Analisi di Reti (mod 2)

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Mechanism Design Without Money

reference:
Twenty Lectures in Algorithmic Game Theory
Tim Roughgarden
Chapters 9 & 10.

Motivations and applications

motivations:

- sometimes the use of money is infeasible or illegal;

applications:

- voting;
- organ donation;
- school choice;
- ...

what can be done:

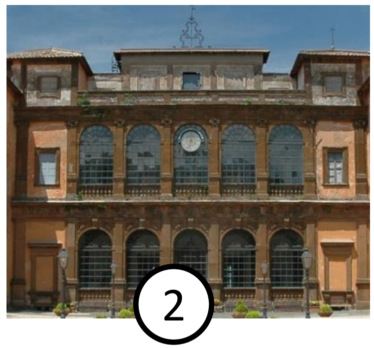
- strong impossibility results in general
- still some of mechanism design's greatest hits

House allocation problem
&
Top Trading Cycle algorithm

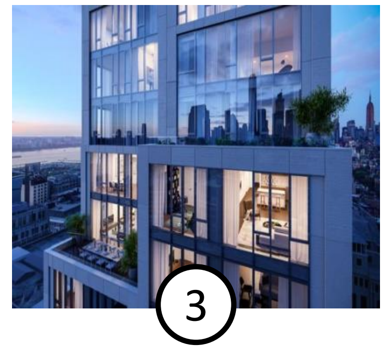
House allocation problem

- n agents
- each agent initially owns a house
- preferences (type) of the agent i: a total ordering over the n houses
  - an agent need not prefer her own house over the others

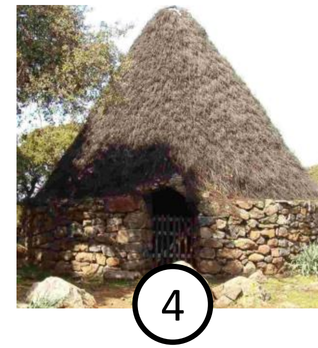
1's preferences:
1,3,2,4



2's preferences:
1,3,2,4



3's preferences:
1,4,3,2



4's preferences:
1,2,3,4

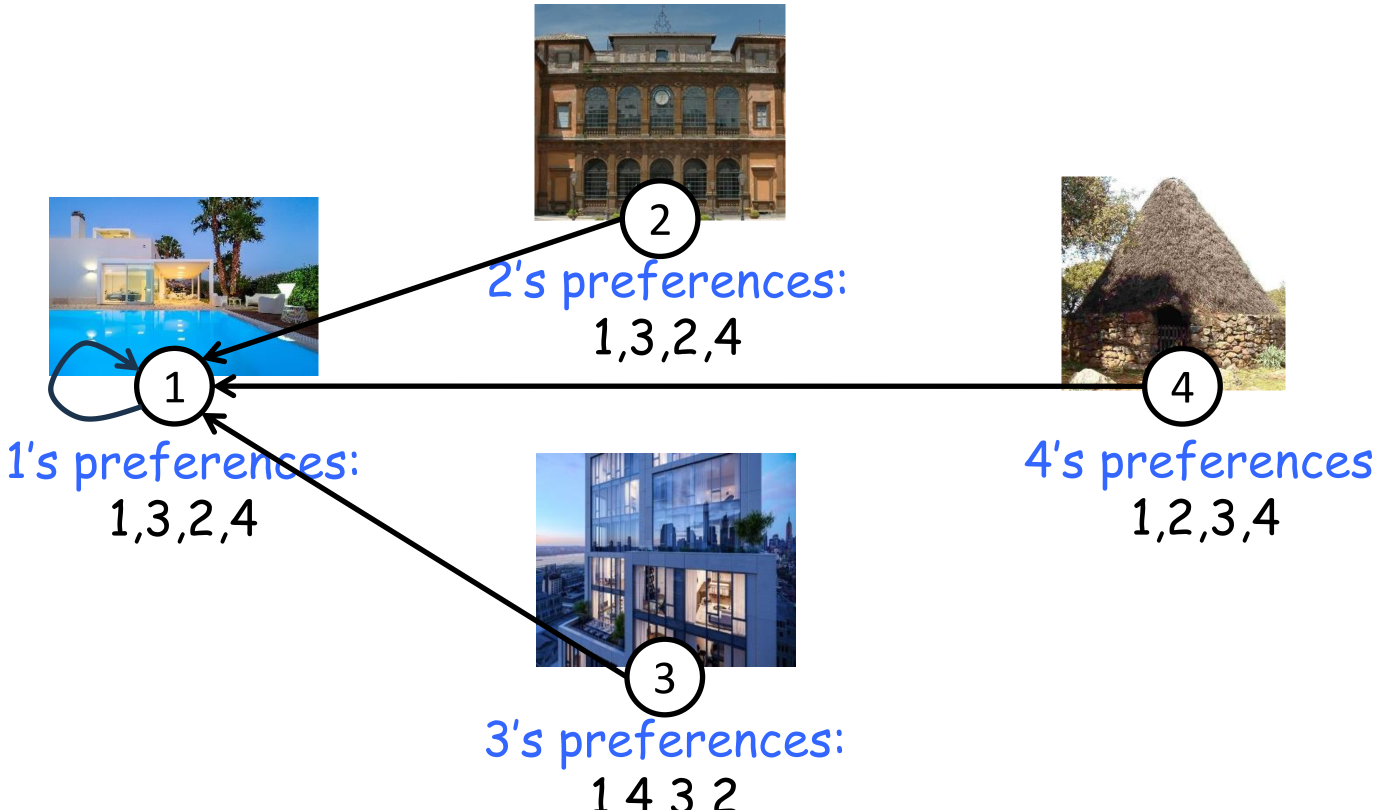
goal 1: reallocate the houses to make the agents better off

goal 2: do it in a way agents cannot manipulate the allocation

⇨ Top Trading Cycle (TTC) algorithm

TTC algorithm (idea)

- allocation proceeds in iterations
- at each iteration:
  - each remaining agent participates with her own house
  - each remaining agent points to her favorite still available house
  - look at (disjoint) cycles formed and perform the reallocation suggested by the cycles
  - remove the agents of the cycles



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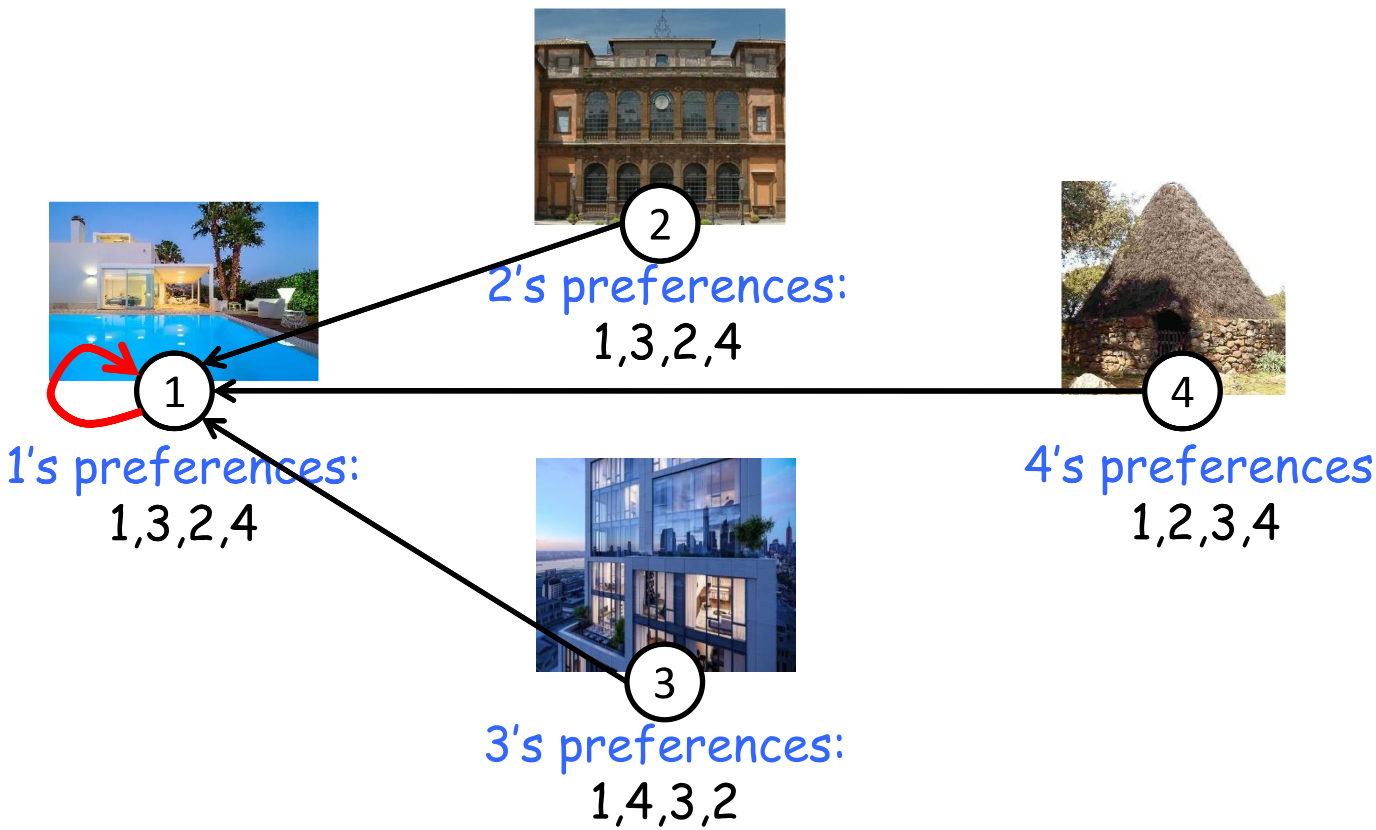

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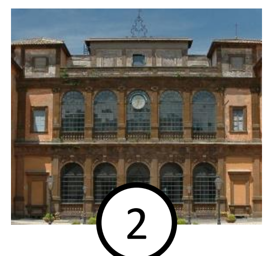

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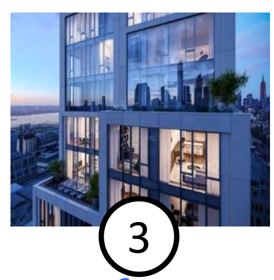
1,3,2,4

1's preferences:
1,3,2,4

4's preferences:
1,2,3,4



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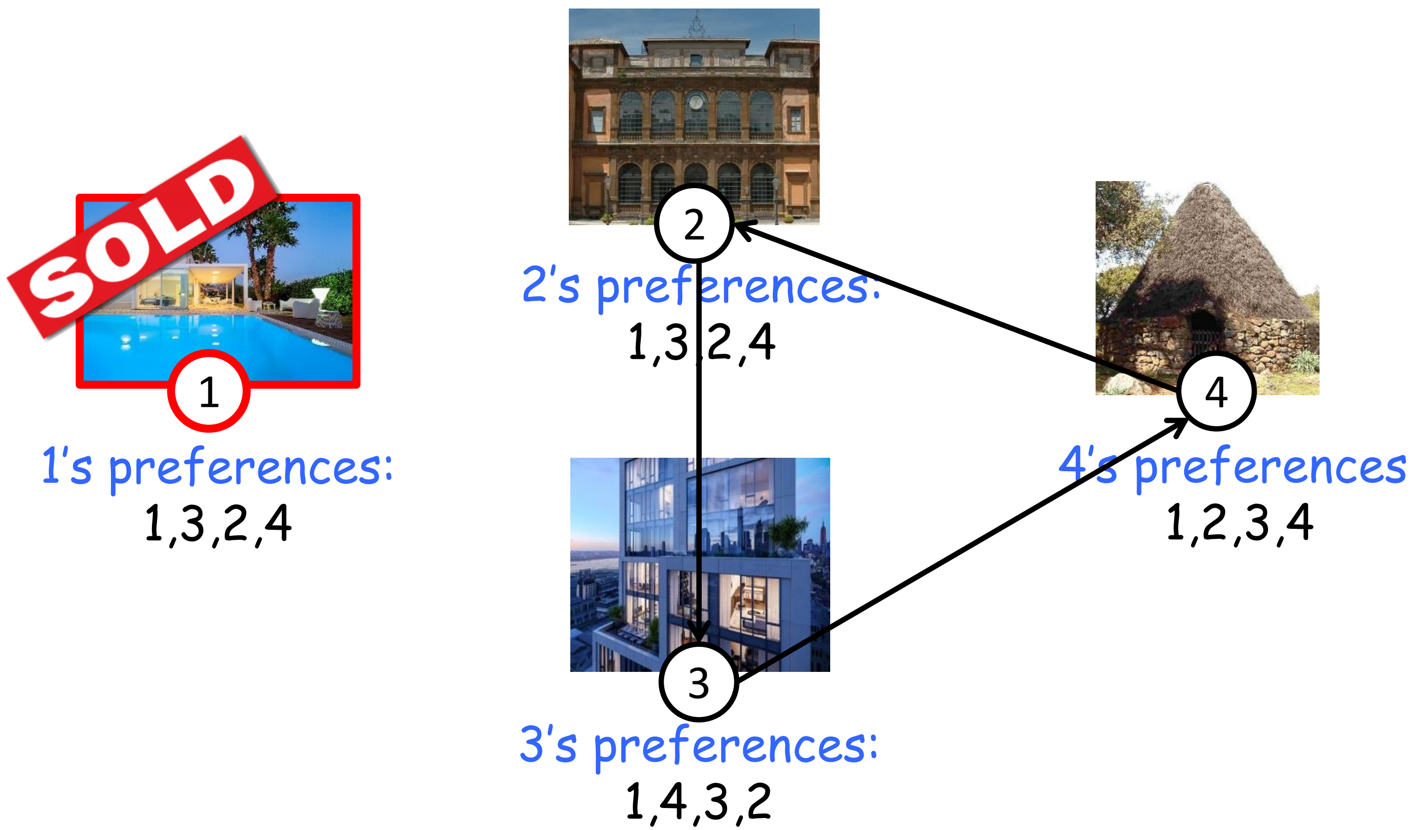

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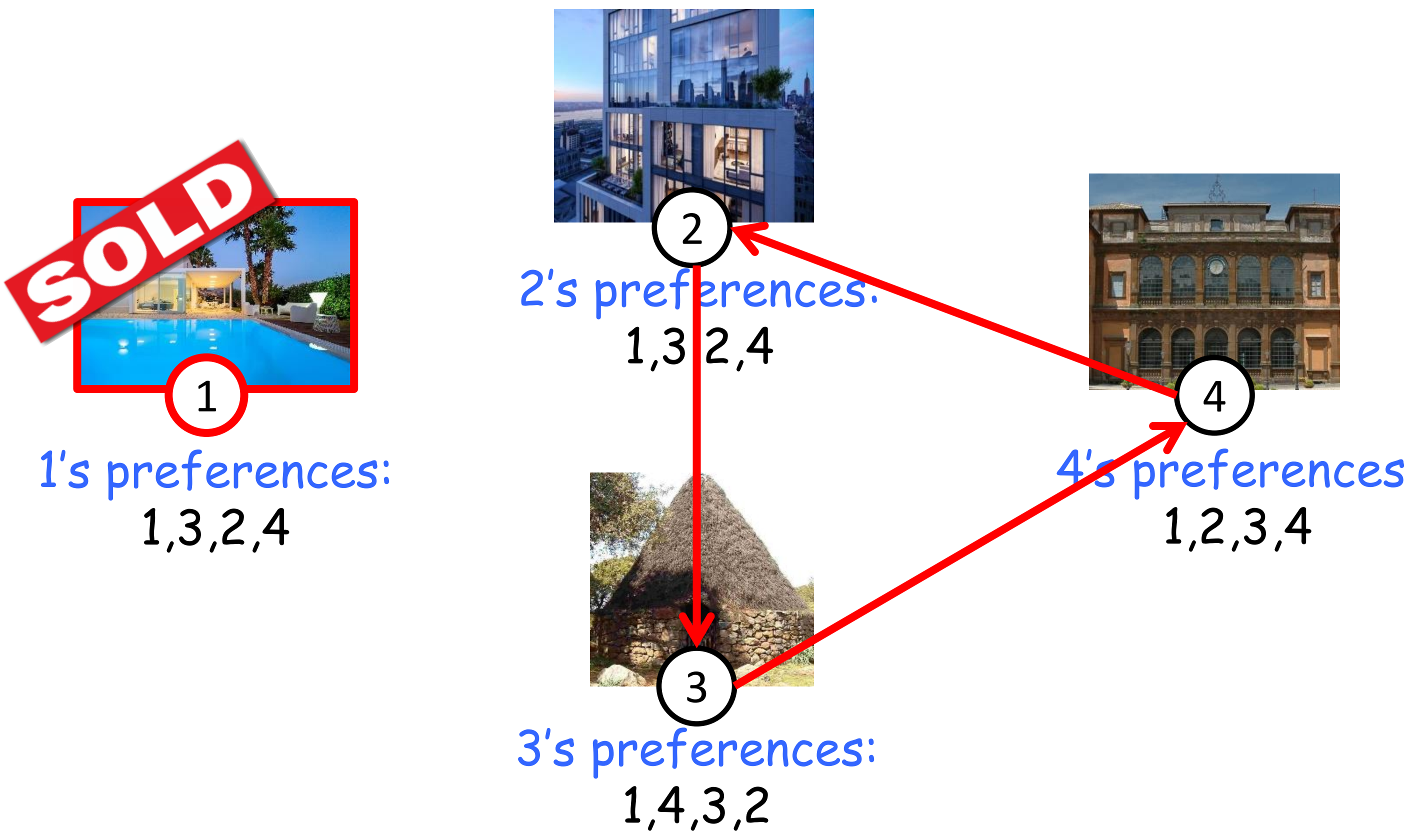

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Top Trading Cycle (TTC) Algorithm

initialize N to the set of all agents

while $N \neq \theta$ **do**

form the directed graph G with vertex set N and edge set $\{(i, \ell)$: i's favorite house within N is owned by $\ell\}$

compute the directed cycles $C_1, \ldots, C_h$ of G

```
// self-loops count as directed cycles
// cycles are disjoint
```

for each edge (i, ℓ) of each cycle $C_1, \ldots, C_h$ **do** reallocate ℓ's house to agent i

remove the agents of $C_1, \ldots, C_h$ from N

- G has at least one directed cycle, since traversing a sequence of outgoing edges must eventually repeat a vertex
- Because all out-degrees are 1, these cycles are disjoint

Some properties of the TTC algorithm

Lemma

Let N_k denote the set of agents removed in the k-th iteration of the TTC algorithm. Every agent of N_k receives her favorite house outside of those owned by $N_1 \cup \ldots \cup N_{k-1}$, and the original owner of this house is in N_k.

Theorem

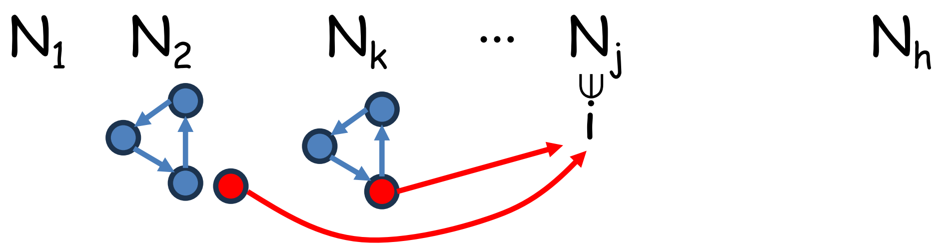
When the TTC algorithm is used for the reallocation, for every agent, it is a dominant strategy to report truthfully.

proof

Fix an agent i and reports by the others.

Assume i reports truthfully.

Let N_k be the set of agents removed in the k-th iteration.



By the previous lemma it suffices to prove:

Claim: no misreport can net i a house of an agent in $N_1 \cup \ldots \cup N_{j-1}$.

For each k<j:

- no agent in N_k point to i at iteration k

  otherwise i would belong to N_k

- no agent in N_k point to i at iteration <k

  otherwise that agent would point to i at iteration k

■

Why the previous theorem is nice?

Notice: the mechanism that never reallocates anything is also truthful

Consider a reallocation of the houses. A subset of agents forms a blocking coalition for this reallocation if they can internally reallocate their initial houses to make some member better off while making no member worse off (w.r.t. the proposed reallocation).

A core allocation is a reallocation with no blocking coalitions.

Theorem

For every house allocation problem, the allocation computed by the TTC algorithm is the unique core allocation.

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 1

every agent in N_1 receives her first choice

⇒ N_1 is a blocking coalition for any allocation $\neq$ from the TTC one

⇒ every core allocation must agree with the TTC one on agents in N_1

⇒ every agent in N_2 receives her first choice outside N_1

⇒ N_2 is a blocking coalition for any allocation $\neq$ from the TTC one that agrees with TTC on N_1

⇒ every core allocation must agree with the TTC one on agents in N_1 and N_2

⇒ ... ■

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 2

consider an arbitrary subset S of agents and an internal reallocation of their houses

the reallocation partitions S into directed cycles

consider any such cycle *C*

- if *C* contains two agents $i \in N_j$ and $t \in N_k$ with $j<k$

  ⇒ i is worse off than in the TTC allocation

- if *C* contains agents all in N_k but there is an agent i that does not receives her favorite choice in N_k

  ⇒ i is worse off than in the TTC allocation

⇒ the TTC allocation has no blocking coalitions. ■

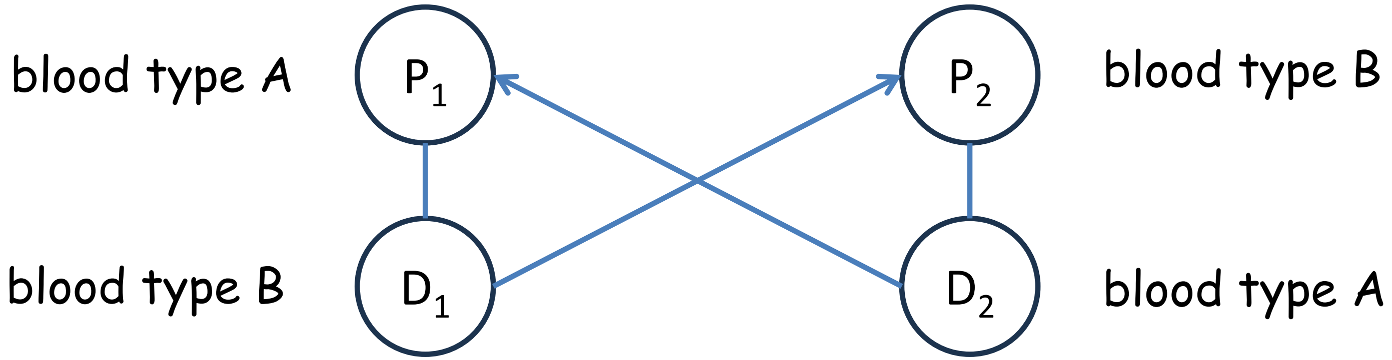
Kidney Exchange

Background

- many people suffer from kidney failure and need a kidney transplant
- In US more than 100.000 people are on the waiting list for such a transplant
- old idea (used also for other organs): deceased donors
- special feature for kidneys: living donors
  (a healthy person can survive just fine with a single kidney)

compatibility issues:
- having a living kidney donors is not always enough
- a patient-donor pair can be incompatible
  (primary culprits for incompatibility: blood and tissue types need to match)

idea: swap donors!



kidney exchange

- initially, few kidney exchanges were done on an ad-hoc basis
- This made clear the need of a system to organize kidney exchanges
- a system where patient-donor pairs can register and be matched with others

goal: how such an exchange system can be designed in order to enable as many matches as possible?

- currently, monetary compensation is illegal in most of the countries

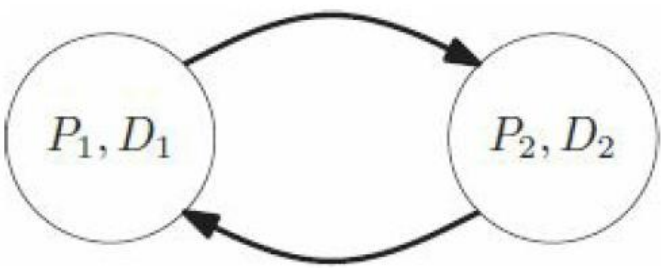
⇒ problem naturally modeled as a mechanism design problem without money

first idea: model the problem as a house allocation problem

- each patient-donor pair treated as a agent-house
  - patient=agent
  - donor=house
- a patient's total ordering over the donors can be defined according to the estimated probability of a successful kidney transplant

- use the TCC algorithm to find kidney exchanges

- the reallocation of donors suggested by the TCC algorithm can only improve every patient's probability of a successful transplant



Good case for the TTC algorithm.
- each circle represents an incompatible patient-donor pair
- each arrow represents a kidney transplant from the donor in the first pair to the patient of the second pair.

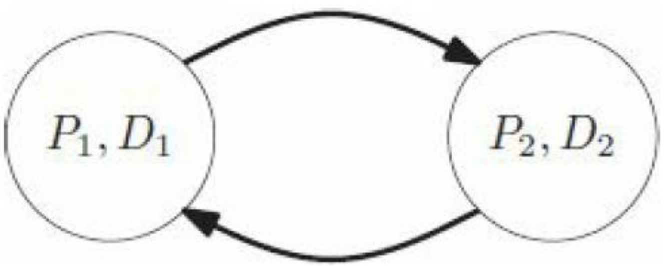
first idea: model the problem as a house allocation problem

some technical issues:
- need to manage patients without a donor (agent without a house)
- need to manage deceased donors (house without an agent/owner)

the TTC algorithm & its incentive guarantee can be extended to this more general setting (with some non-trivial extra work)

some more important issues:
- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)

how many surgeries? 4



what if P_1-D_2 surgeries today and P_2-D_1 surgeries tomorrow?

D_1 could renege on her offer
- P_1 unfairly got a kidney for free
- P_2 is still sick and can no longer participate in a kidney exchange

first idea: model the problem as a house allocation problem

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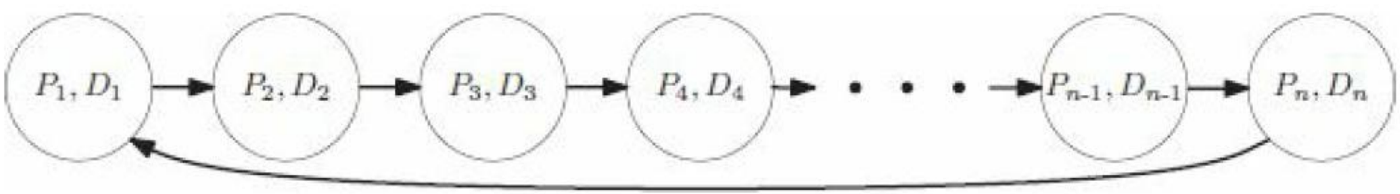

really bad case

first idea: model the problem as a house allocation problem

some technical issues:

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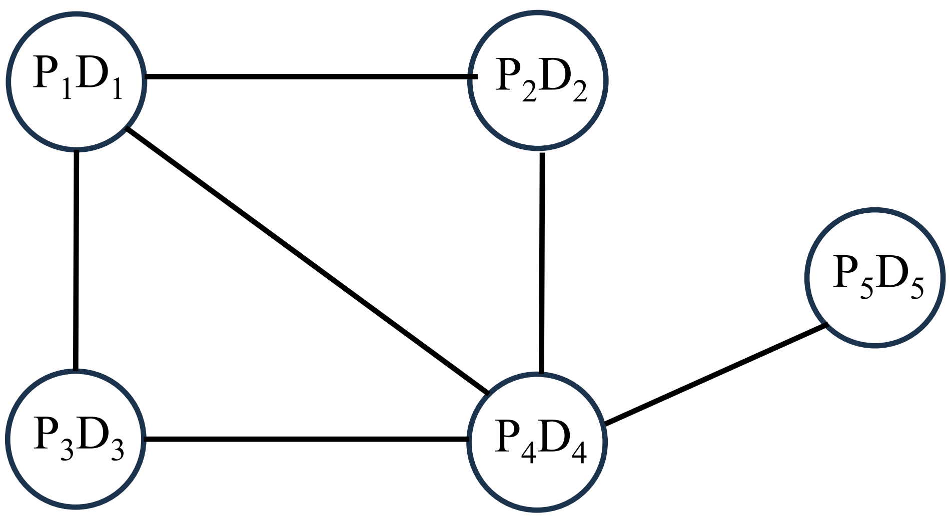
some more important issues:

- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)
- modeling patient's preferences as a total order over donors is overkill (binary preferences over donors are more appropriate)

⇒ change the model: use graph matching!

A matching of an undirected graph is a subset of the edges that share no endpoints.

The relevant graph for kidney exchanges:
- we have a vertex for each incompatible patient-donor pair
- there is an edge between (P_i,D_i) and (P_j,D_j) if and only if P_i and D_j are compatible & P_j and D_i are compatible



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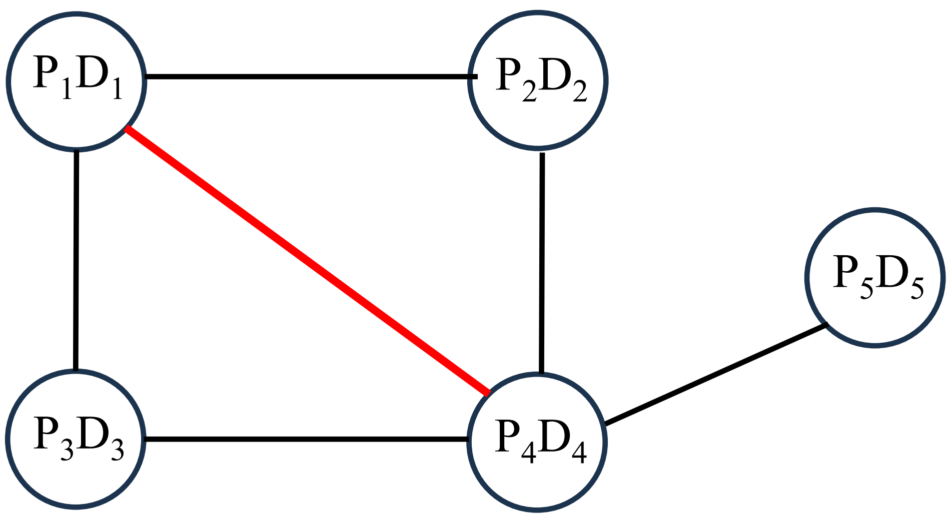

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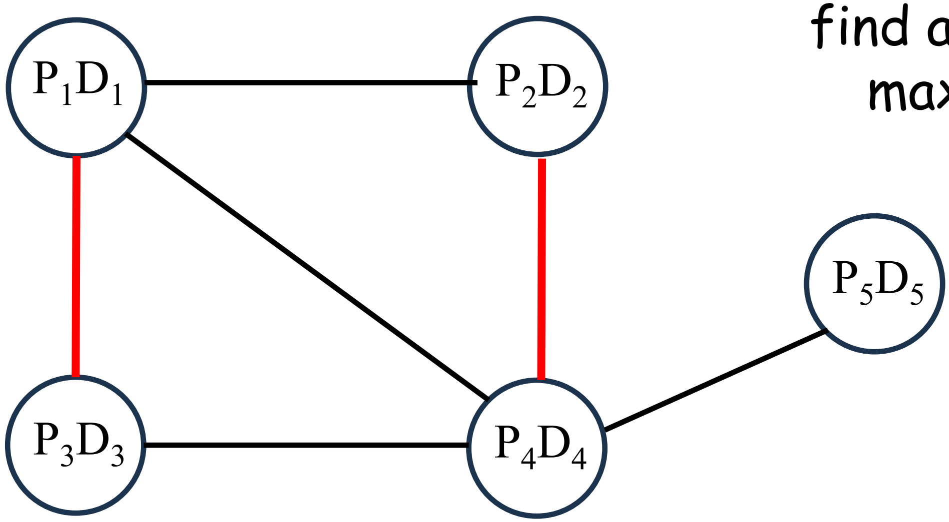


goal:
find a matching of maximum size

notice: we are restricting ourselves to 2-length cycles

How do incentives come into play?

we assume that each patient i

- has a set E_i of compatible donors belonging to other patient-donor pairs
- can report any subset $F_i \subseteq E_i$

it makes sense since:

- proposed kidney exchange can be refused by a patient for any reason
- a patient cannot credibly misreport extra donors with whom she is incompatible

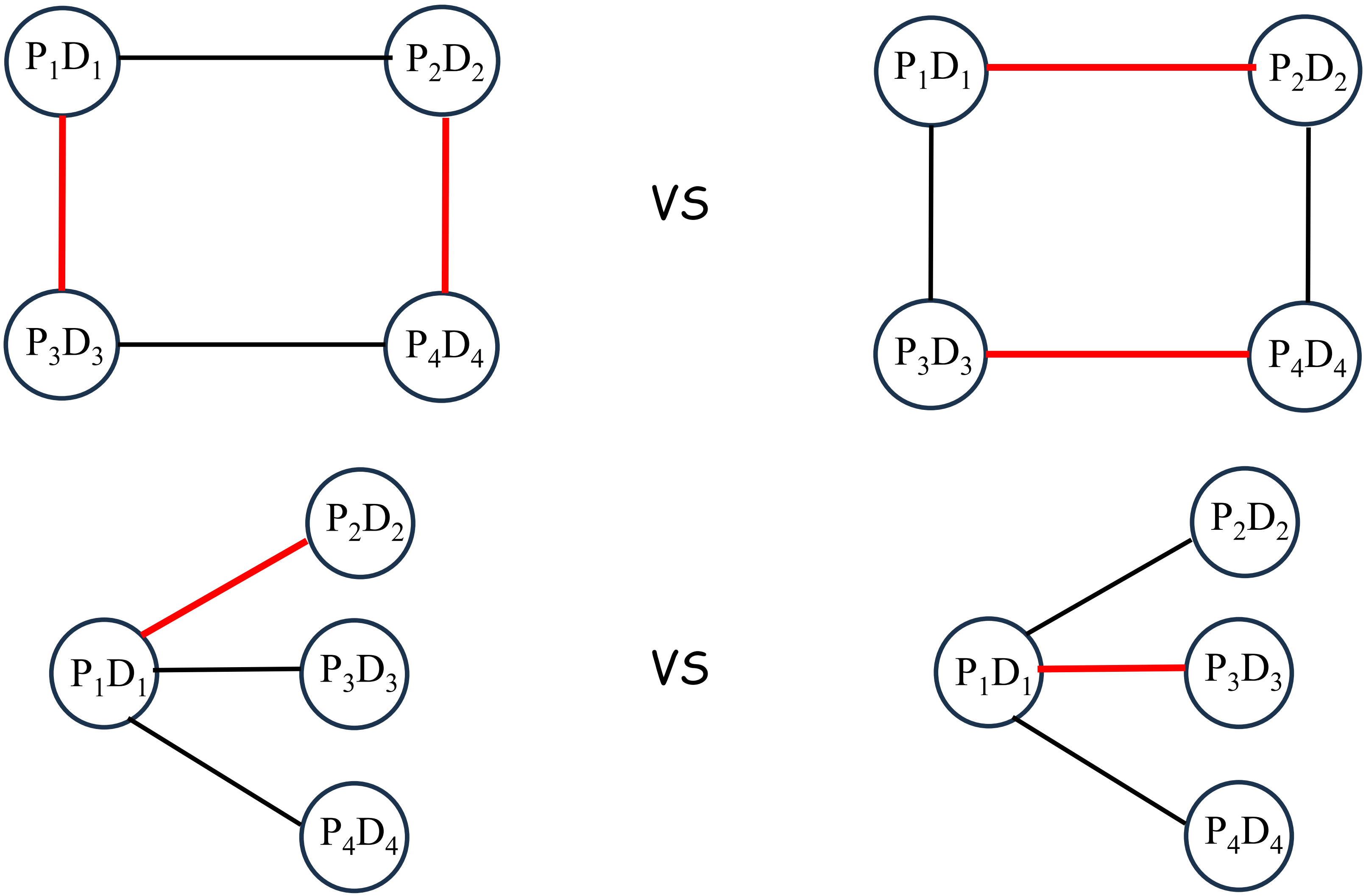
A Mechanism for Pairwise Kidney Exchange

1. Collect a report F_i from each agent i.
2. Form the graph $G = (V, E)$, where V corresponds to agent-donor pairs and $(i, j) \in E$ if and only if the patients corresponding to i and j report as compatible the donors corresponding to j and i, respectively.
3. Return a maximum-cardinality matching of the graph G.

is this mechanism truthful?

It depends on how ties are broken between different maximum matchings

two types of ties



we will manage ties by prioritizing the patient-donor pairs

notice: most hospitals already rely on priority schemes to manage their patients re-index the vertices of G such that $V=\{1,2,\ldots,n\}$ are ordered from highest to lowest priority

Priority Mechanism for Pairwise Kidney Exchange

initialize M_0 to the set of maximum matchings of G

for $i = 1, 2, \ldots, n$ **do**

let Z_i denote the matchings in M_{i-1} that match vertex i

if $Z_i \neq \theta$ **then**

set $M_i = Z_i$

else if $Z_i = \theta$ **then**

set $M_i = M_{i-1}$

return an arbitrary matching of M_n

Theorem
In the priority mechanism for pairwise kidney exchange, for every agent i, it is a dominant strategy to truthfully report E_i.

Exercise 1: Prove it.

Exercise 2:
Exhibit a tie-breaking rule between maximum-cardinality matchings such that the corresponding mechanism is not truthful.

Some other remarks and further directions

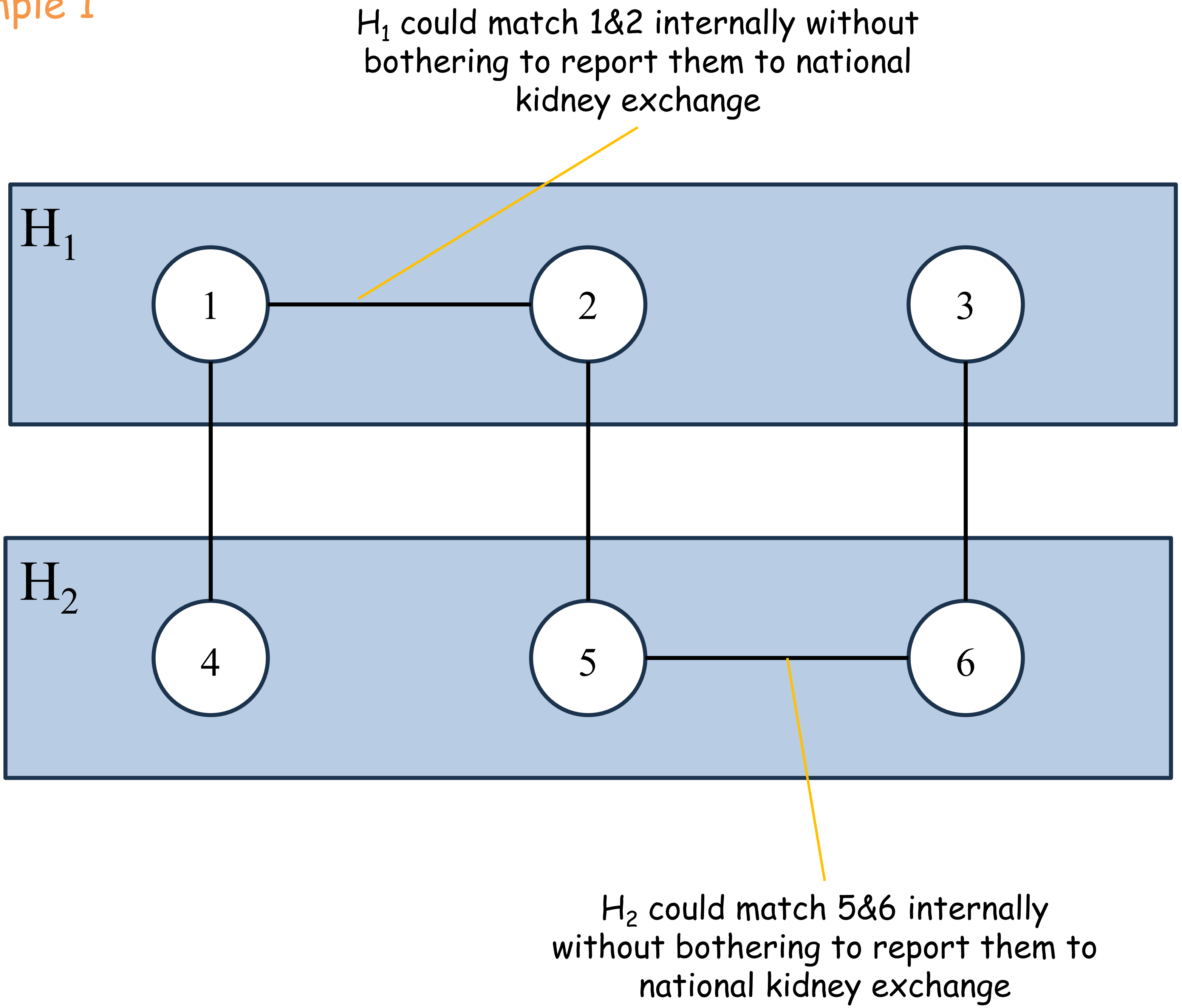
length of the cycles:

- by using matching we are restricting ourselves to 2-length cycles
- actual algorithms allow 3-way exchanges
  (it can significantly increase the number of matched patients)
- 4-way exchanges does not seem to lead to significant further improvements

Incentives for hospitals:

- many patient-donor pairs are reported to national kidney exchanges by hospitals
- the objective of a hospital, to match as many of its patients as possible, is not perfectly aligned with the societal objective of matching as many patients as possible

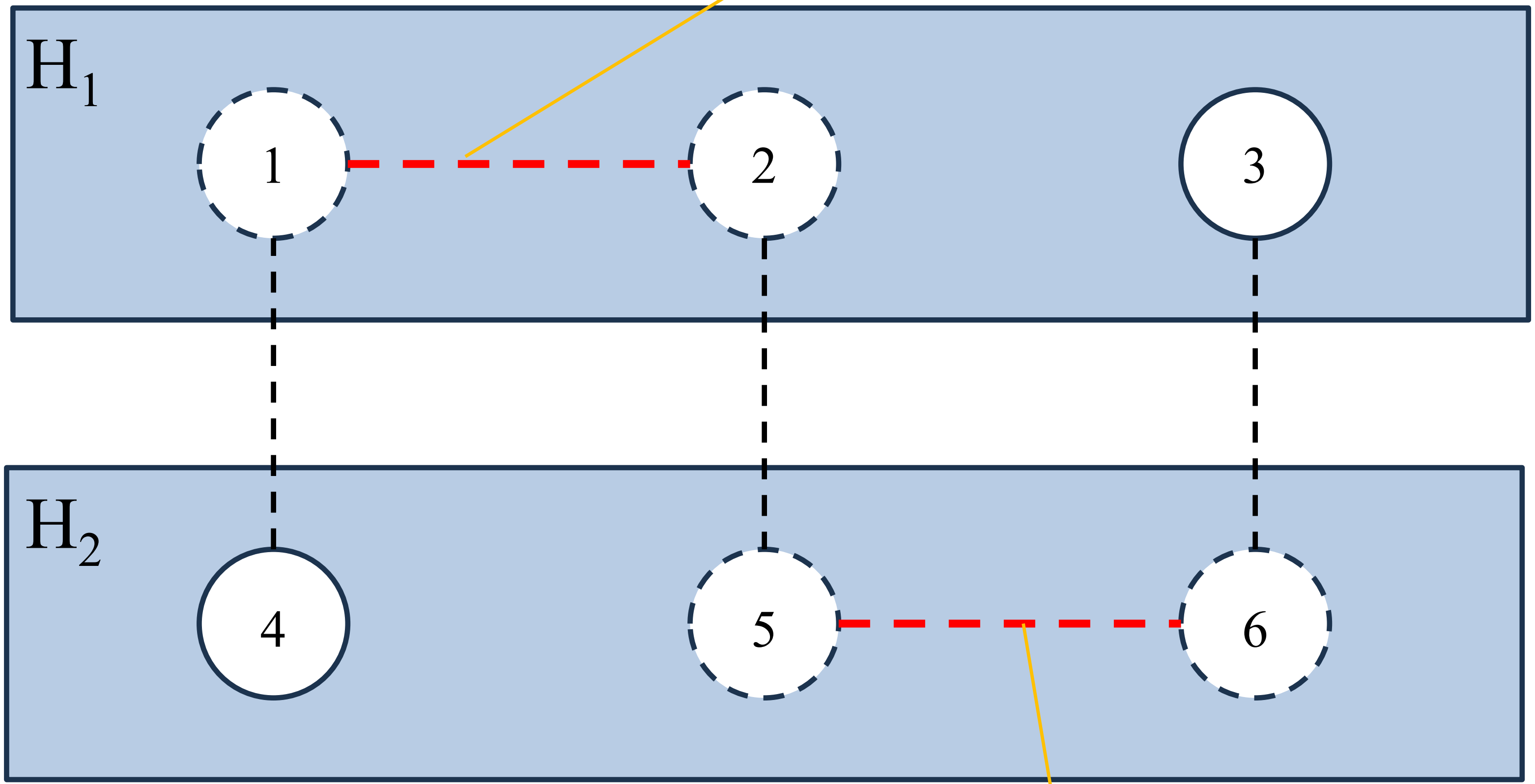
Example 1



H_1 could match 1&2 internally without bothering to report them to national kidney exchange
H_1
1
2
3
H_2
4
5
6
H_2 could match 5&6 internally without bothering to report them to national kidney exchange

Example 1

H_1 could match 1&2 internally without bothering to report them to national kidney exchange



H_2 could match 5&6 internally without bothering to report them to national kidney exchange

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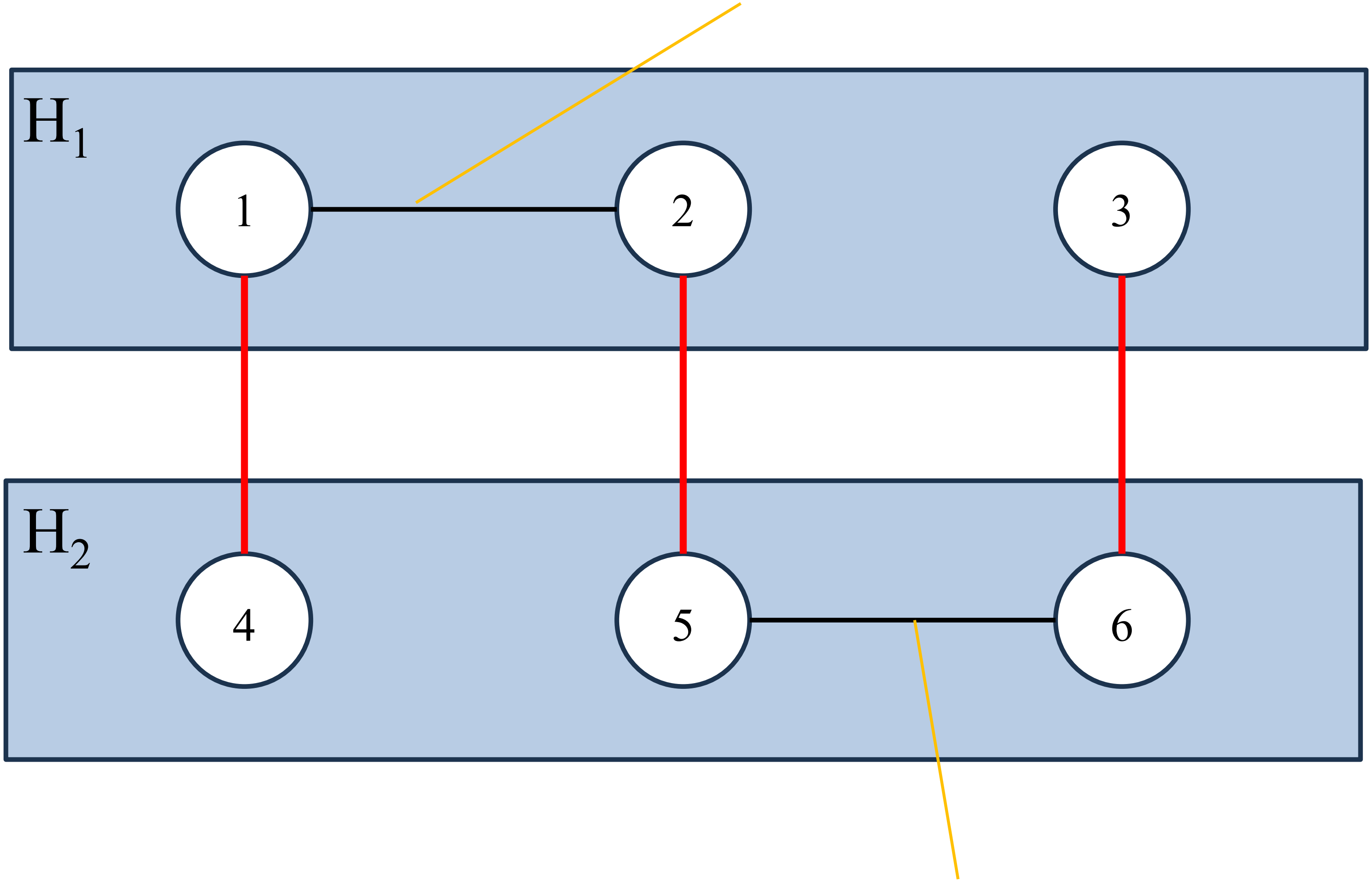
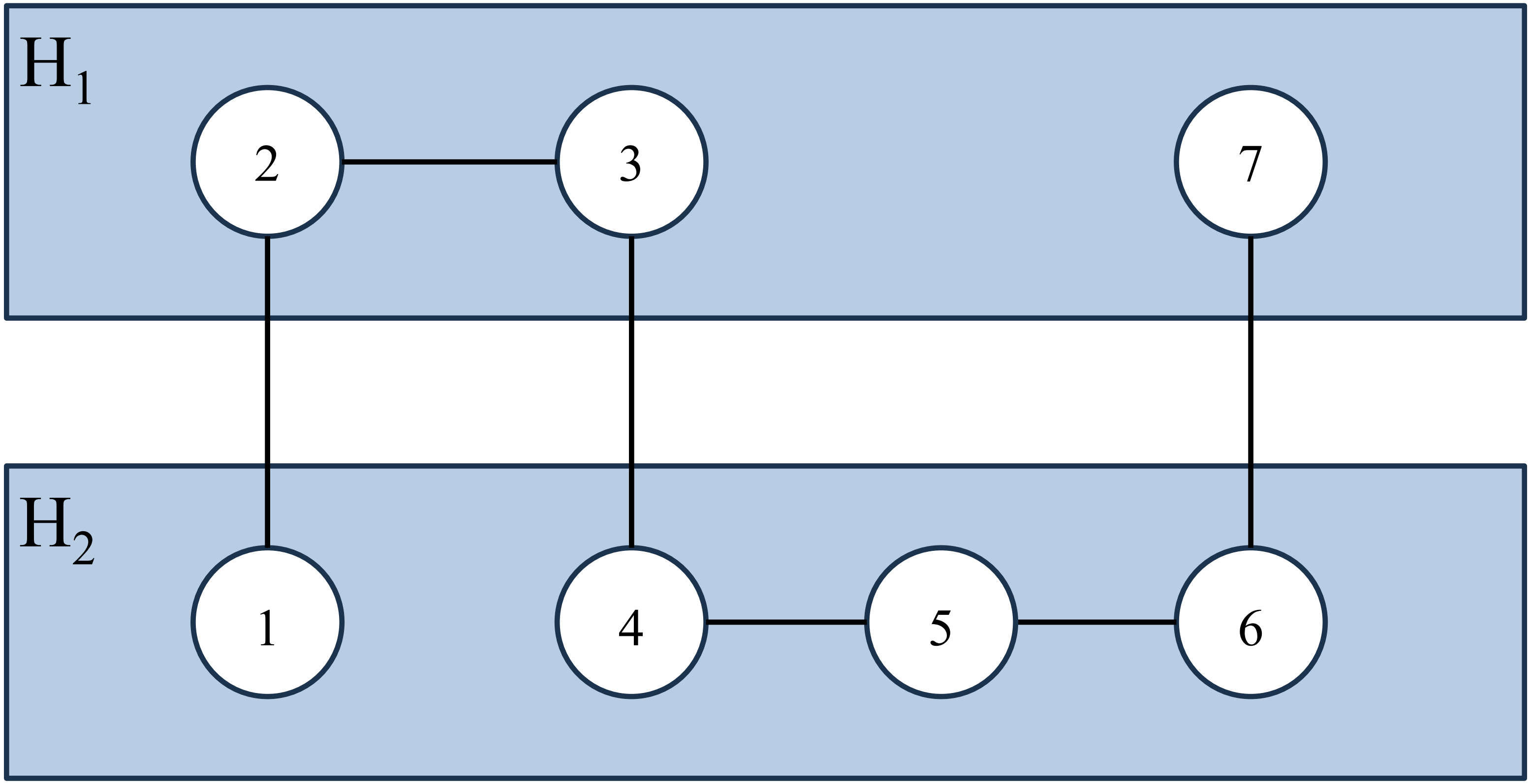


Full reporting by hospitals leads to more matches

H_2 could match 5&6 internally without bothering to report them to national kidney exchange

Example 2 (impossibility result)



Assume truthful reporting. A pair x will not be matched.

If x ∈ H_1, H_1 has convenience to not report 2&3 (and get all its pairs matched)

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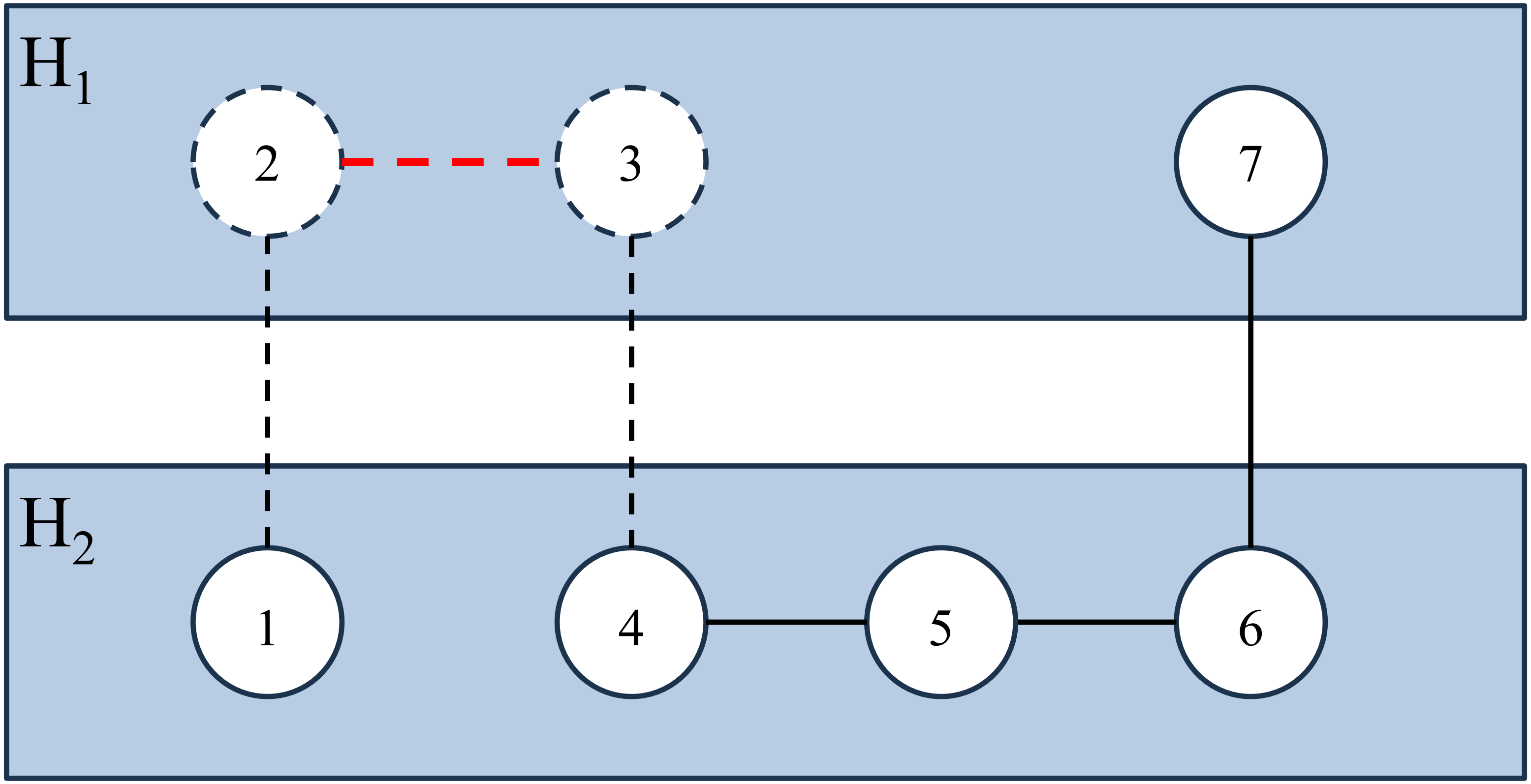

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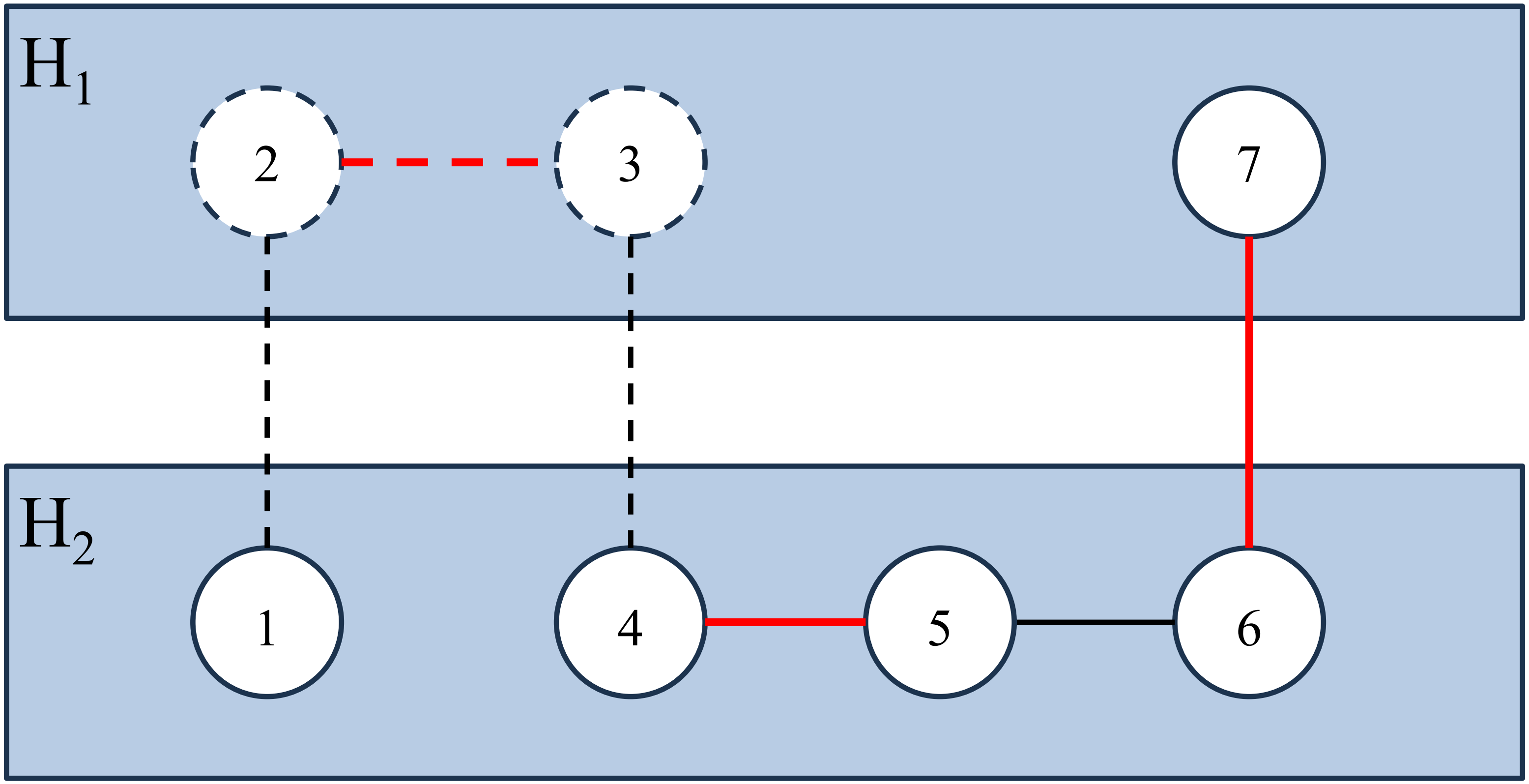

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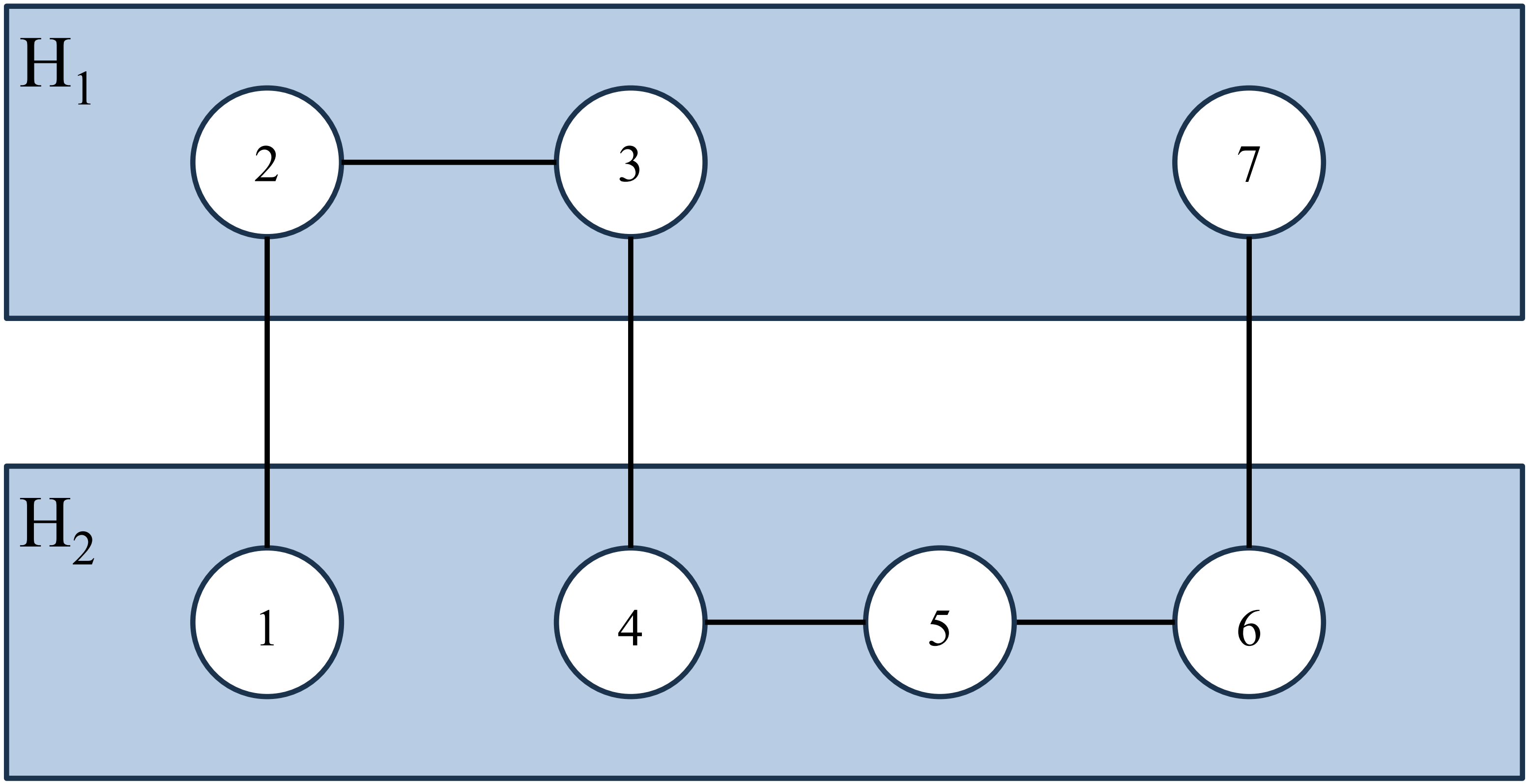

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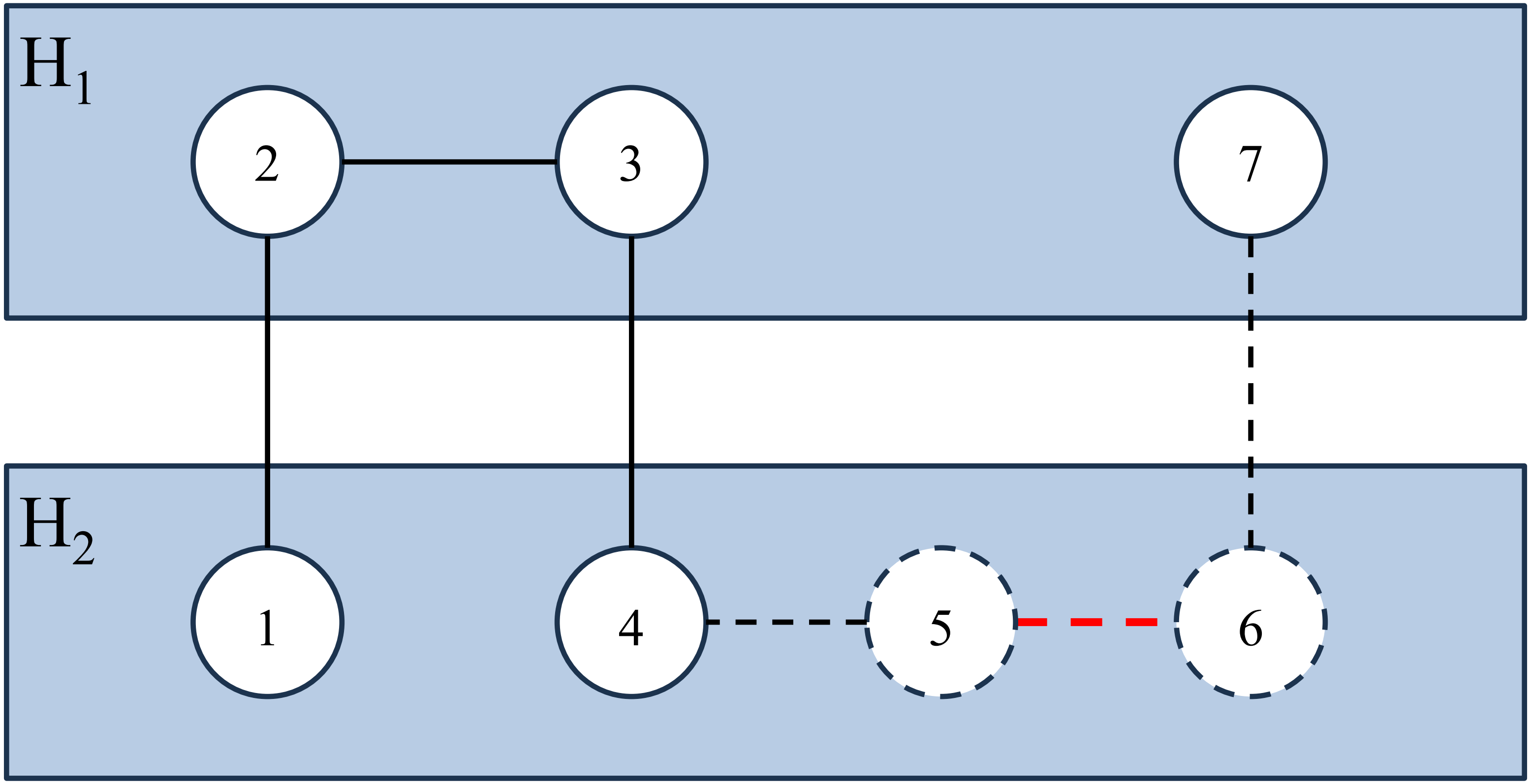

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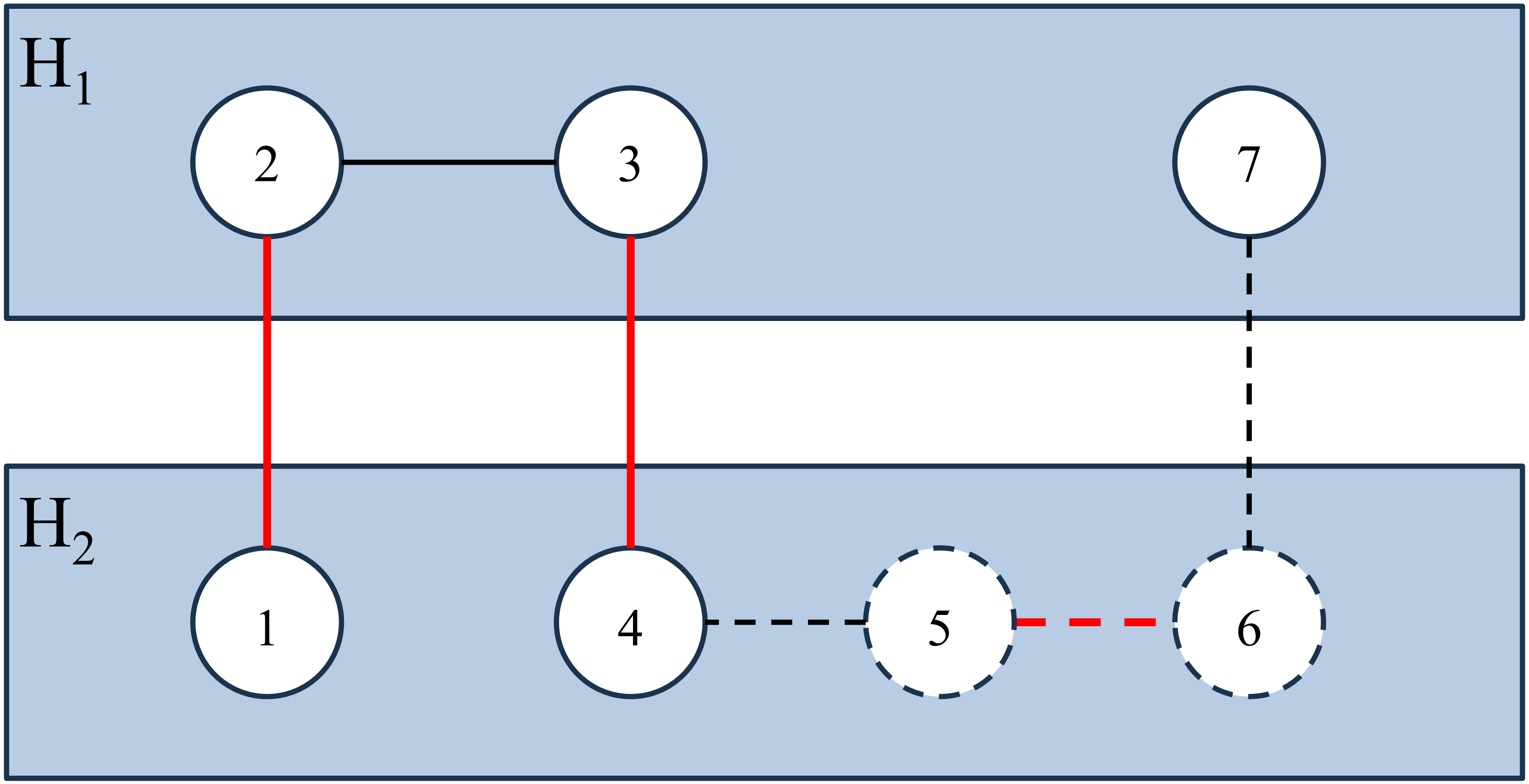

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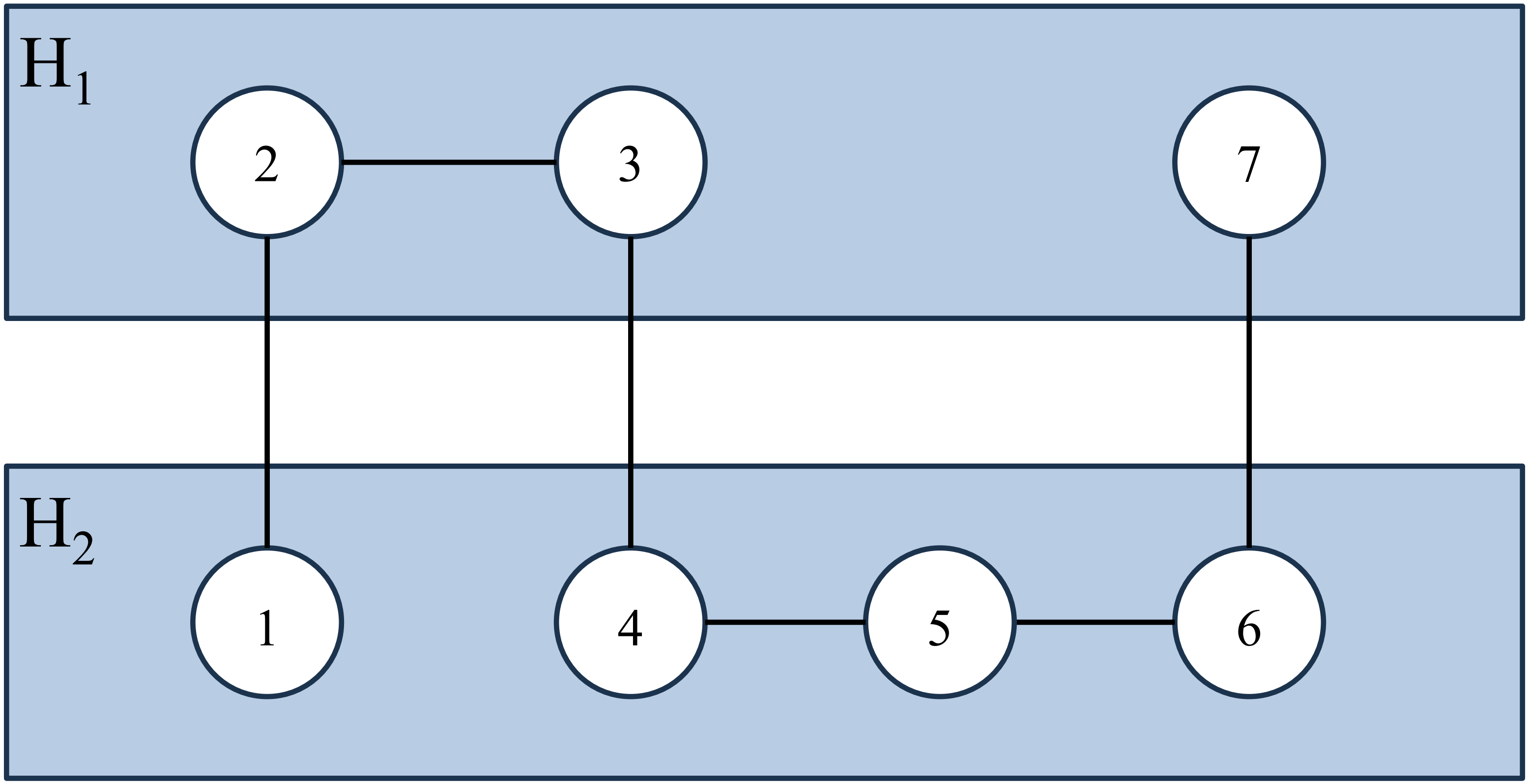

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no truthful maximum matching is possible!