Advanced topics on Algorithms

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Approximation algorithms: Episode IV

(the final one)

Primal-dual schema

High-level idea of the approach

Algorithm

start with:

- an infeasible integral primal solution, and
- a dual feasible solution

Iteratively:

- improve the dual solution
- improve the feasibility of the integral primal solution

Until a feasible integral primal solution is obtained

analysis: prove the approximation guarantee using the value of the dual solution as a lower bound

minimum Set Cover problem

Input:

- universe U of n elements
- a collection of subsets of U, $\mathcal{S}=\{S_1,\dots,S_k\}$
- each $S \in \mathcal{S}$ has a positive cost c(S)

Feasible solution:

a subcollection $\mathcal{C} \subseteq \mathcal{S}$ that covers U (whose union is U)

measure (min):

cost of $\mathcal{C}$: $\sum_{S \in \mathcal{C}} c(S)$

frequency of an element e: number of sets e belongs to

f: frequency of the most frequent element

ILP:

$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c(S) x_S$$

$$\text{subject to} \quad \sum_{S: e \in S} x_S \geq 1 \qquad e \in U$$

$$x_S \in \{0,1\} \qquad S \in \mathcal{S}$$

LP-relaxation

$$\text{minimize} \quad \sum_{S \in \mathcal{S}} c(S) x_S$$

$$\text{subject to} \quad \sum_{S: e \in S} x_S \geq 1 \qquad e \in U$$

$$x_S \geq 0 \qquad S \in \mathcal{S}$$

dual program

$$\text{maximize} \quad \sum_{e \in U} y_e$$

$$\text{subject to} \quad \sum_{e: e \in S} y_e \leq c(S) \qquad S \in \mathcal{S}$$

$$y_e \geq 0 \qquad e \in U$$

Given a dual solution y, we say that a set S is tight if $\sum_{e \in S} y_e = c(S)$

idea: pick only tight sets & do not overpack any set.

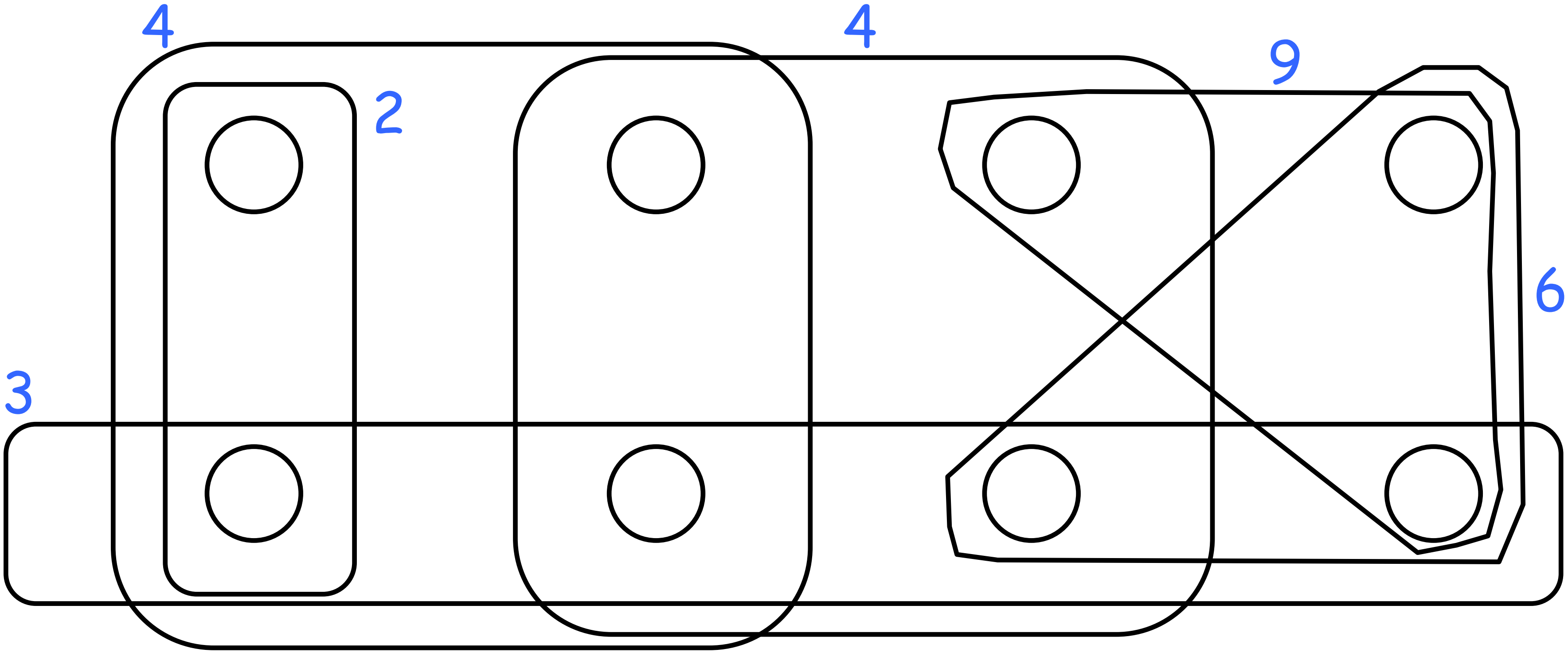
Algorithm 15.2 (Set cover – factor f)

1. **Initialization:** $x \leftarrow 0$; $y \leftarrow 0$
2. Until all elements are covered, do:

   Pick an uncovered element, say e, and raise y_e until some set goes tight.

   Pick all tight sets in the cover and update x.

   Declare all the elements occurring in these sets as "covered".
3. Output the set cover x.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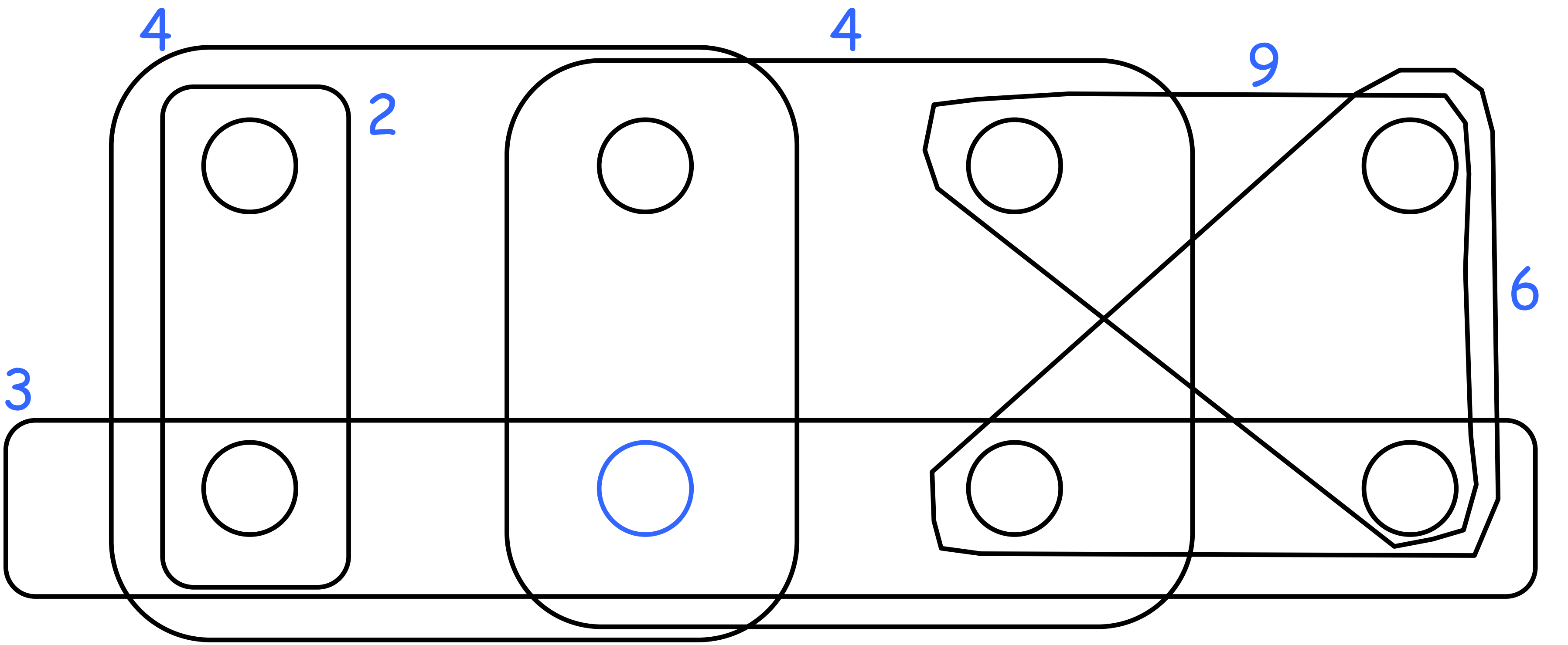
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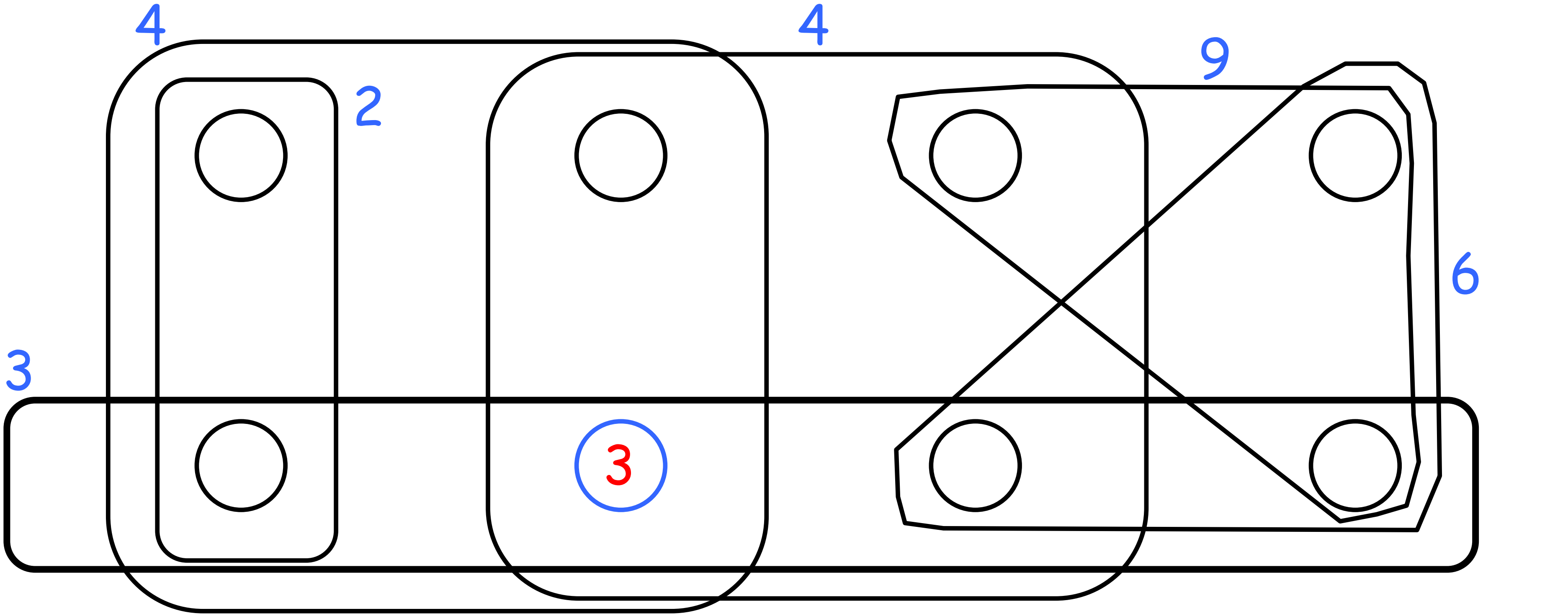
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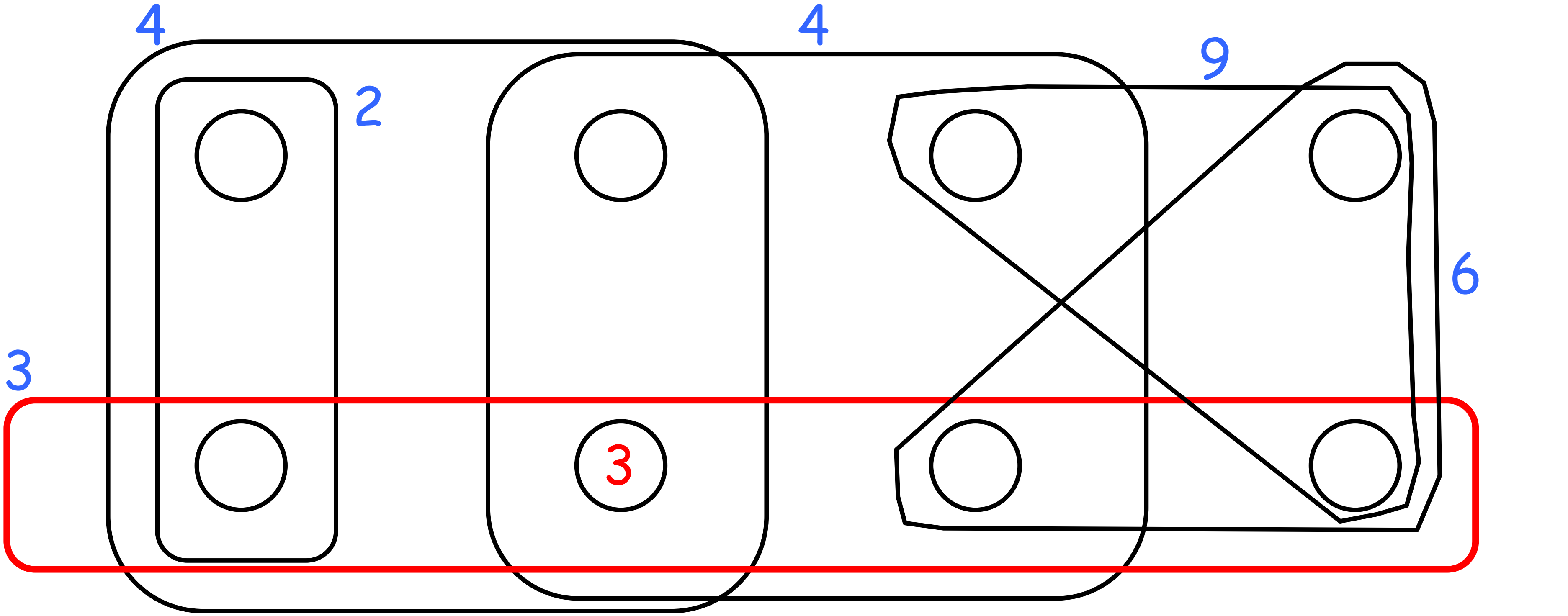
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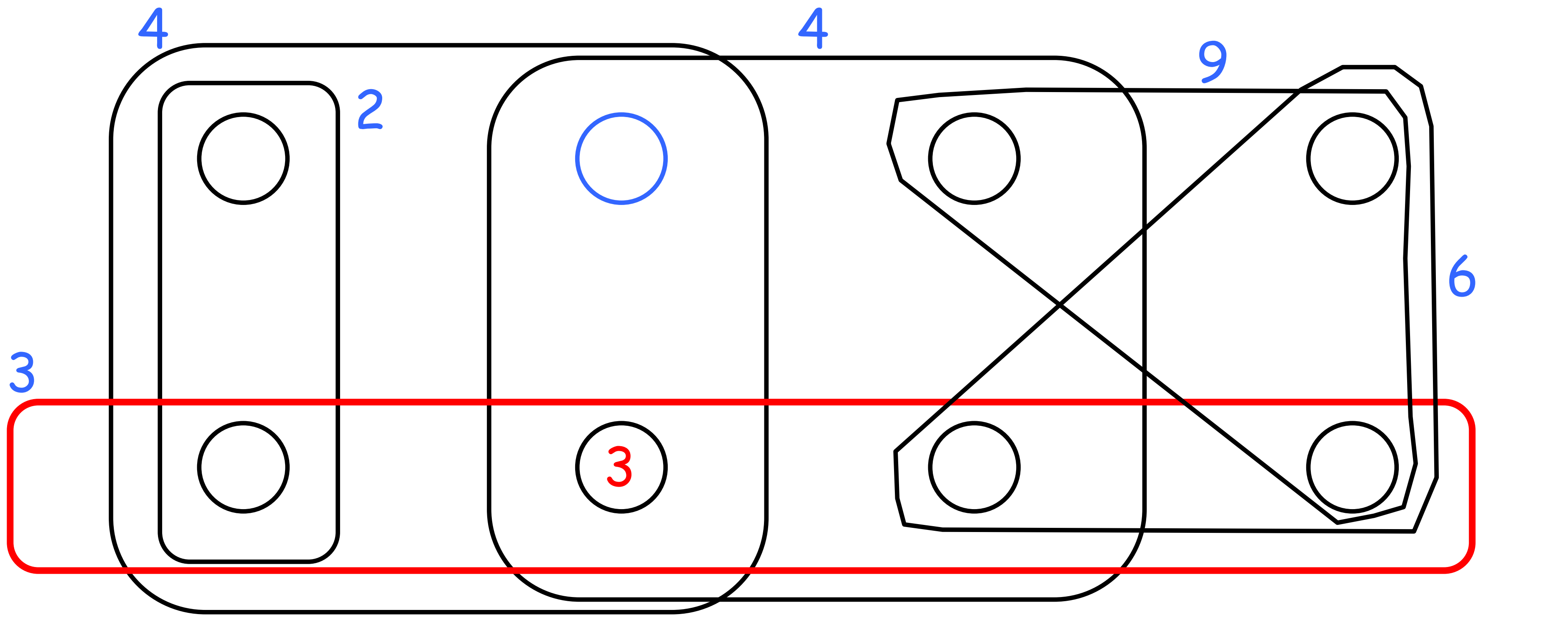
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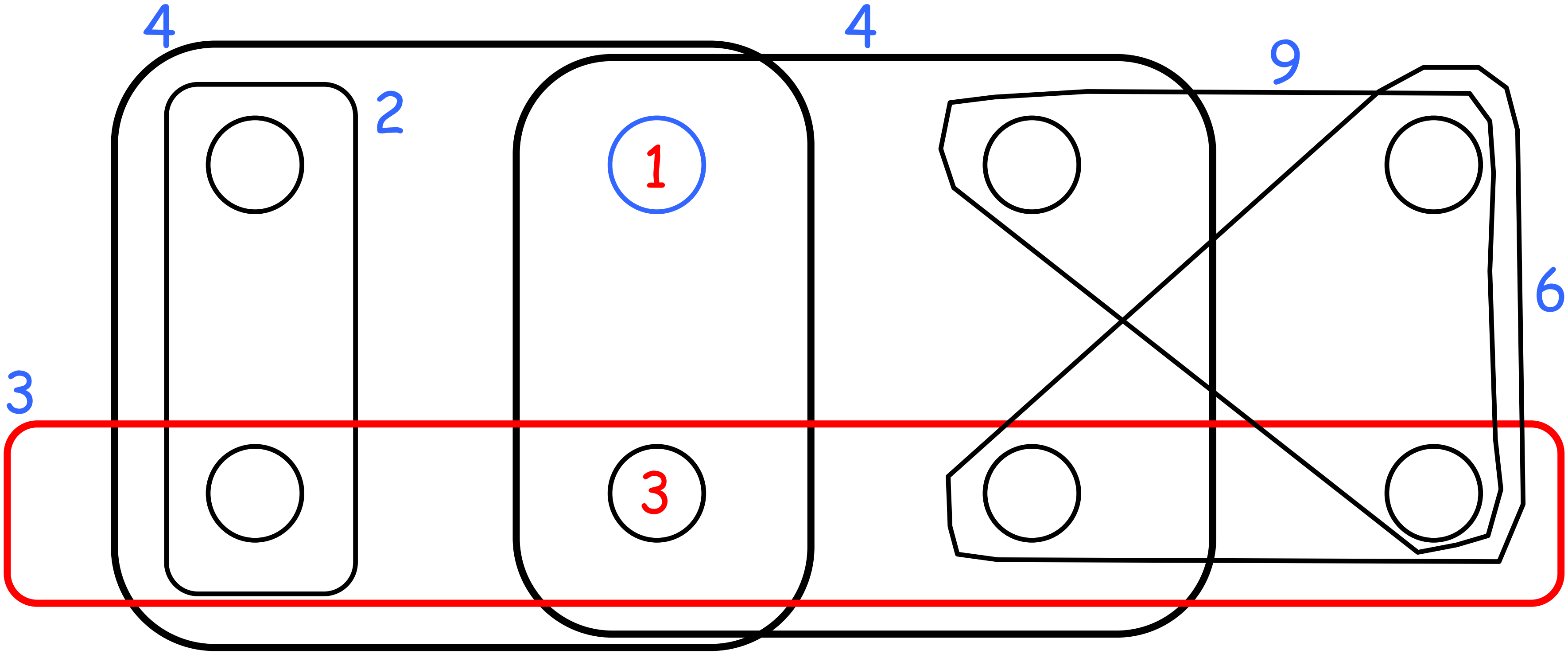
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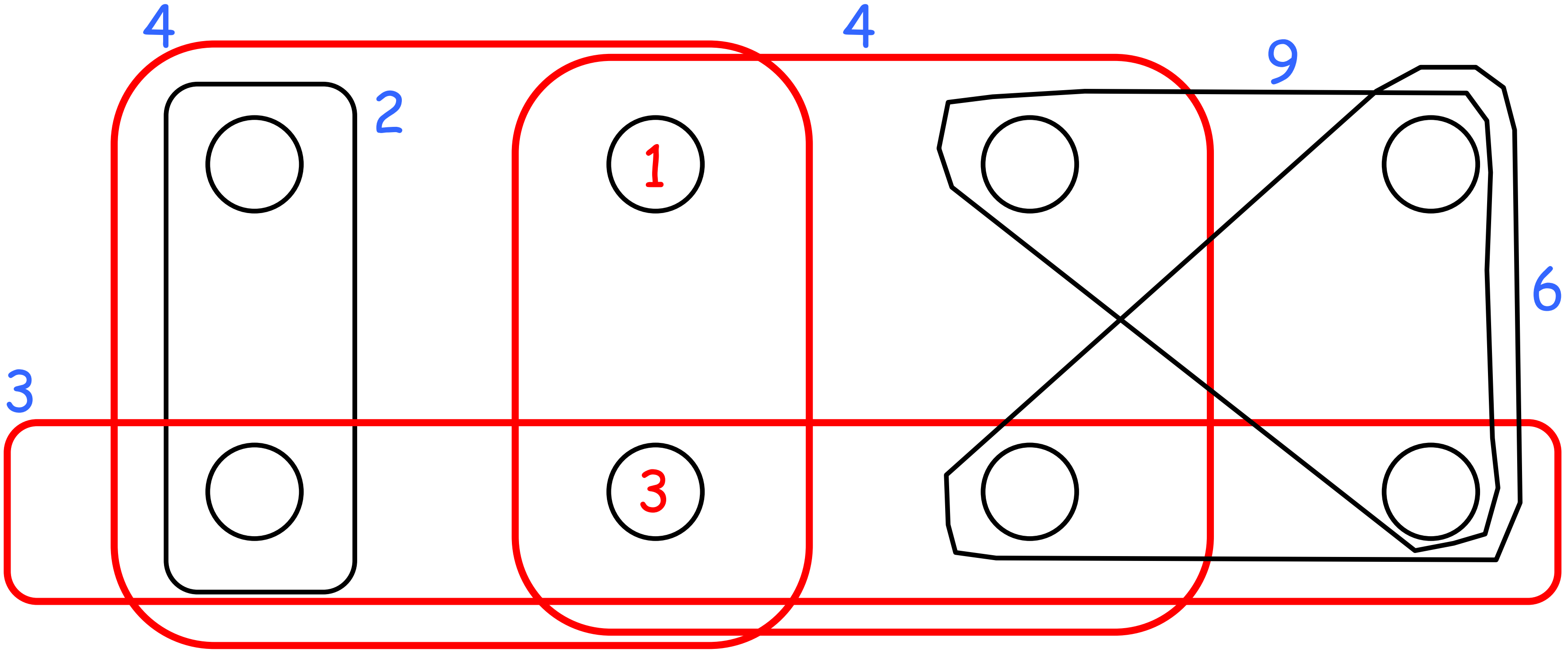

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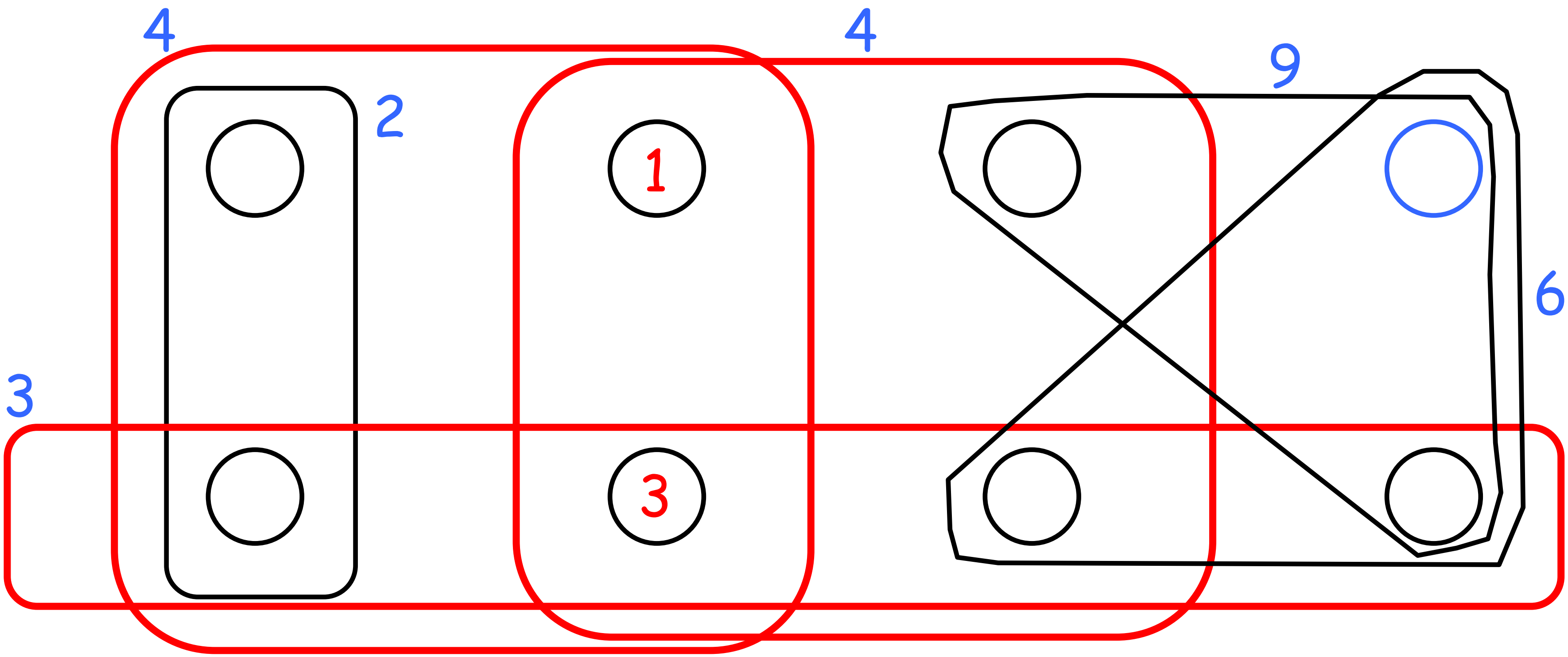
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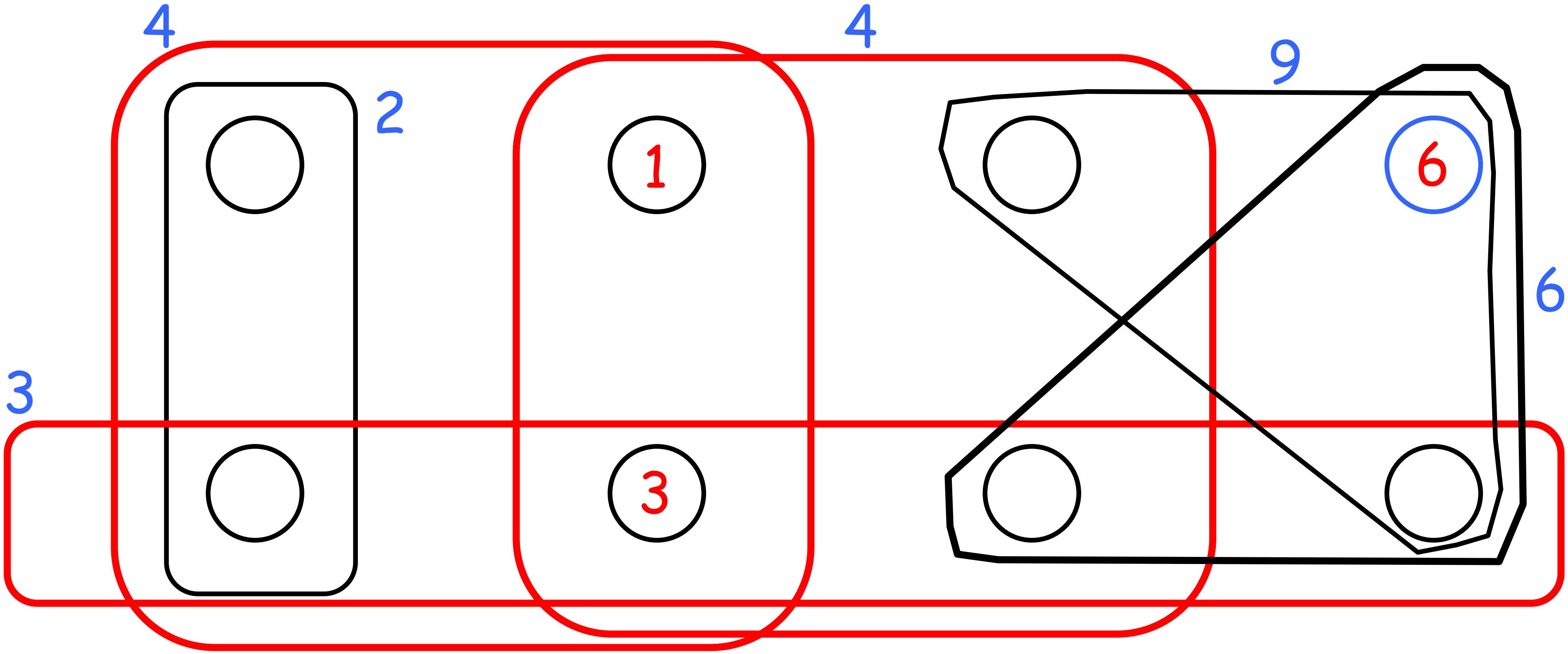

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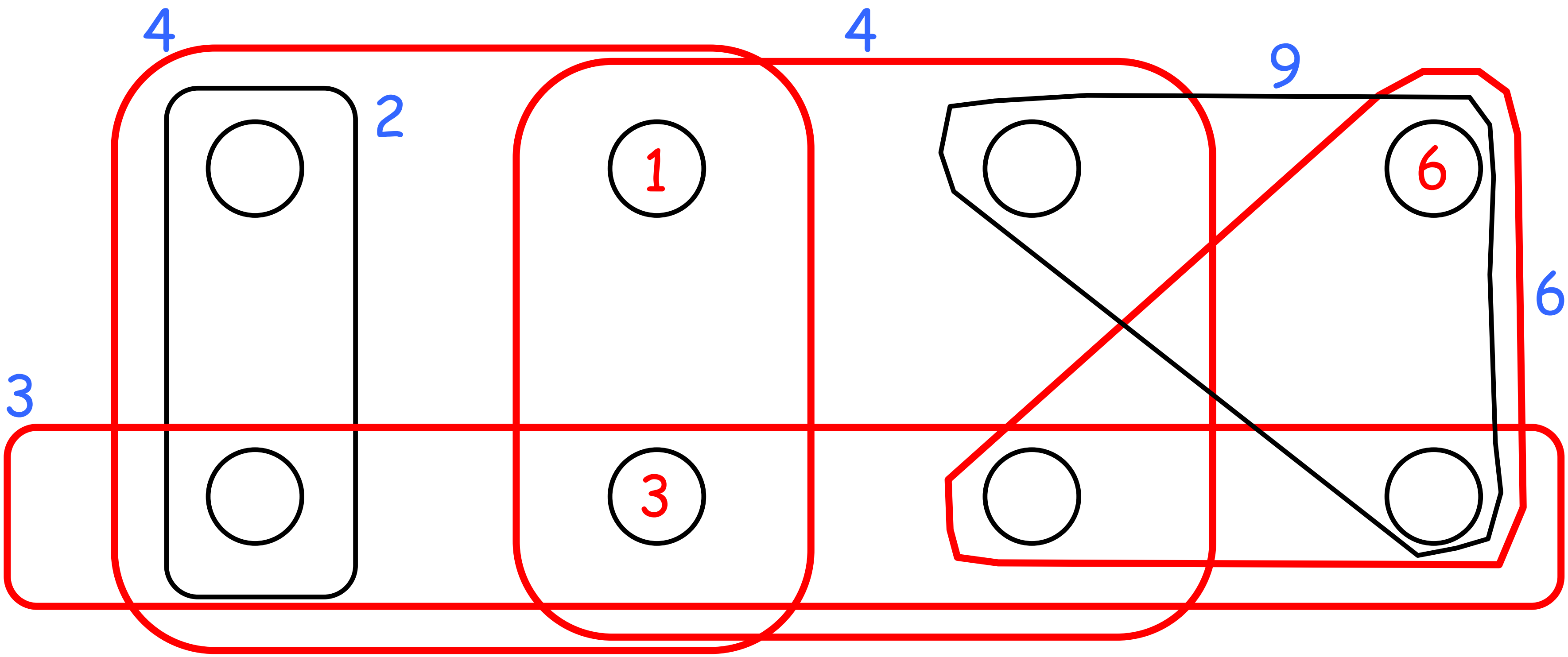

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Theorem

The algorithm is an f-approximation algorithm for the SC problem.

proof

the computed cover is clearly feasible.

we claim that: $$\sum_{S \in \mathcal{S}} c(S) x_S \le f \sum_{e \in U} y_e$$

Think of it as money you can use to buy the picked primal solution

each element e:

- has $f \cdot y_e$ amount of money
- pays y_e for each picked set S containing e

since each e is in at most f sets, e has enough money for its payments

since each picked set S is tight, S is fully paid for by the elements it contains

since **y** is feasible: $$\sum_{e \in U} y_e \le OPT$$

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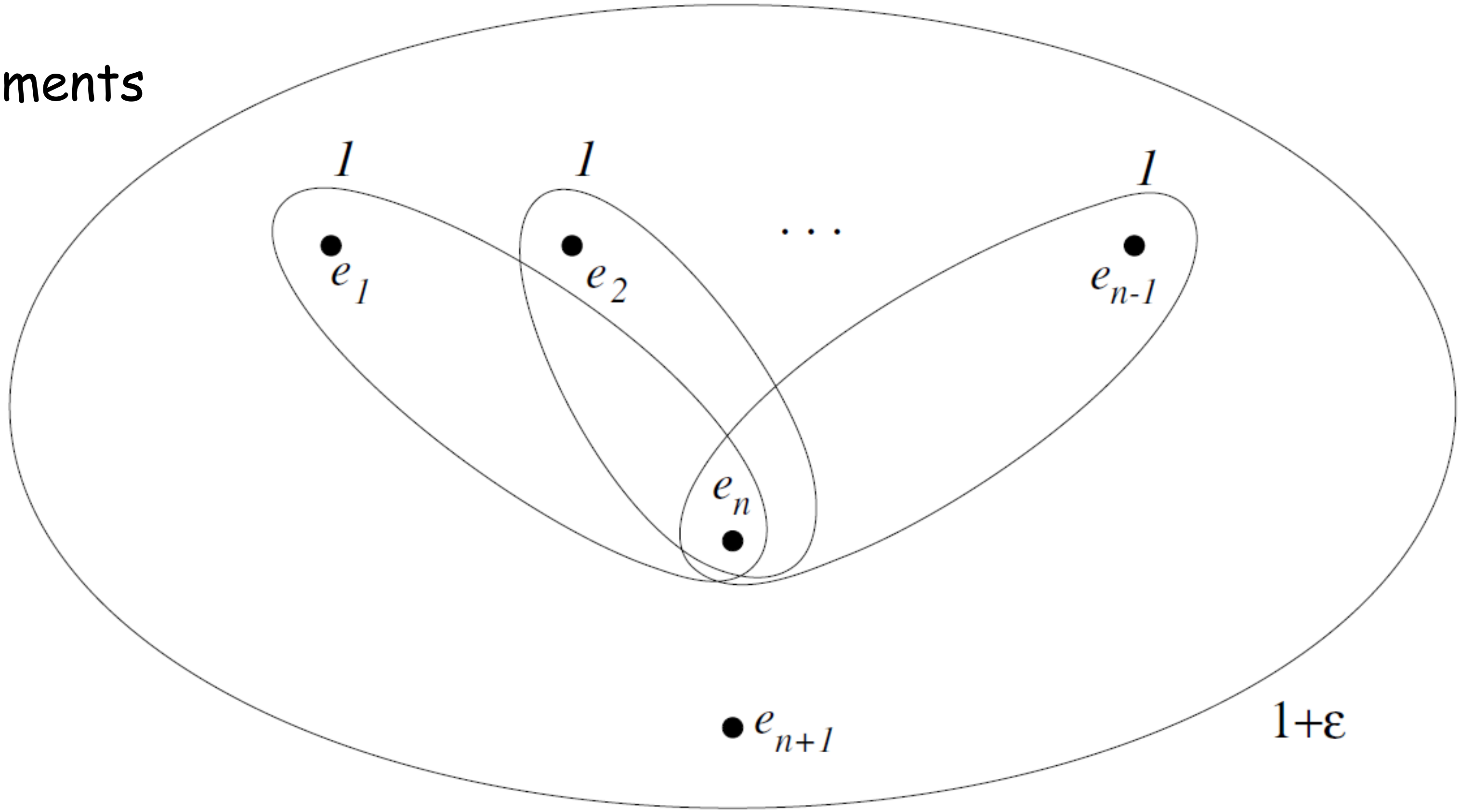
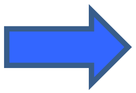
tight example

n+1 elements
$f=n$



suppose the algorithm raises first variable y_{e_n}



returned solution has cost n+ε

OPT=1+ε

The Steiner Forest problem

minimum Steiner Forest problem

Input:

- undirected graph $G=(V,E)$ with non-negative edge costs
- collection of disjoint subsets of V, $S_1,\dots,S_k$

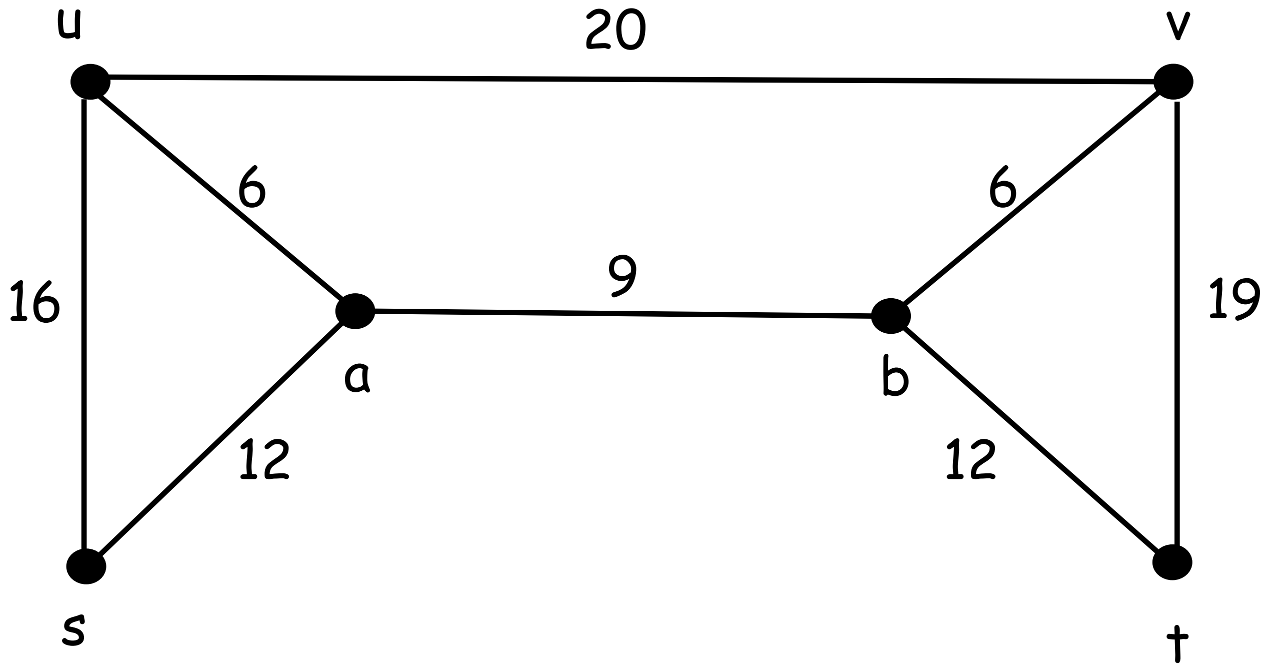
Feasible solution:

a forest F in which each pair of vertices belonging to the same set S_i is connected

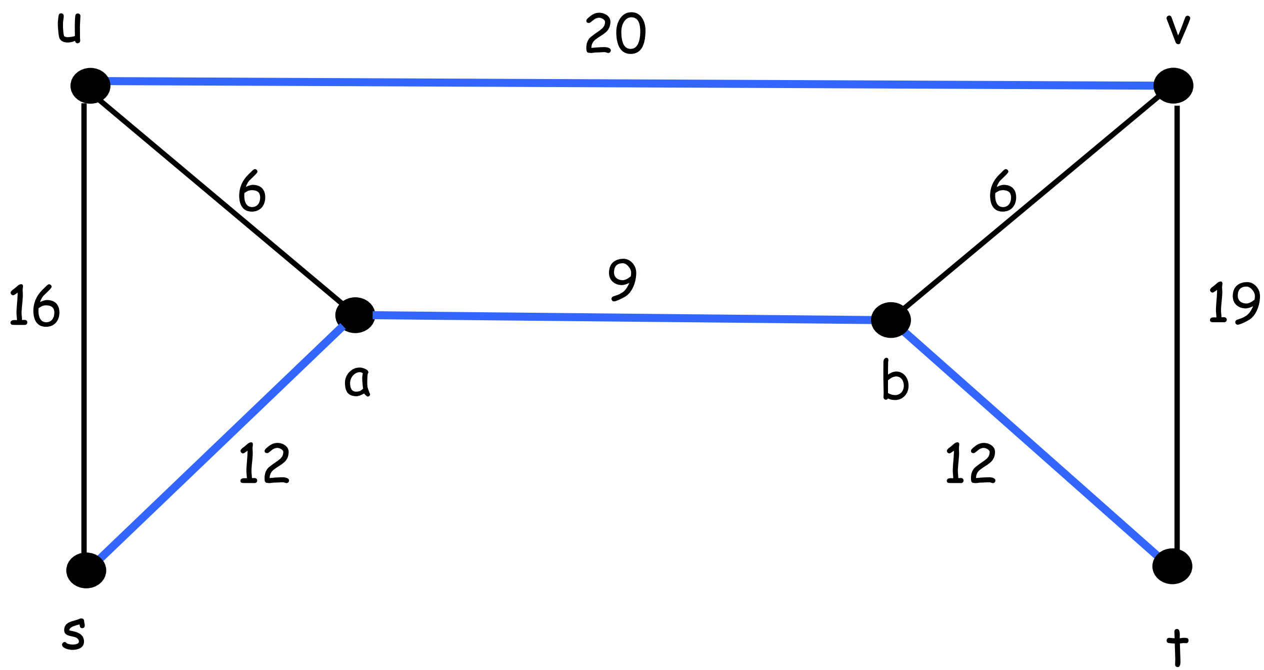
measure (min):

cost of F : $\sum_{e \in E(F)} c(e)$

$S_1=\{u,v\}$ $S_2=\{s,t\}$

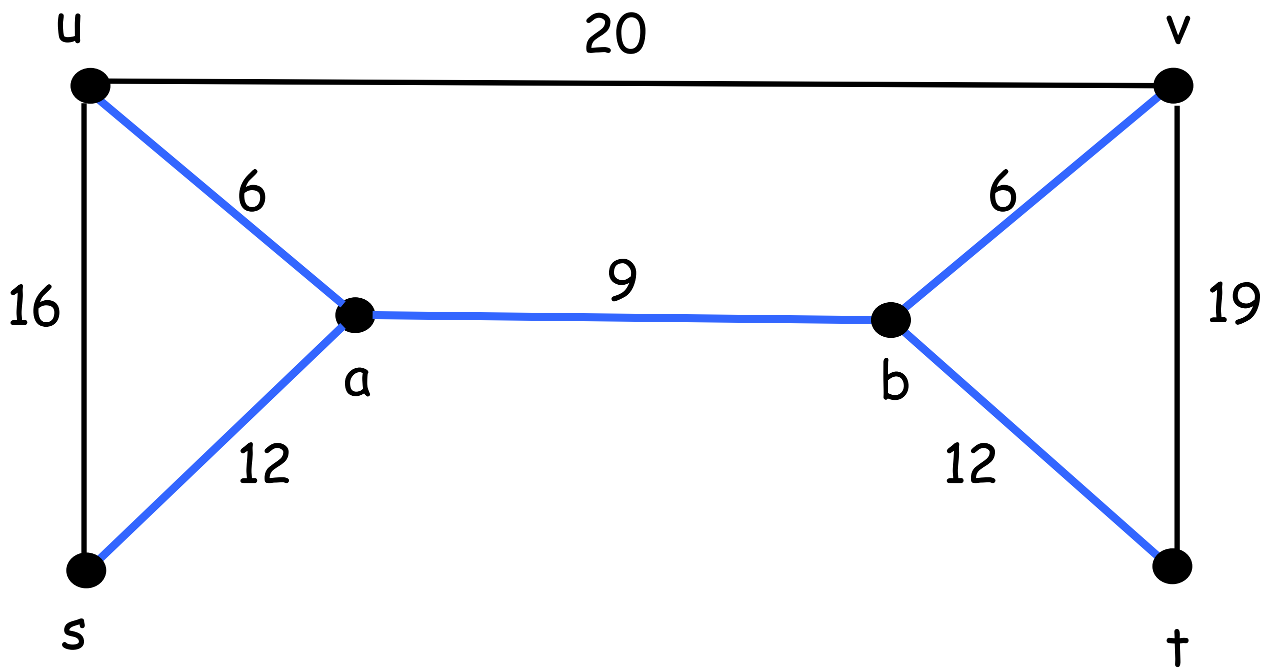


S_1={u,v} S_2={s,t}



a Steiner forest of cost 53

$S_1=\{u,v\}$ $S_2=\{s,t\}$



a better Steiner forest of cost 45

minimum Steiner Forest problem

Input:

- undirected graph $G=(V,E)$ with non-negative edge costs
- collection of disjoint subsets of V, $S_1,\dots,S_k$

Feasible solution:

a forest F in which each pair of vertices belonging to the same set S_i is connected

measure (min):

cost of F : $\sum_{e \in E(F)} c(e)$

metric Steiner forest problem:

- G is complete, and
- edge costs satisfy the *triangle inequality*
  for every u,v,w : $c(u,v) \leq c(u,w)+c(w,v)$

minimum Steiner Forest problem

Input:

- undirected graph $G=(V,E)$ with non-negative edge costs
- collection of disjoint subsets of V, $S_1,\dots,S_k$

Feasible solution:

a forest F in which each pair of vertices belonging to the same set S_i is connected

measure (min):

cost of F : $\sum_{e \in E(F)} c(e)$

a connectivity requirement function r

$$r(u,v) = \begin{cases} 1 & \text{if } u \text{ and } v \text{ belong to the same } S_i \\ 0 & \text{otherwise} \end{cases}$$

a function f on all cuts in G, for each $S \subseteq V$ i.e. cut $(S, S' = V \setminus S)$:

$$f(S) = \begin{cases} 1 & \text{if } \exists\, u \in S \text{ and } v \in S' \text{ such that } r(u,v)=1 \\ 0 & \text{otherwise} \end{cases}$$

an Integer Linear Programming (ILP) formulation of SF

$$\text{minimize} \quad \sum_{e \in E} c_e x_e$$

$$\text{subject to} \quad \sum_{e: e \in \delta(S)} x_e \geq f(S) \qquad S \subseteq V$$

$$x_e \in \{0,1\} \qquad e \in E$$

relax with
$x_e \geq 0$ & $x_e \leq 1$

redundant

LP-relaxation

$$\text{minimize} \quad \sum_{e \in E} c_e x_e$$

$$\text{subject to} \quad \sum_{e: e \in \delta(S)} x_e \geq f(S) \qquad S \subseteq V$$

$$x_e \geq 0 \qquad e \in E$$

$\delta(S)$: edges crossing the cut $(S, S' = V \setminus S)$

LP-relaxation

$$\text{minimize} \quad \sum_{e \in E} c_e x_e$$

$$\text{subject to} \quad \sum_{e: e \in \delta(S)} x_e \geq f(S) \quad S \subseteq V$$

$$x_e \geq 0 \quad e \in E$$

dual program

$$\text{maximize} \quad \sum_{S \subseteq V} f(S)\, y_S$$

$$\text{subject to} \quad \sum_{S: e \in \delta(S)} y_S \leq c_e \quad e \in E$$

$$y_S \geq 0 \quad S \subseteq V$$

edge e *feels* dual y_S if $y_S > 0$ and $e \in \delta(S)$

S has been *raised* in a dual solution if $y_S > 0$

obs: raising S or S′ has the same effect

obs: no advantage in raising a set S with $f(S)=0$

⇒ assume never raise such sets

edge e is *tight* if the total amount of dual it feels equals its cost

obs: dual program tries to maximize the sum of the duals subject to no edge is *overtight* (i.e., feels more than its cost) at any point, the currently picked edges form a forest F

S is unsatisfied if f(S)=1 but there is no picked edge crossing the cut (S,S')

S is active if it is a minimal (w.r.t. inclusion) unsatisfied set in F

obs: if F is not feasible then there must be an active set

Lemma

Set S is active iff it is a connected component in the currently picked forest and f(S)=1.

proof

Let S be an active set

S cannot contain part of a connected component because otherwise there will already be a picked edge in the cut (S,S')

⇒ S is the union of connected components

Since f(S)=1, there is a vertex u∈S and v∈S' such that r(u,v)=1

Let S' be the connected component containing u

minimality of S implies S'=S.

■

Algorithm 22.3 (Steiner forest)

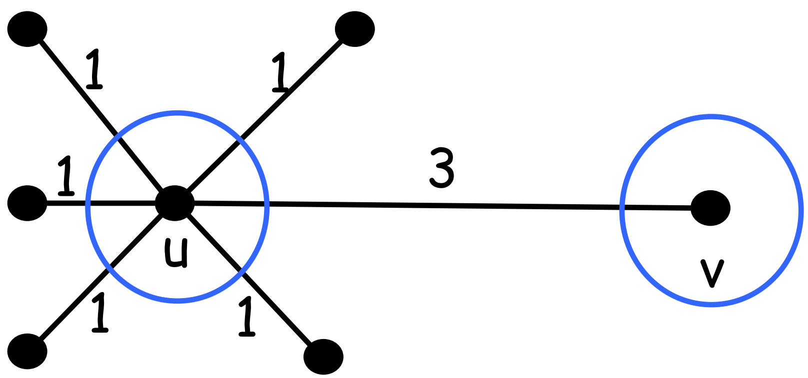
1. **(Initialization)** $F \leftarrow \emptyset$; for each $S \subseteq V$, $y_S \leftarrow 0$.
2. **(Edge augmentation)** while there exists an unsatisfied set do:
   simultaneously raise y_S for each active set S, until some edge e goes tight;
   $F \leftarrow F \cup \{e\}$.
3. **(Pruning)** return $F' = \{e \in F | \; F - \{e\}$ is primal infeasible$\}$

discard all redundant edges

an edge $e \in F$ is redundant if $F-\{e\}$ is also a feasible solution

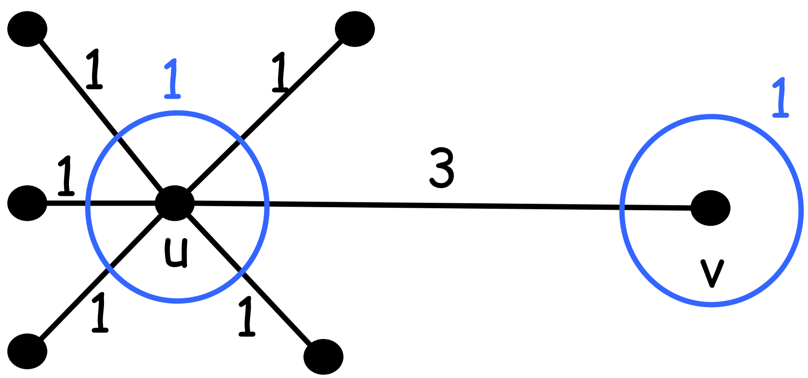
example: pruning step is needed

$S_1=\{u,v\}$



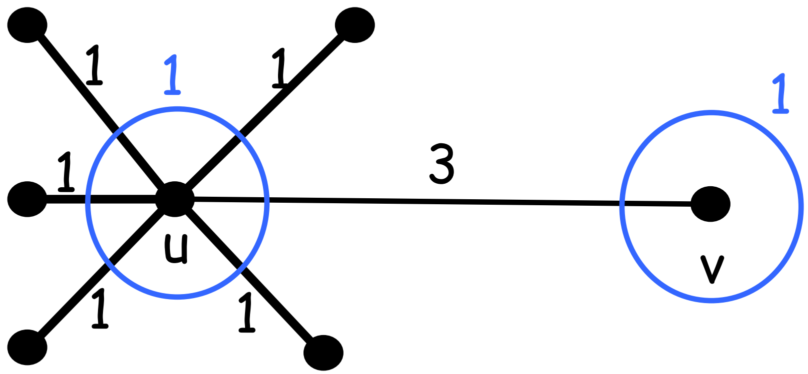
example: pruning step is needed

S_1={u,v}



example: pruning step is needed

$S_1=\{u,v\}$



example: pruning step is needed

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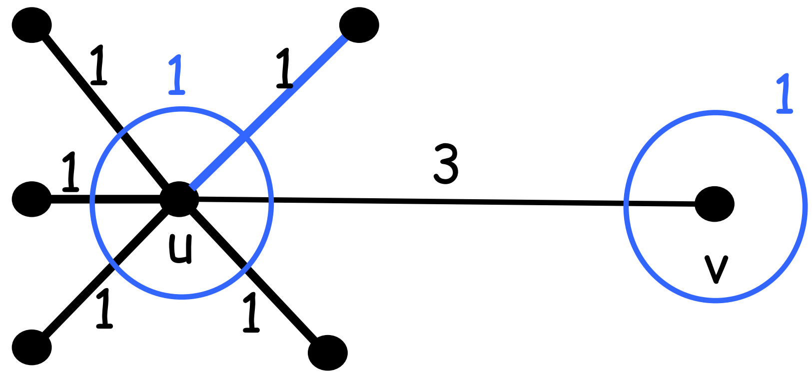

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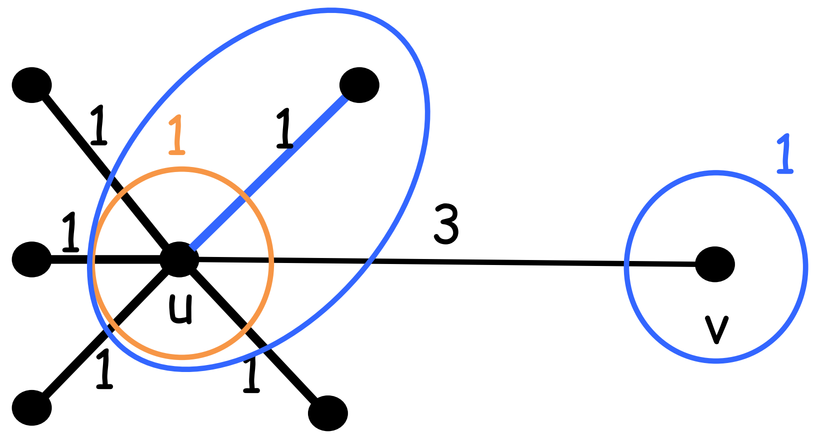

example: pruning step is needed

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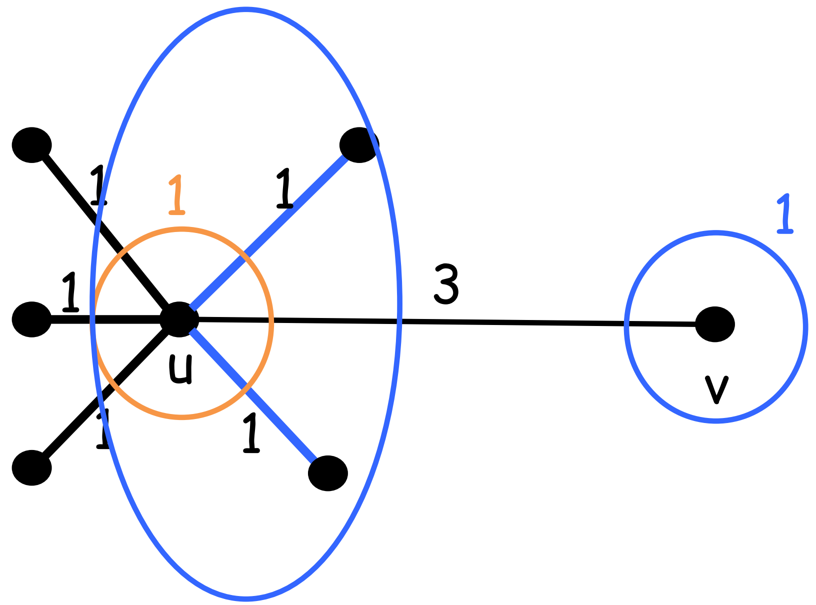

example: pruning step is needed

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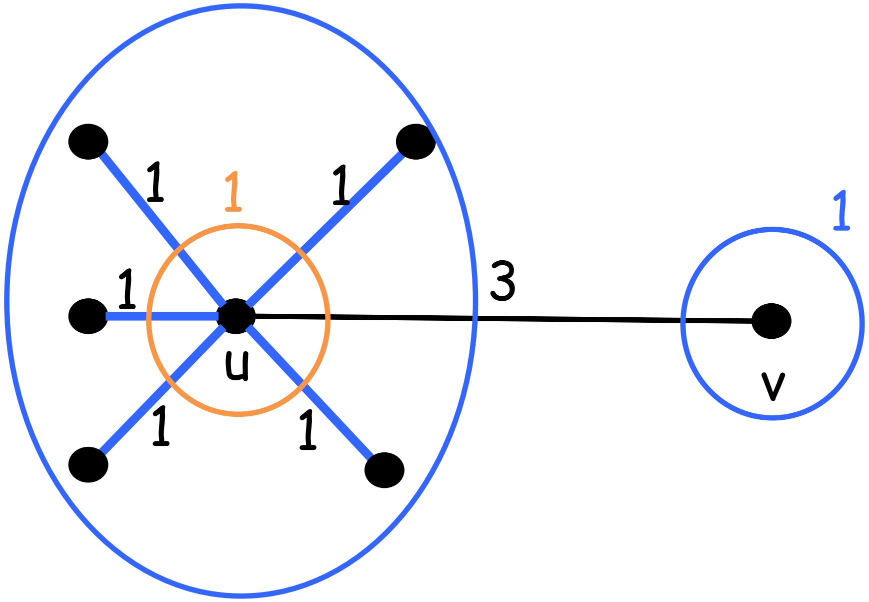
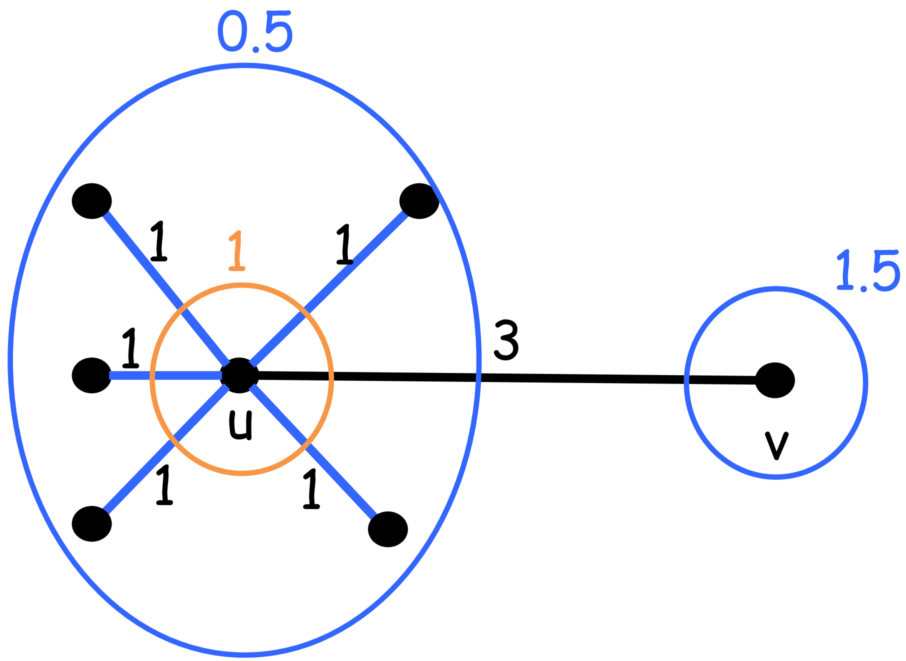


example: pruning step is needed

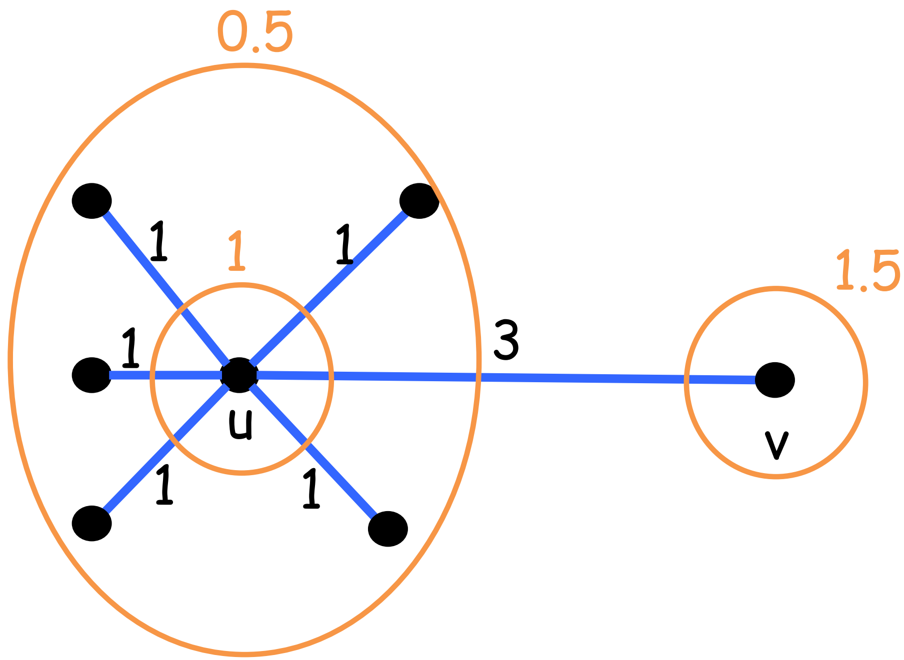
S_1={u,v}



example: pruning step is needed

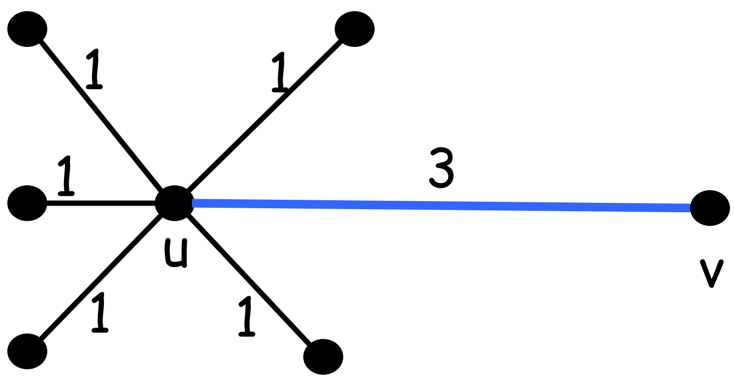
$S_1=\{u,v\}$

...pruning...



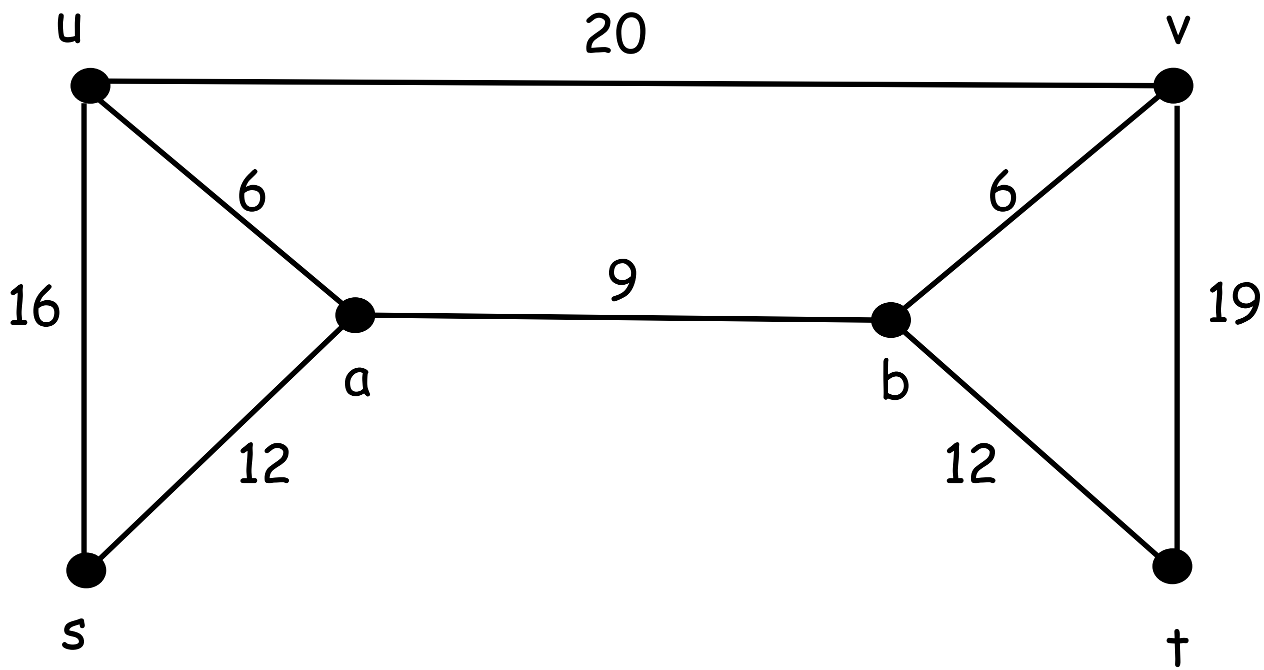
example: pruning step is needed

$S_1=\{u,v\}$



computed solution

$S_1=\{u,v\}$ $S_2=\{s,t\}$



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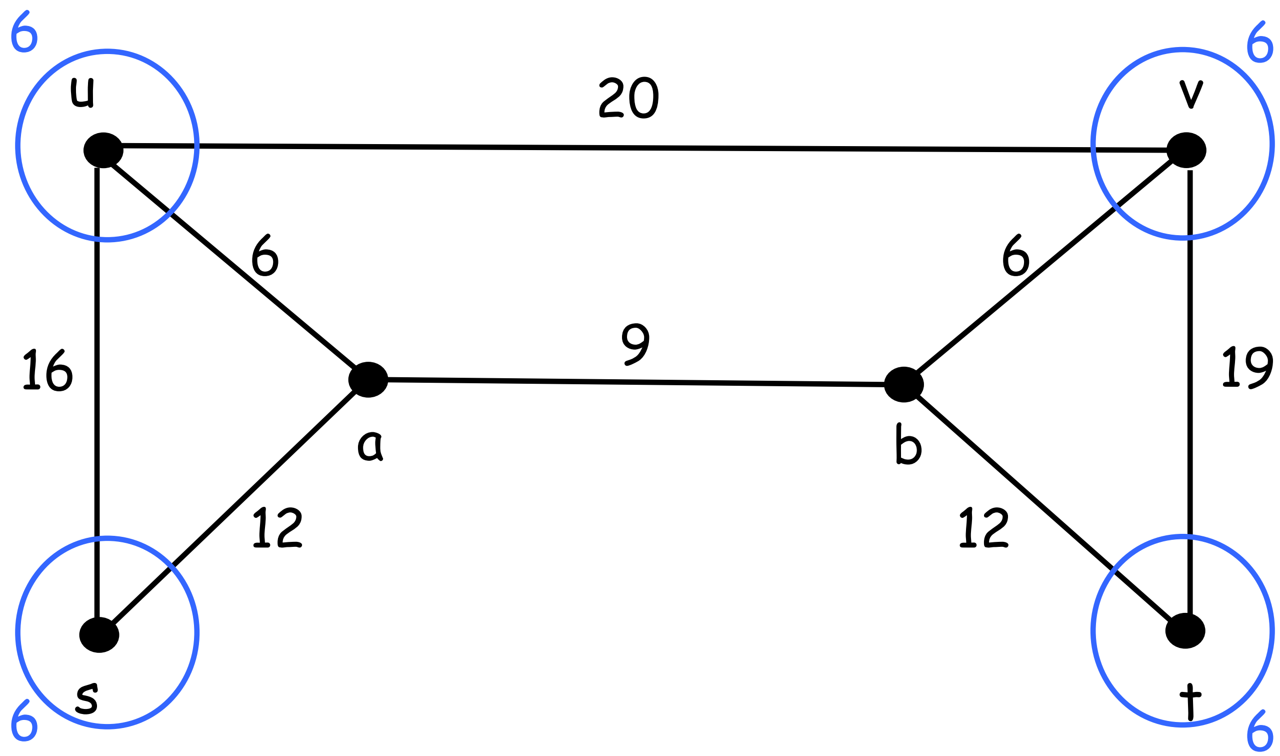

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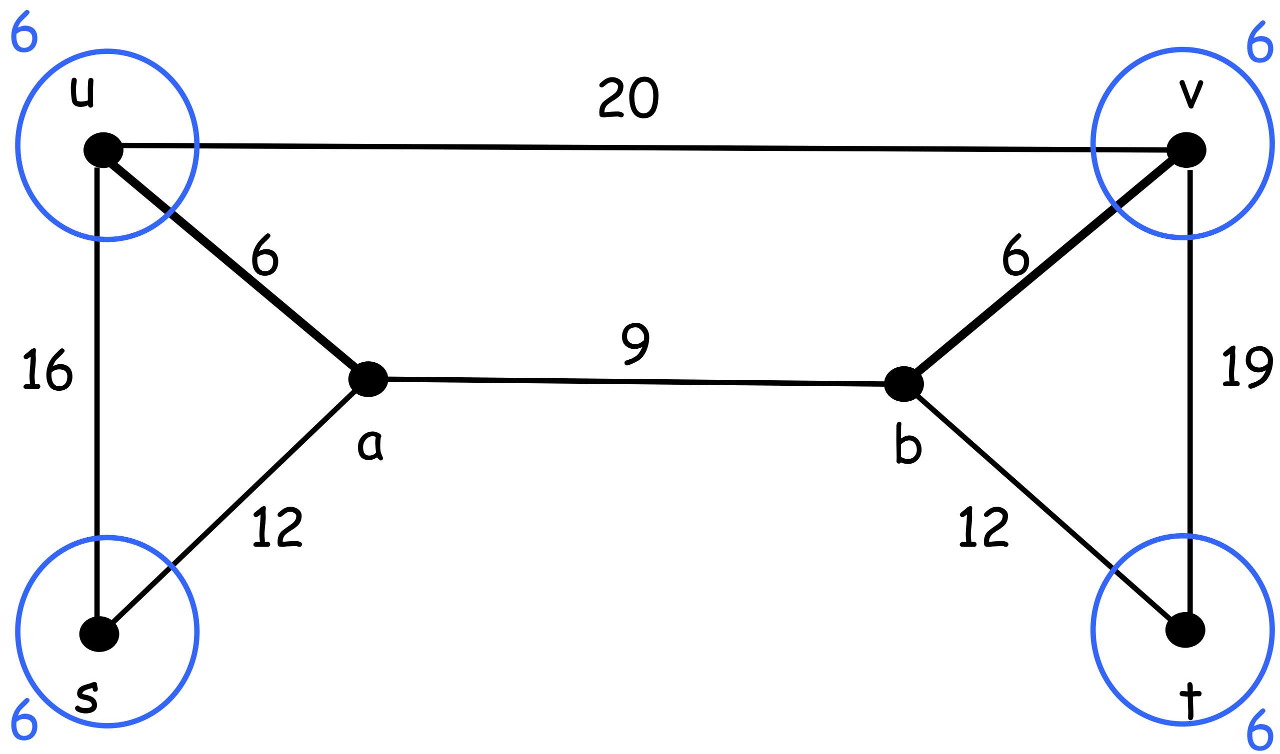

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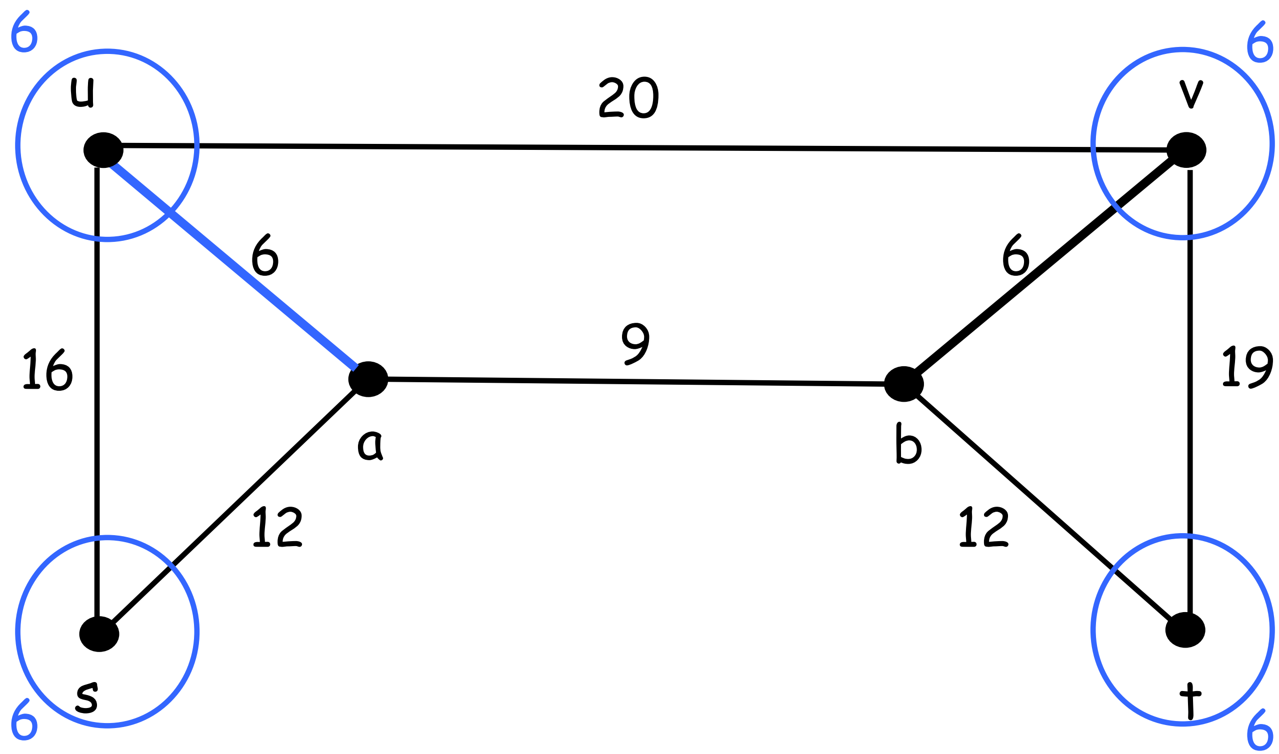

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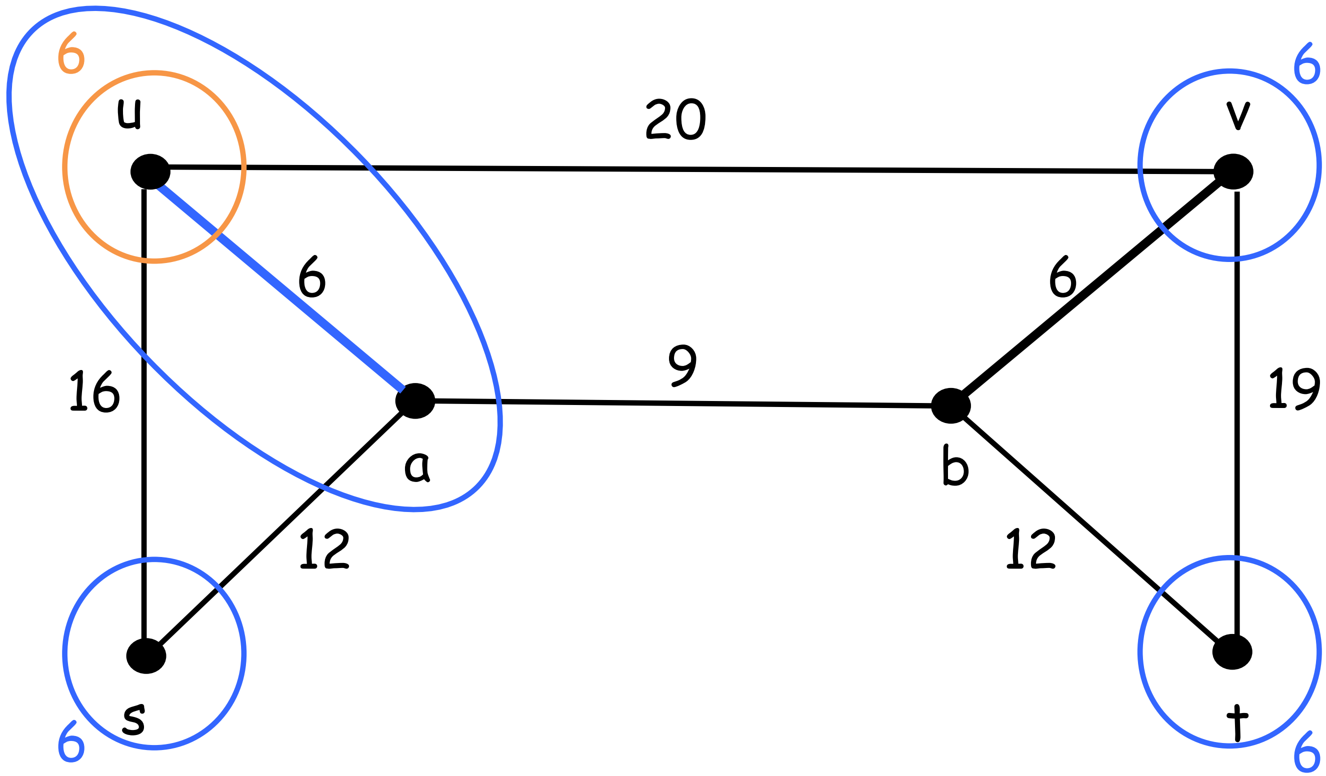

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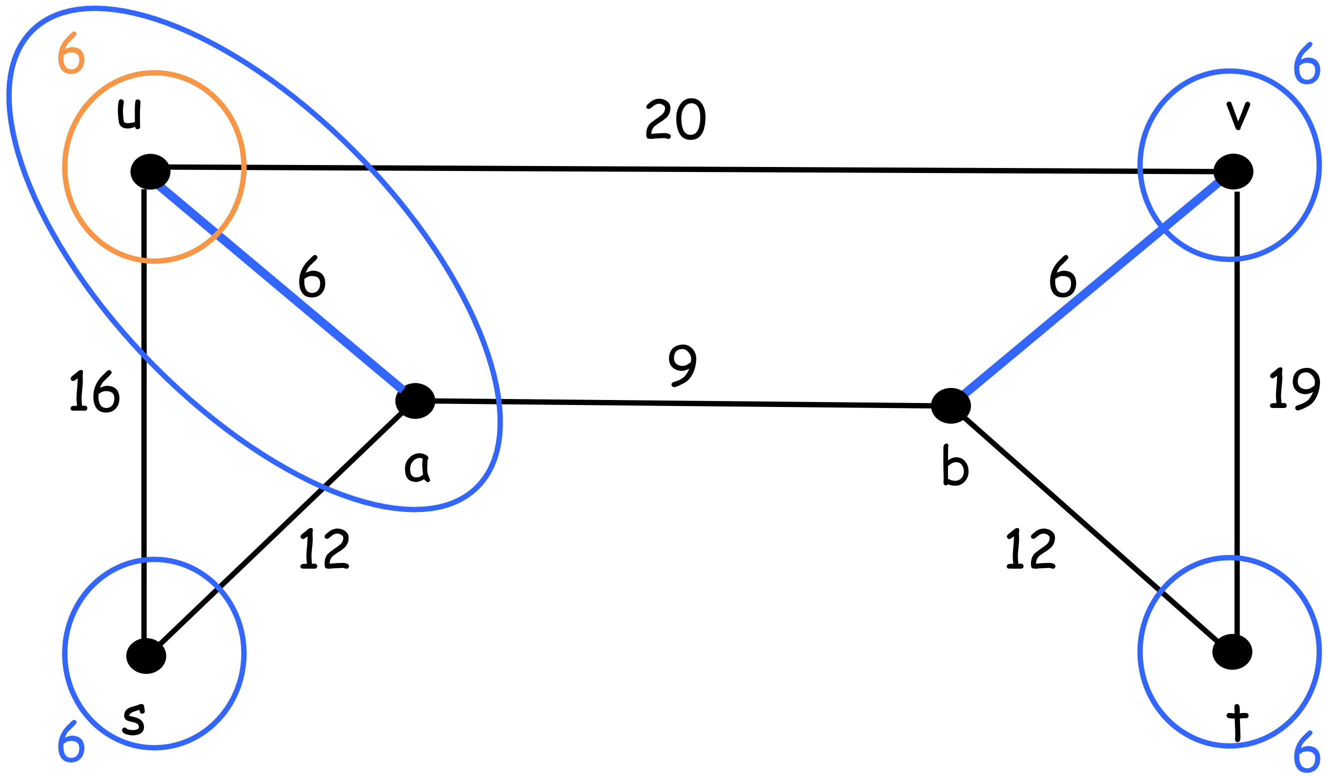

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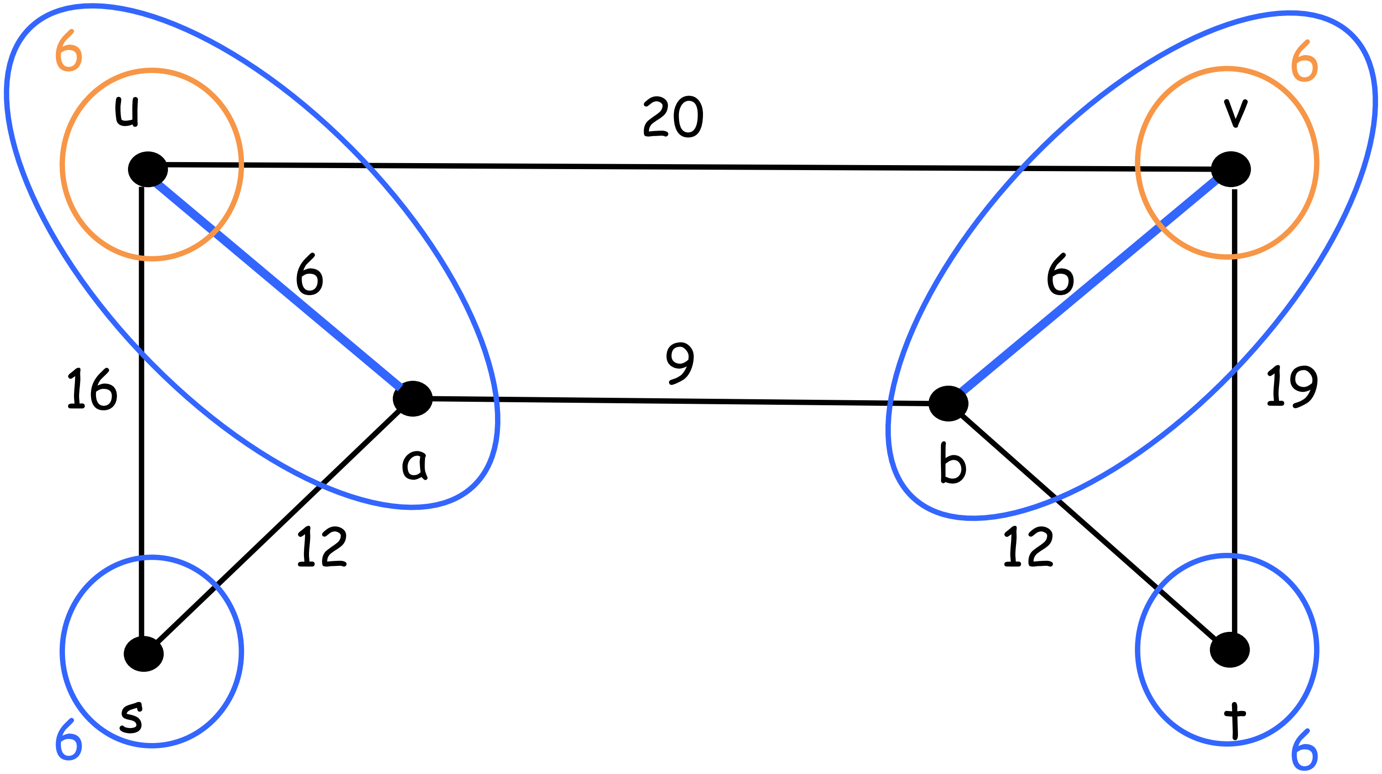

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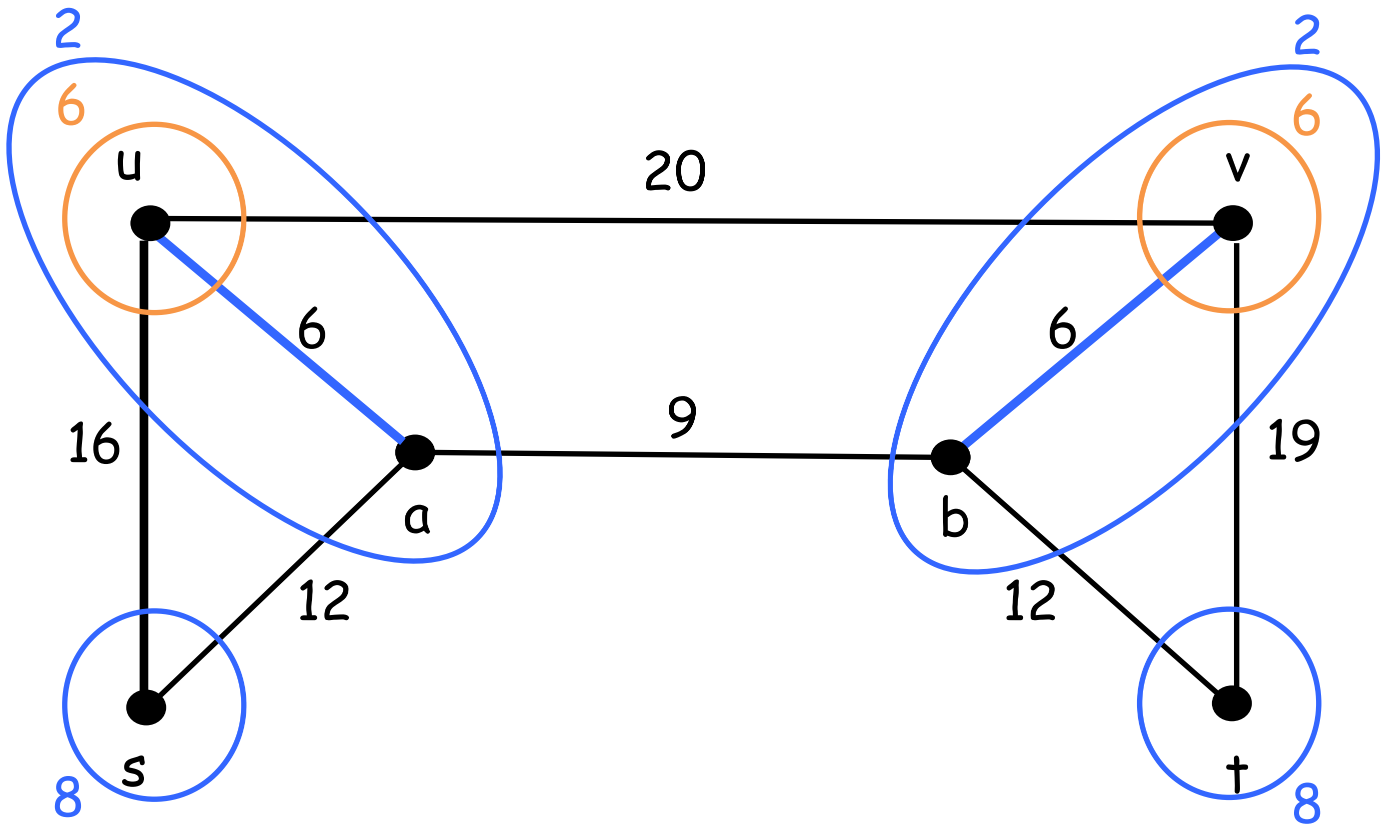

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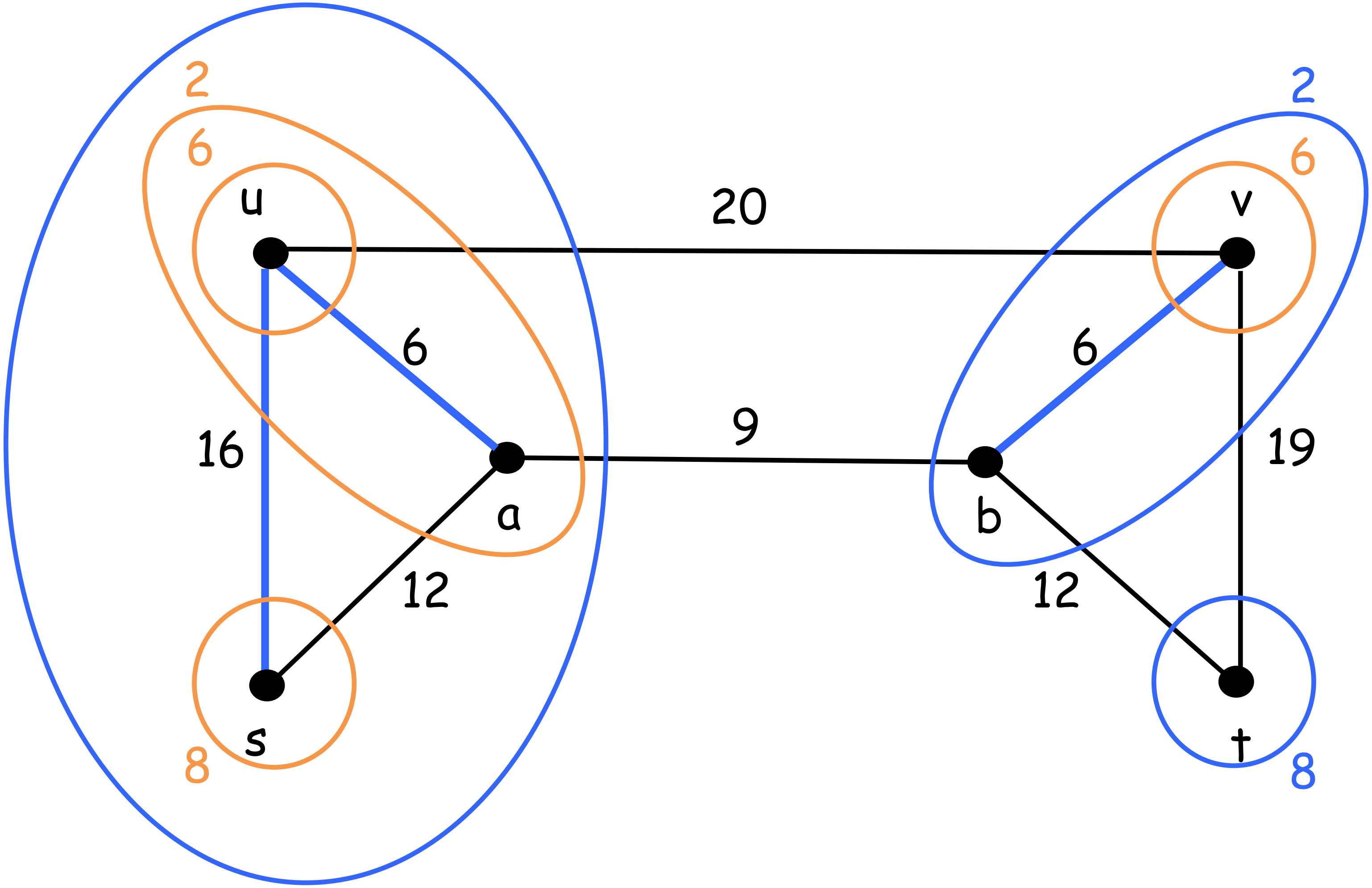

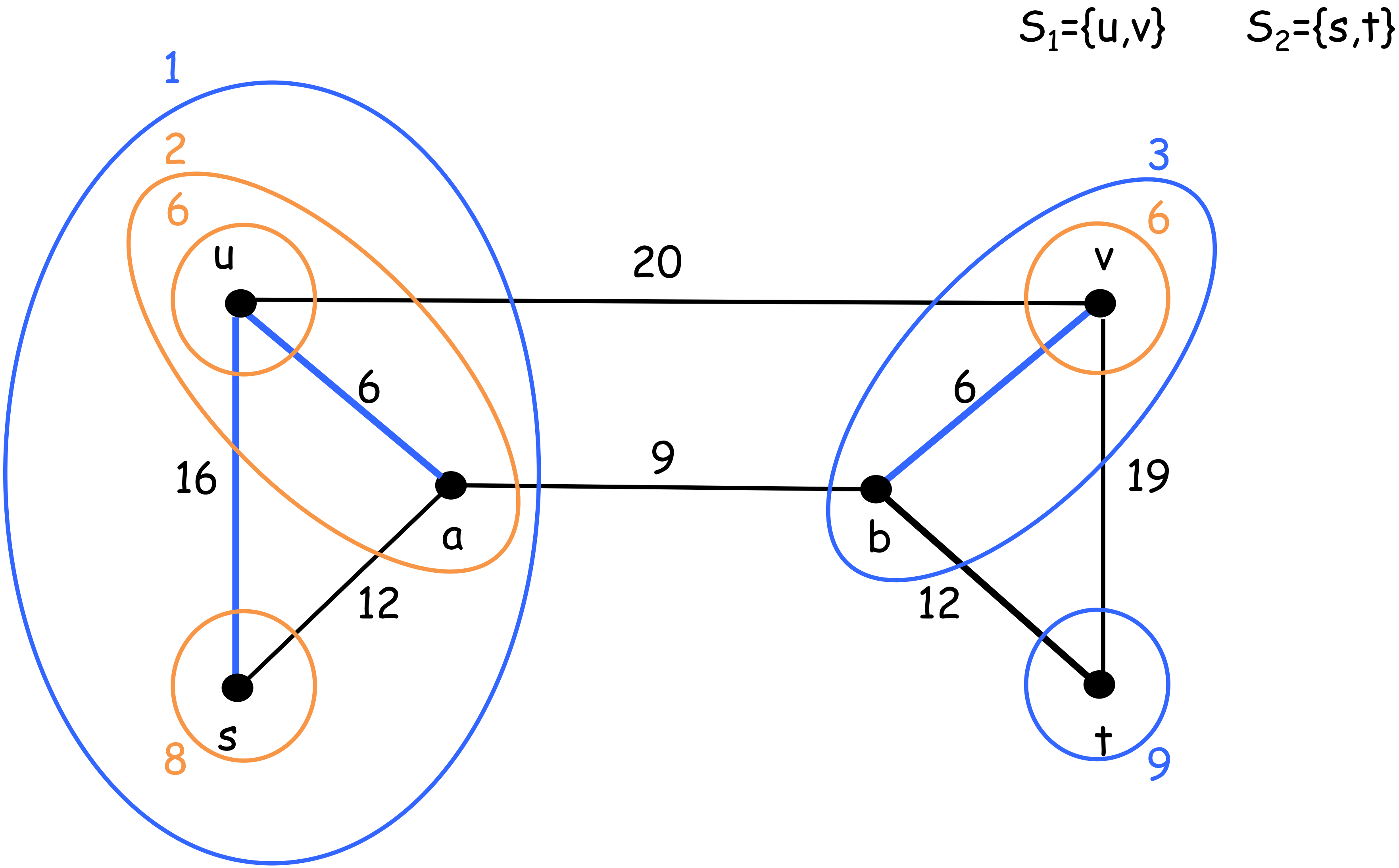
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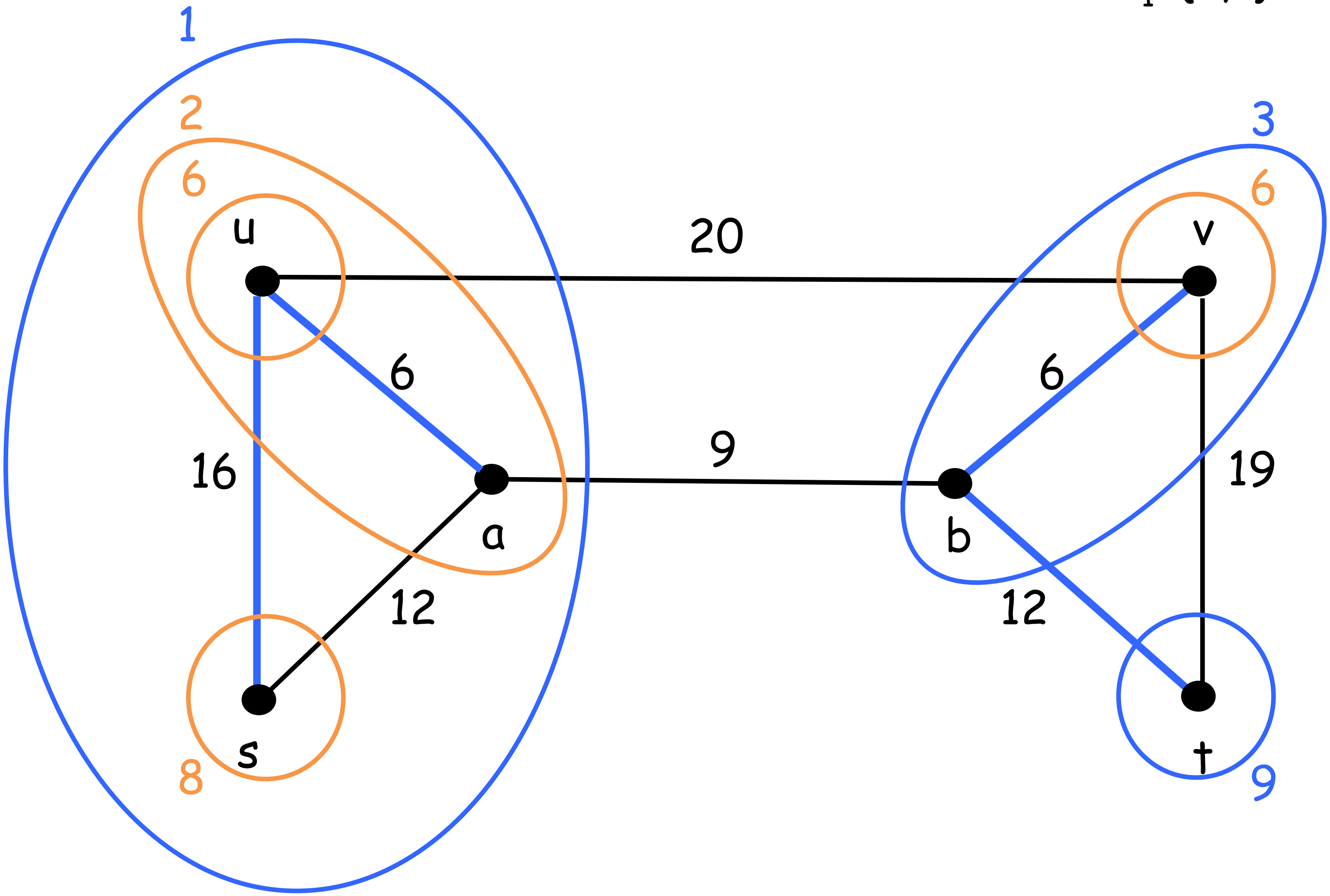
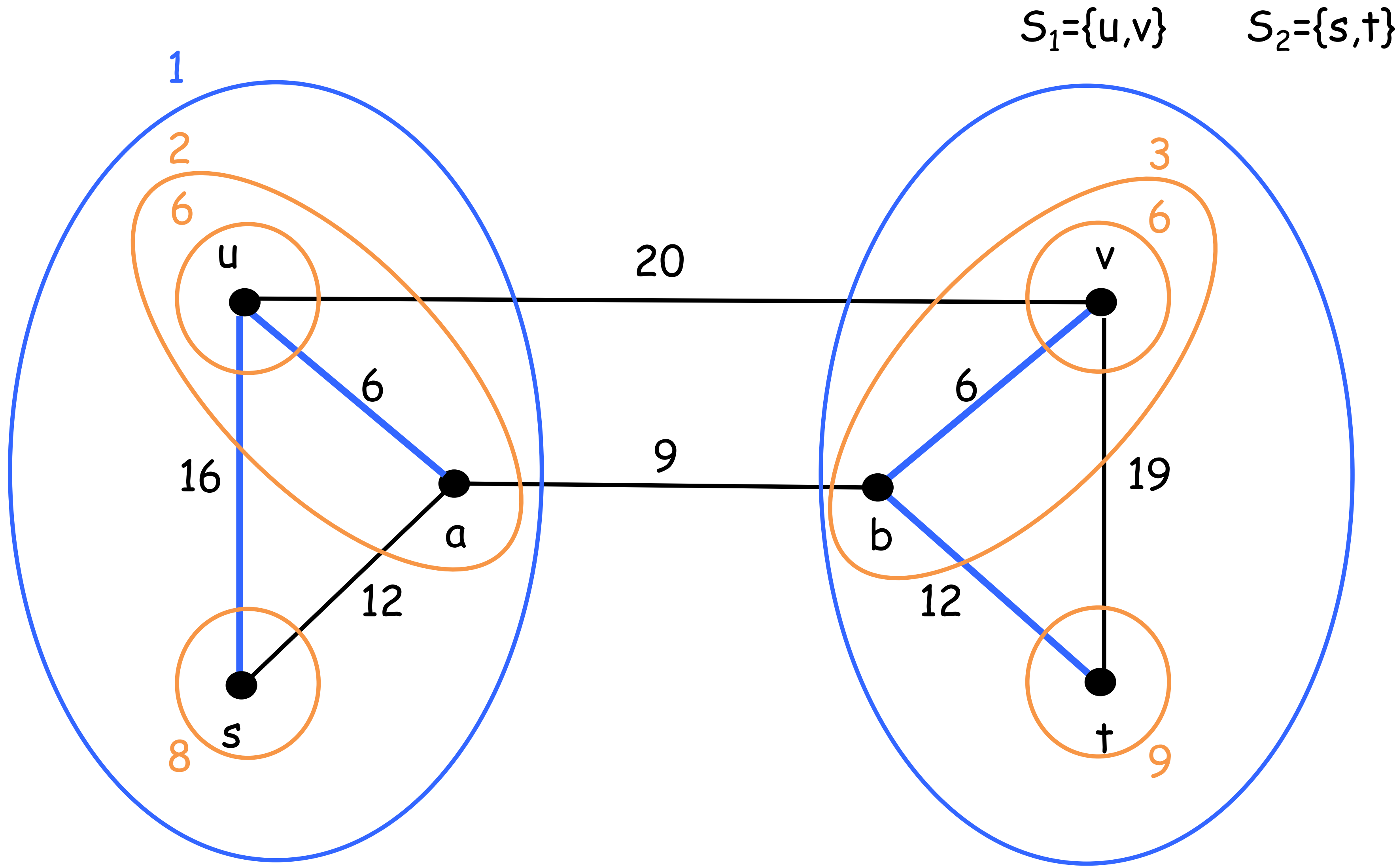
$S_1 = \{u, v\}$ $S_2 = \{s, t\}$





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S_2={s,t}
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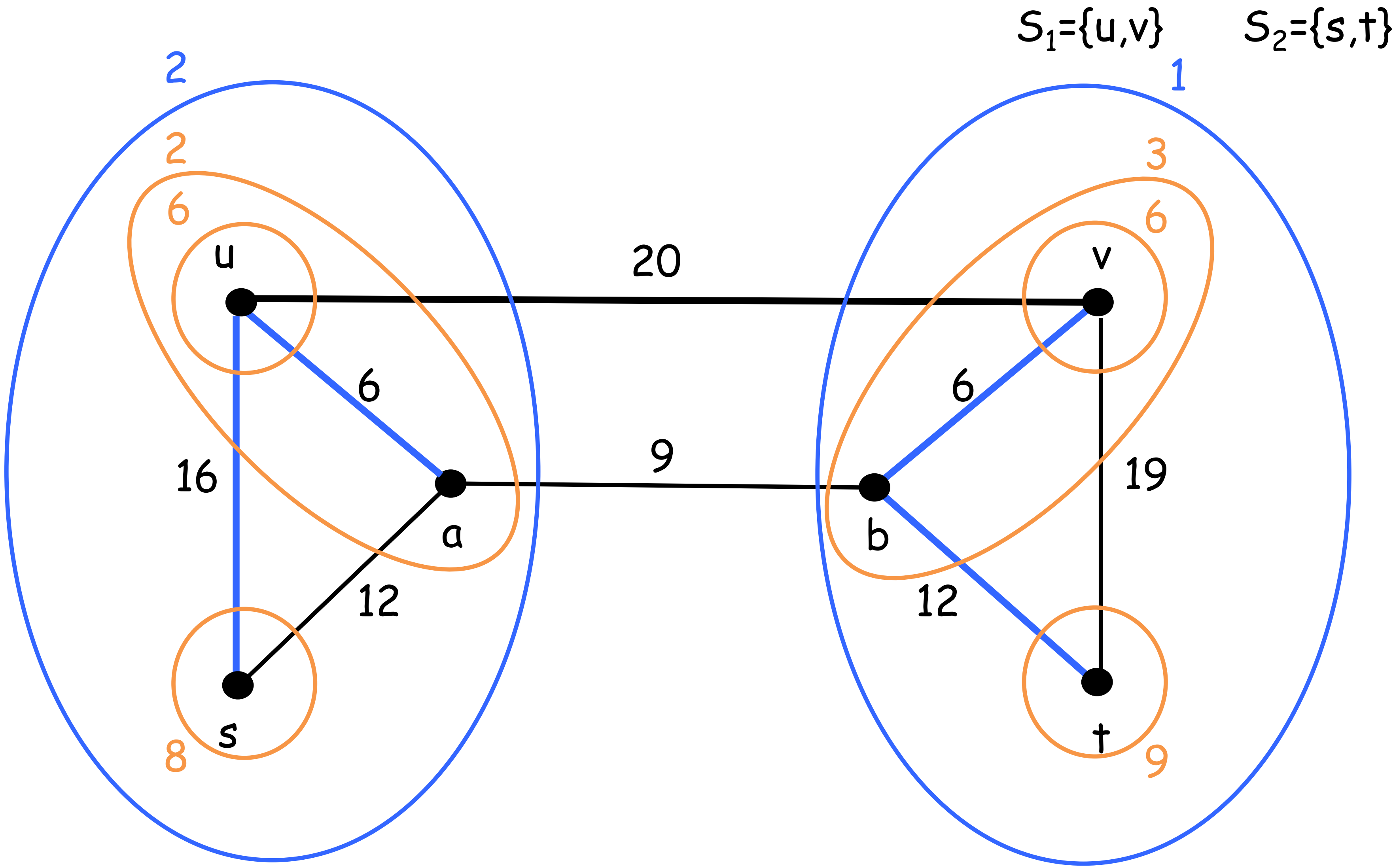

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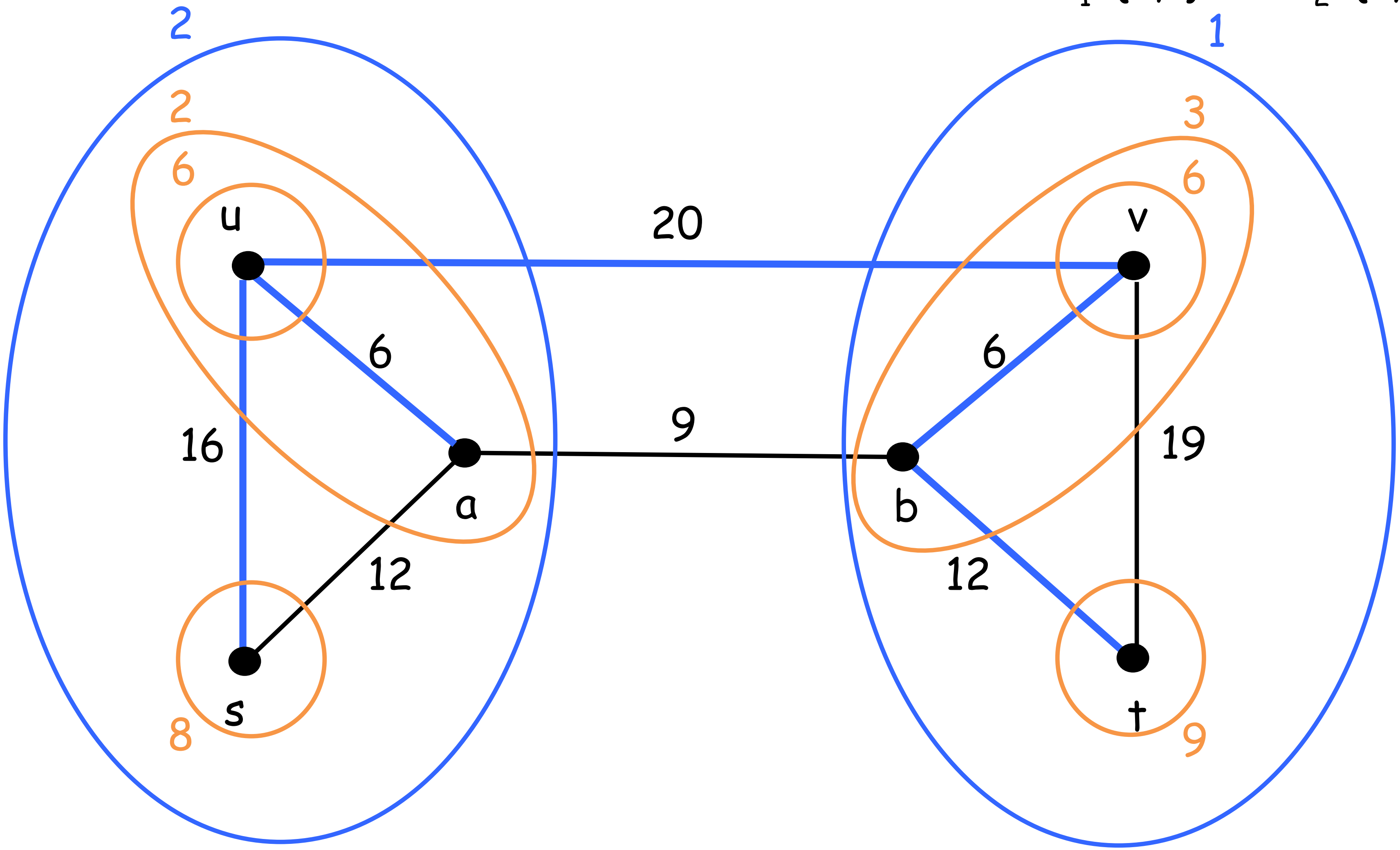
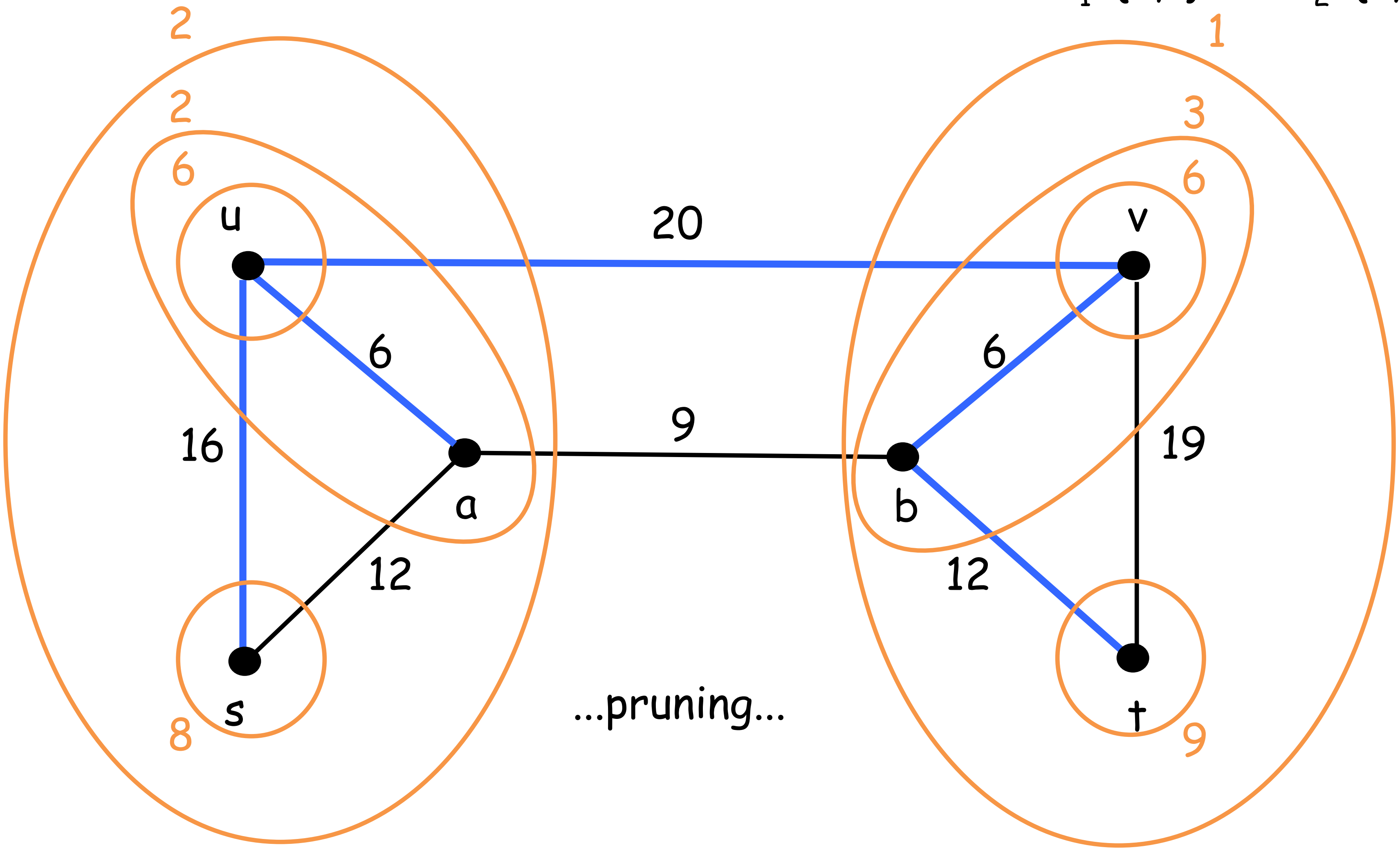
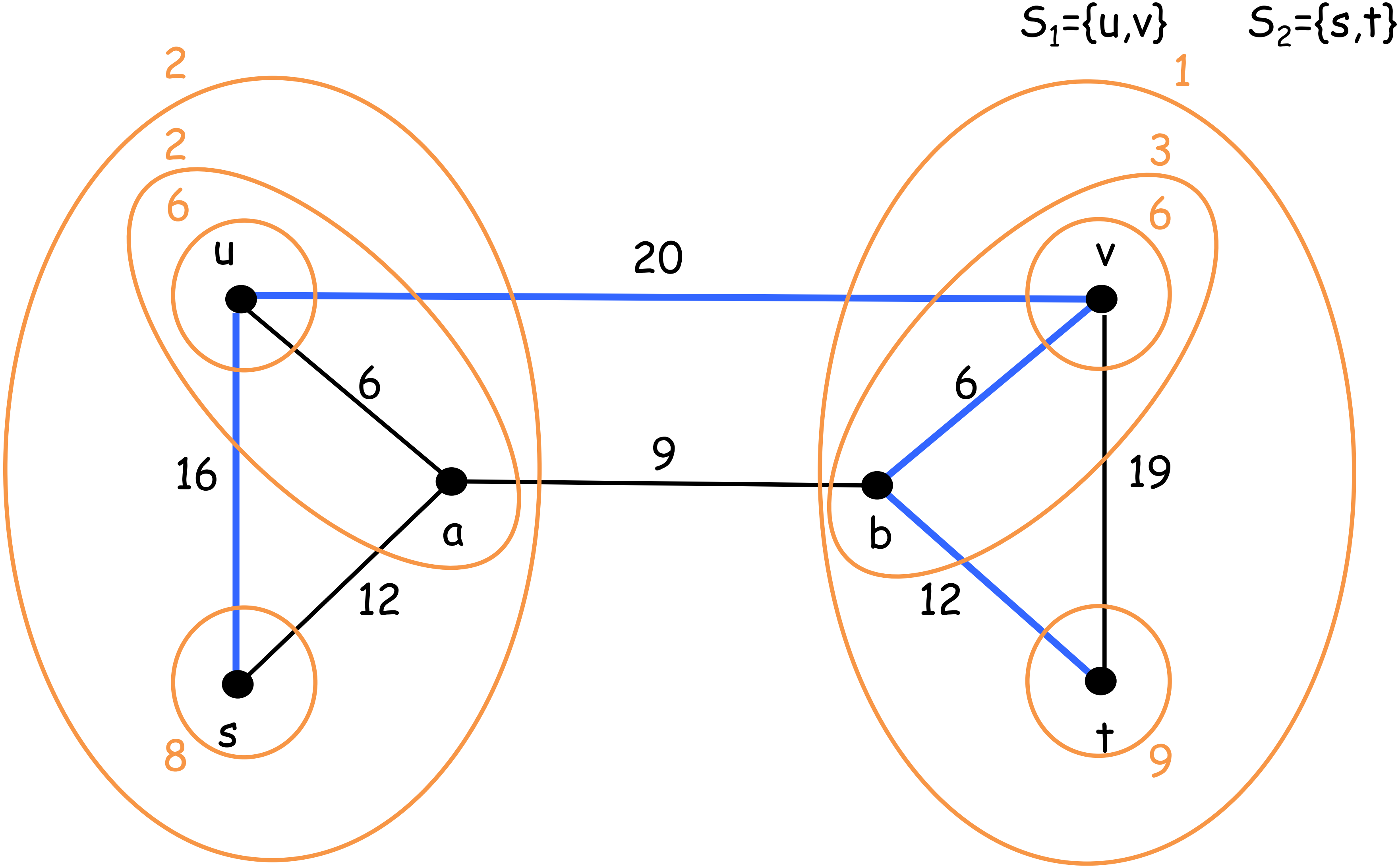
$S_1=\{u,v\}$ $S_2=\{s,t\}$



S_1={u,v} S_2={s,t}





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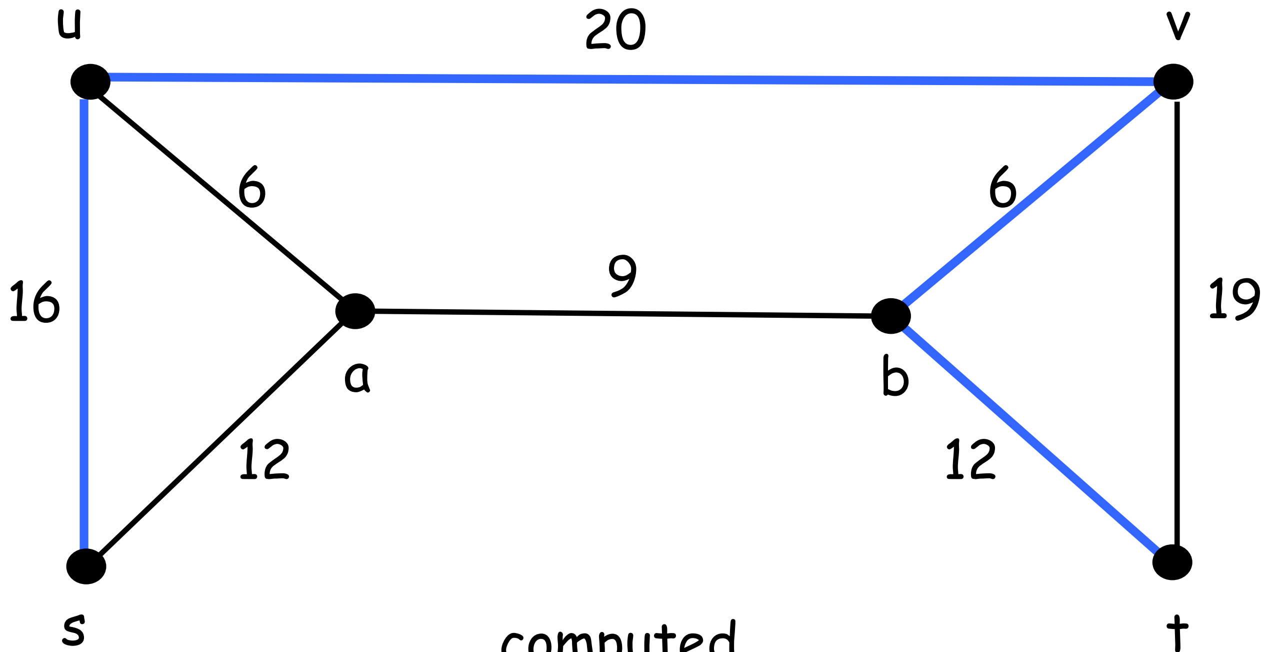
$S_1=\{u,v\}$ $S_2=\{s,t\}$



computed solution

Theorem

The algorithm is a 2-approximation algorithm for the SF problem.

proof

The primal computed solution F' is feasible

The dual solution is feasible, since there is no overtight edge

We claim that: $\sum_{e \in F'} c_e \leq 2 \sum_{S \subseteq V} y_S$

$$\sum_{e \in F'} c_e = \sum_{e \in F'} \left(\sum_{S: e \in \delta(S)} y_S \right) = \sum_{S \subseteq V} \left(\sum_{e \in \delta(S) \cap F'} y_S \right) = \sum_{S \subseteq V} \deg_{F'}(S)\, y_S$$

since every picked edge is tight

changing the order of summation

$\deg_{F'}(S)$ = # of picked edges crossing the cut (S,S'=V\S) we need to show that: $\sum_{S \subseteq V} \deg_{F'}(S)\, y_S \le 2 \sum_{S \subseteq V} y_S$

We prove a stronger claim:
- in each iteration the increase in the l.h.s. $\le$ the increase of in r.h.s.

Consider an iteration, and let Δ be the extent to which active sets were raised in this iteration.

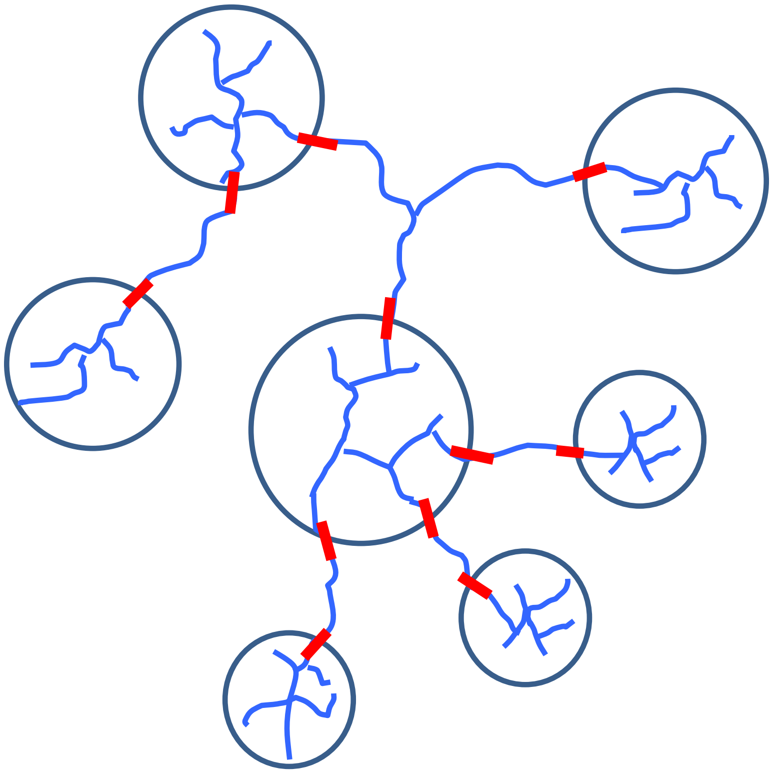
we need to show that:

$$\Delta \times \left(\sum_{S \text{ active}} \deg_{F'}(S) \right) \le 2\,\Delta \times (\#\text{ of active sets})$$

$$\sum_{S \text{ active}} \deg_{F'}(S) \le 2\,(\#\text{ of active sets})$$

$$\sum_{S \text{ active}} \deg_{F'}(S) \leq 2\ (\#\text{ of active sets})$$

F' is a forest with no redundant edges



#of red sticks ≤ 2 #of blue circles

shrink blue circles and root the obtained tree arbitrarily

every shrunk circle pays for:

- its red stick towards its parent
- parent's red stick towards it

Thus:

$$\sum_{e \in F'} c_e \leq 2 \sum_{S \subseteq V} y_S \leq 2\ OPT$$