TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Advanced Topics on Algorithms
2025/2026

Clustering is Hard Only When it Doesn't Matter

Lecturer: Alessandro Straziota

In previous episodes...

Why do we need algorithm analysis?

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1. Performance Prediction

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2. Identify Optimal Algorithms

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$\Longrightarrow$ the theory cannot differentiate `LRU` from `FWF` or `FIFO`.

But `FWF` is clearly worse than `LRU`!

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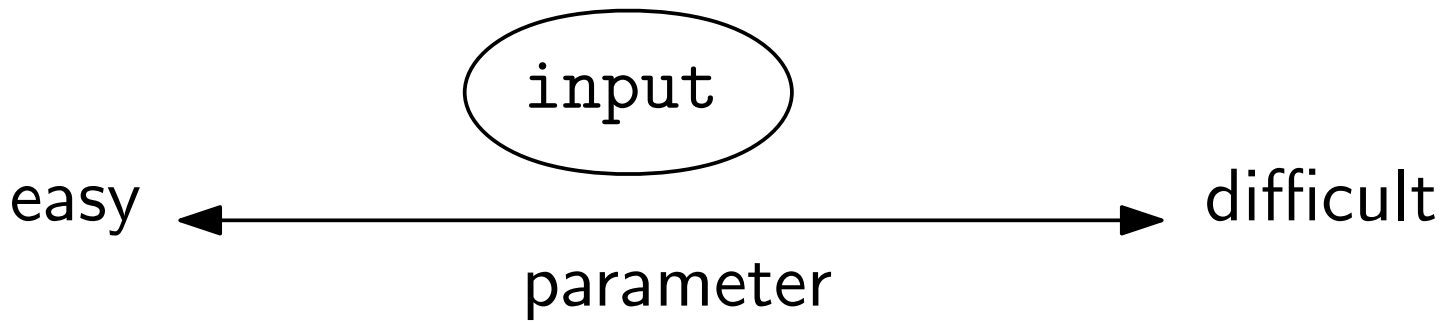

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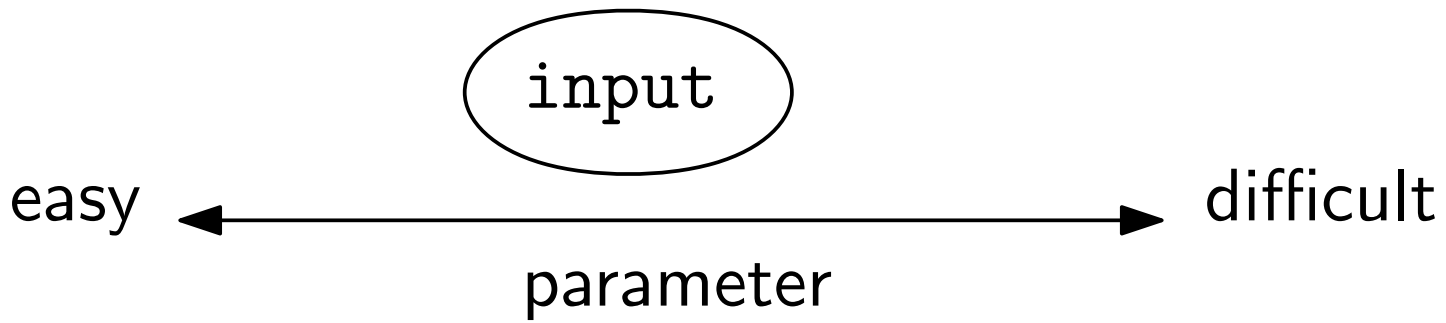

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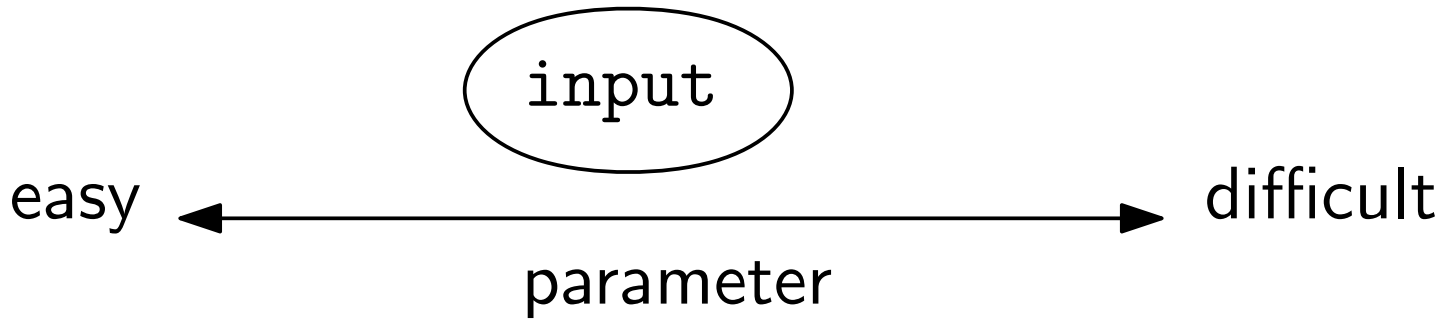

Example (Working Set Model): the maximum number of **distinct** page requests in every window length w.

Claim. `LRU` has a **page fault rate** strictly lower than `FIFO`.

In this episode...

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Insight: in most applications we only care about instances in which there is a "meaningful solution", that are computationally easier to solve than worst-case instances.

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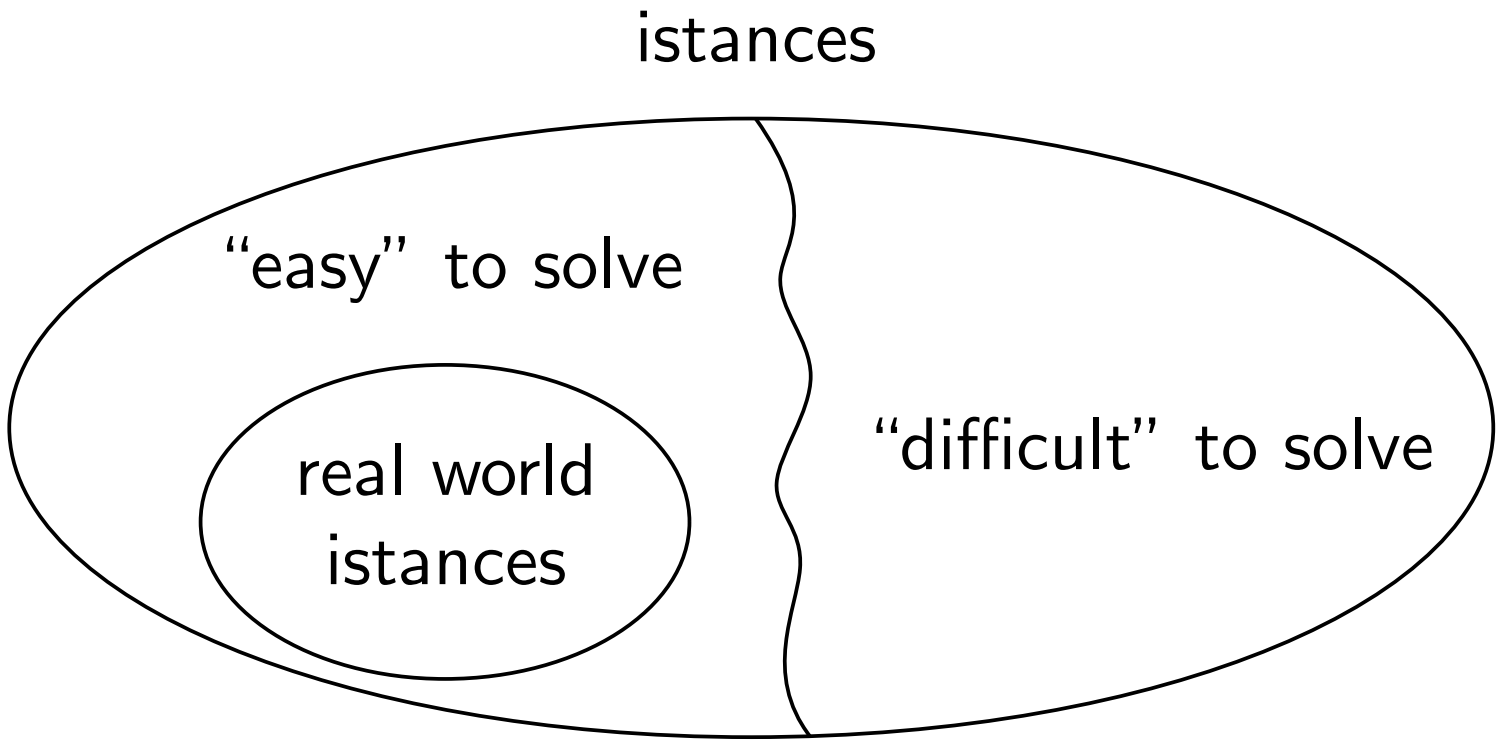

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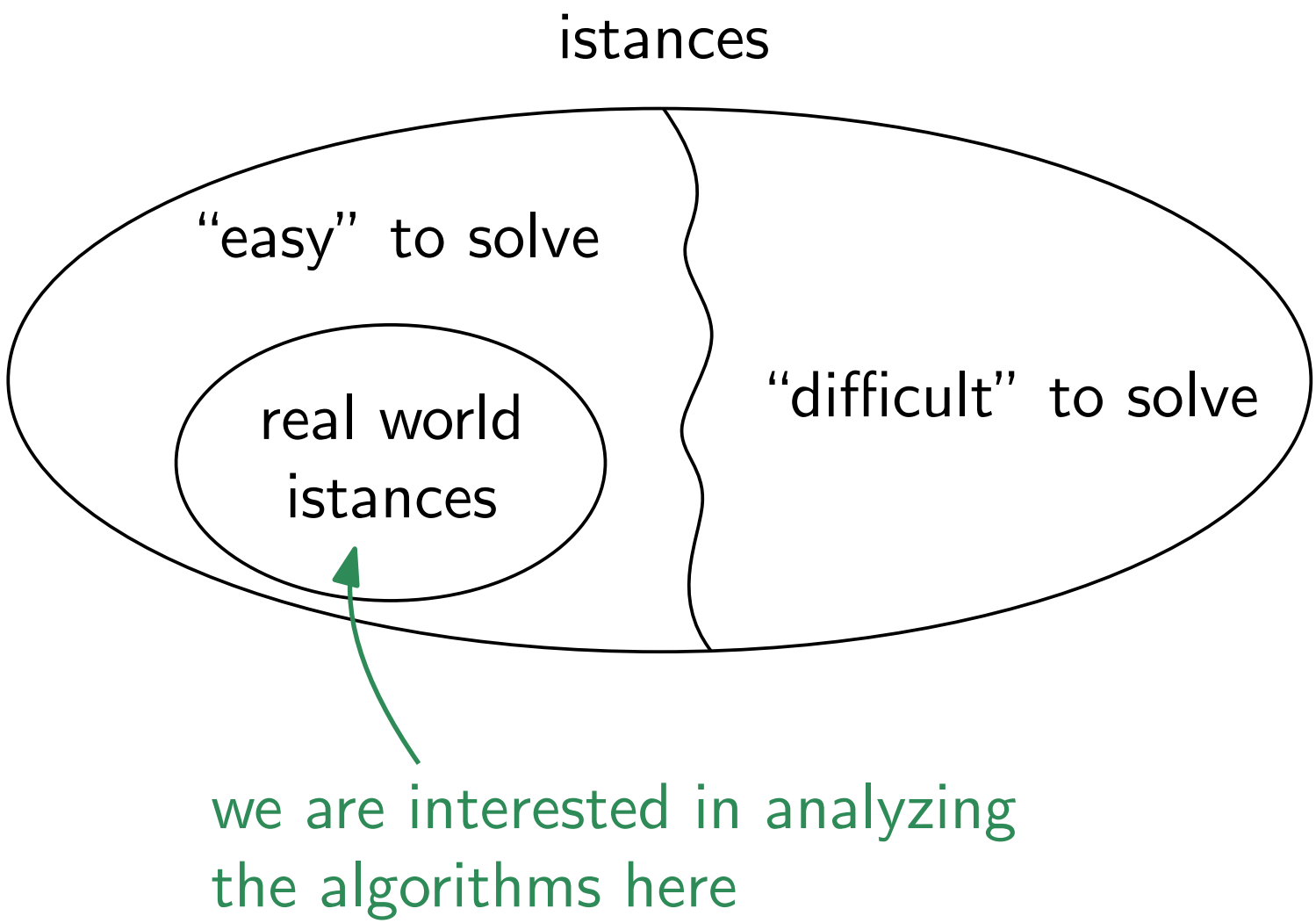

Beyond Worst-Case Analysis
Episode III

Clustering is Hard Only When it Doesn't Matter

Clustering

Input: a set of data.

Clustering

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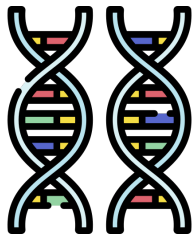
documents

figures



genomes

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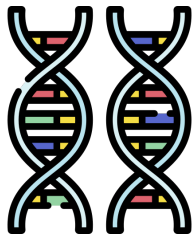

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Goal: group homogeneous data into sets (**clusters**) based on their similarity.

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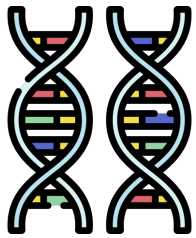

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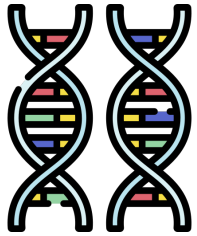

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Cluster 1

Cluster 2

Cluster 3

Clustering

Is it a simple problem?

Clustering

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Clustering

Is it a simple problem?

... and this?

Clustering

Is it a simple problem?

... and what about this ???

Clustering

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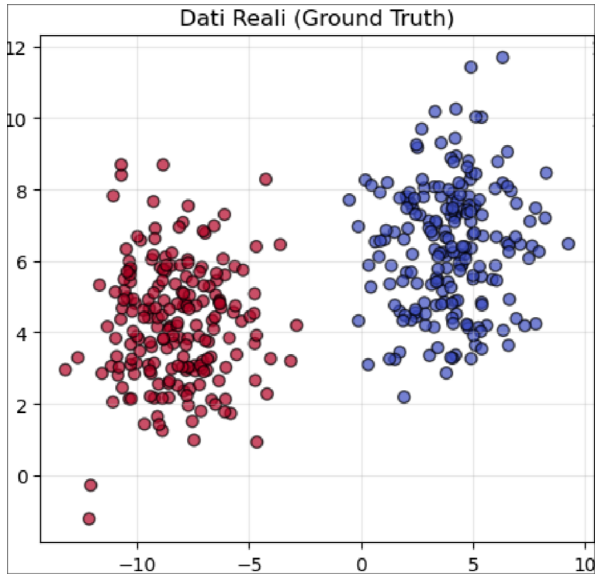

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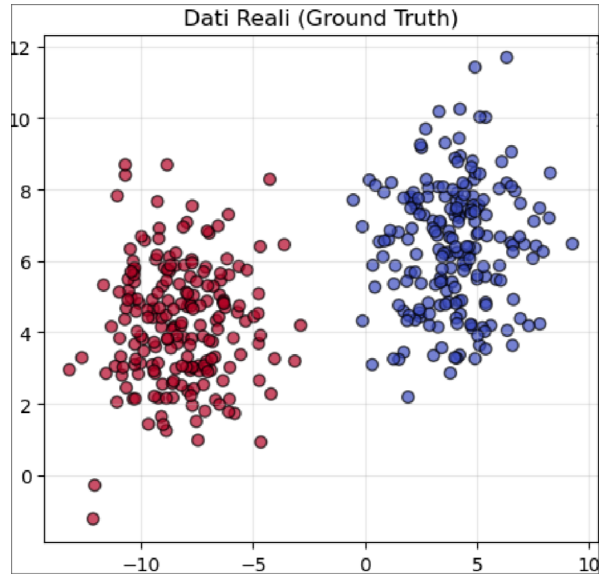

vector representation

Distance function: $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ such that

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

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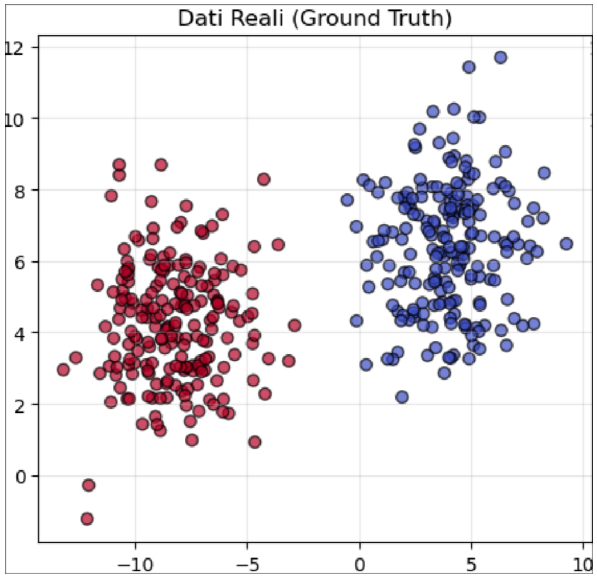

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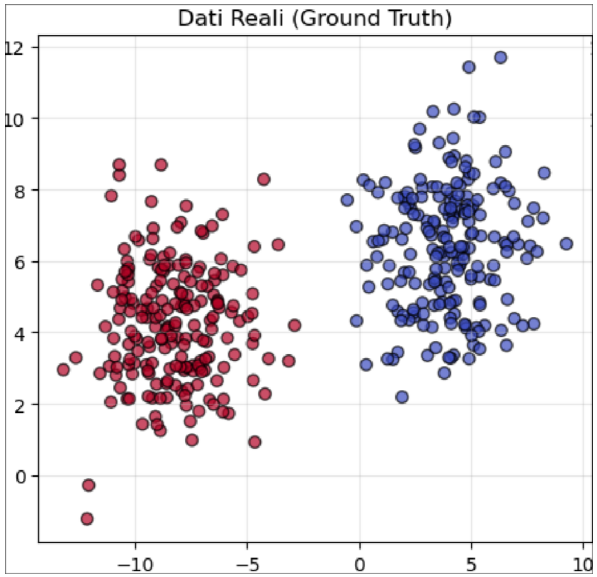

vector representation

How to group similar points?

Through an optimization problem!

k-Median Clustering

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it depends...

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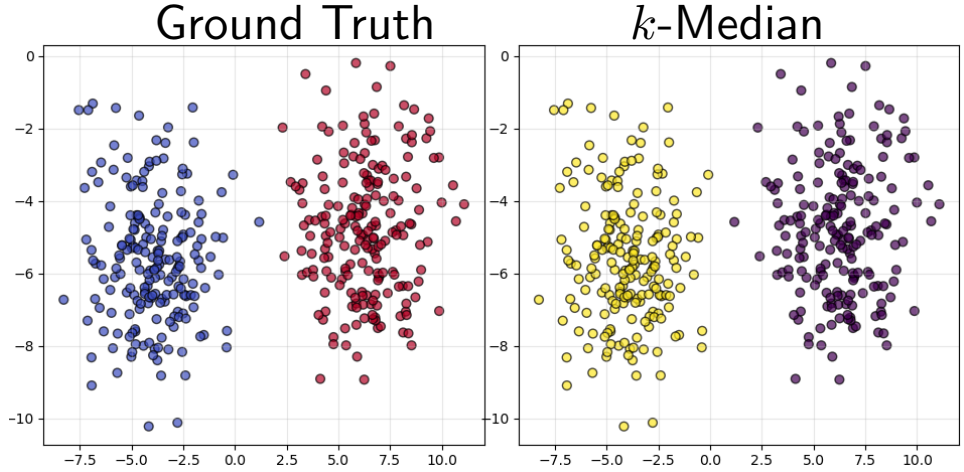

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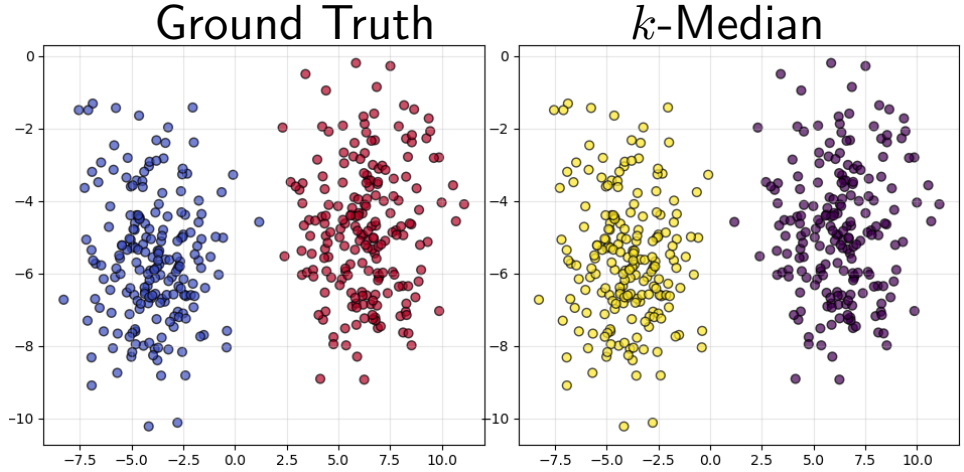

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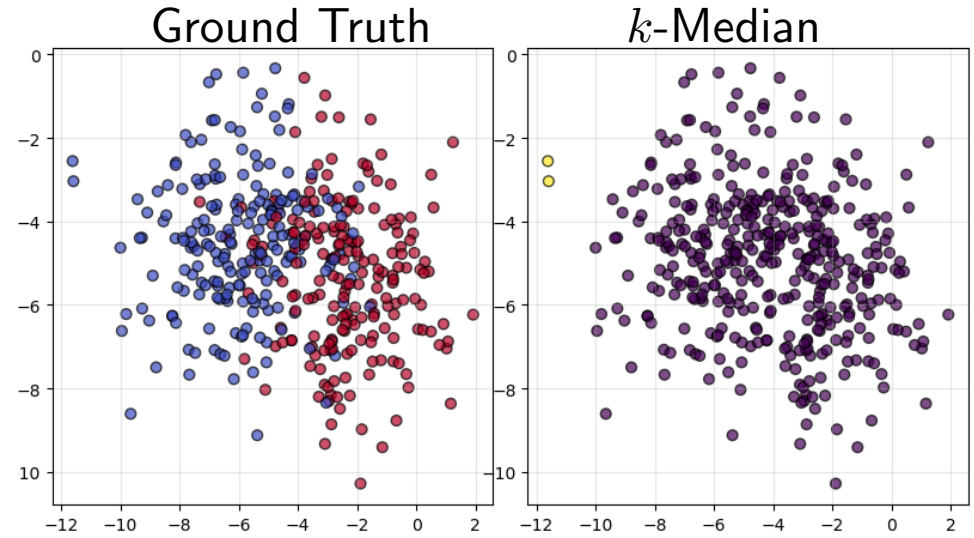

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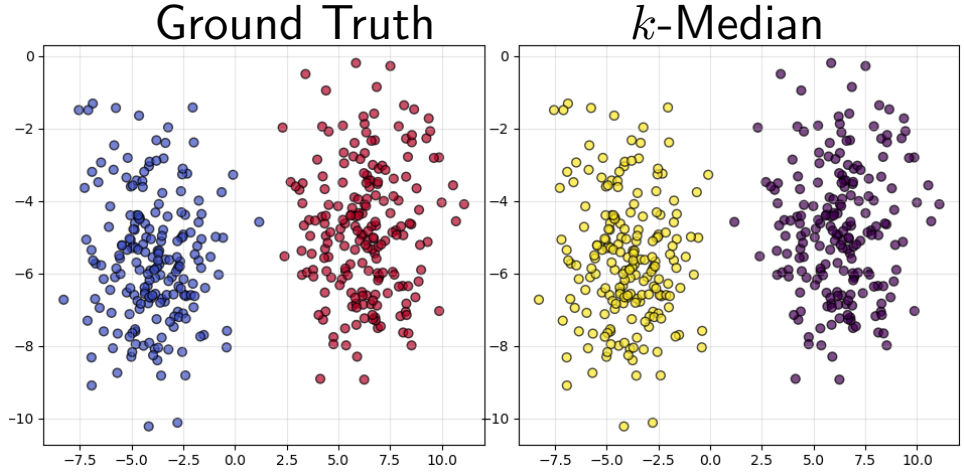

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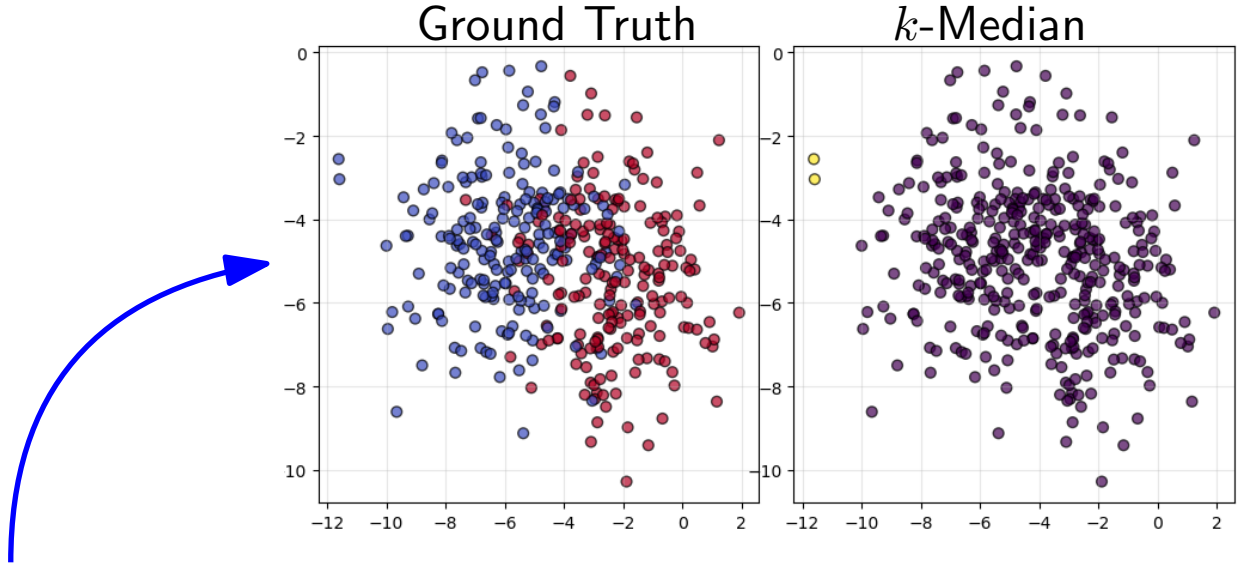

Good news: it's not an interesting case :)

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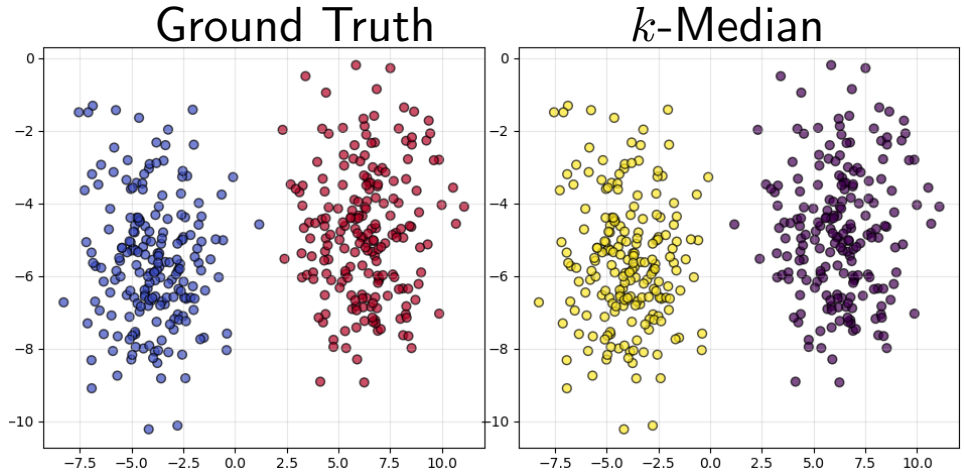

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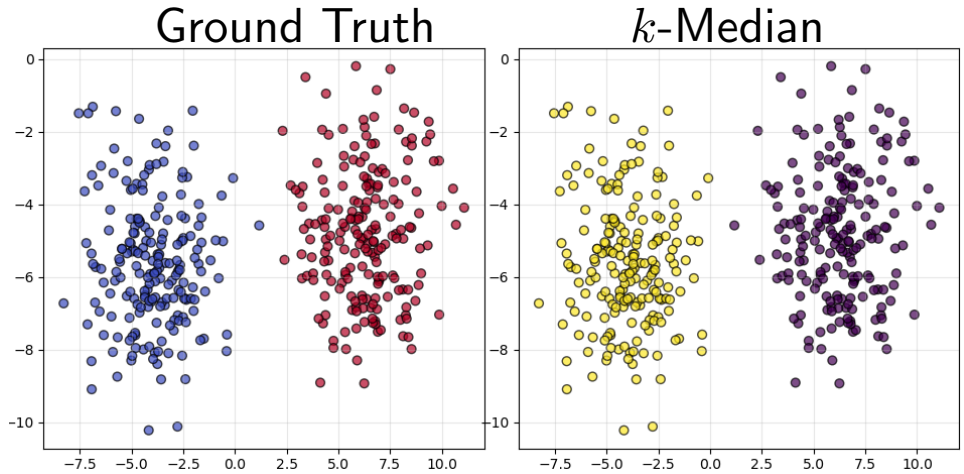

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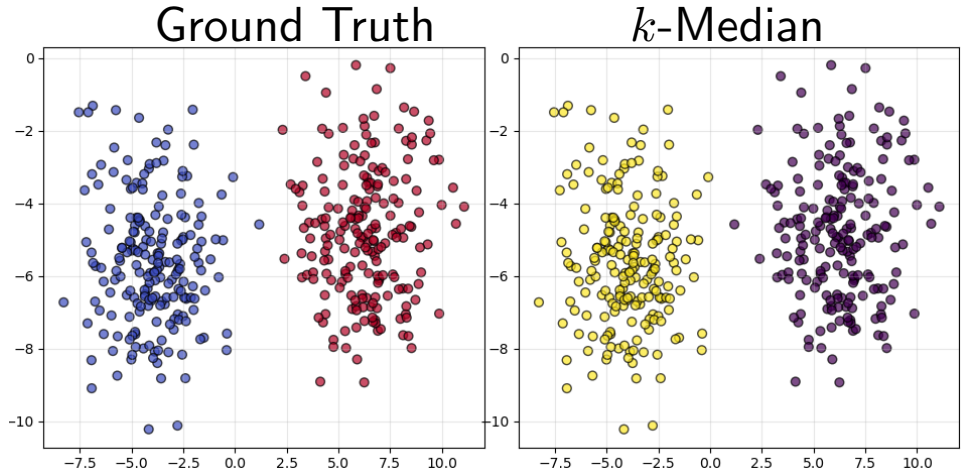

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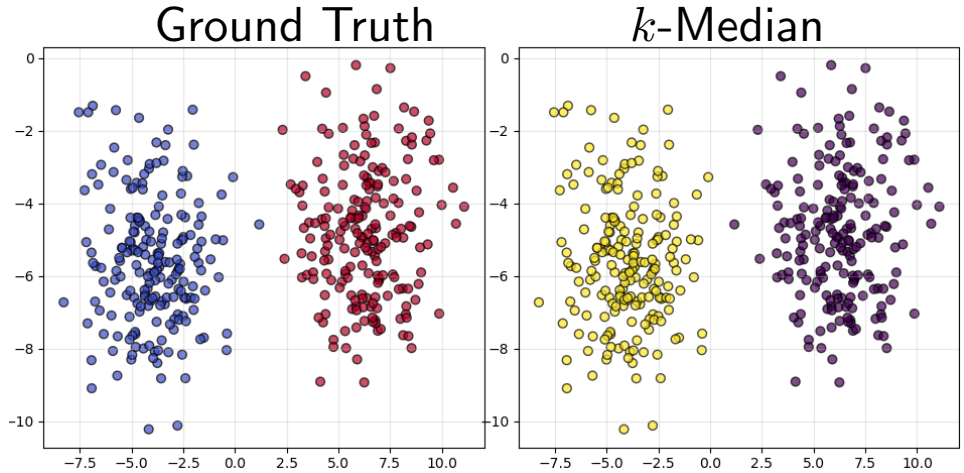

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clustering is hard only when it doesn't matter!

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Def. (γ-perturmation) For $\gamma \geq 1$, a γ-perturbation of a metric space $(\mathcal{M}, d)$ is another metric space $(\mathcal{M}, d')$ such that

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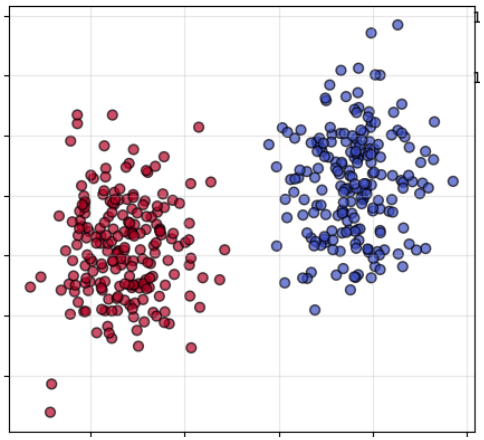
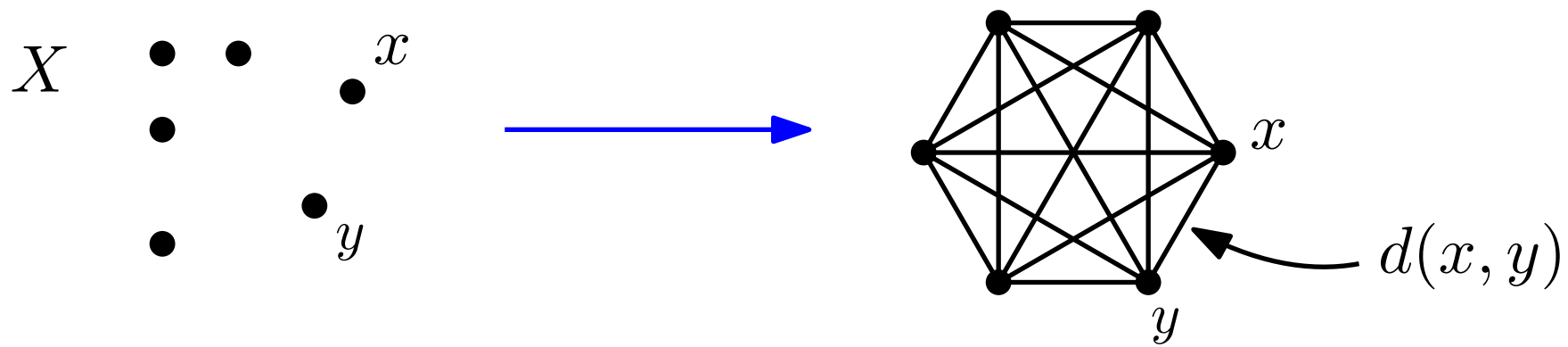
Intuition: the clusters are well separated, so even changing the metric (provided it is not too different) the solution S^* remains optimal (and unique).

interesting istance



Single-Link Algorithm

Setp 1. Think the input (X, d) as a complete graph with vertices X and edge weights given by d.



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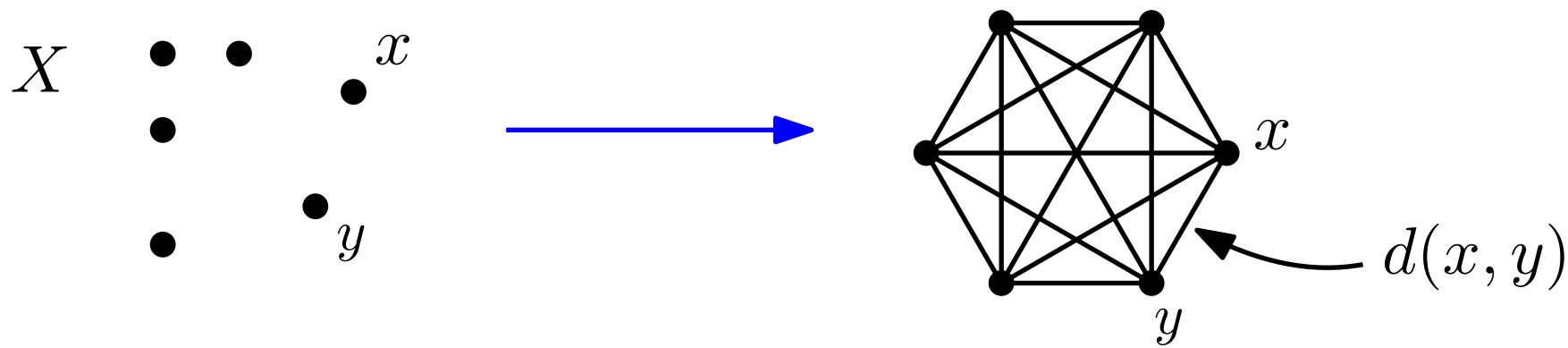
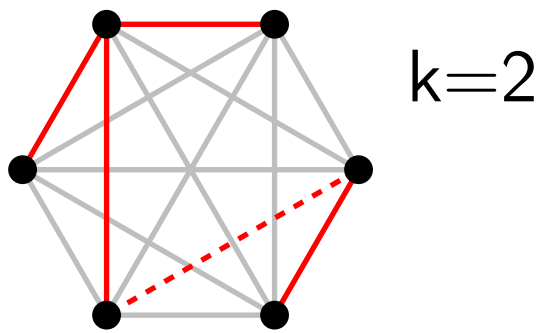


Step 2. runs Kruskal's minimum spanning tree (MST) algorithm, but stop when there are k connected components. That is, we skip the final $k-1$ edge additions of Kruskal's algorithm.



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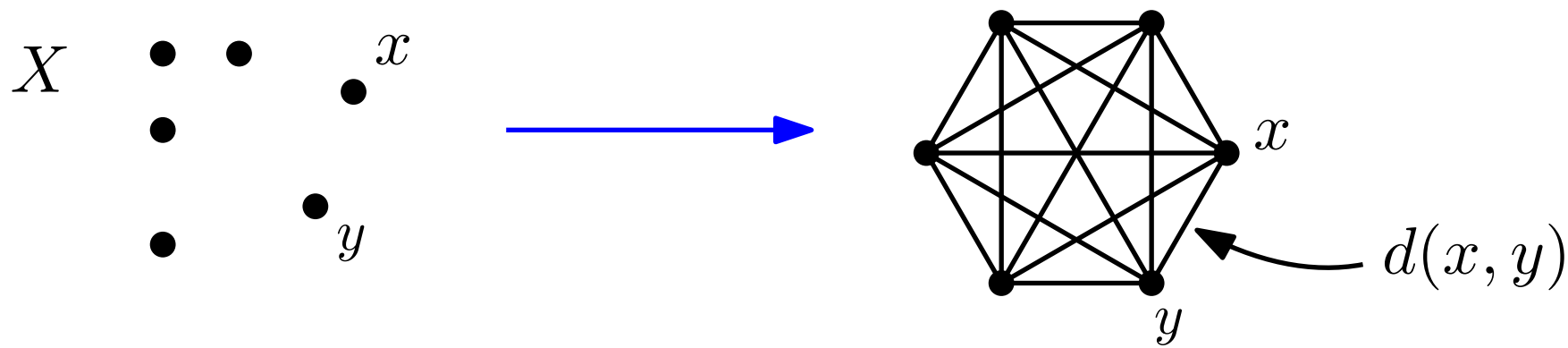

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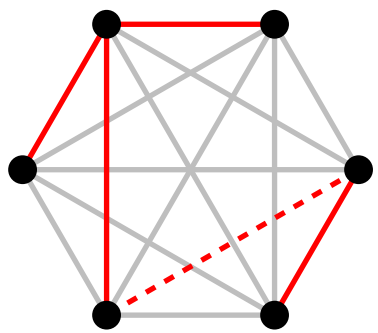
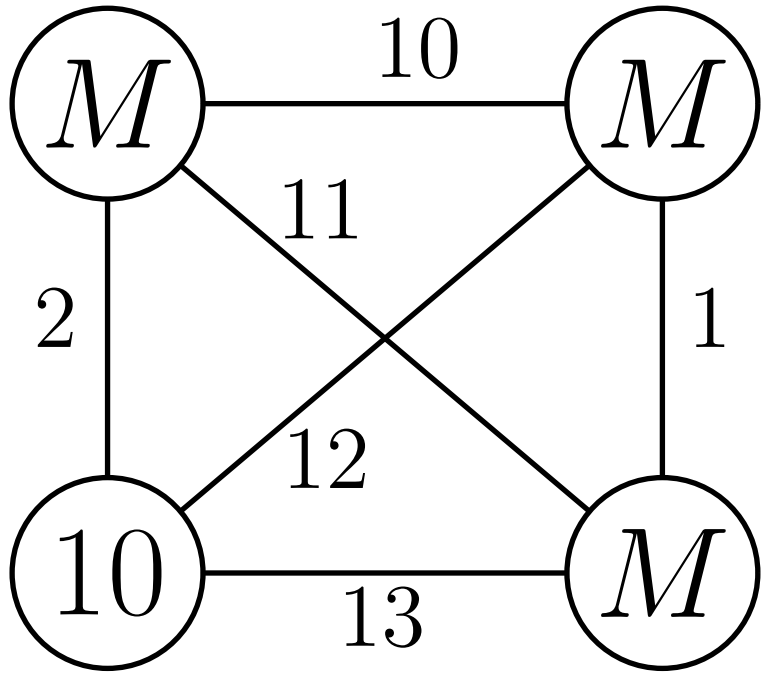


Does it give a goos solution?

Counterexample

$k = 3, M \gg 10$

The circle represent M (or 10) co-located points



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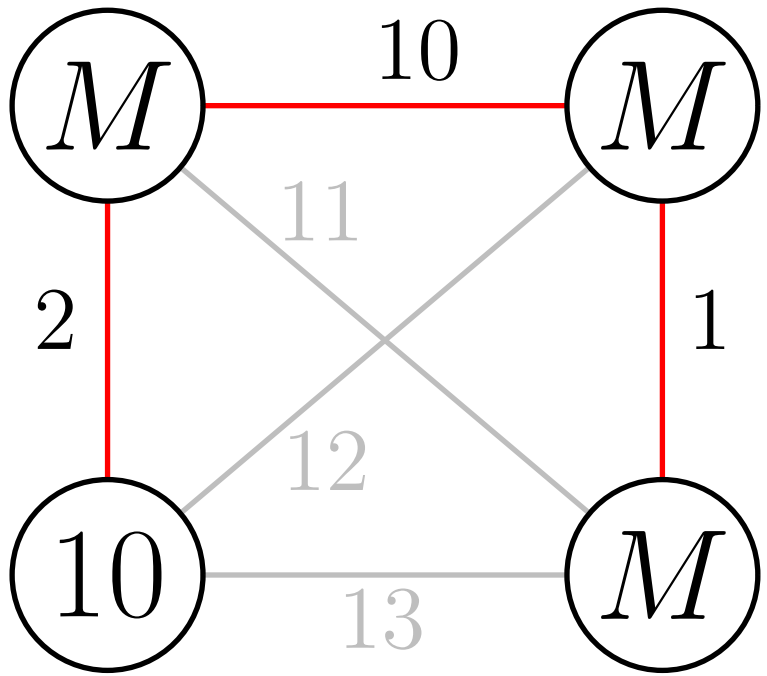

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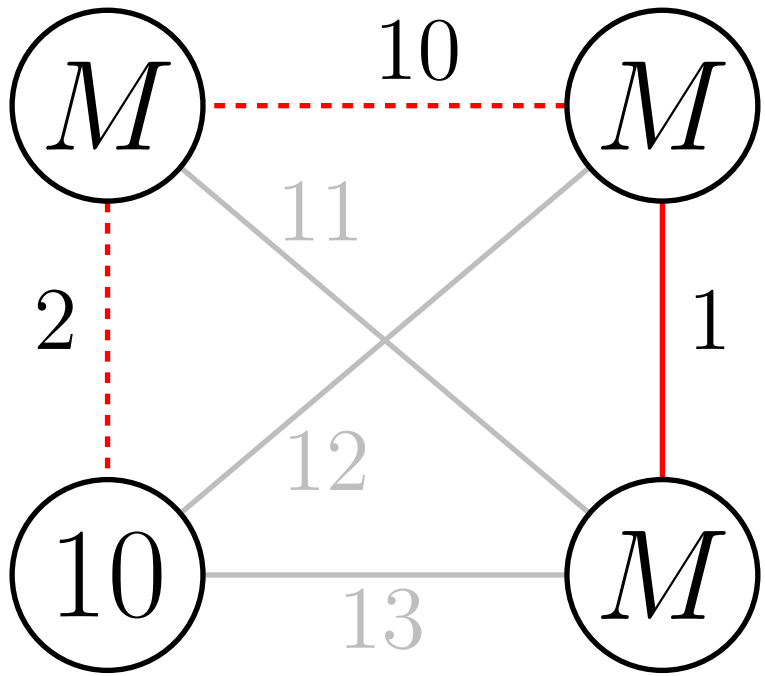

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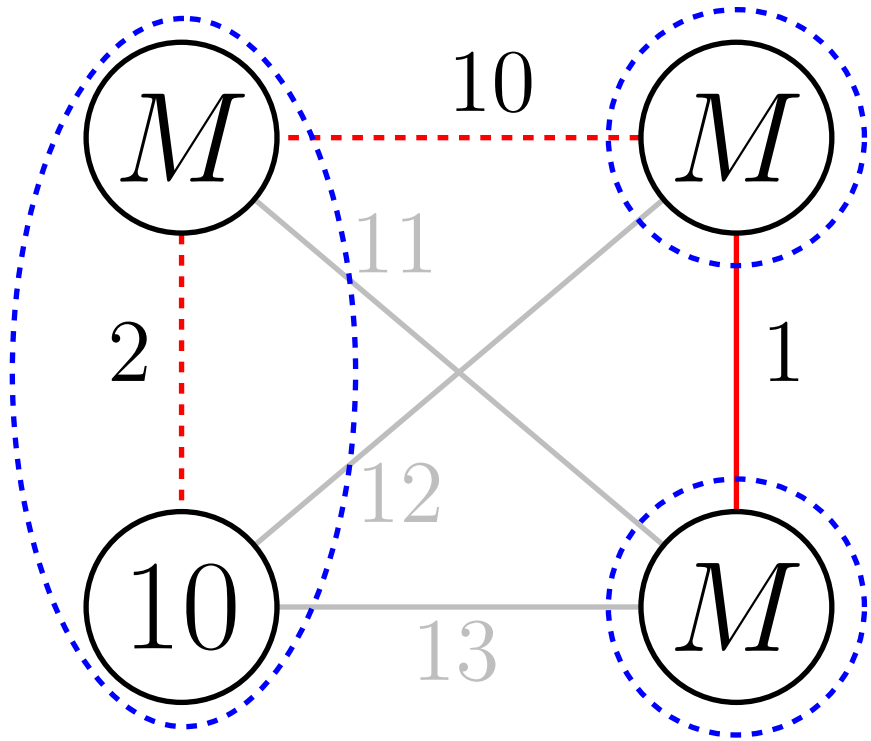
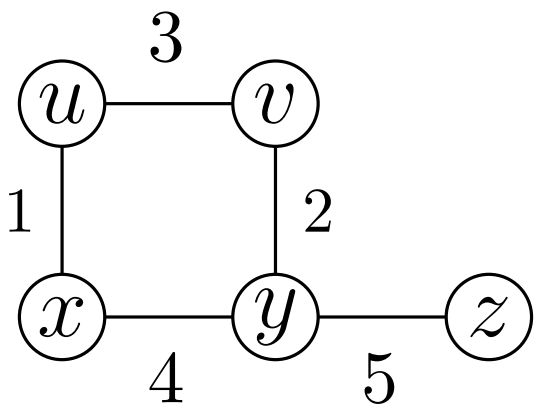


optimal solution

Hierarchical Cluster Tree



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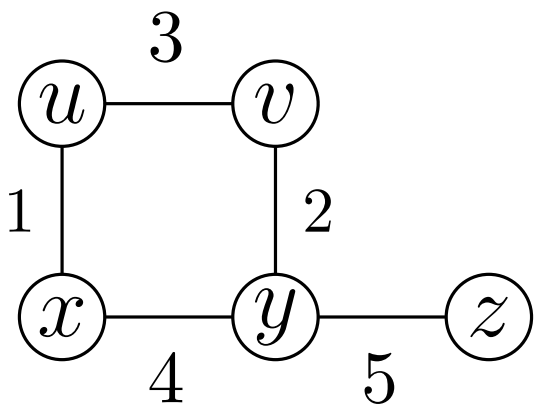

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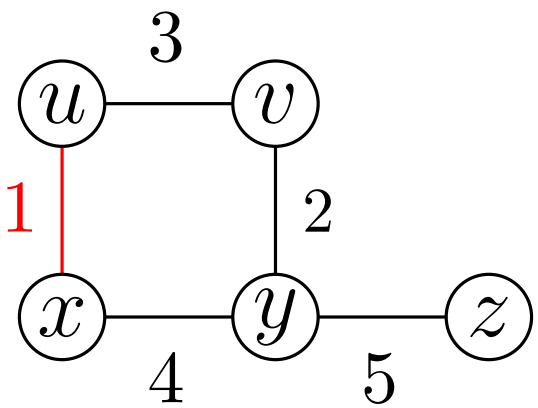
H

Hierarchical Cluster Tree



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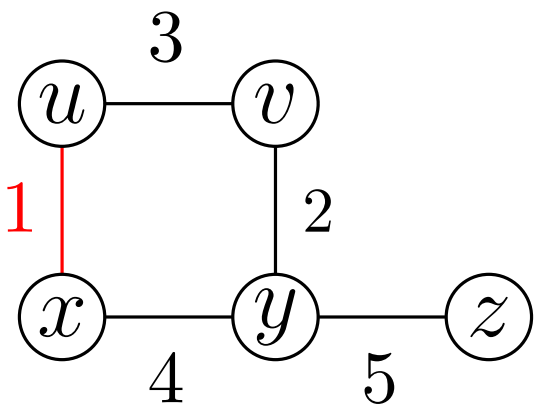

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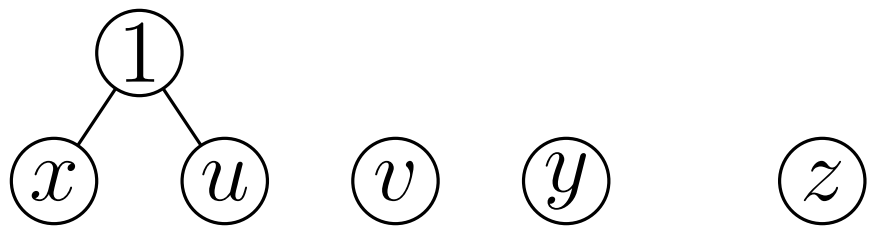

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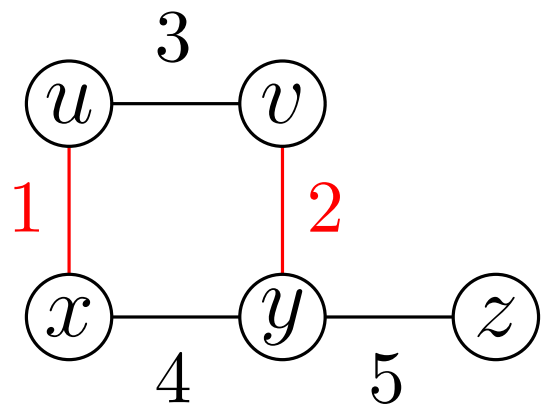

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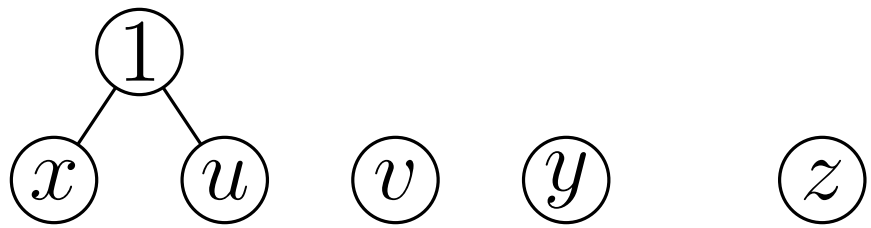

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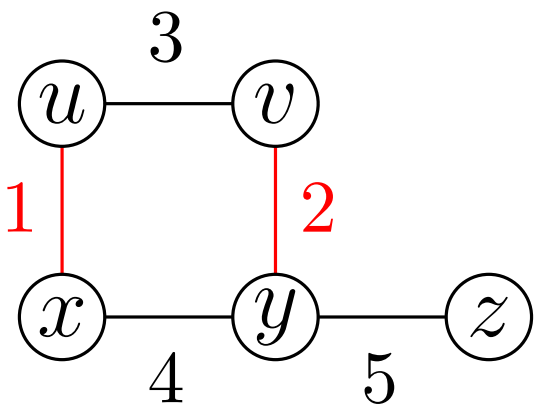

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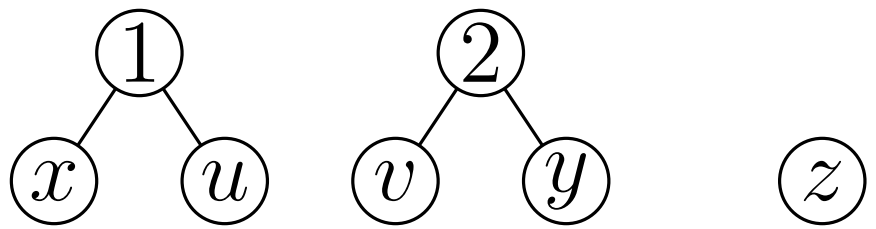

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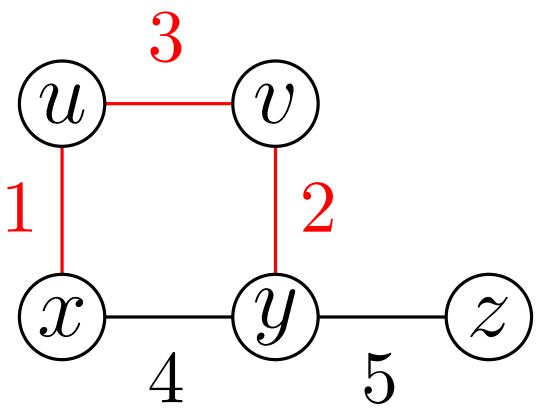

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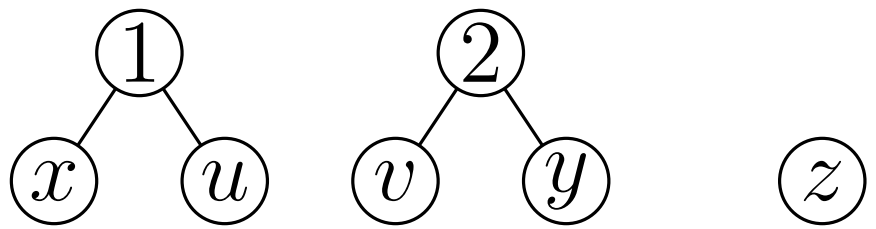

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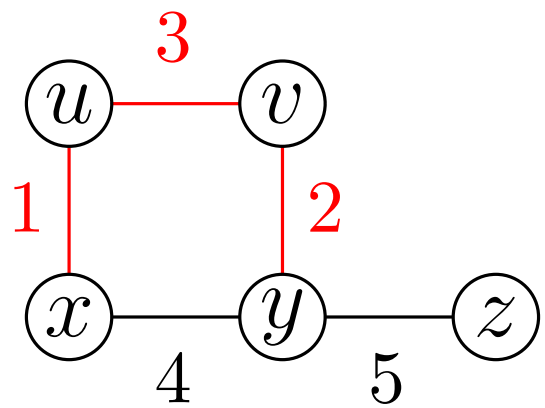

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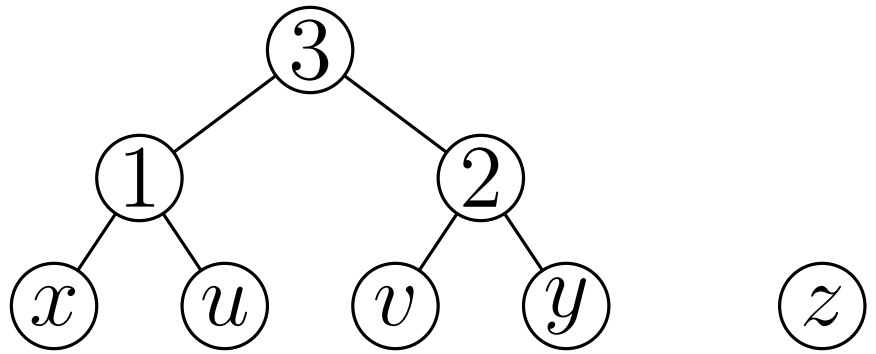

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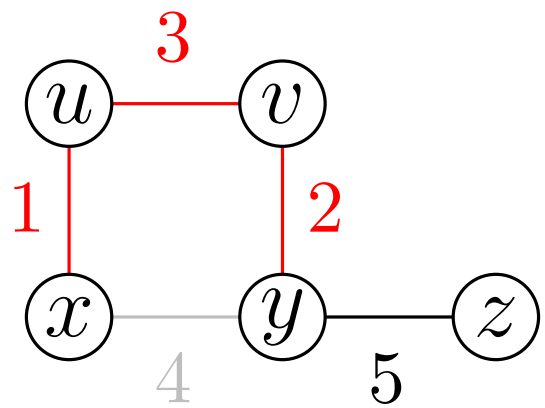

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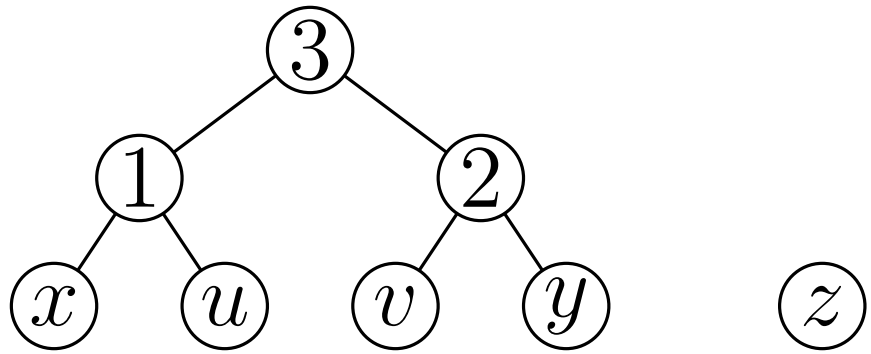

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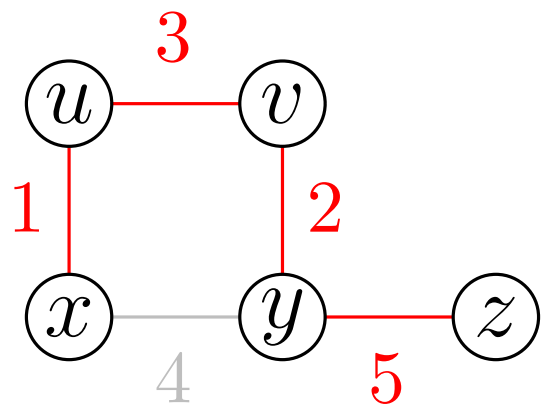

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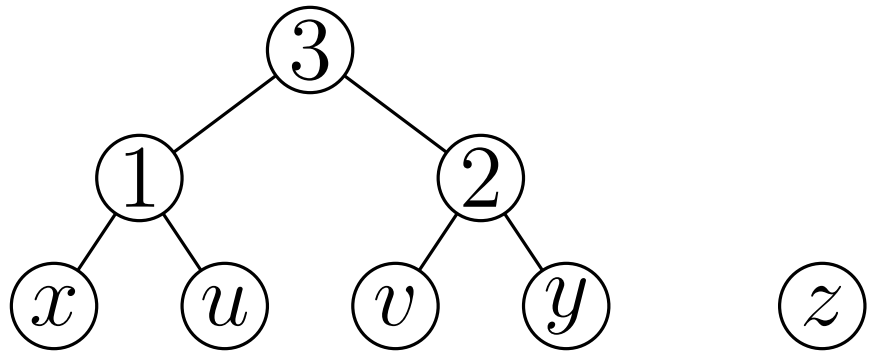

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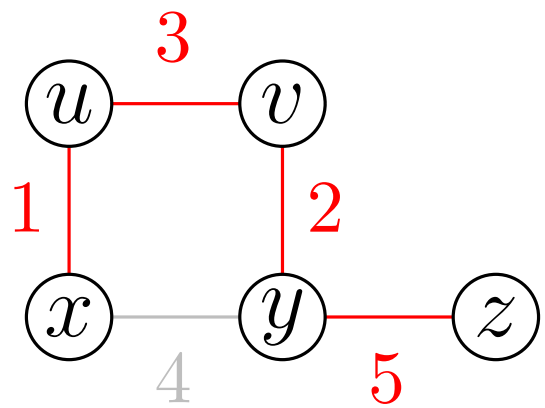

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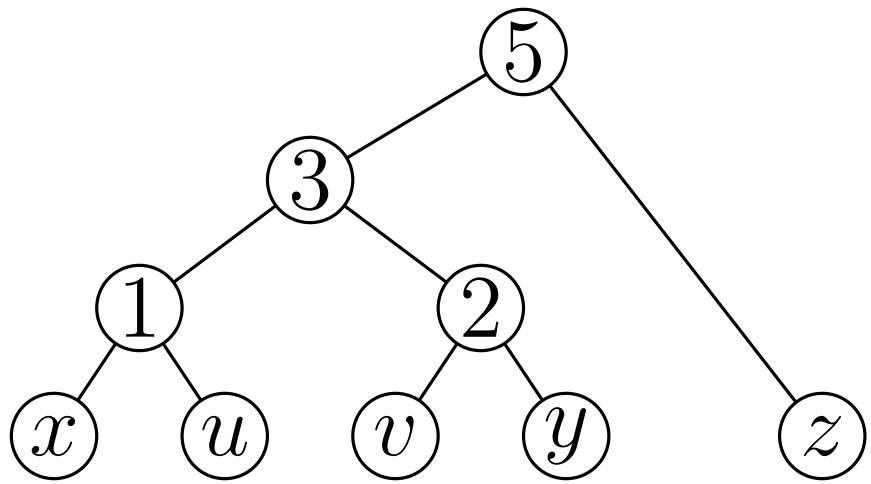

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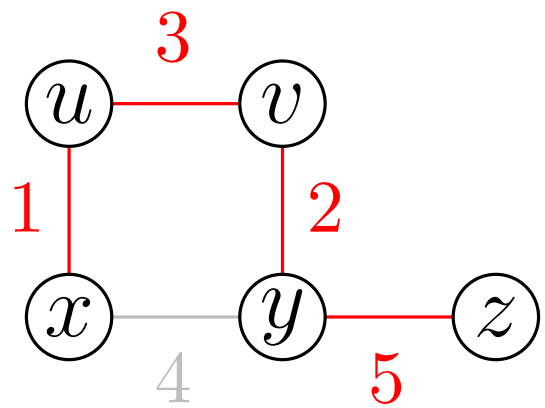
Property 1: Each of the $n-1$ edges added by Kruskal's algorithm corresponds to an internal node of H.



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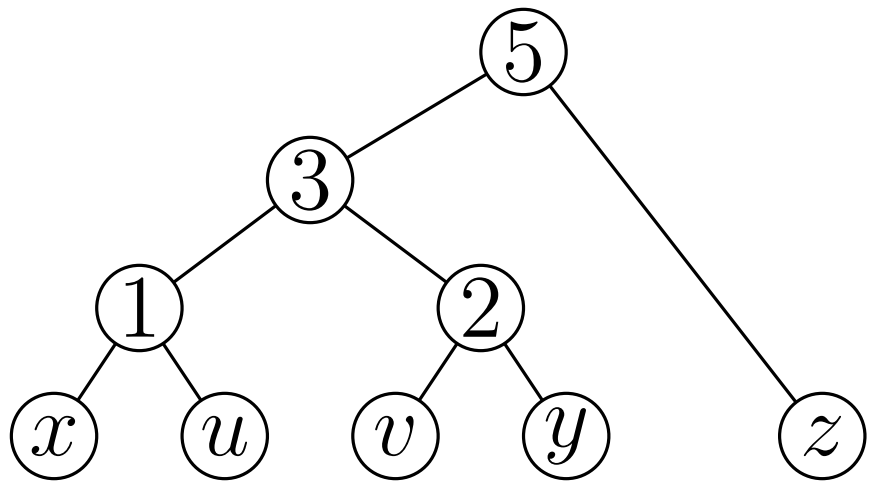

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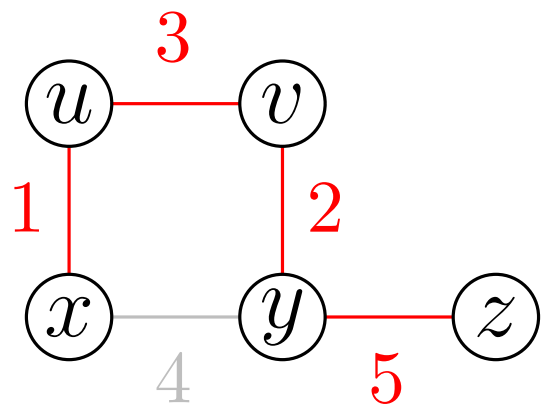

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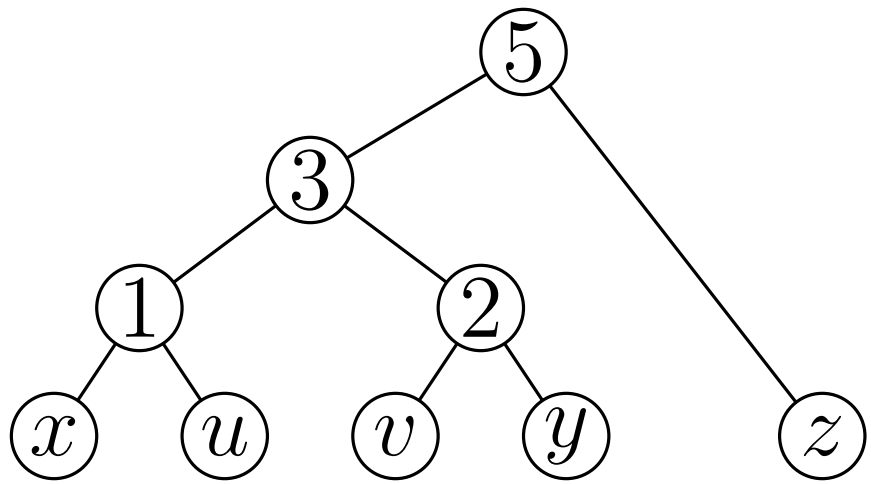

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Where Does Single-Link Fail?

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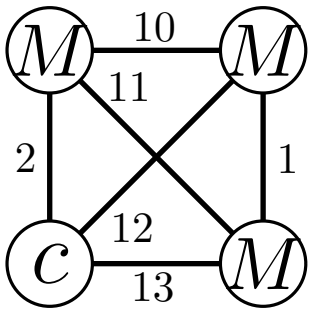
$c \ll M$



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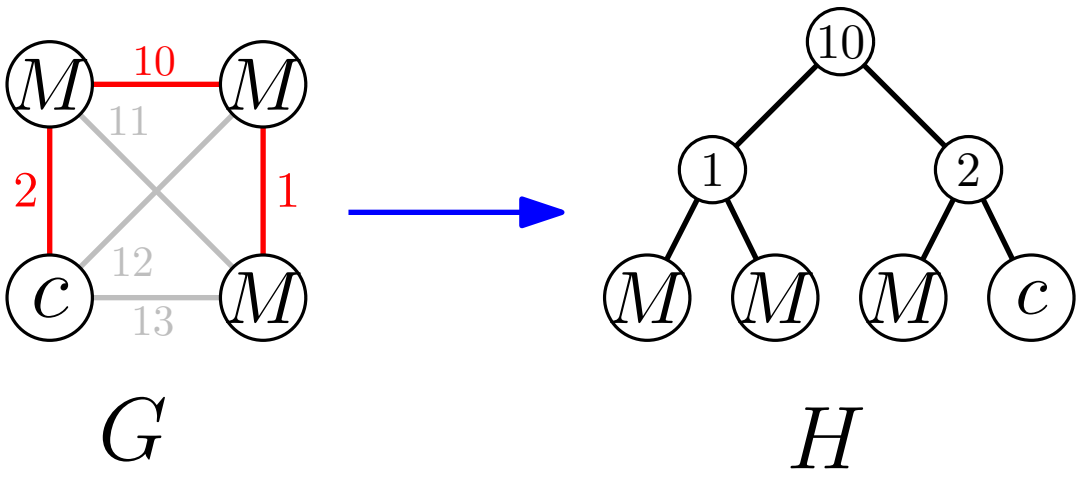

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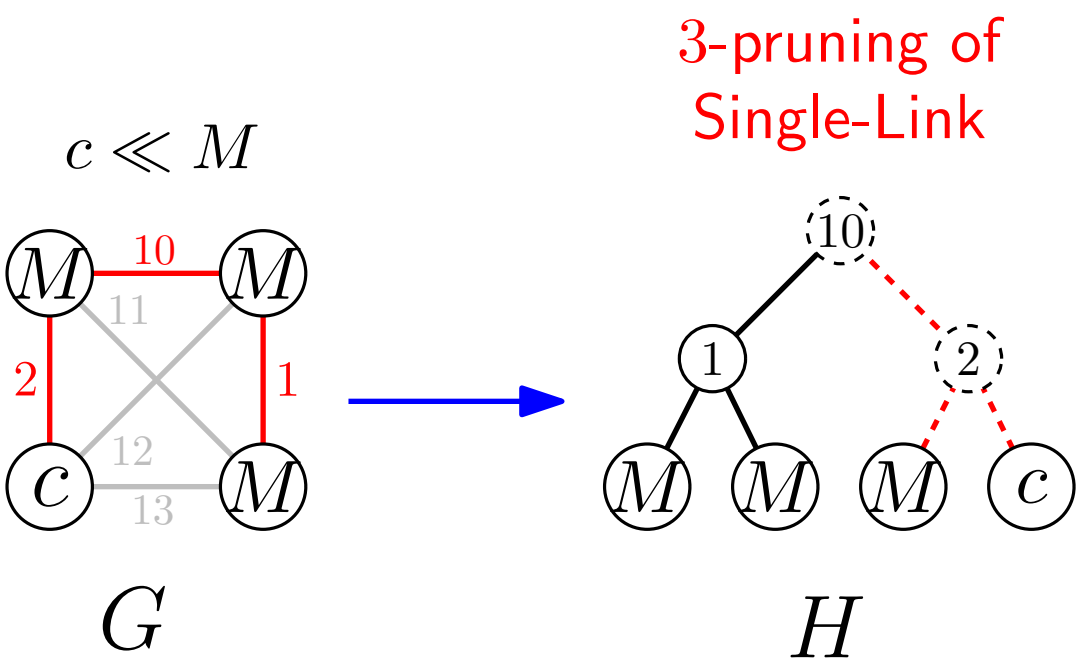

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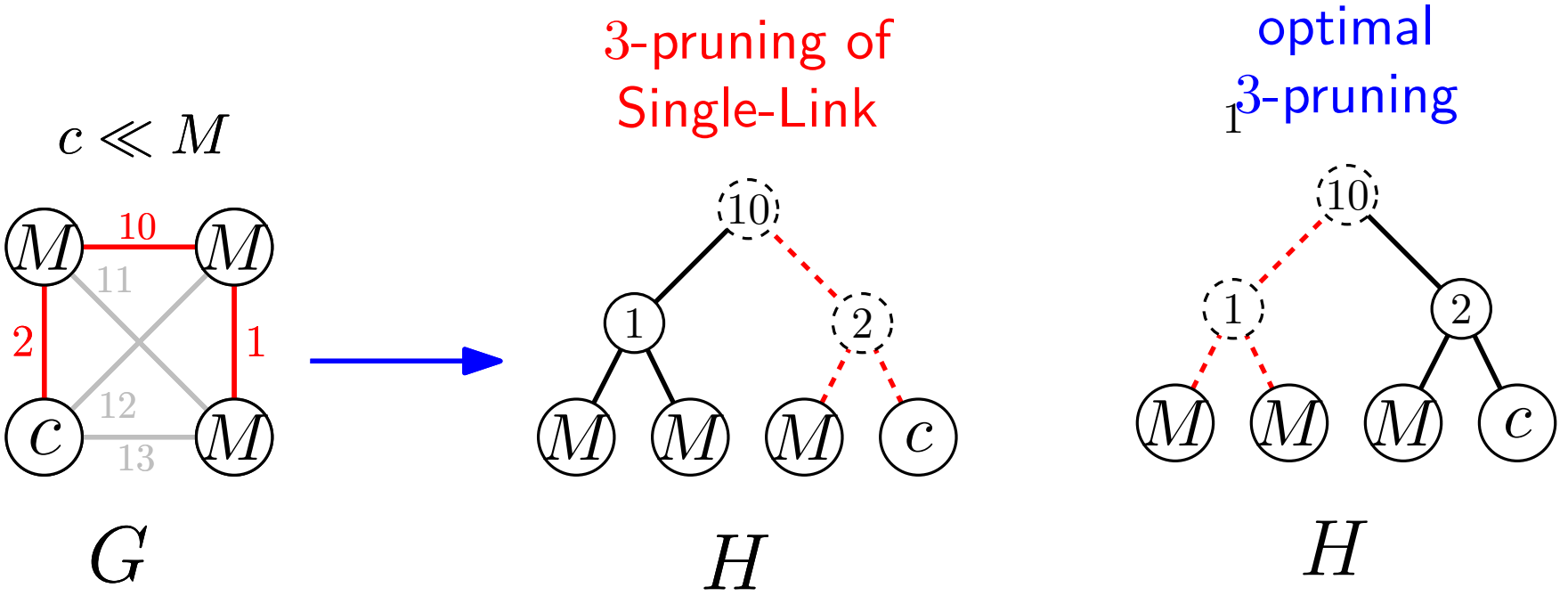

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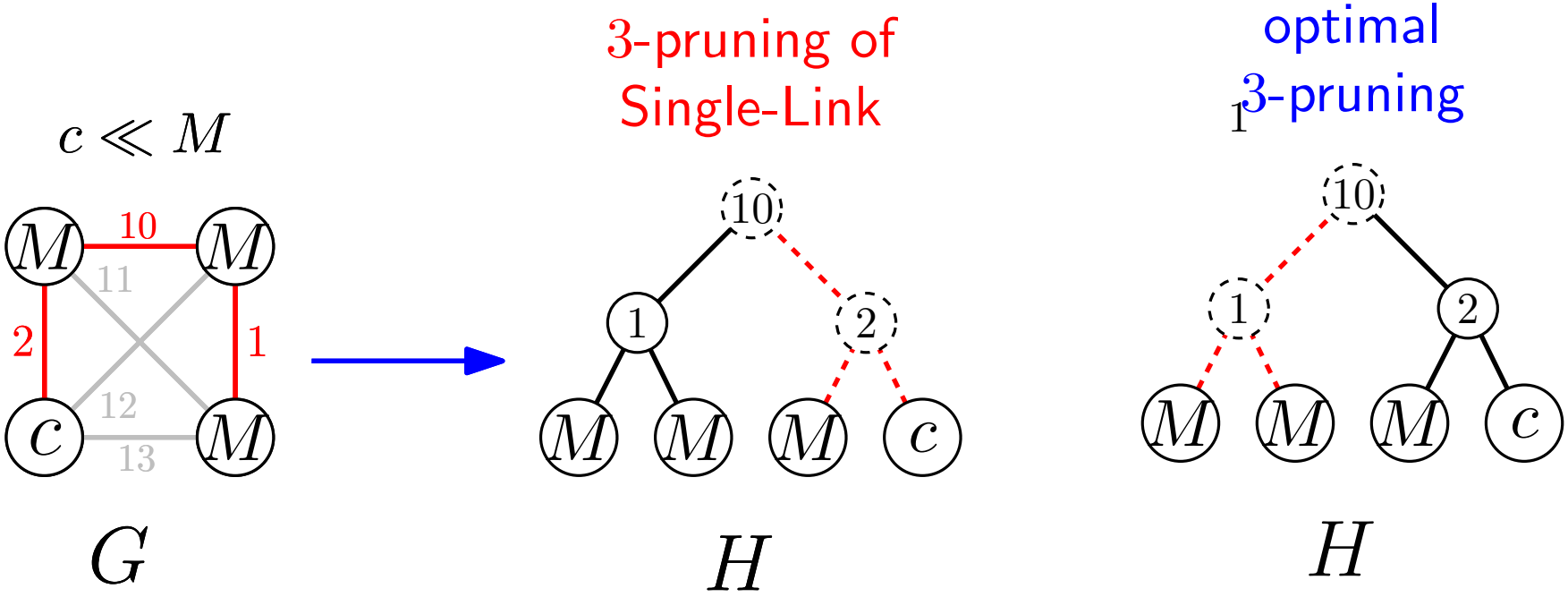

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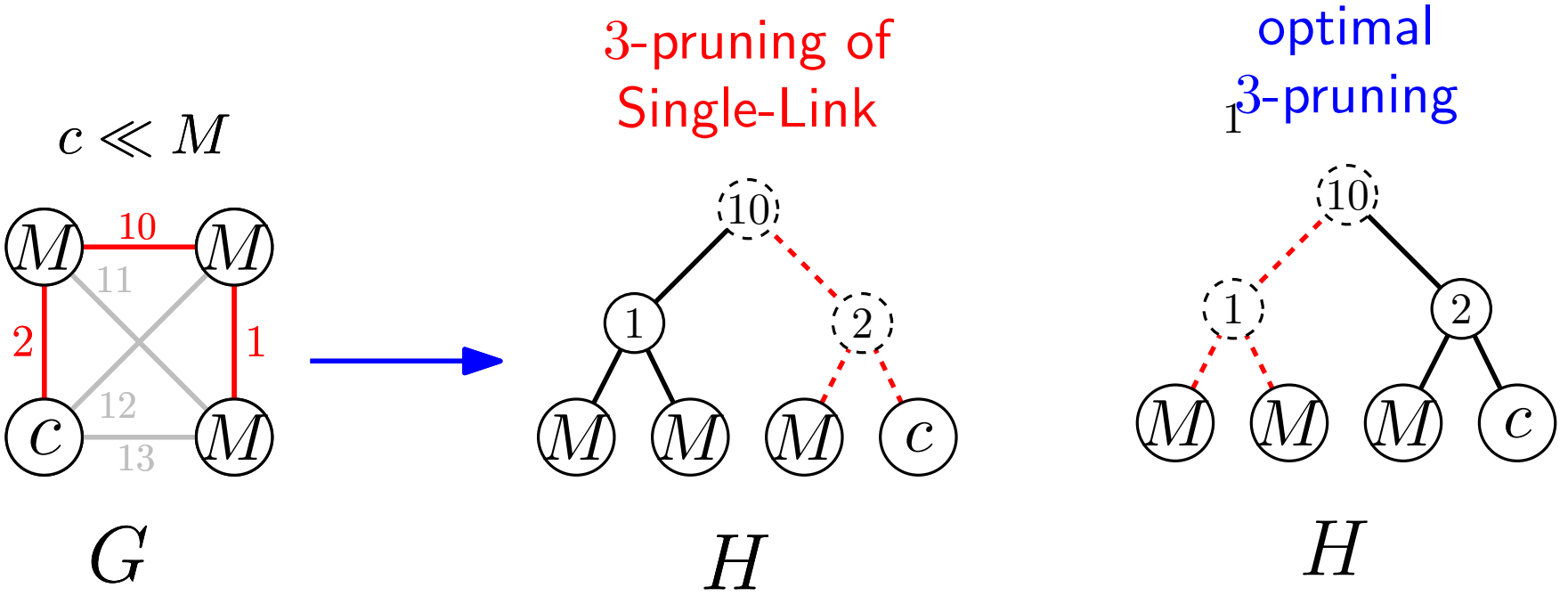

Good news: it can be done in polynmial time vial **dynamic programming**! (exercise)

Single-Link++ (SL++)

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Analysis

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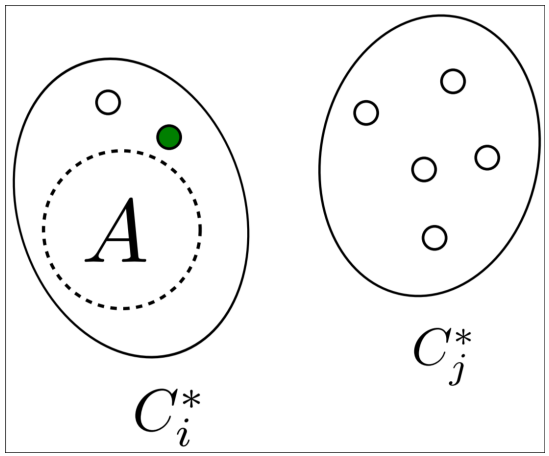
(a) $\implies$ SL++ computes the optimal solution

(b) for every i, and every strict subset $A \subset C_i^*$, the point $x^* = \arg\max_{x \in X \setminus A} d(x, A)$ of $X \setminus A$ closest to A lies in C_i^*



(b) $\implies$ *(a)* $\implies$ SL++ computes the optimal solution

$(b) \implies (a)$

Assume (b) is true.

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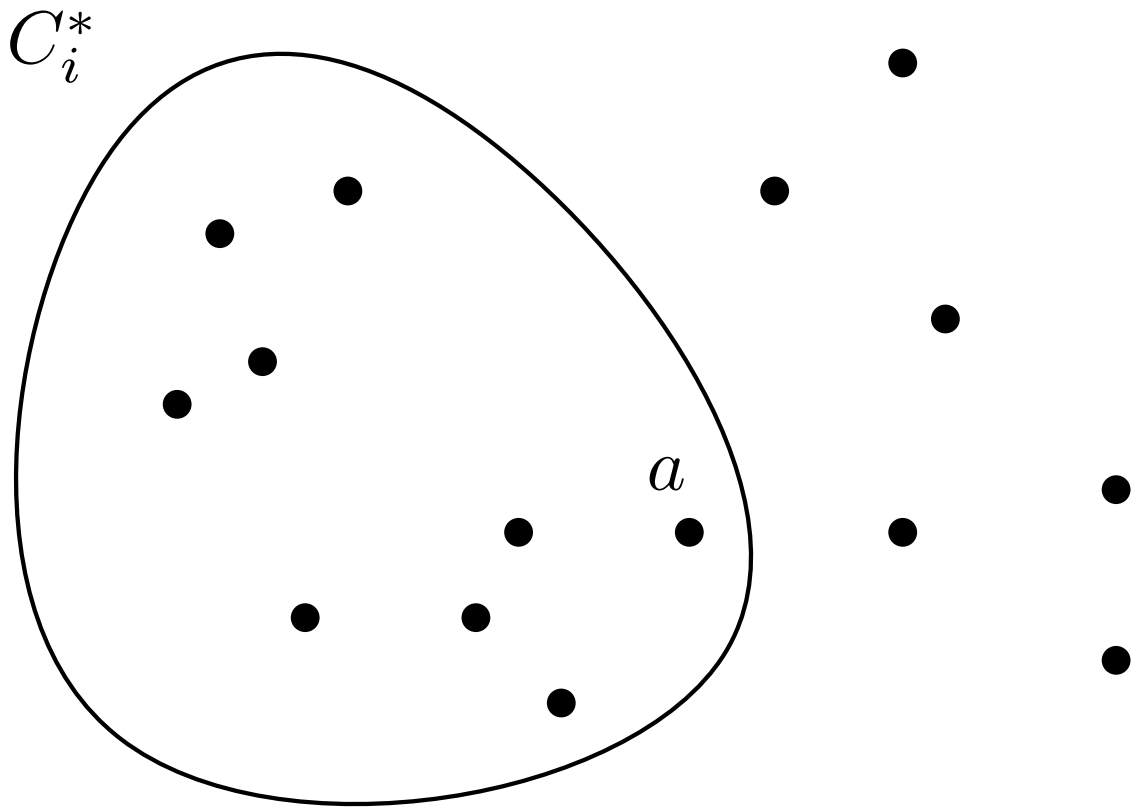

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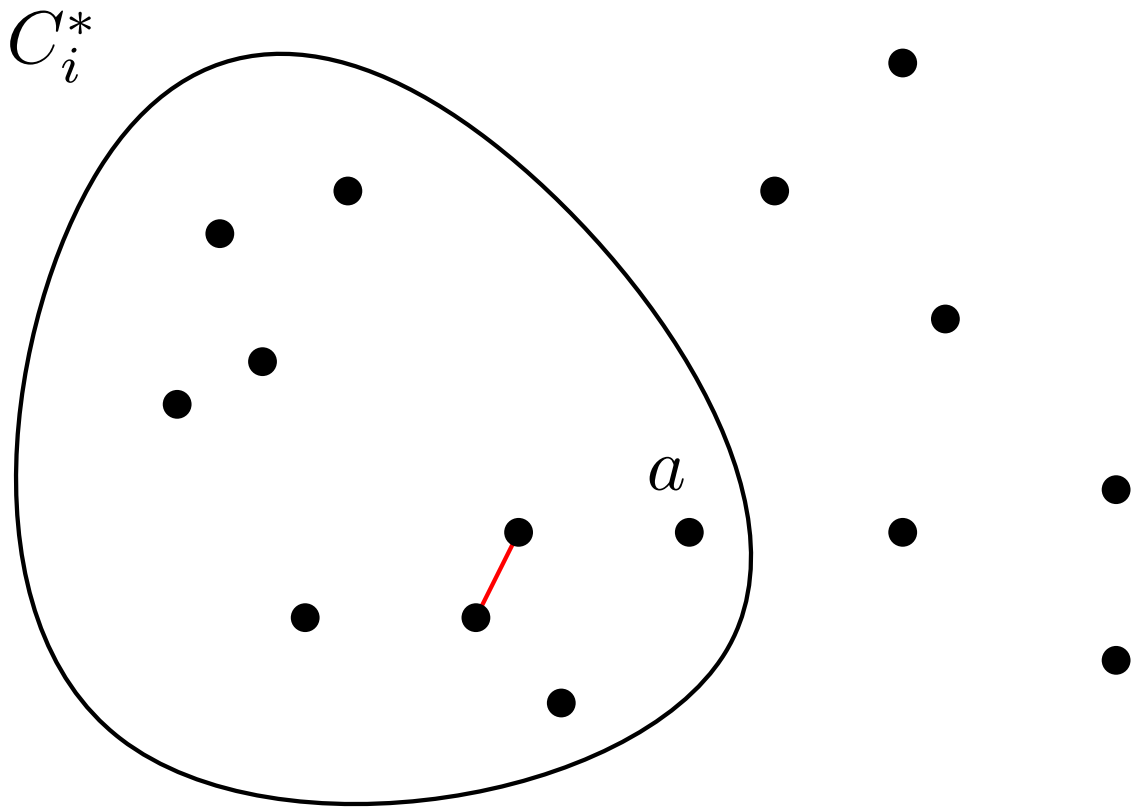

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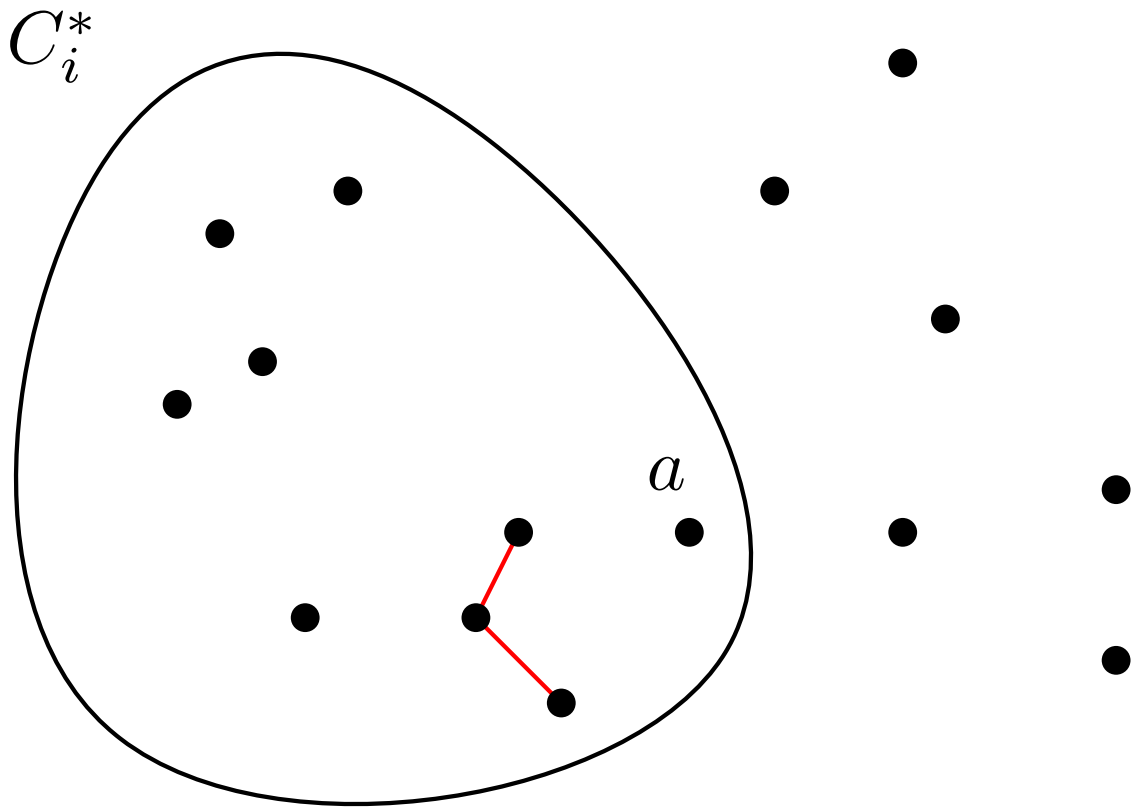

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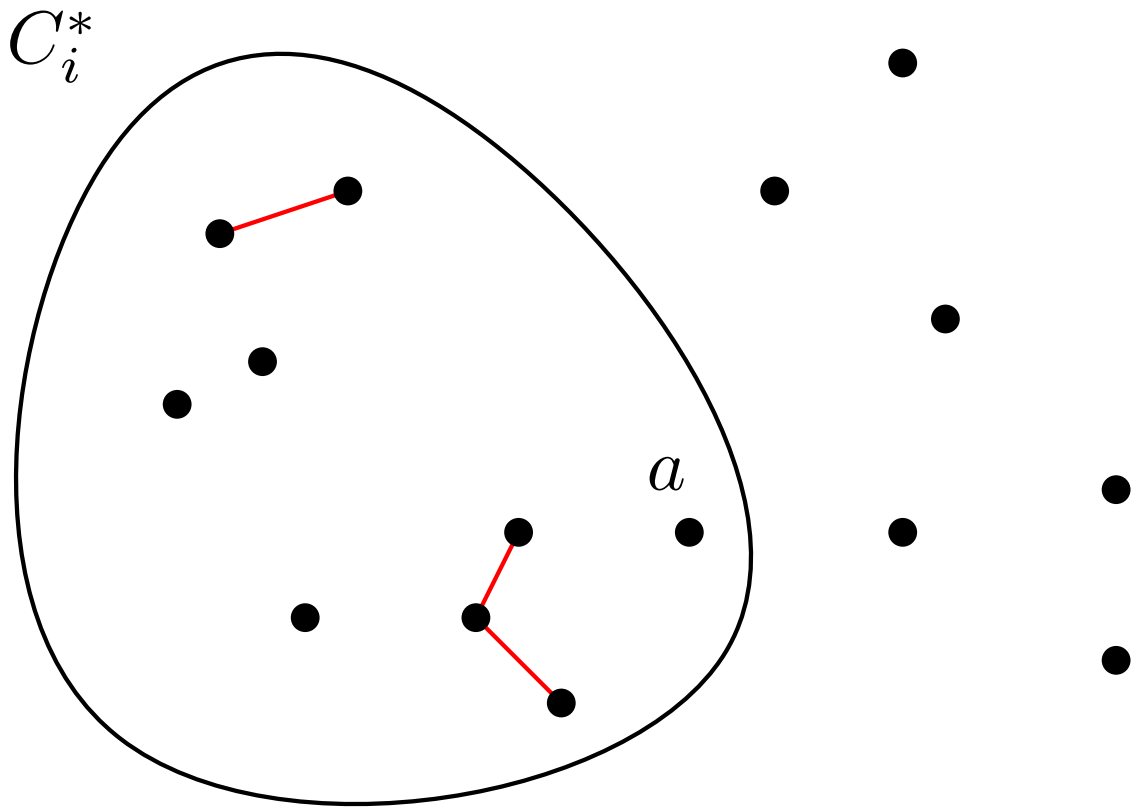

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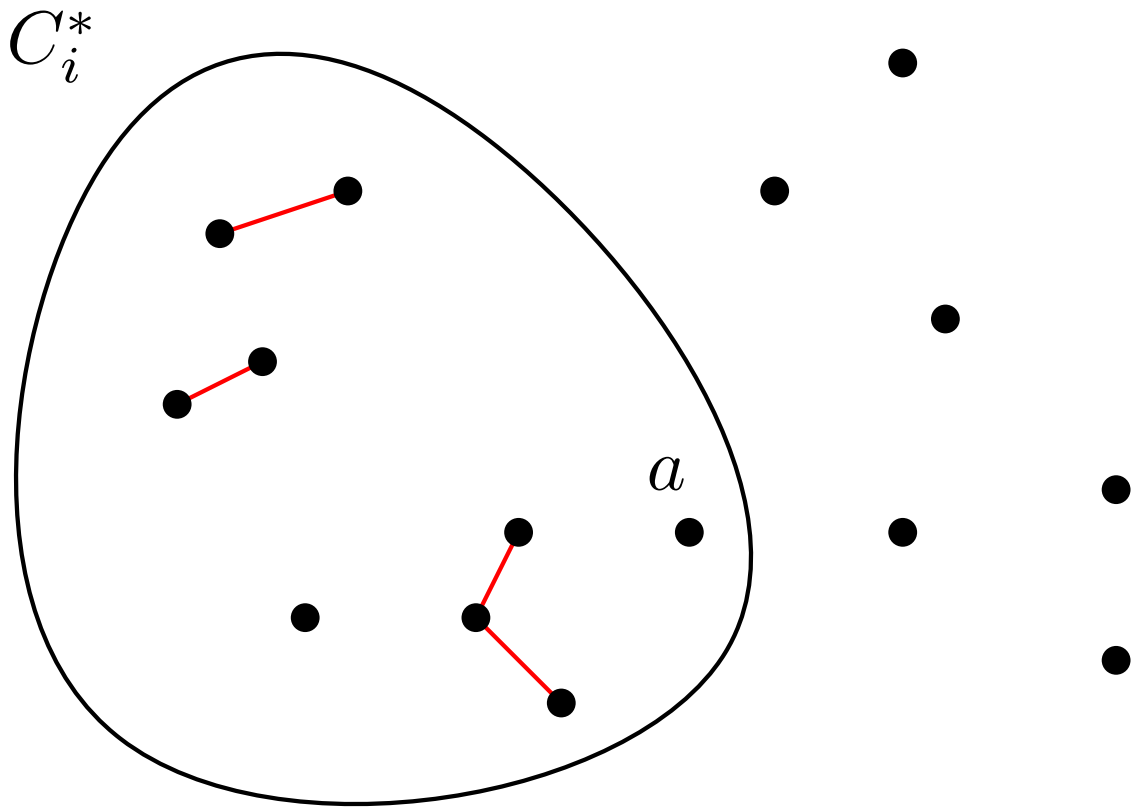

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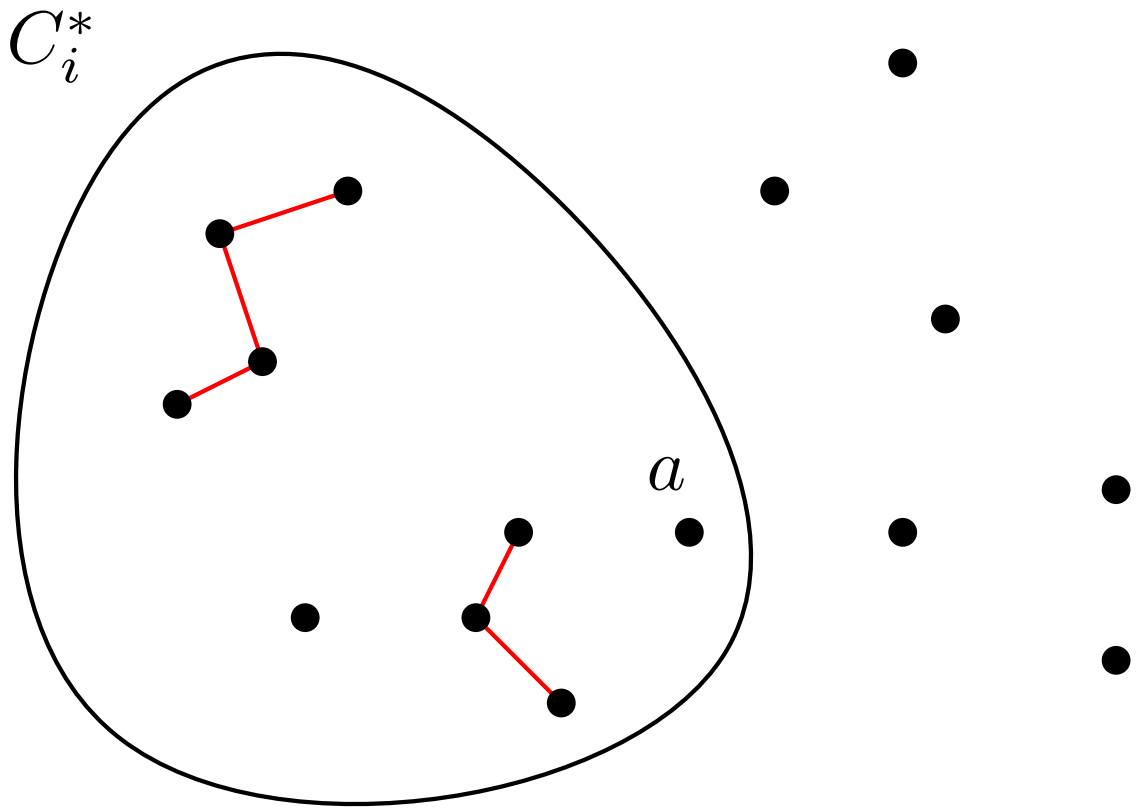

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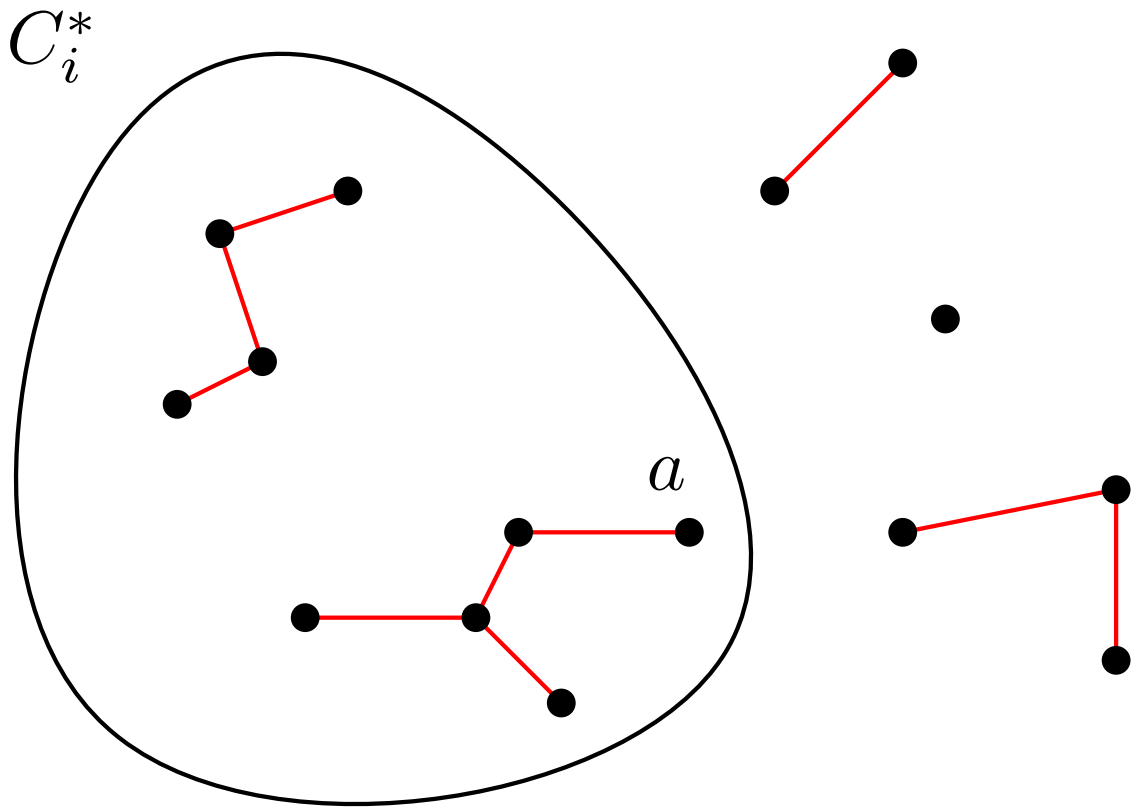

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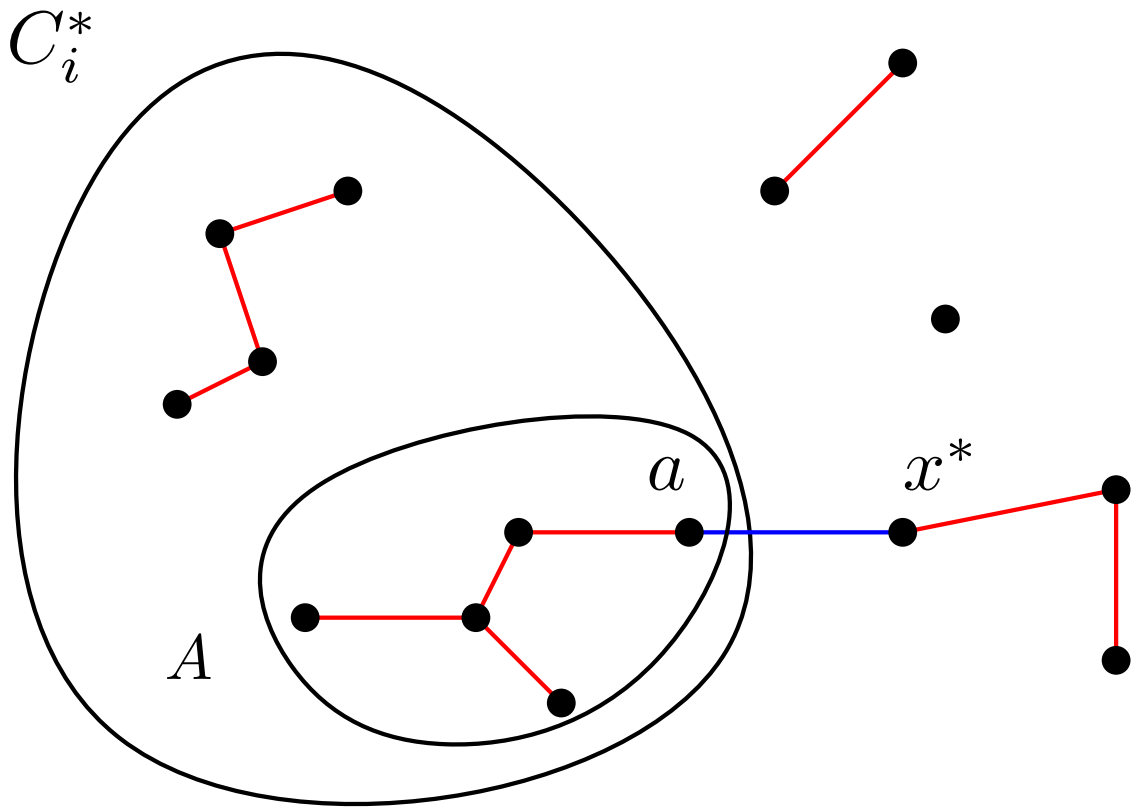

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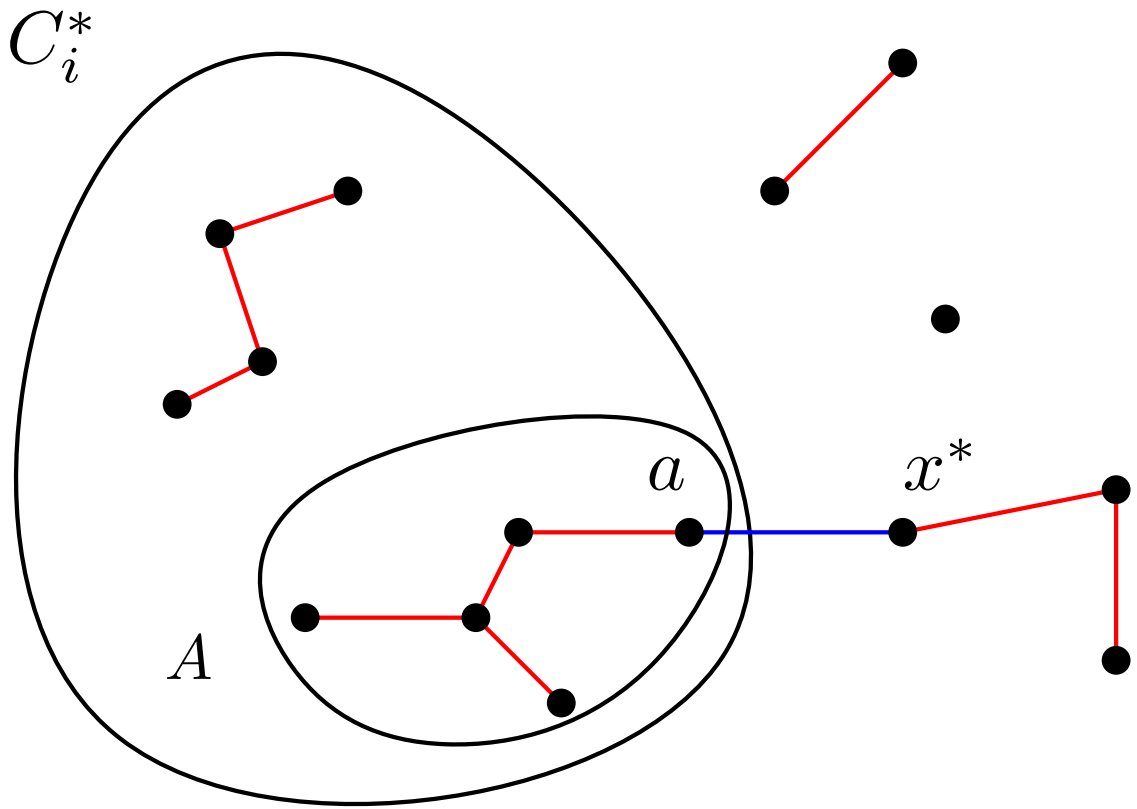

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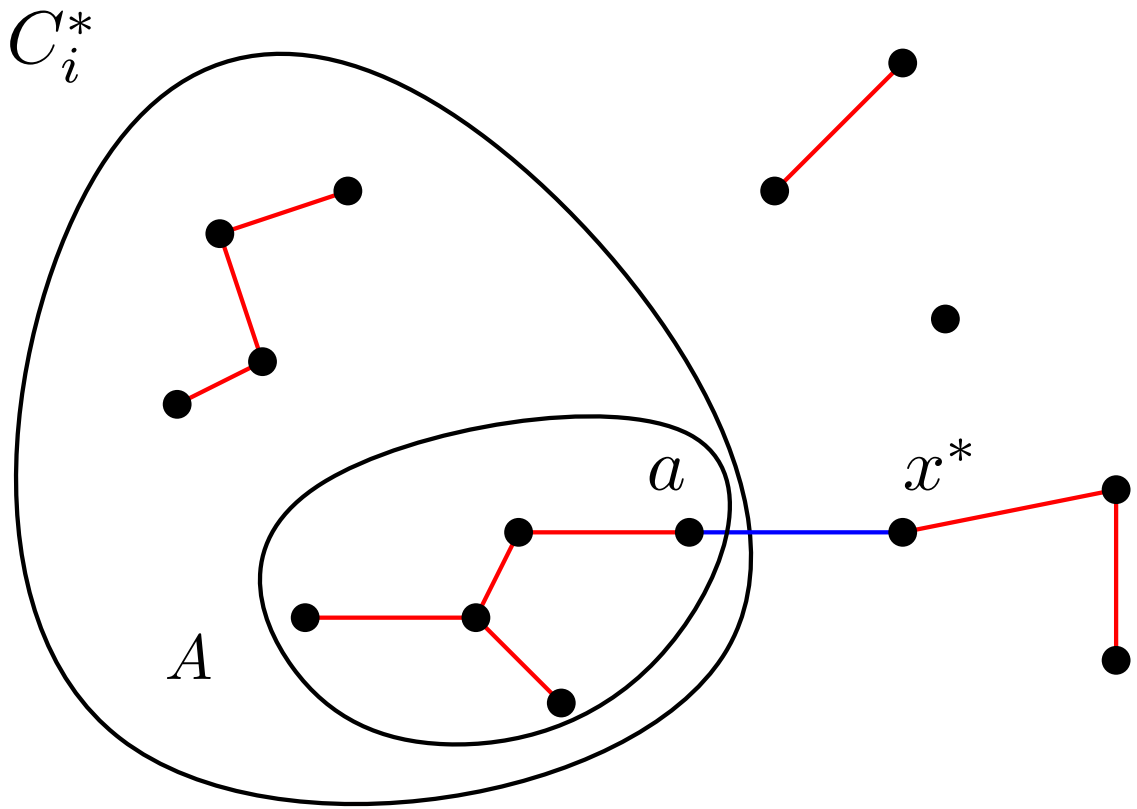

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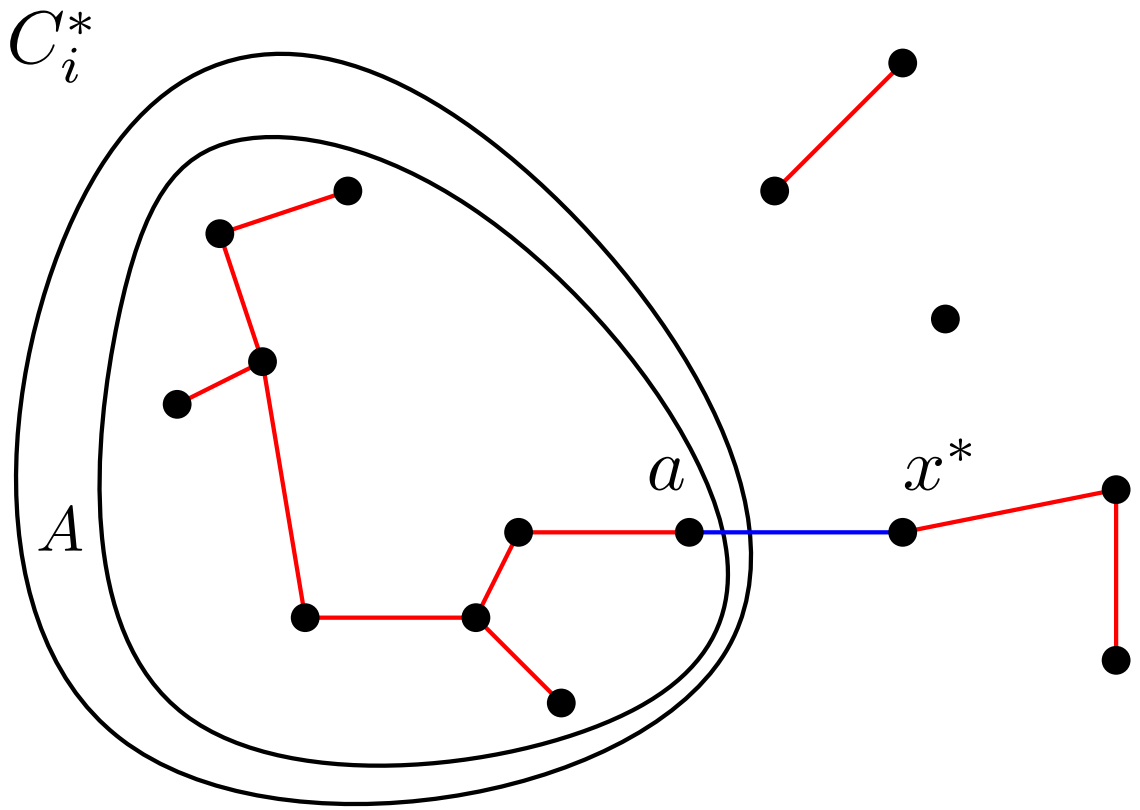

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$\Longrightarrow$ $A \not\subset C_i^*$ $\Longrightarrow$ $A = C_i^*$ $\Longrightarrow$ C_i^* appears as a connected component during Kruskal.

Proof of *(b)*

Claim 1. In every γ-stable instance with $\gamma \geq 3$, for every point $x \in C_i^*$ we have

$$d(x, c_j) > 3d(x, c_i), \quad \forall j \neq i.$$

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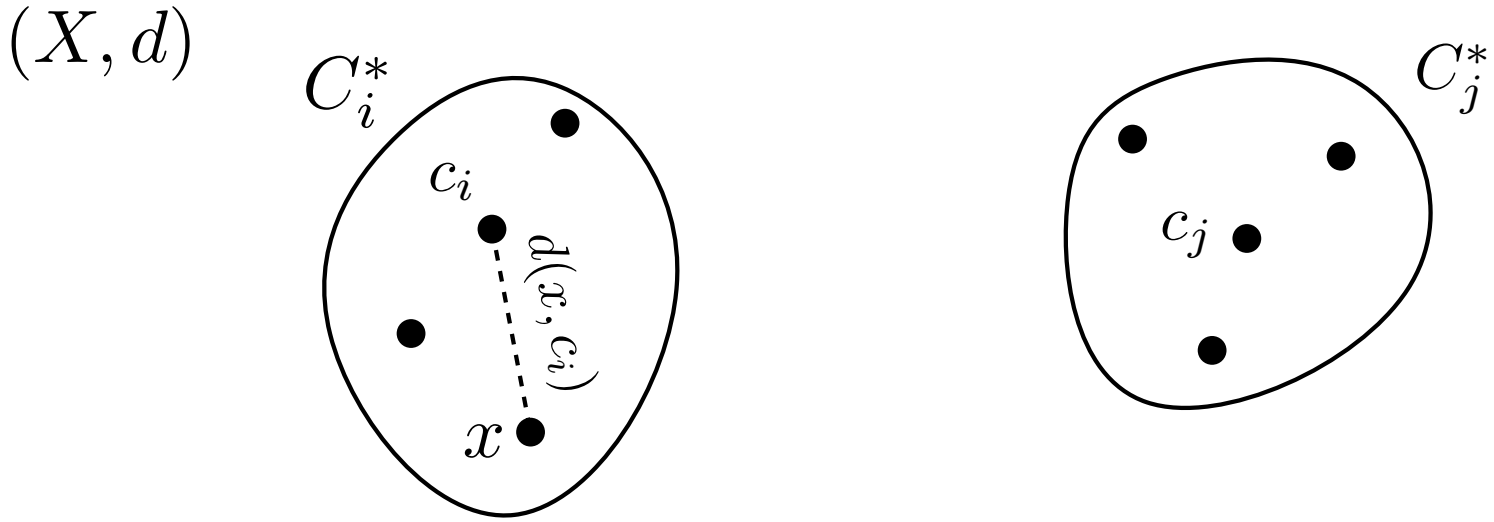

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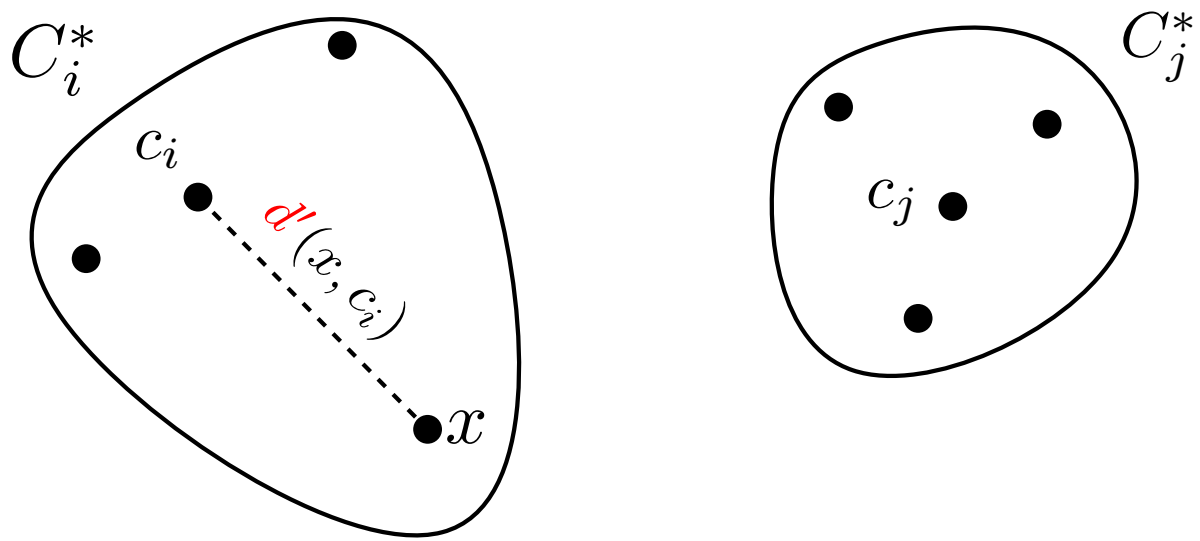

Blow up C_i^* by a factor γ, leaving all the other distances the same.

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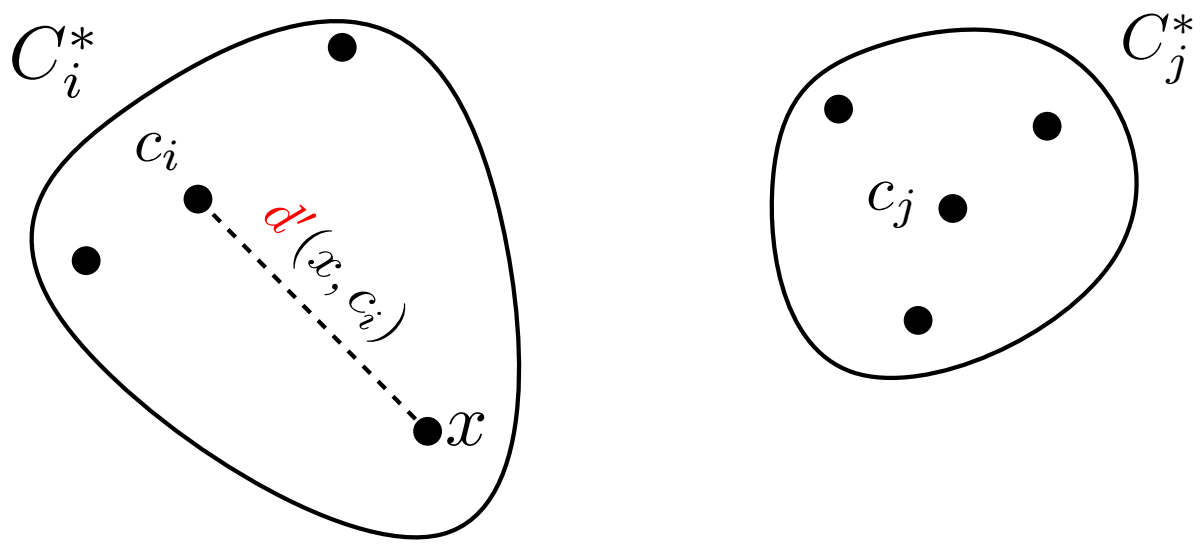

Blow up C_i^* by a factor γ, leaving all the other distances the same.

$$d(x, c_j) = d'(x, c_j) > d'(x, c_i) = \gamma d(x, c_i) \geq 3d(x, c_i)$$

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Proof of *(b)*

Claim 2. For every $x \in C_i^*$ and $y \in C_j^*$, with $j \neq i$, we have

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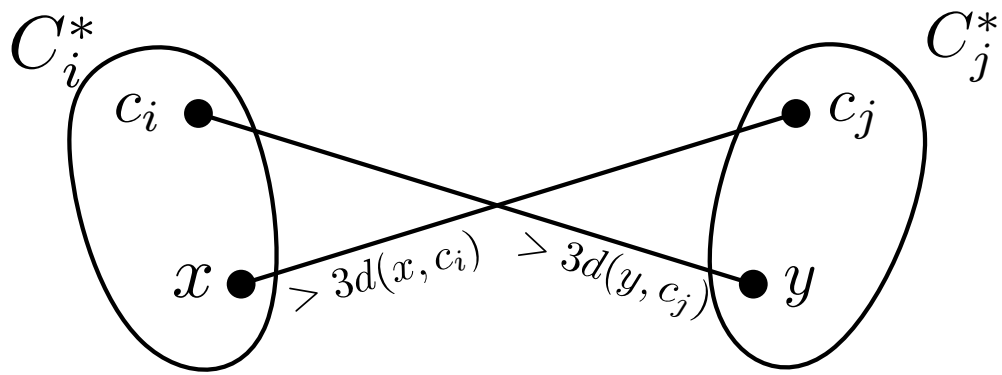

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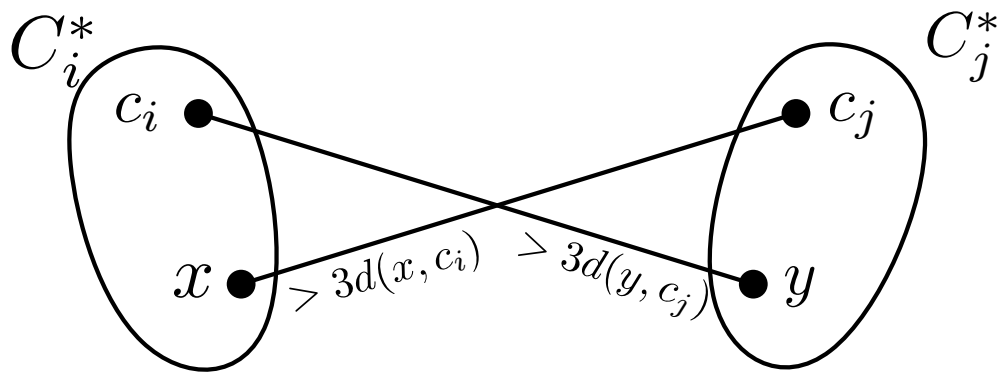

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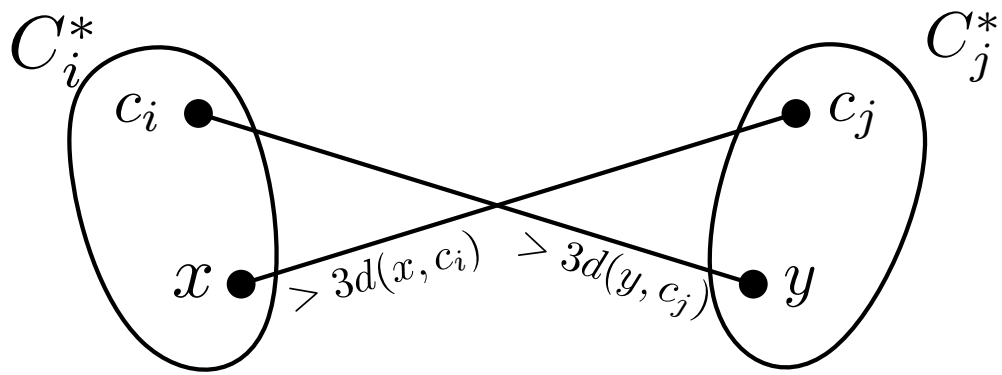

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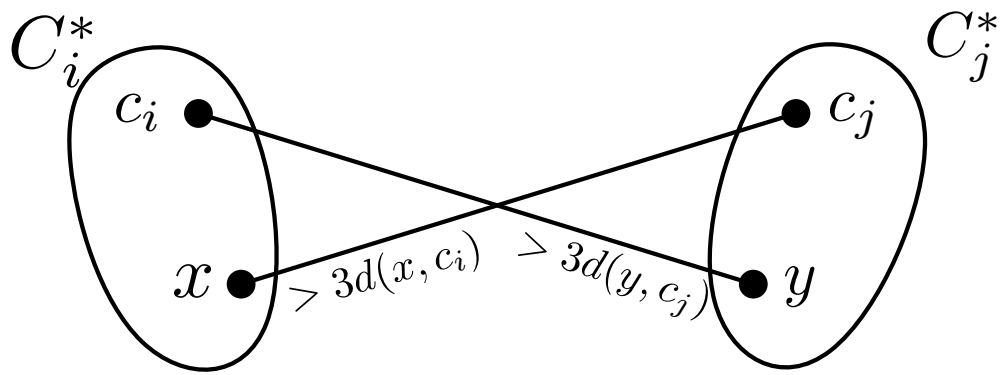

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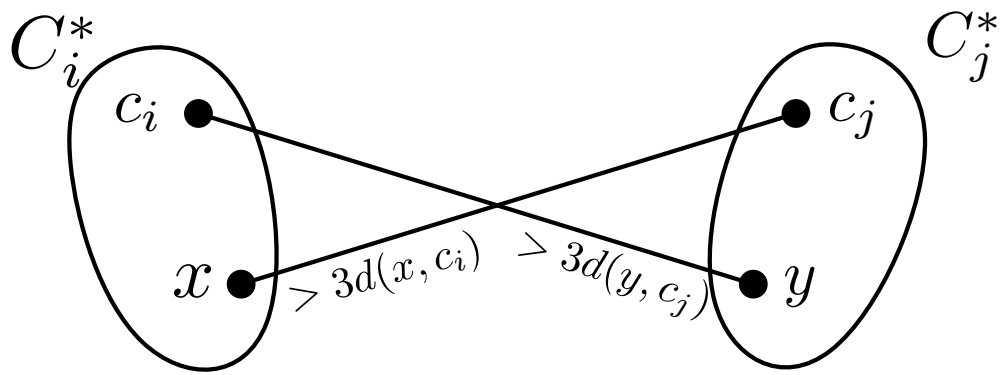

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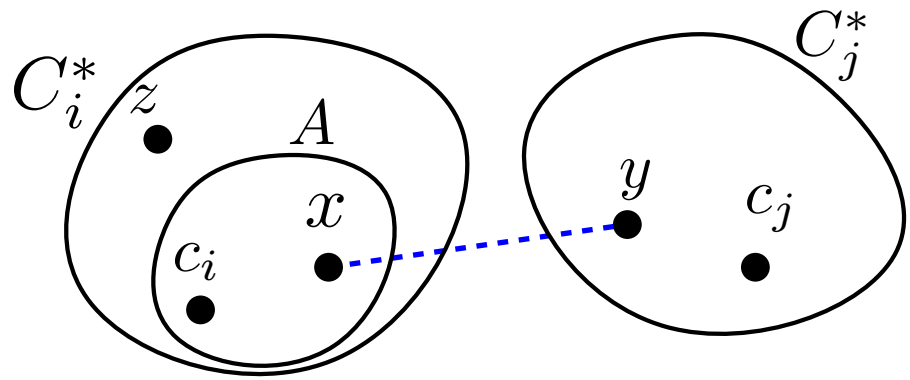

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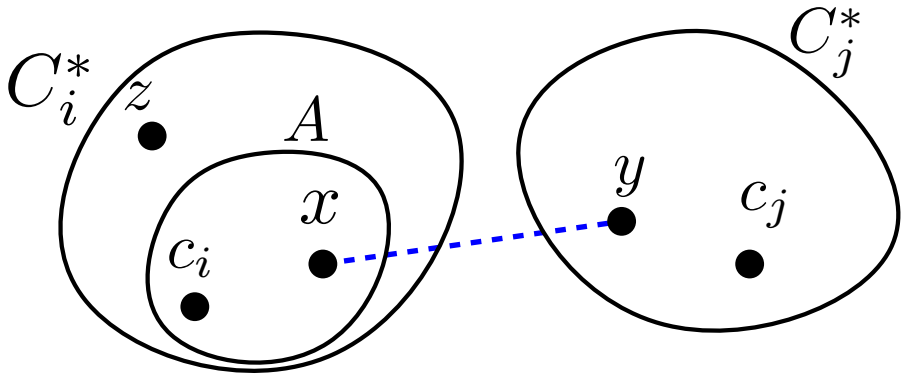

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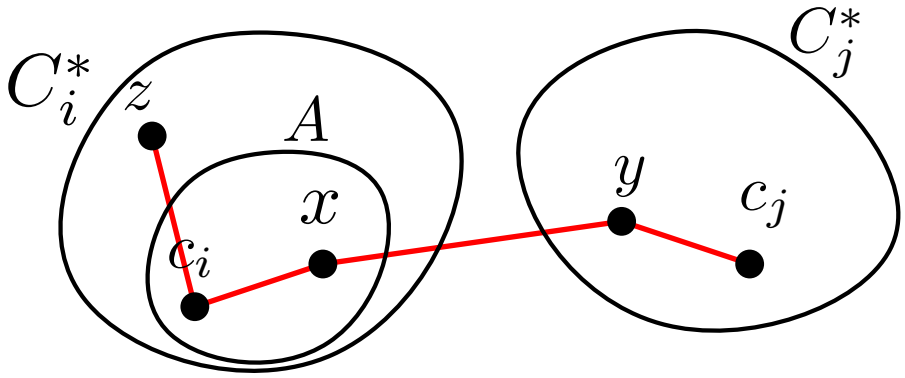

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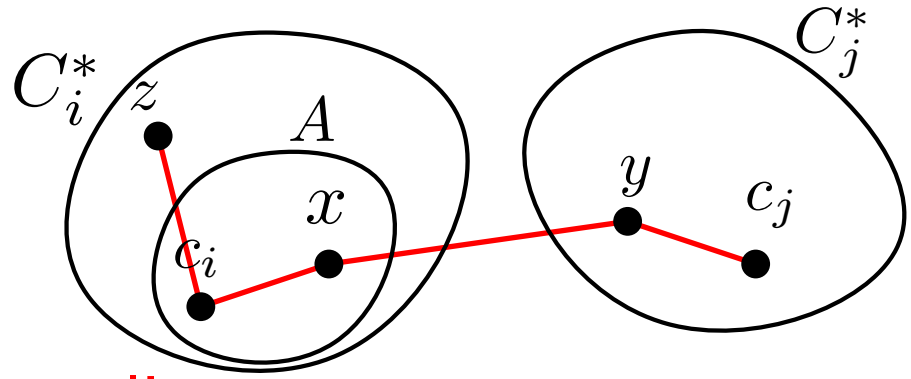

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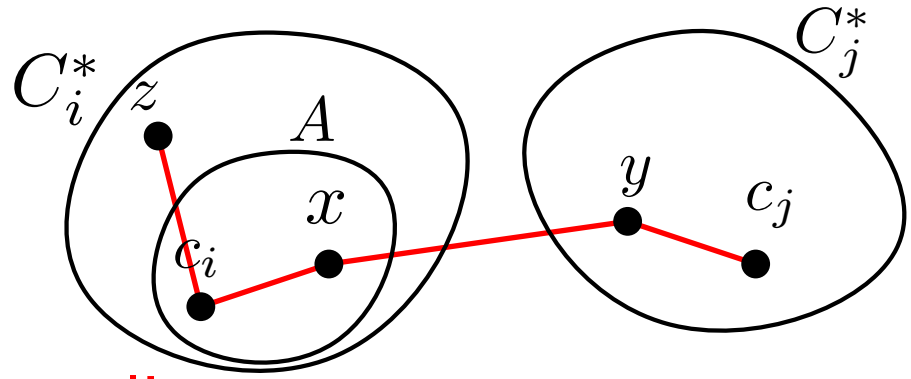

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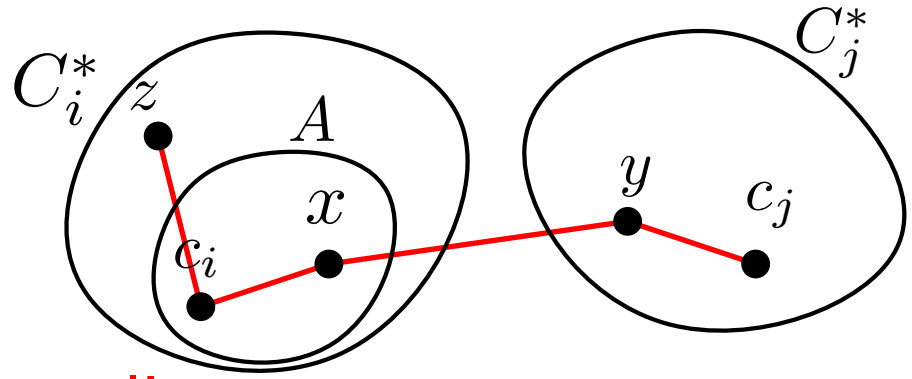

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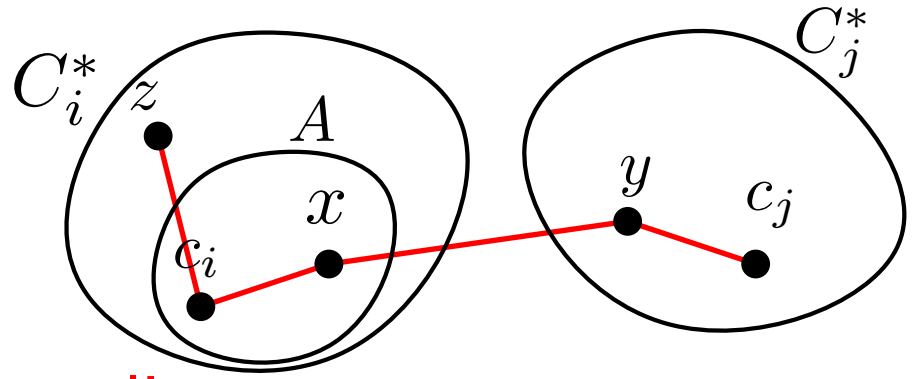

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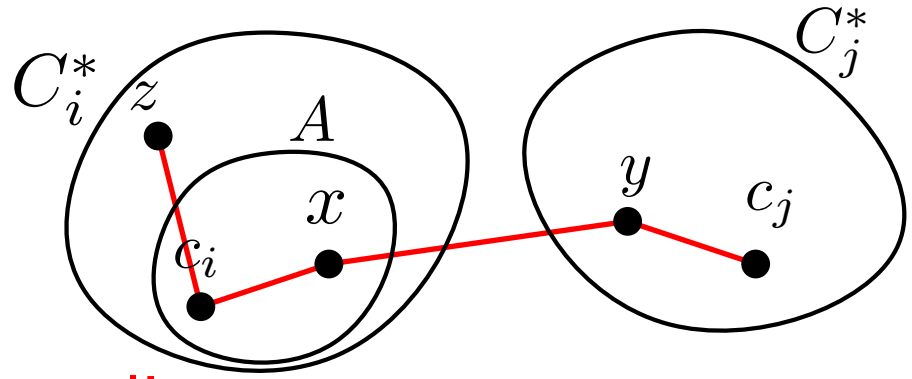

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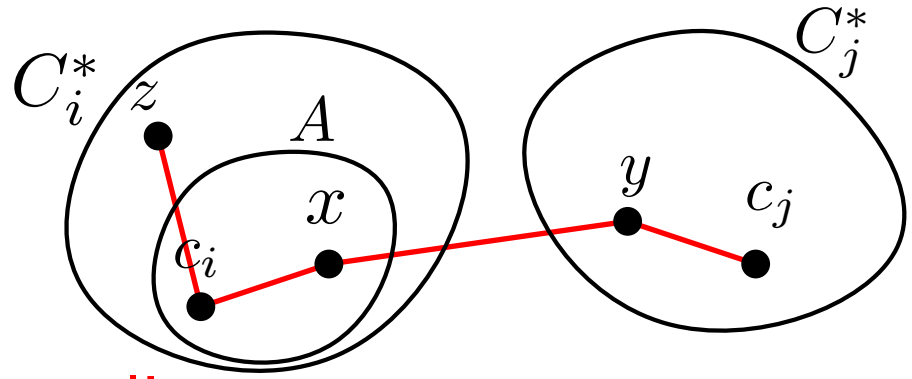

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$\implies$ (X,d) is not γ-stable (absurd!)

■

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Def. (Approximation Stability) The target clustering of a k-median instance is (c, ε)-approximation stable if every c-approximate k-clustering is ε-accurate, meaning that it agrees with the target clustering on at least a $1 - \varepsilon$ fraction of the points.

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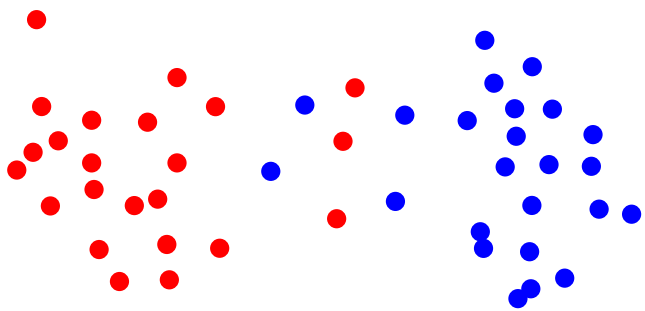
Traget clusters



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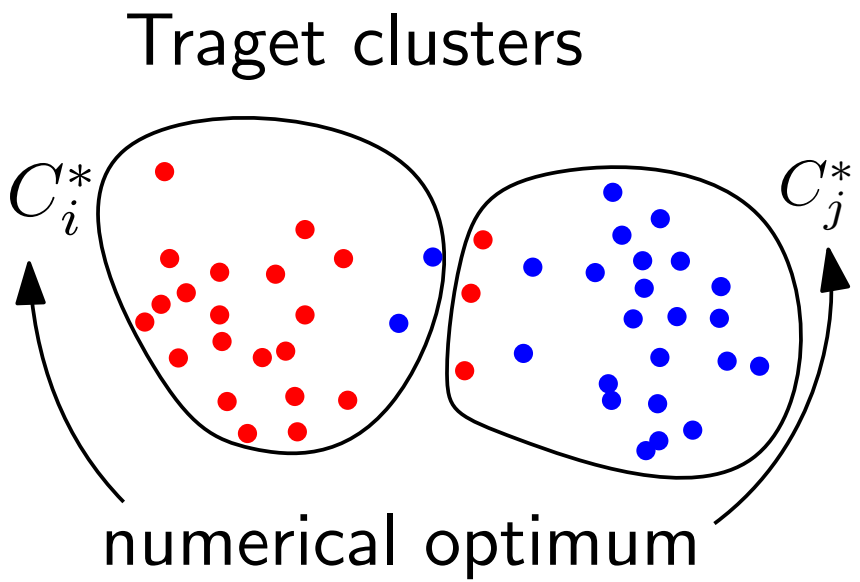

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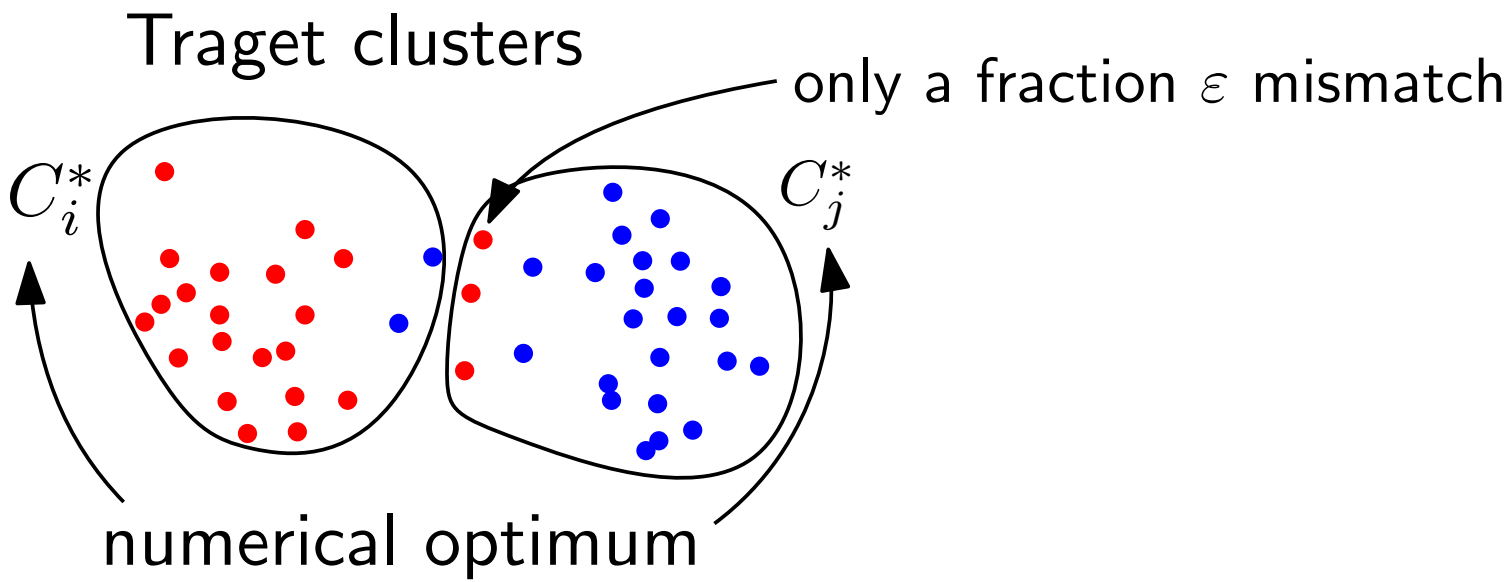

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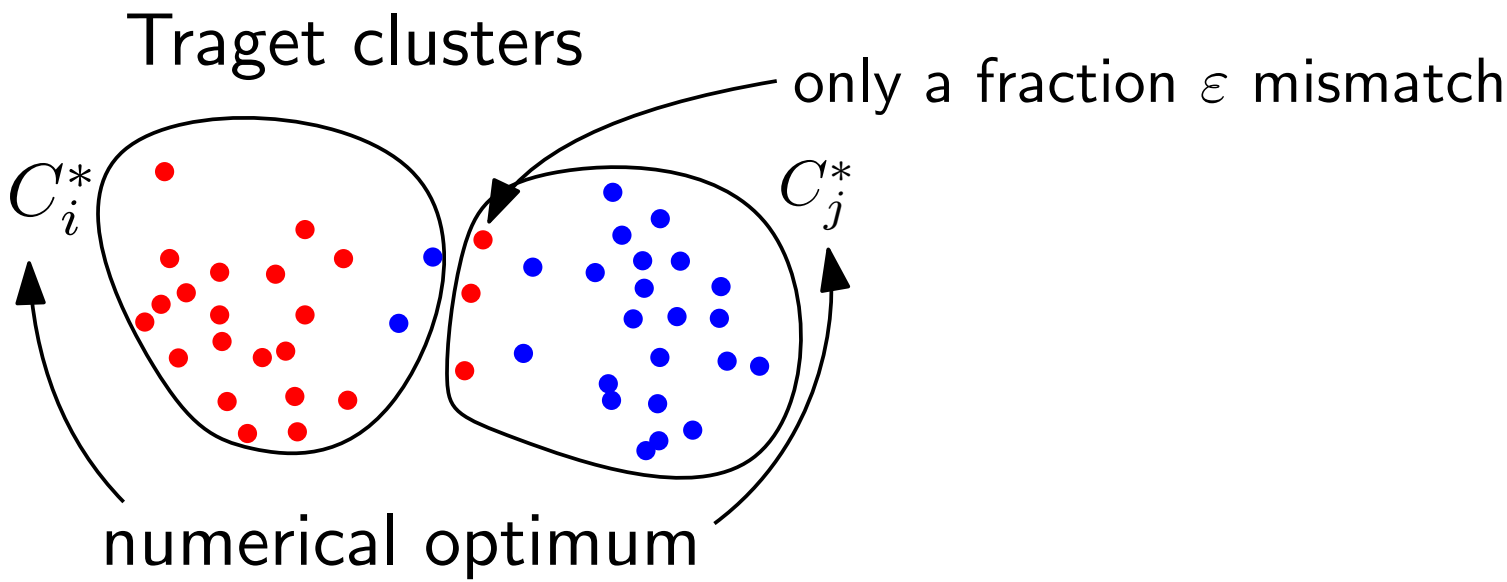

Theorem. For every pair $\alpha, \varepsilon > 0$ of constants, there is a polynomial-time algorithm that, for every $(1 + \alpha, \varepsilon)$-approximation stable k-median instance, recovers a k-clustering that agrees with the target clustering on all but an $O(\varepsilon/\alpha)$ fraction of the points.

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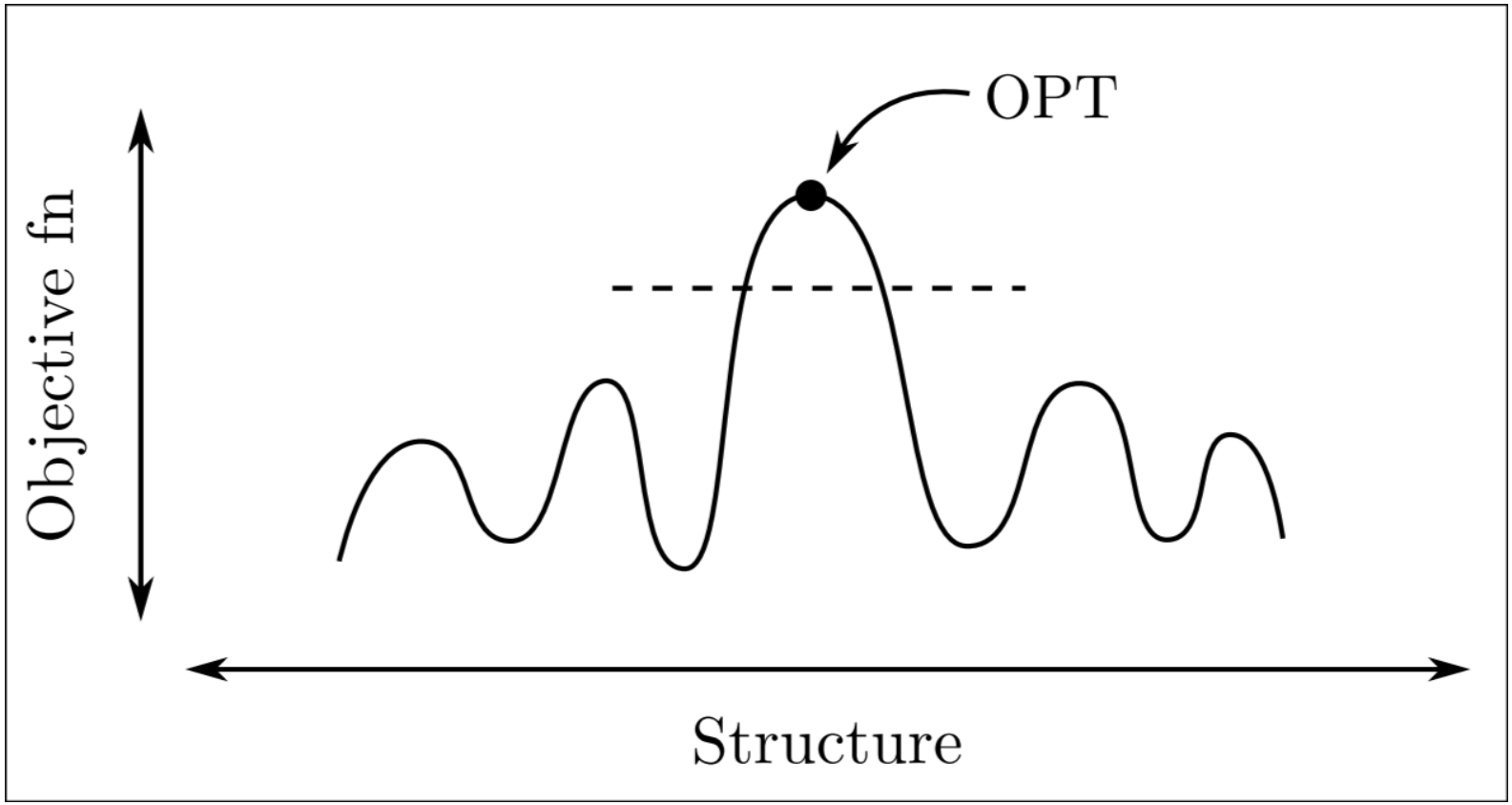

Intuition: Numerical near-optimality implies structural near-optimality.