Analisi di Reti (mod 2)

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Mechanism Design Without Money

reference:
Twenty Lectures in Algorithmic Game Theory
Tim Roughgarden
Chapters 9 & 10.

Motivations and applications

motivations:
- sometimes the use of money is infeasible or illegal;

applications:
- voting;
- organ donation;
- school choice;
- ...

what can be done:
- strong impossibility results in general
- still some of mechanism design's greatest hits

House allocation problem
&
Top Trading Cycle algorithm

House allocation problem

- n agents
- each agent initially owns a house
- preferences (type) of the agent i: a total ordering over the n houses
  - an agent need not prefer her own house over the others

1

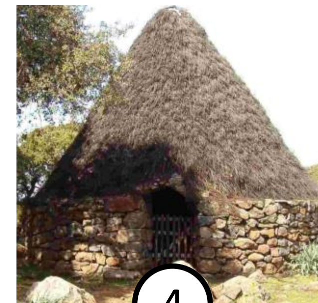
1's preferences:
1,3,2,4

2

2's preferences:
1,3,2,4

3

3's preferences:
1,4,3,2



4

4's preferences:
1,2,3,4

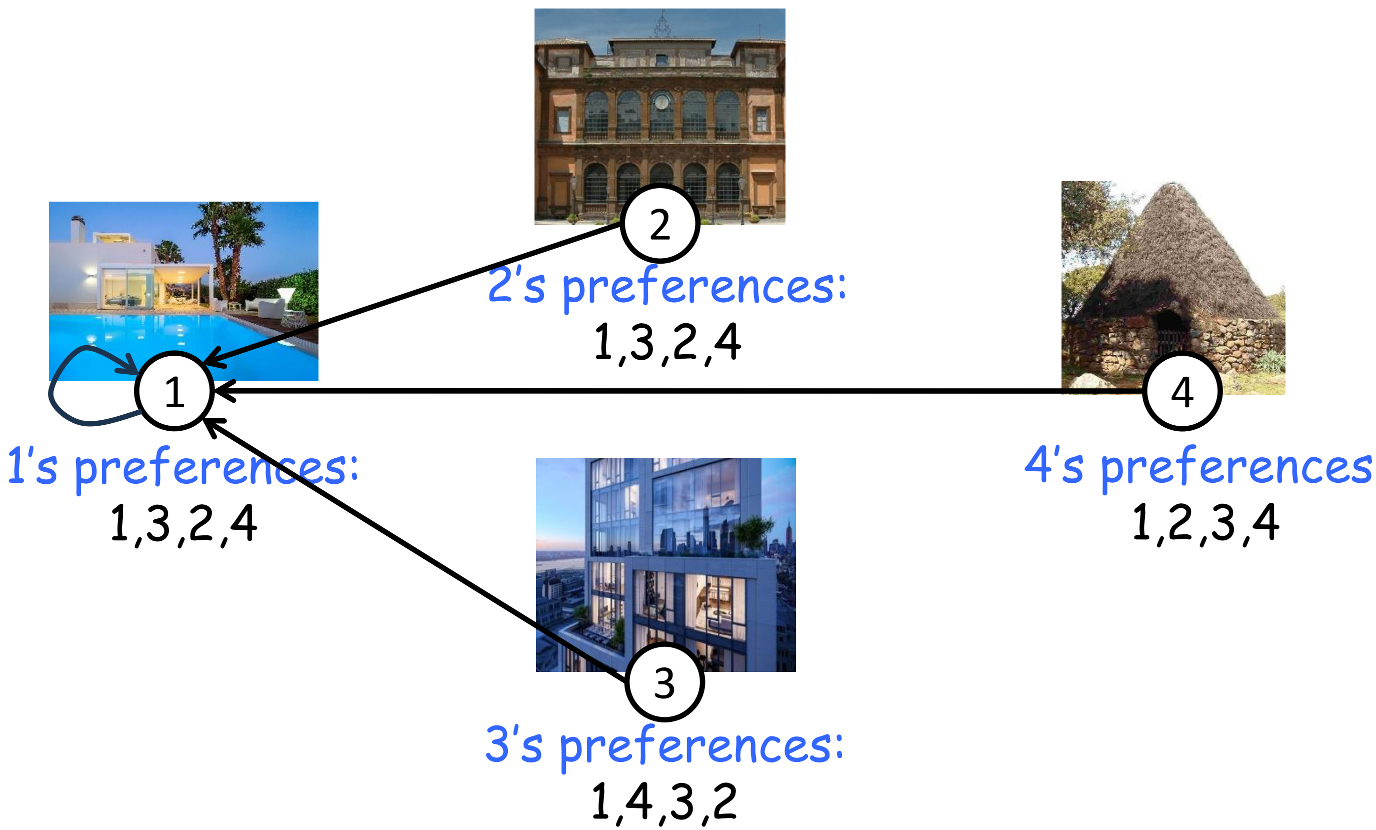
goal 1: reallocate the houses to make the agents better off

goal 2: do it in a way agents cannot manipulate the allocation

⇒ Top Trading Cycle (TTC) algorithm

TTC algorithm (idea)

- allocation proceeds in iterations
- at each iteration:
  - each remaining agent participates with her own house
  - each remaining agent points to her favorite still available house
  - look at (disjoint) cycles formed and perform the reallocation suggested by the cycles
  - remove the agents of the cycles



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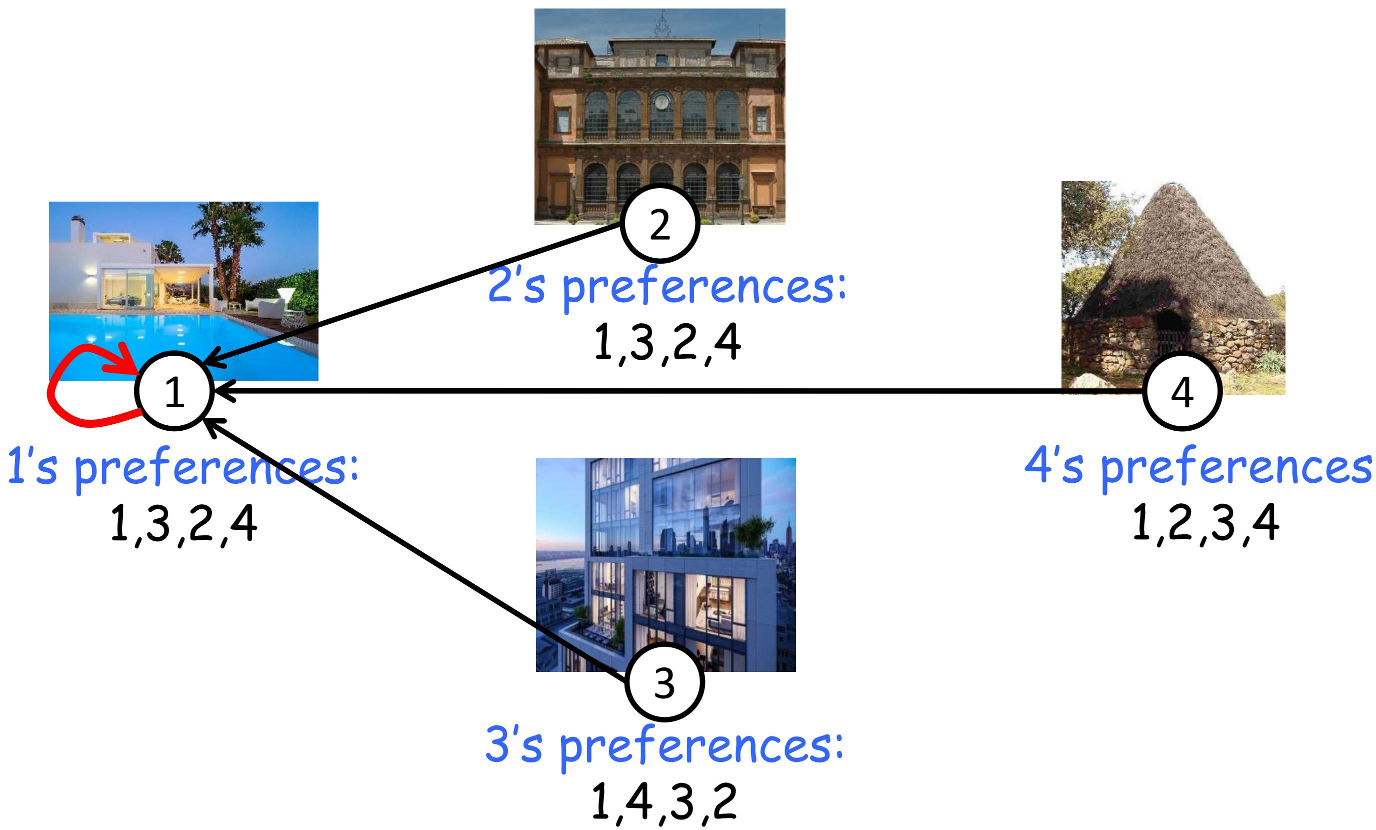

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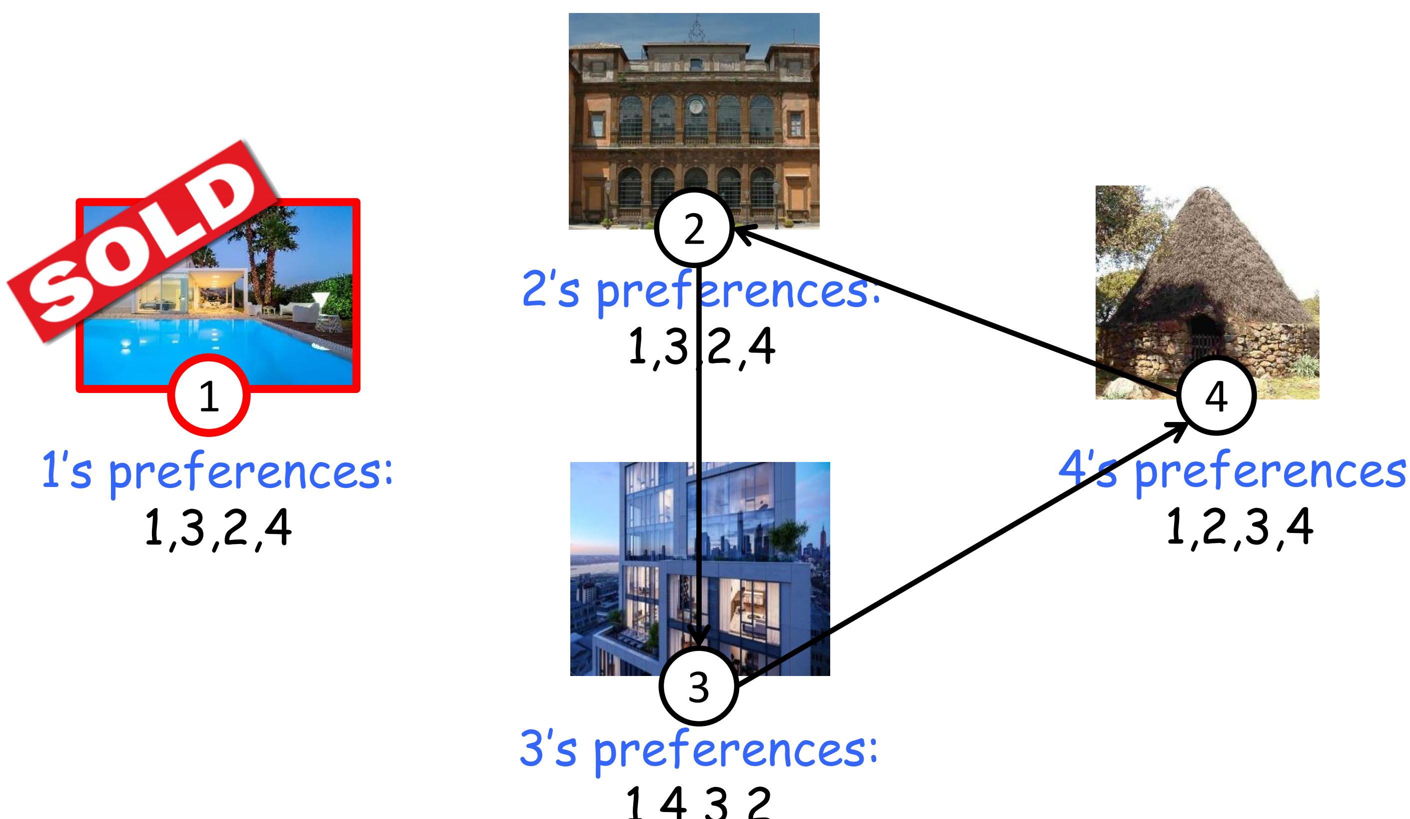

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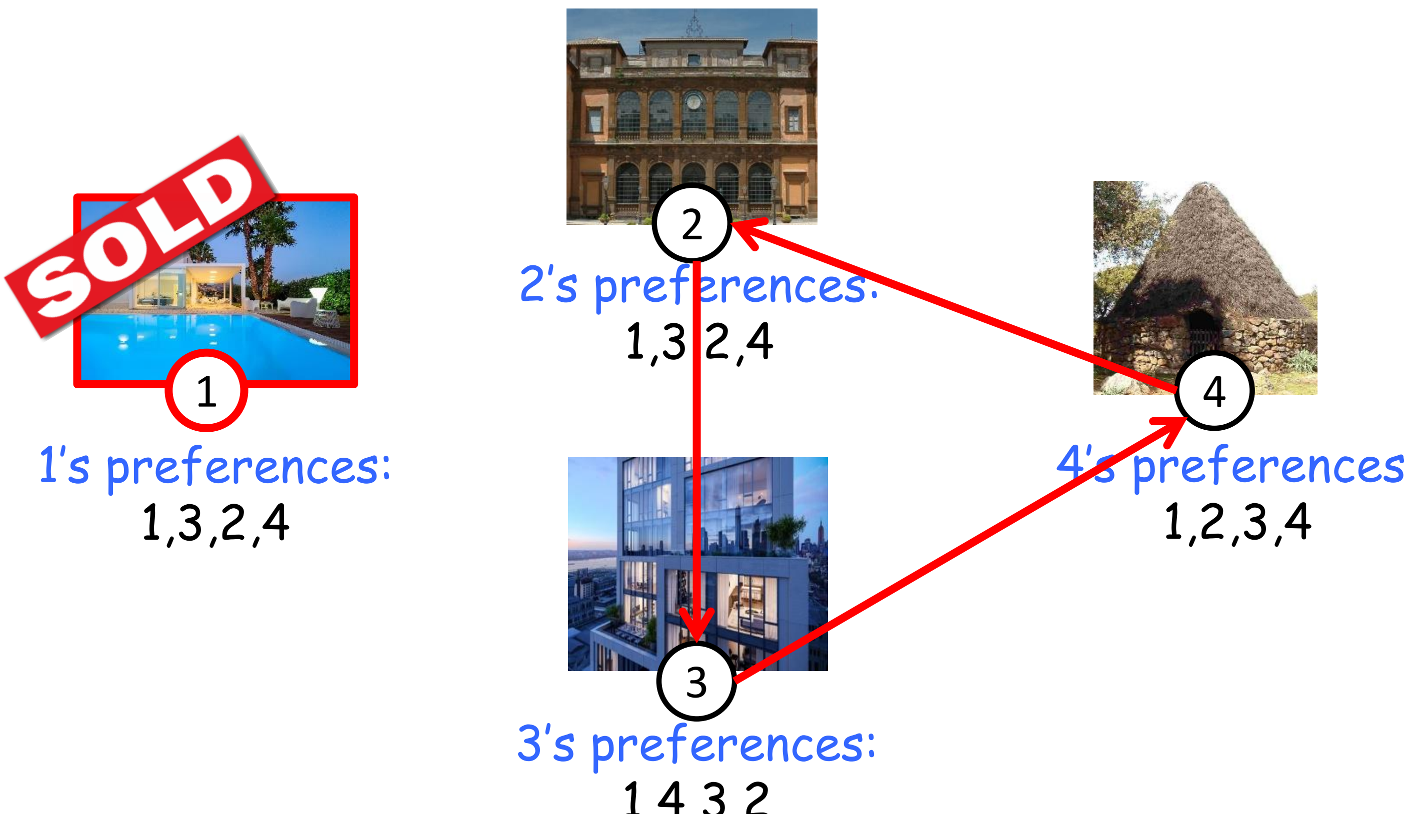

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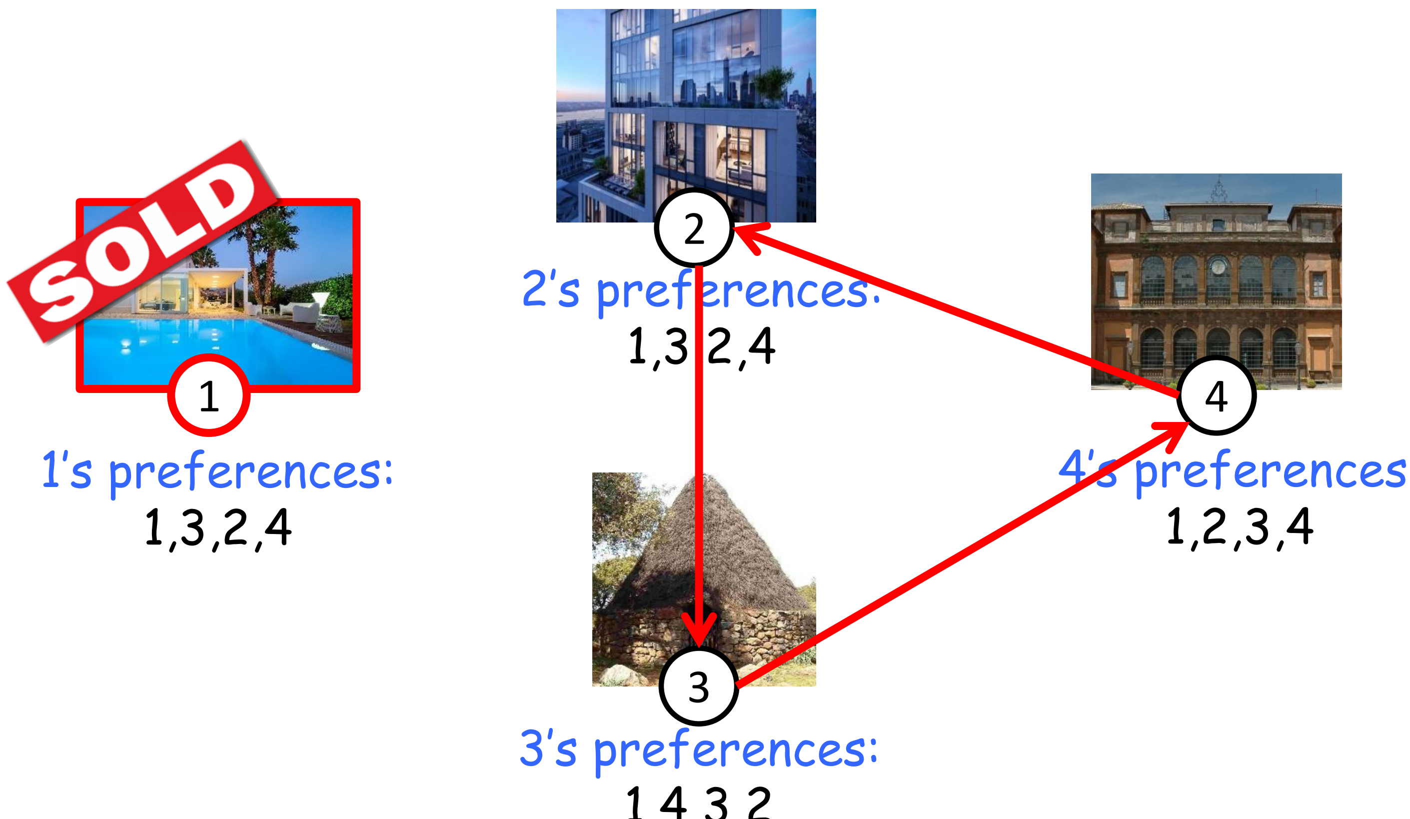

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Top Trading Cycle (TTC) Algorithm

initialize N to the set of all agents
while $N \neq \theta$ **do**

form the directed graph G with vertex set N and edge set $\{(i, \ell)$: i's favorite house within N is owned by $\ell\}$

compute the directed cycles $C_1, \ldots, C_h$ of G

```
// self-loops count as directed cycles
// cycles are disjoint
```

for each edge (i, ℓ) of each cycle $C_1, \ldots, C_h$ **do** reallocate ℓ's house to agent i

remove the agents of $C_1, \ldots, C_h$ from N

- G has at least one directed cycle, since traversing a sequence of outgoing edges must eventually repeat a vertex
- Because all out-degrees are 1, these cycles are disjoint

Some properties of the TTC algorithm

Lemma
Let N_k denote the set of agents removed in the k-th iteration of the TTC algorithm. Every agent of N_k receives her favorite house outside of those owned by $N_1 \cup \ldots \cup N_{k-1}$, and the original owner of this house is in N_k.

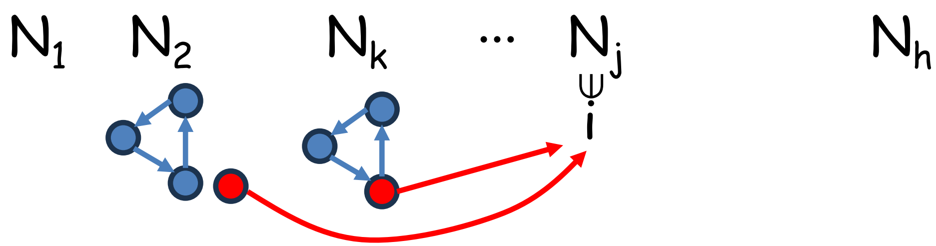
Theorem
When the TTC algorithm is used for the reallocation, for every agent, it is a dominant strategy to report truthfully.

proof

Fix an agent i and reports by the others.

Assume i reports truthfully.

Let N_k be the set of agents removed in the k-th iteration.



By the previous lemma it suffices to prove:

Claim: no misreport can net i a house of an agent in $N_1 \cup \ldots \cup N_{j-1}$.

For each k<j:

- no agent in N_k point to i at iteration k

  otherwise i would belong to N_k

- no agent in N_k point to i at iteration <k

  otherwise that agent would point to i at iteration k

Why the previous theorem is nice?

Notice: the mechanism that never reallocates anything is also truthful

Consider a reallocation of the houses. A subset of agents forms a blocking coalition for this reallocation if they can internally reallocate their houses to make some member better off while making no member worse off.

A core allocation is a reallocation with no blocking coalitions.

> **Theorem**
> For every house allocation problem, the allocation computed by the TTC algorithm is the unique core allocation.

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 1

every agent in N_1 receives her first choice

⇨ N_1 is a blocking coalition for any allocation $\neq$ from the TTC one

⇨ every core allocation must agree with the TTC one on agents in N_1

⇨ every agent in N_2 receives her first choice outside N_1

⇨ N_2 is a blocking coalition for any allocation $\neq$ from the TTC one that agrees with TTC on N_1

⇨ every core allocation must agree with the TTC one on agents in N_1 and N_2

⇨ ... ■

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 2

consider an arbitrary subset S of agents and an internal reallocation of their houses

the reallocation partitions S into directed cycles

consider any such cycle C

- if C contains two agents $i \in N_j$ and $t \in N_k$ with $j<k$

  ⇒ i is worse off than in the TTC allocation

- if C contains agents all in N_k but there is an agent i that does not receives her favorite choice in N_k

  ⇒ i is worse off than in the TTC allocation

⇒ the TTC allocation has no blocking coalitions. ■

Kidney Exchange

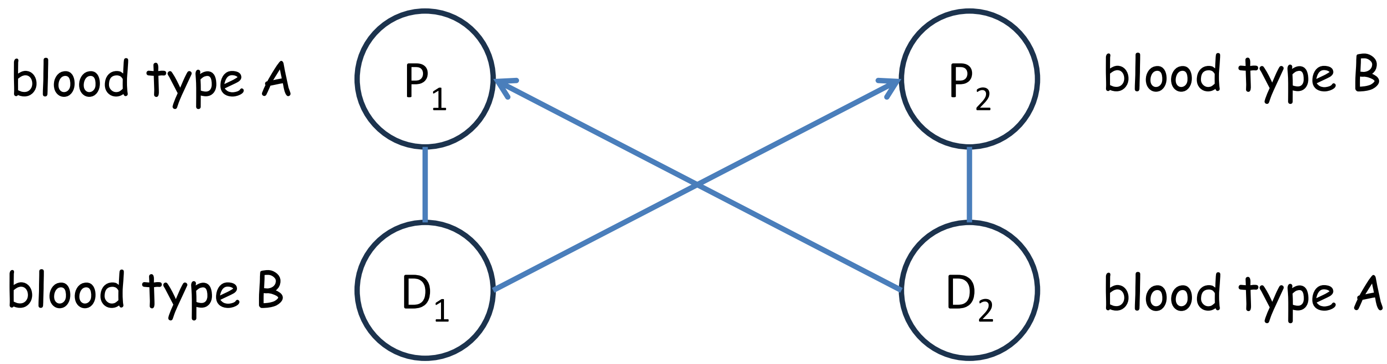
Background

- many people suffer from kidney failure and need a kidney transplant
- In US more than 100.000 people are on the waiting list for such a transplant
- old idea (used also for other organs): deceased donors
- special feature for kidneys: living donors
  (a healthy person can survive just fine with a single kidney)

compatibility issues:

- having a living kidney donors is not always enough
- a patient-donor pair can be incompatible
  (primary culprits for incompatibility: blood and tissue types need to match)

idea: swap donors!



kidney exchange

- initially, few kidney exchanges were done on an ad-hoc basis
- This made clear the need of a system to organize kidney exchanges
- a system where patient-donor pairs can register and be matched with others

goal: how such an exchange system can be designed in order to enable as many matches as possible?

- currently, monetary compensation is illegal in most of the countries

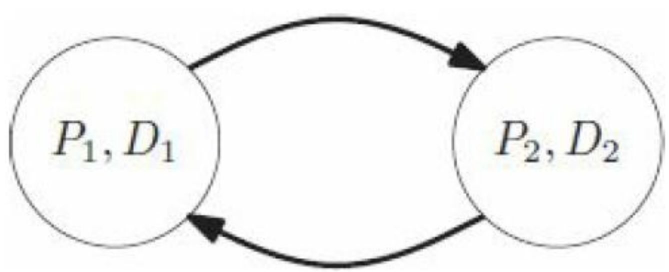
problem naturally modeled as a mechanism design problem without money

first idea: model the problem as a house allocation problem

- each patient-donor pair treated as a agent-house
  - patient=agent
  - donor=house
- a patient's total ordering over the donors can be defined according to the estimated probability of a successful kidney transplant

- use the TCC algorithm to find kidney exchanges

- the reallocation of donors suggested by the TCC algorithm can only improve every patient's probability of a successful transplant



Good case for the TTC algorithm.
- each circle represents an incompatible patient-donor pair
- each arrow represents a kidney transplant from the donor in the first pair to the patient of the second pair.

first idea: model the problem as a house allocation problem

some technical issues:

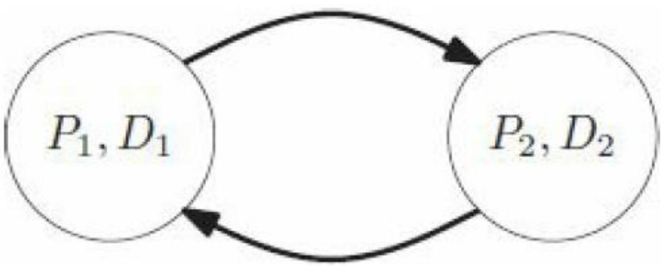
- need to manage patients without a donor (agent without a house)
- need to manage deceased donors (house without an agent/owner)

the TTC algorithm & its incentive guarantee can be extended to this more general setting (with some non-trivial extra work)

some more important issues:

- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)

how many surgeries? 4



what if P_1-D_2 surgeries today and P_2-D_1 surgeries tomorrow?

D_1 could renege on her offer

- P_1 unfairly got a kidney for free
- P_2 is still sick and can no longer participate in a kidney exchange

first idea: model the problem as a house allocation problem

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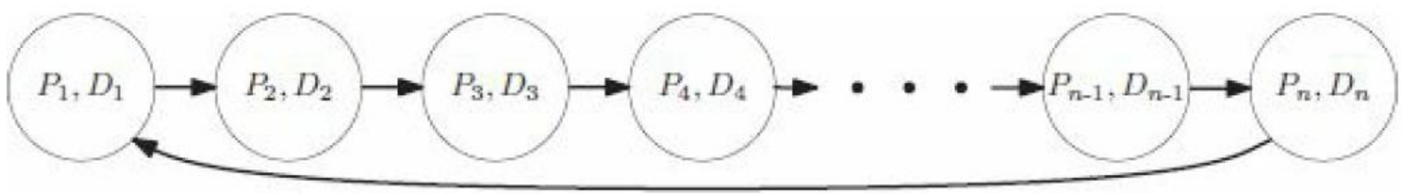

really bad case

first idea: model the problem as a house allocation problem

some technical issues:

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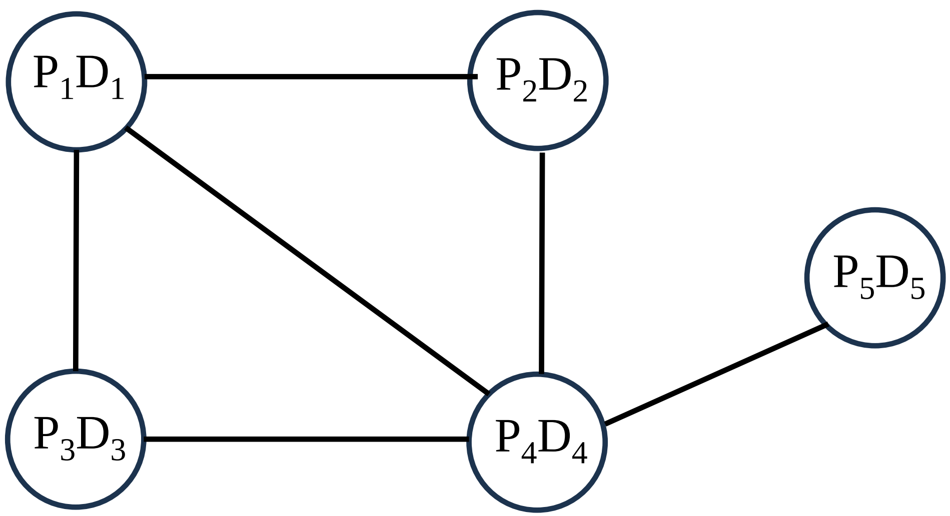
some more important issues:

- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)
- modeling patient's preferences as a total order over donors is overkill (binary preferences over donors are more appropriate)

⇨ change the model: use graph matching!

A matching of an undirected graph is a subset of the edges that share no endpoints.

The relevant graph for kidney exchanges:
- we have a vertex for each incompatible patient-donor pair
- there is an edge between (P_i,D_i) and (P_j,D_j) if and only if P_i and D_j are compatible & P_j and D_i are compatible



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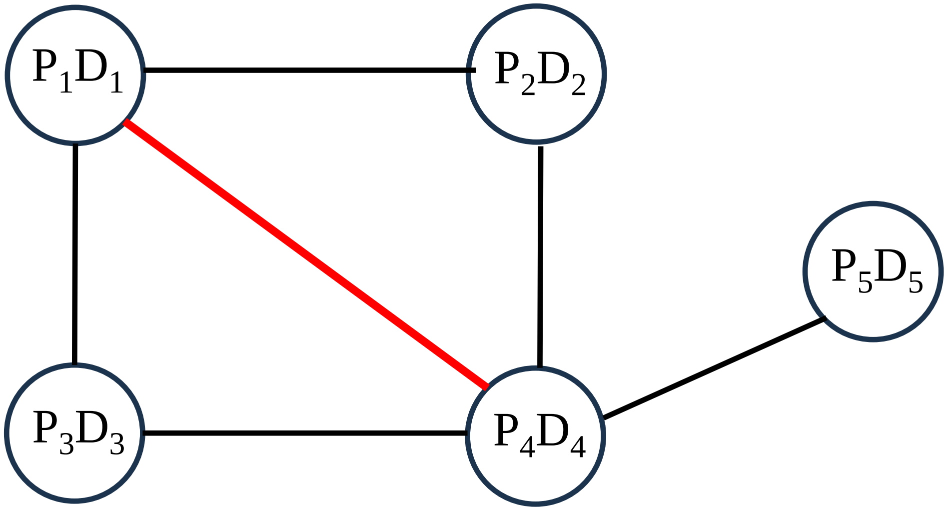

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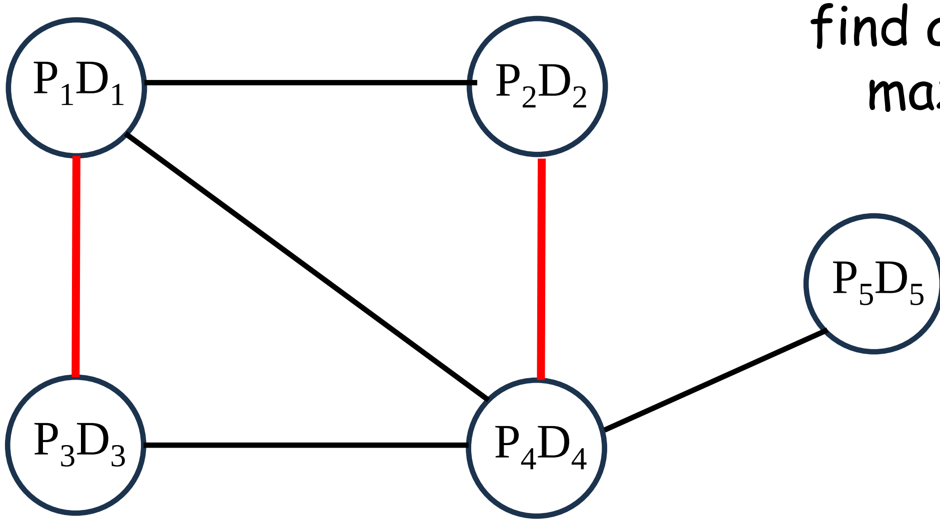


goal:
find a matching of maximum size

notice: we are restricting ourselves to 2-length cycles

How do incentives come into play?

we assume that each patient i

- has a set E_i of compatible donors belonging to other patient-donor pairs
- can report any subset $F_i \subseteq E_i$

it makes sense since:

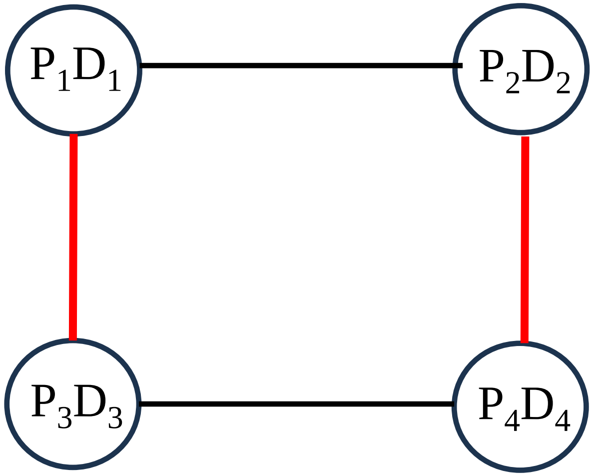
- proposed kidney exchange can be refused by a patient for any reason
- a patient cannot credibly misreport extra donors with whom she is incompatible

A Mechanism for Pairwise Kidney Exchange

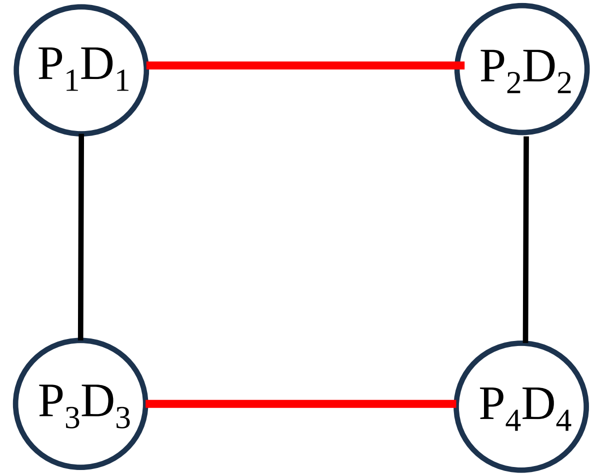
1. Collect a report F_i from each agent i.
2. Form the graph $G = (V, E)$, where V corresponds to agent-donor pairs and $(i, j) \in E$ if and only if the patients corresponding to i and j report as compatible the donors corresponding to j and i, respectively.
3. Return a maximum-cardinality matching of the graph G.

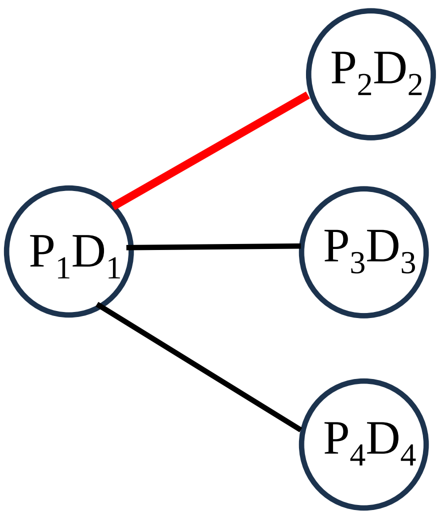
is this mechanism truthful?

It depends on how ties are broken between different maximum matchings two types of ties



VS





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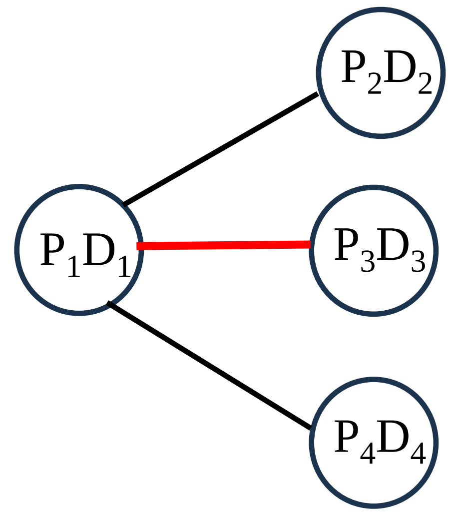


we will manage ties by prioritizing the patient-donor pairs

notice: most hospitals already rely on priority schemes to manage their patients re-index the vertices of G such that $V=\{1,2,\ldots,n\}$ are ordered from highest to lowest priority

Priority Mechanism for Pairwise Kidney Exchange

initialize M_0 to the set of maximum matchings of G

for $\boldsymbol{i} = 1, 2, \ldots, n$ **do**

let Z_i denote the matchings in M_{i-1} that match vertex i

if $Z_i \neq \theta$ **then**

set $M_i = Z_i$

else if $Z_i = \theta$ **then**

set $M_i = M_{i-1}$

return an arbitrary matching of M_n

Theorem
In the priority mechanism for pairwise kidney exchange, for every agent i, it is a dominant strategy to truthfully report E_i.

Exercise 1: Prove it.

Exercise 2:
Exhibit a tie-breaking rule between maximum-cardinality matchings such that the corresponding mechanism is not truthful.

Some other remarks and further directions

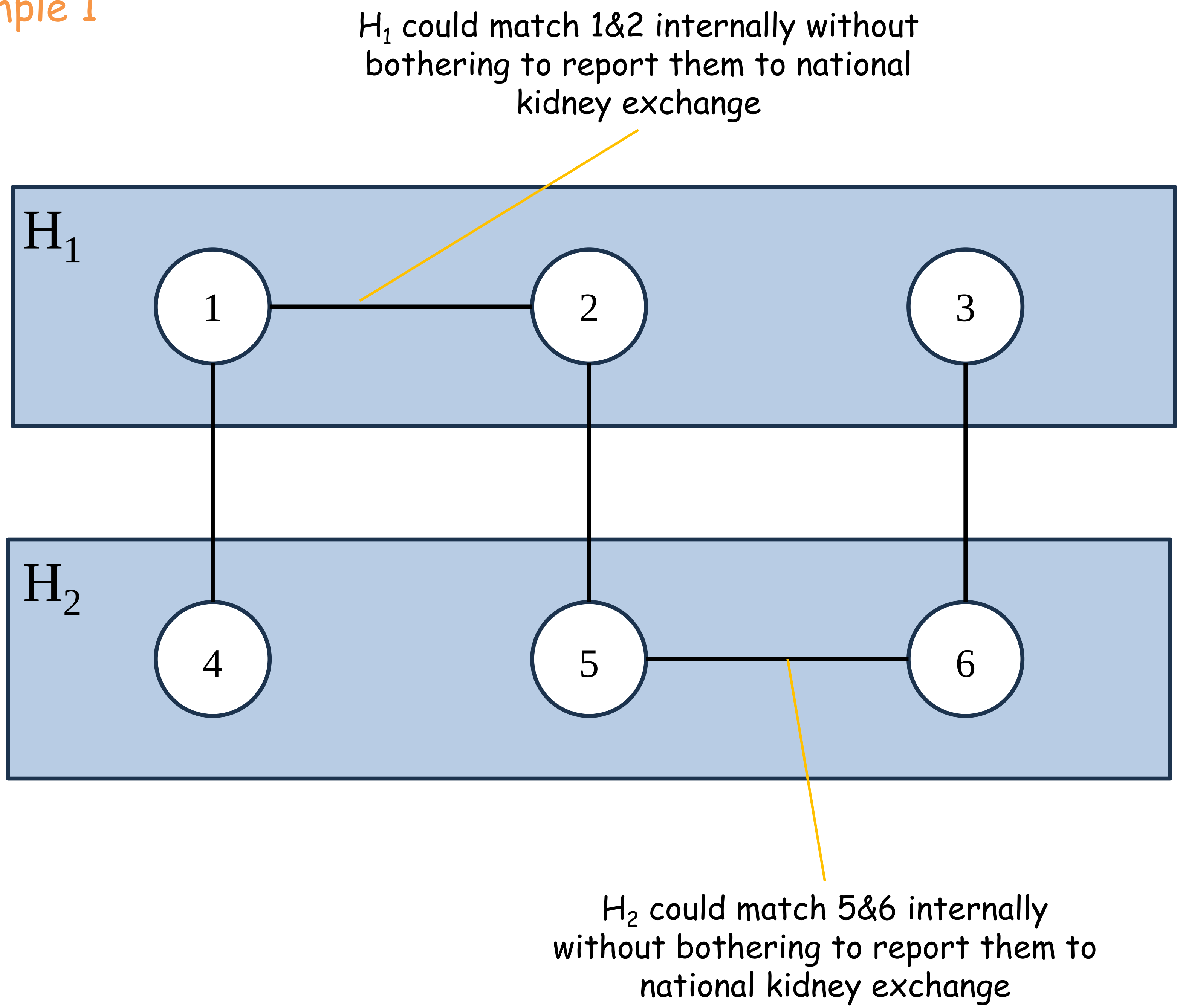
length of the cycles:

- by using matching we are restricting ourselves to 2-length cycles
- actual algorithms allow 3-way exchanges
  (it can significantly increase the number of matched patients)
- 4-way exchanges does not seem to lead to significant further improvements

Incentives for hospitals:

- many patient-donor pairs are reported to national kidney exchanges by hospitals
- the objective of a hospital, to match as many of its patients as possible, is not perfectly aligned with the societal objective of matching as many patients as possible

Example 1



H_1 could match 1&2 internally without bothering to report them to national kidney exchange
H_1
1
2
3
H_2
4
5
6
H_2 could match 5&6 internally without bothering to report them to national kidney exchange

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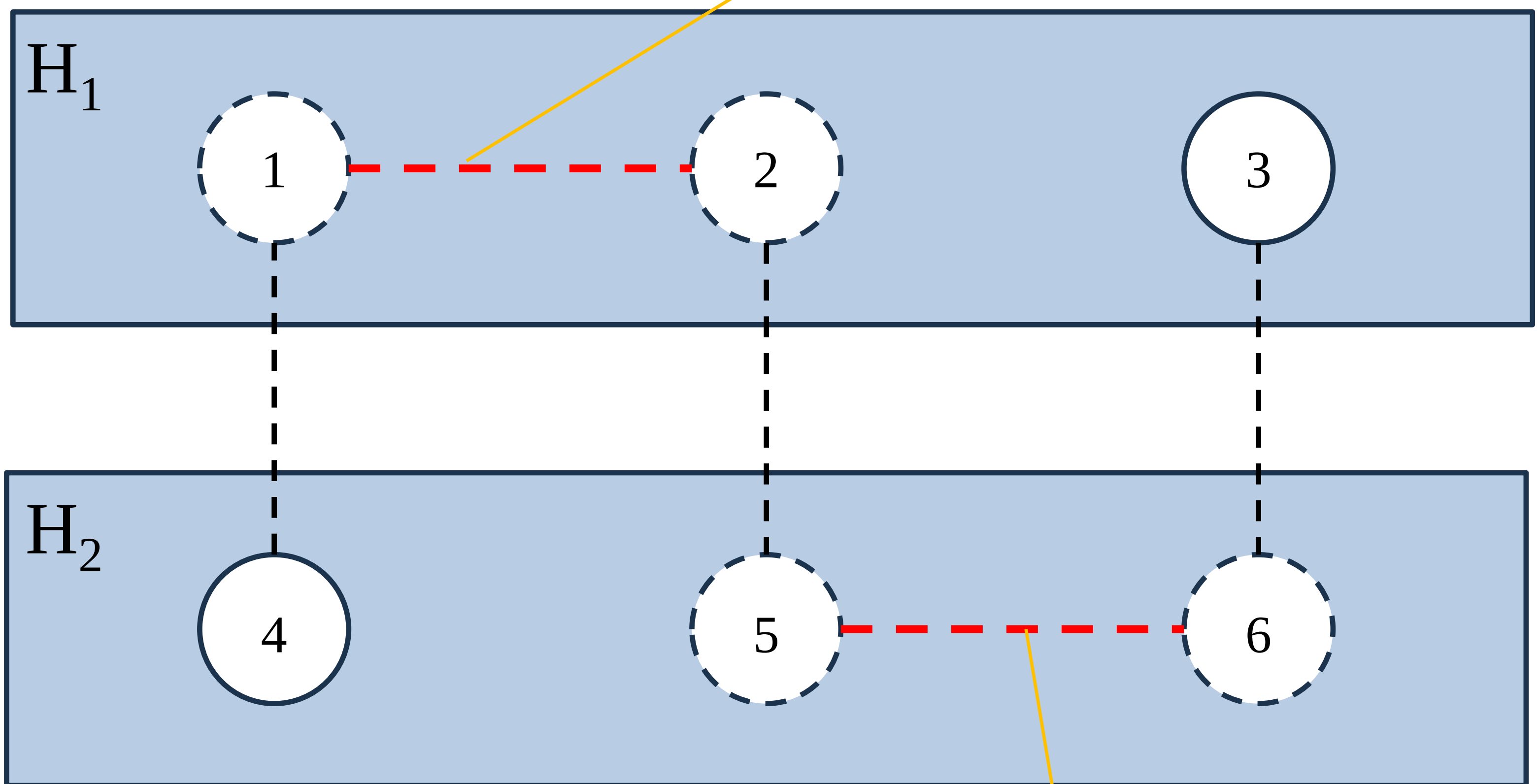
H_1 could match 1&2 internally without bothering to report them to national kidney exchange



H_2 could match 5&6 internally without bothering to report them to national kidney exchange

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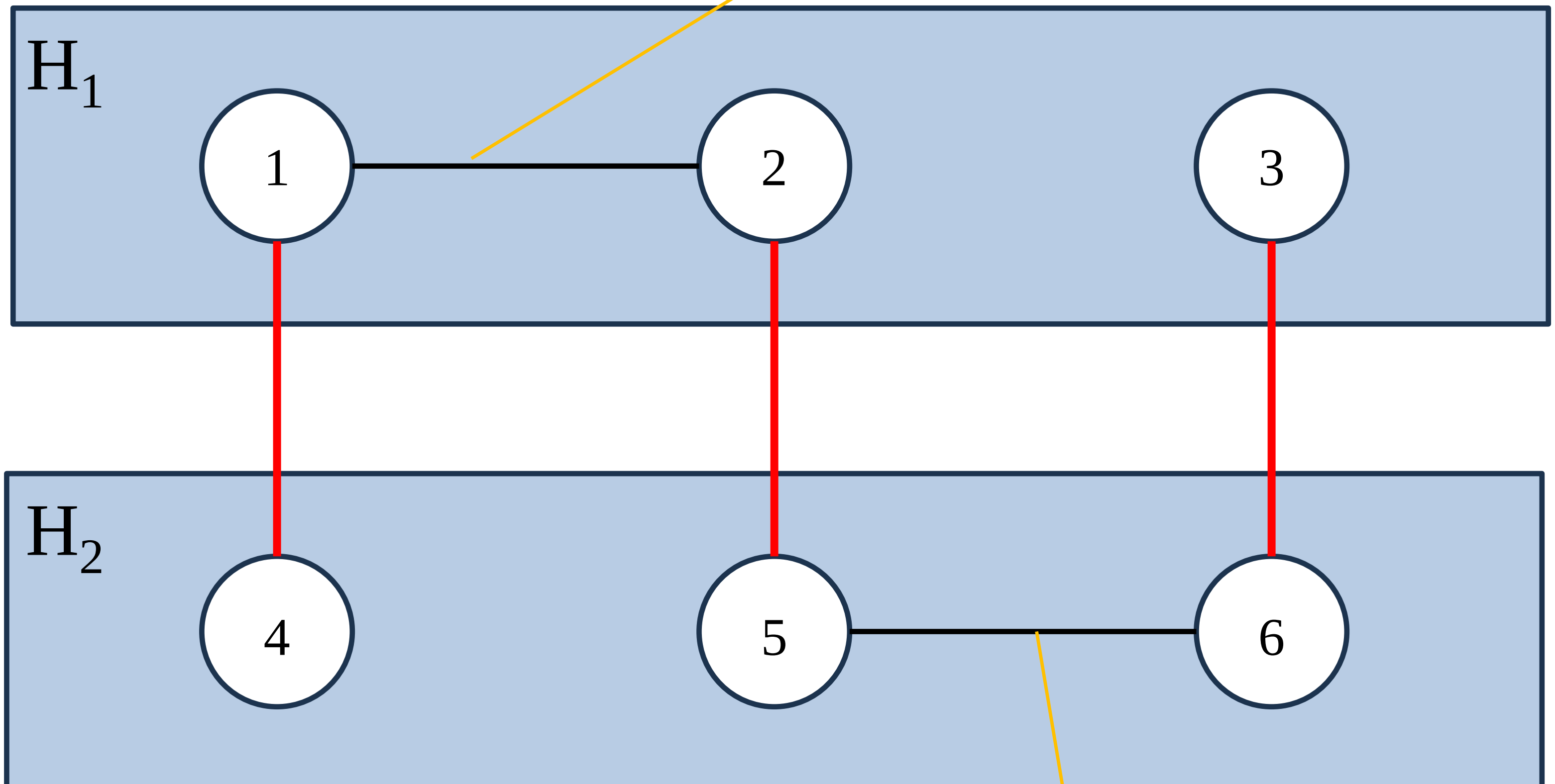
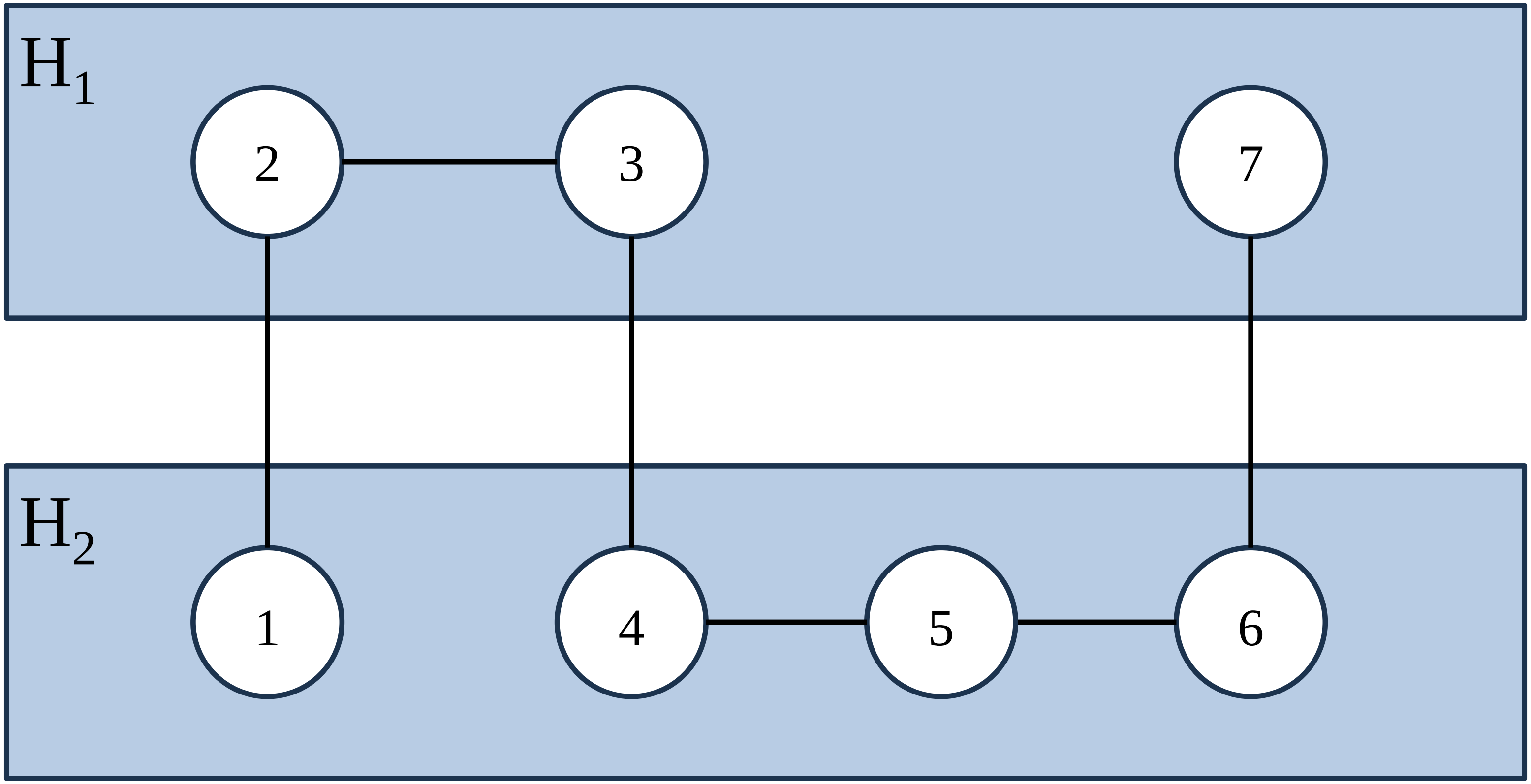


H_2 could match 5&6 internally without bothering to report them to national kidney exchange

Full reporting by hospitals leads to more matches

Example 2 (impossibility result)



Assume truthful reporting. A pair x will not be matched.

If x $\in H_1$, H_1 has convenience to not report 2&3 (and get all its pairs matched)

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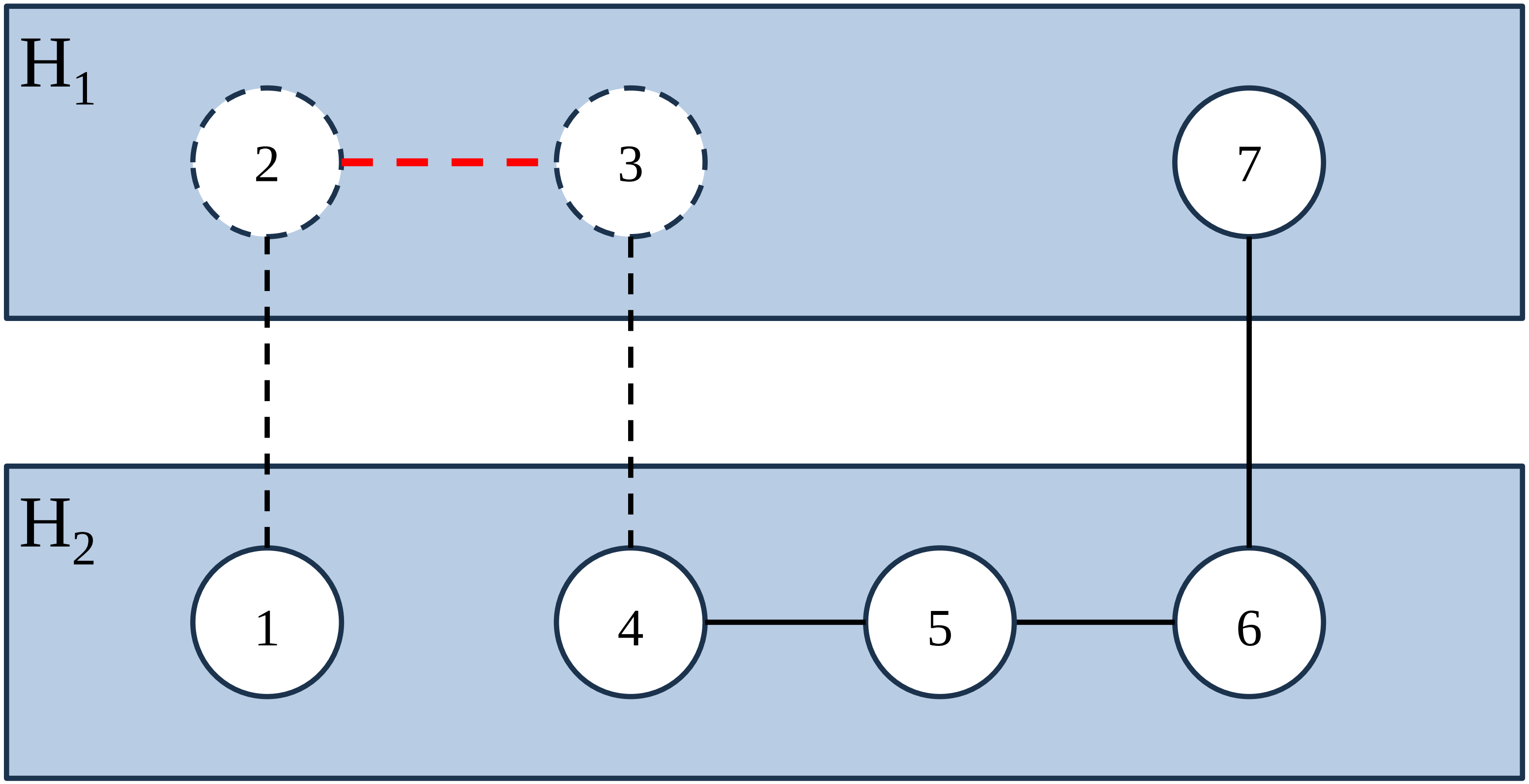

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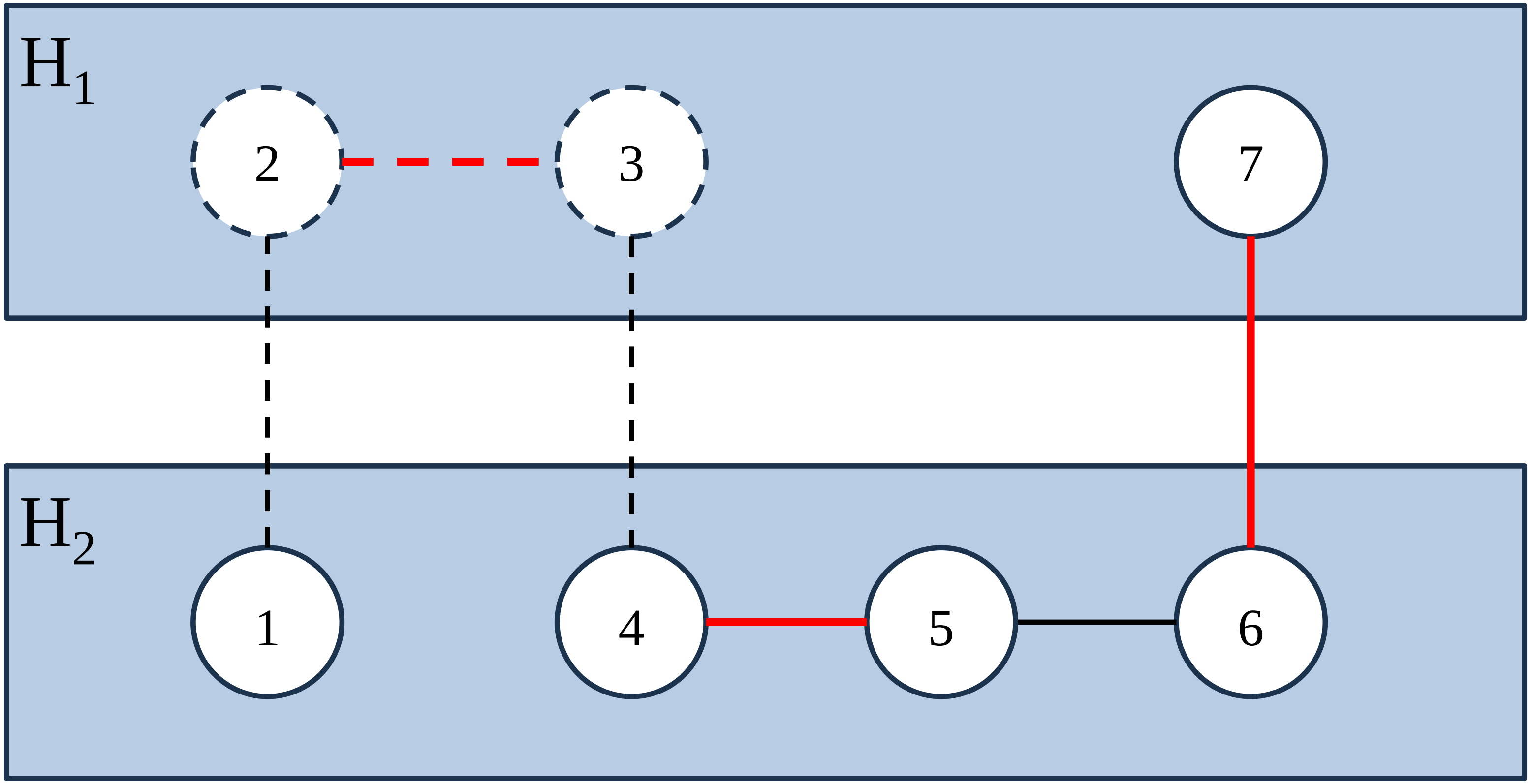

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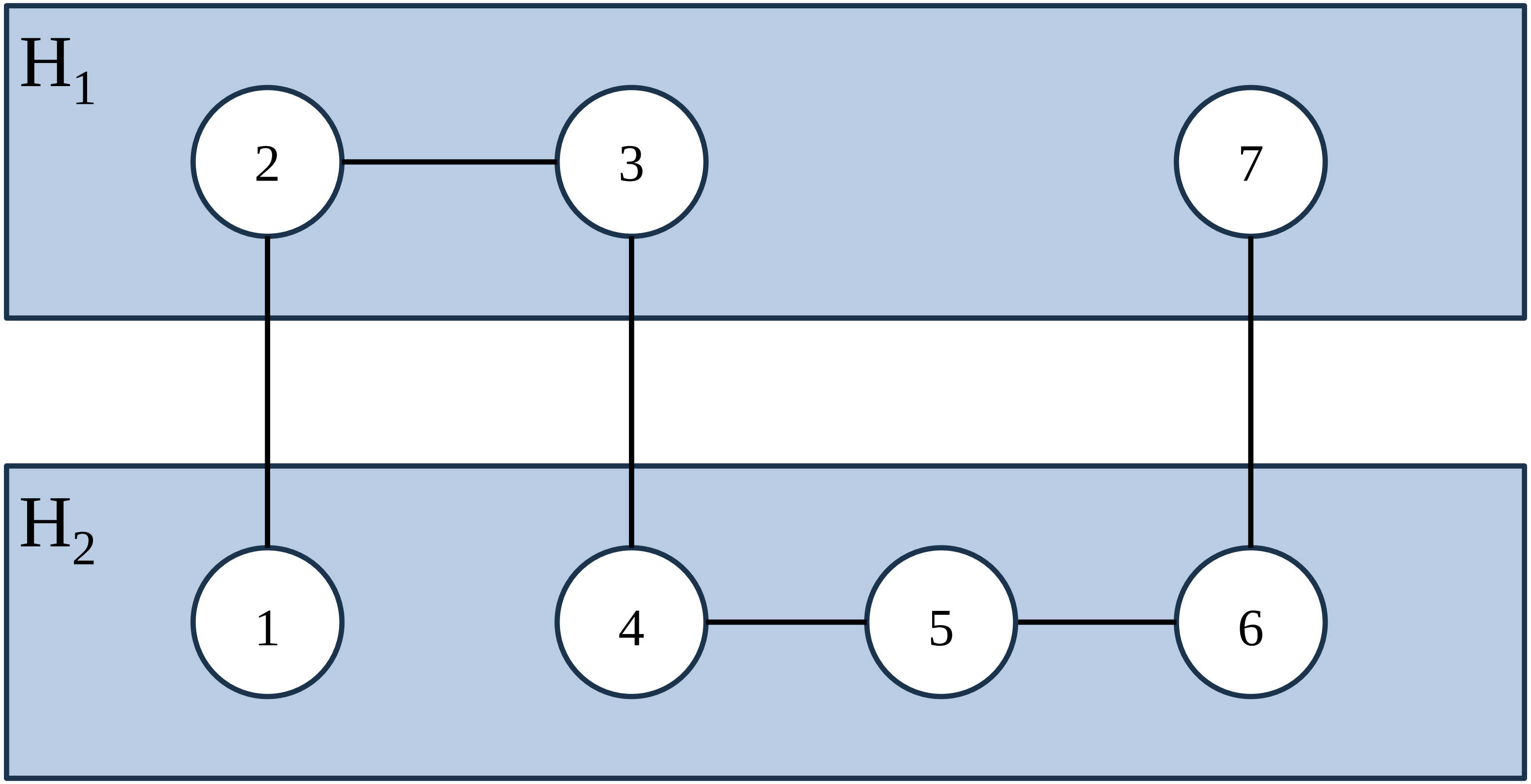

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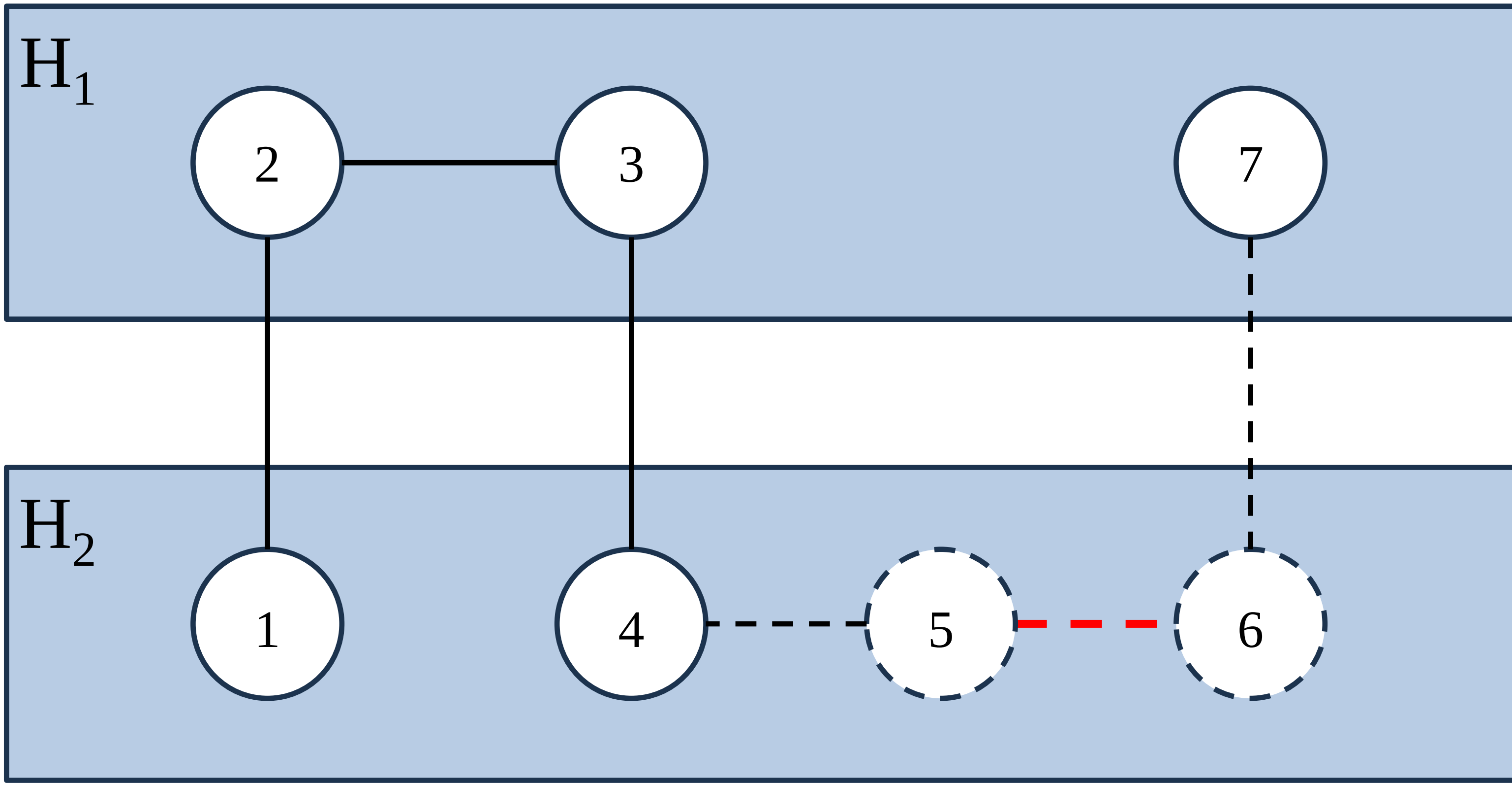

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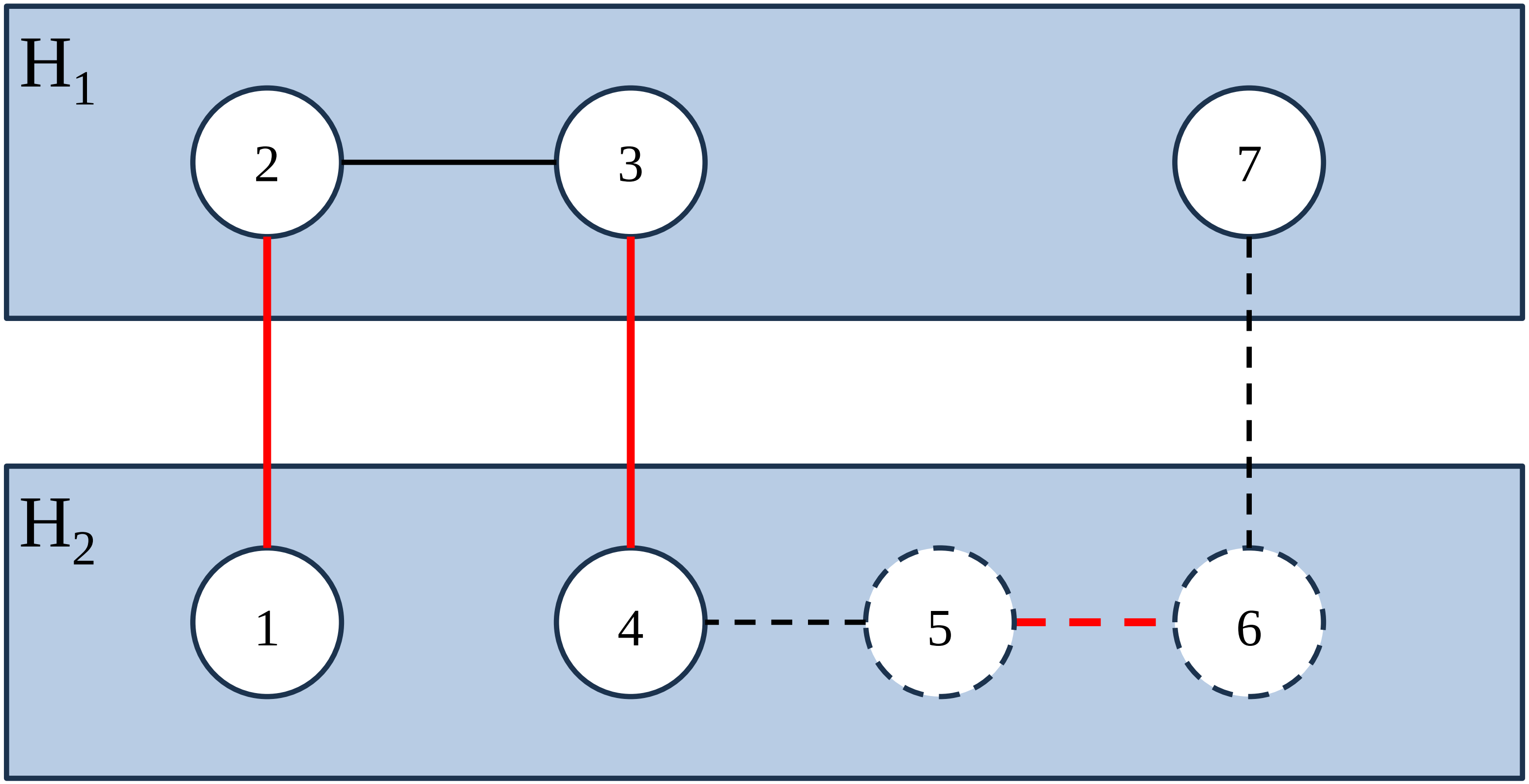

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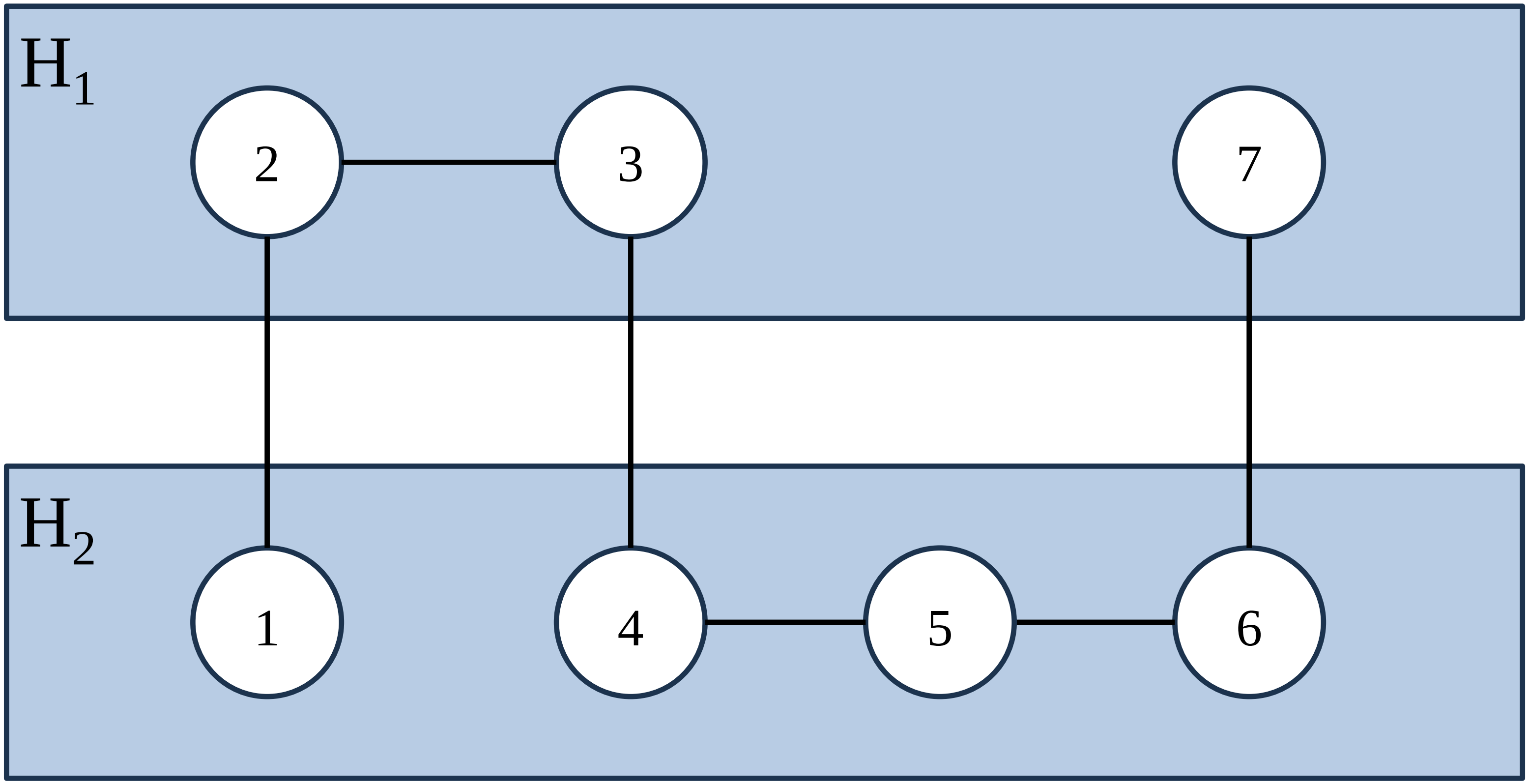

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no truthful maximum matching is possible!