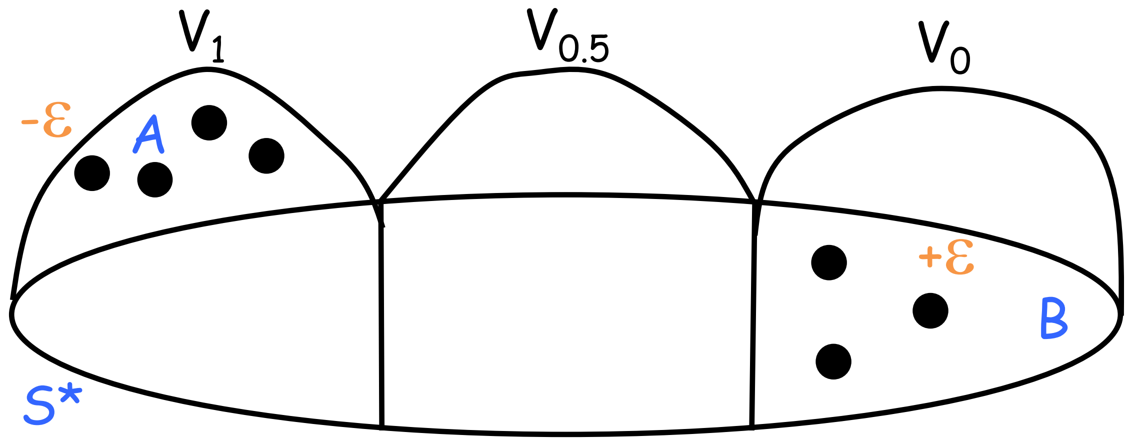
$$V_1 \subseteq S \subseteq V_1 \cup V_{0.5}$$

Let S^* be a minimum VC   Let $S=(S^*\setminus V_0)\cup V_1$ proof

S is a VC   since every adjacent vertex of V_0 must be in V_1

claim: S is minimum   assume by contradiction that $|S|>|S^*|$

⇒ $\underbrace{|S^*\cap V_0|}_{B} < \underbrace{|V_1\setminus S^*|}_{A}$



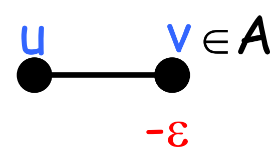
$$y_v = \begin{cases} x_v - \varepsilon & \text{if } v\in A \\ x_v + \varepsilon & \text{if } v\in B \\ x_v & \text{otherwise} \end{cases}$$

$\varepsilon = \min\{|x_v - \tfrac{1}{2}| : v\in V_0\cup V_1\}$

claim:

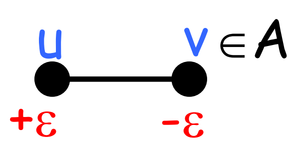
- y is strictly better that x
- y is feasible

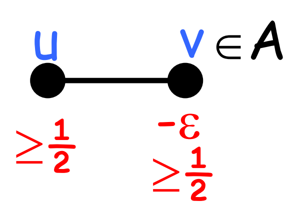
} contradicts optimality of x



interesting case: u or v ∈ A

since S^* is a VC then $u\in B$ or $u\in S^*\setminus B$





kernelization

compute an optimal fractional solution x of the LP-relaxation for the VC instance (G,k). Define V_0,$V_{0.5}$, V_1 as before.

if $\sum_{v \in V} x_v > k$ conclude that (G,k) is a No-instance.

Otherwise, greedily pick V_1 in the VC, delete vertices in V_1 and V_0 (and all their incident edges).
The new instance is $(G'=G-(V_1 \cup V_0), k'=k-|V_1|)$.

Theorem

k-Vertex Cover admits a kernel of at most 2k vertices.

proof

(G,k) is a YES-instance iff (G',k') is a YES-instance

$$|V(G')| = |V_{0.5}| = \sum_{v \in V_{0.5}} 2x_v \le 2 \sum_{v \in V} x_v \le 2k$$

■

...in the next episode...

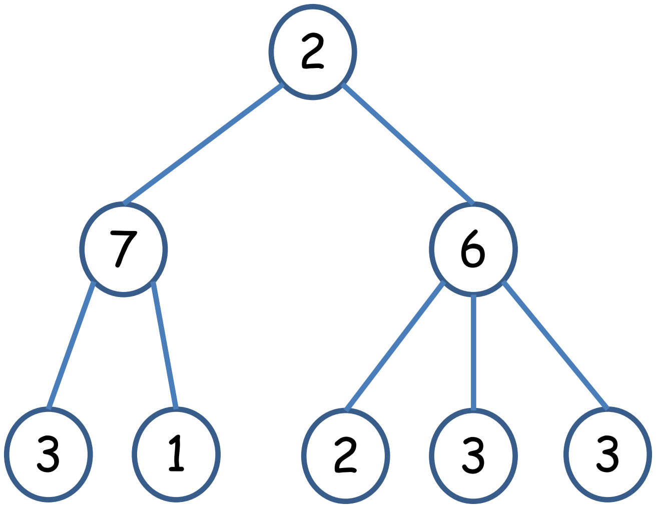
The party problem

problem: invite people to a party

maximize: total fun factor of the invited people

constraint: everyone should be having fun

→ do not invite a colleague and his direct boss at the same time!



input: a tree with weights on the nodes

goal: an independent set of maximum total weight

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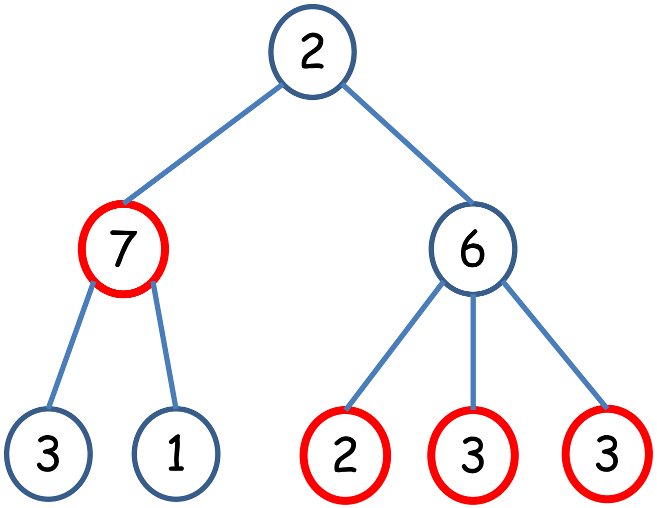

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OPT= 15

Exercise: give a polynomial time algorithm for it