

Analisi di Reti (mod 2)

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Mechanism Design Without Money

reference:

Twenty Lectures in Algorithmic Game Theory
Tim Roughgarden
Chapters 9 & 10.

Motivations and applications

motivations:

- sometimes the use of money is infeasible or illegal;

applications:

- voting;
- organ donation;
- school choice;
- ...

what can be done:

- strong impossibility results in general
- still some of mechanism design's greatest hits

House allocation problem
&
Top Trading Cycle algorithm

House allocation problem

- n agents
- each agent initially owns a house
- preferences (type) of the agent i : a **total ordering** over the n houses
 - an agent need not prefer her own house over the others



1

1's preferences:

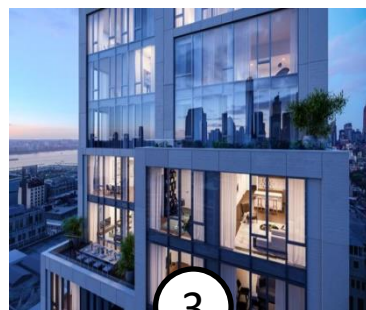
1,3,2,4



2

2's preferences:

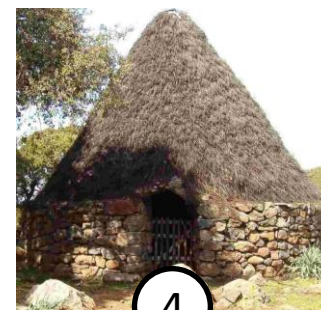
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3

3's preferences:

1,4,3,2



4

4's preferences:

1,2,3,4

goal 1: reallocate the houses to make the agents better off

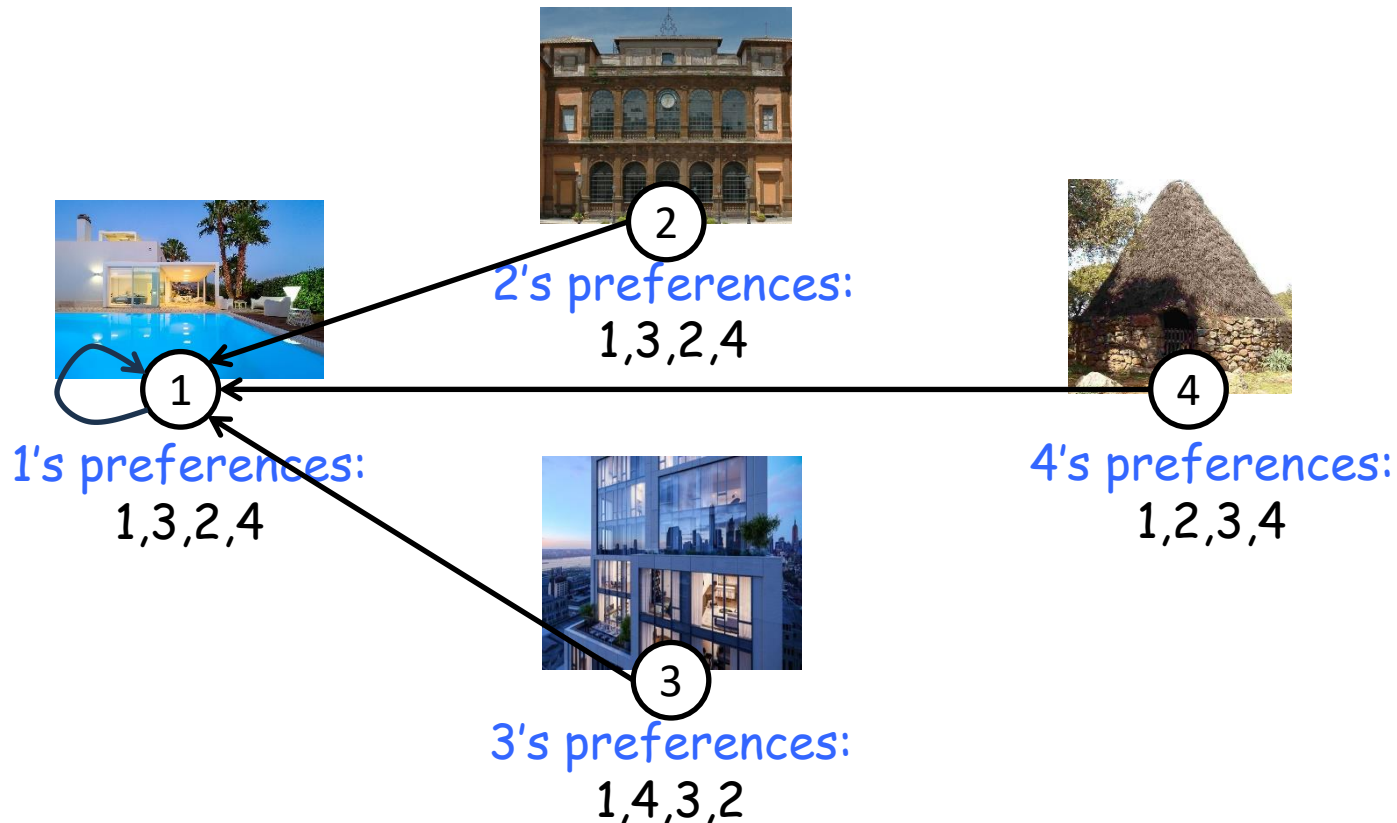
goal 2: do it in a way agents cannot manipulate the allocation



Top Trading Cycle (TTC) algorithm

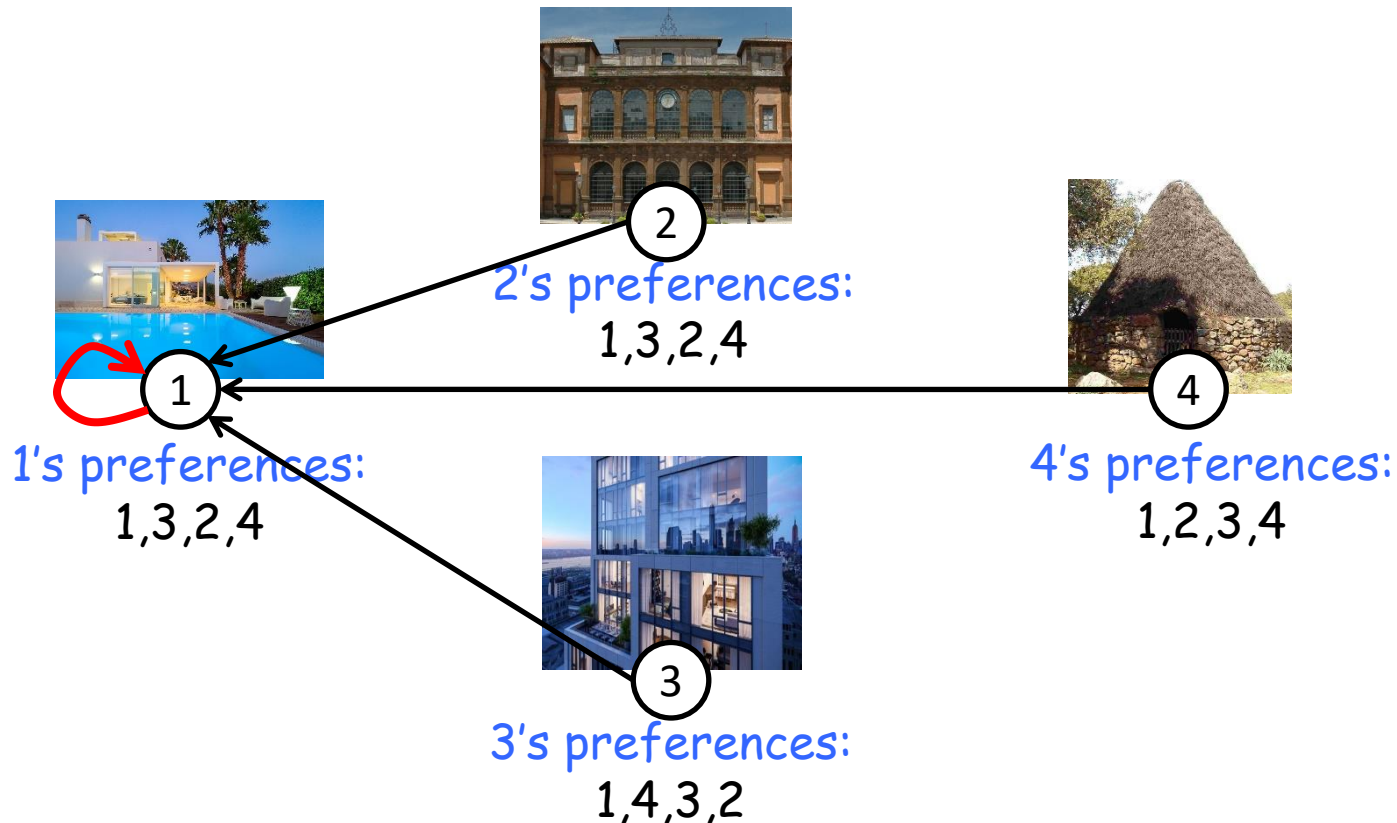
TTC algorithm (idea)

- allocation proceeds in iterations
- at each iteration:
 - each remaining agent participates with her own house
 - each remaining agent points to her favorite still available house
 - look at (disjoint) cycles formed and perform the reallocation suggested by the cycles
 - remove the agents of the cycles



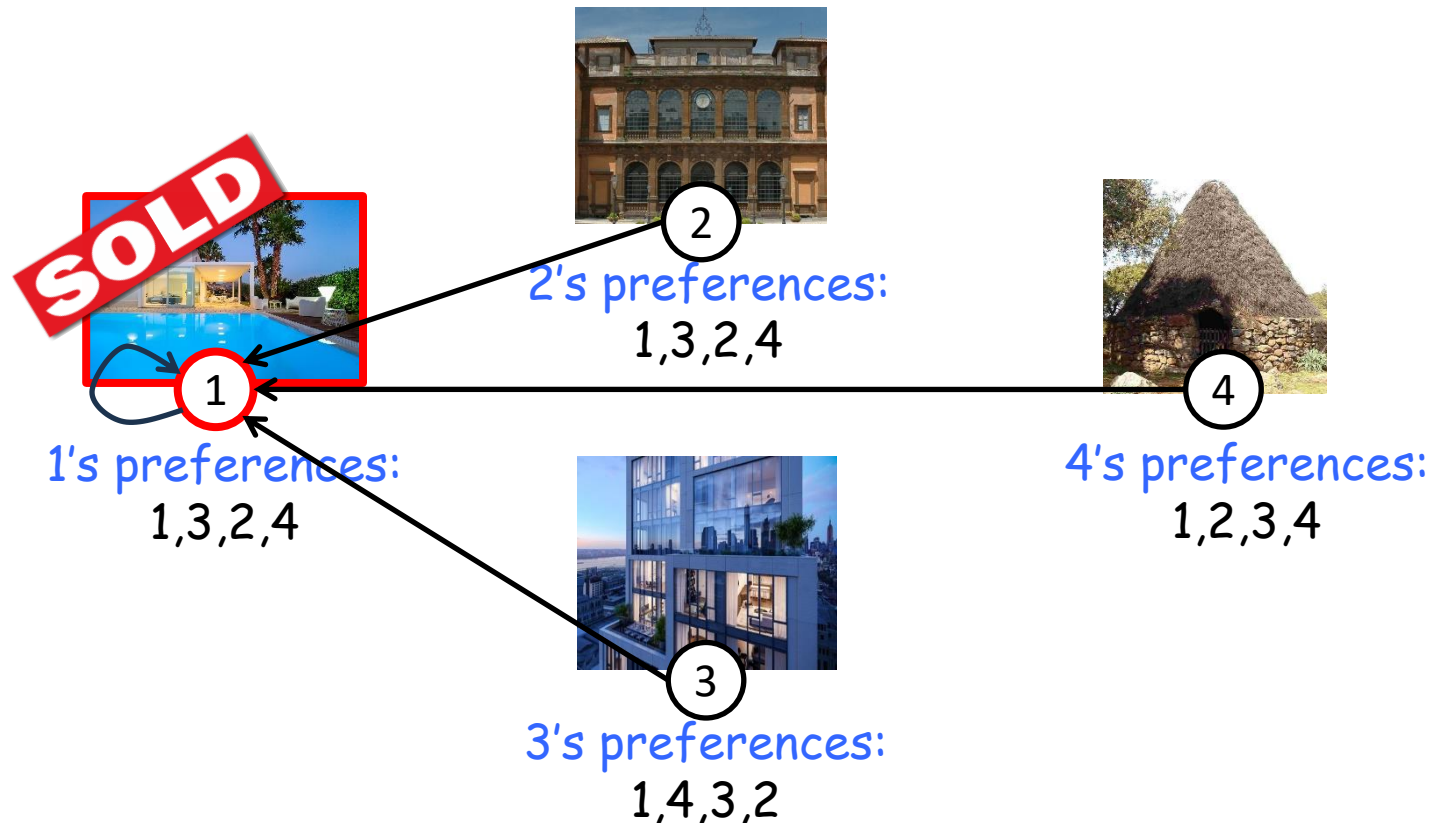
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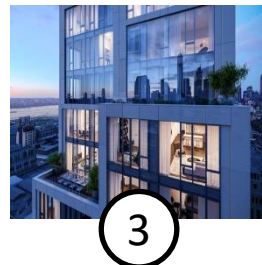
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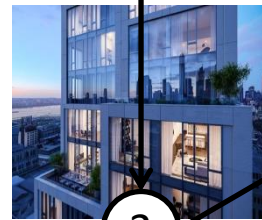
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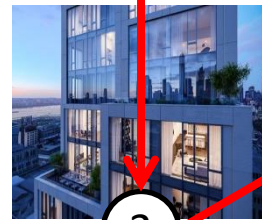
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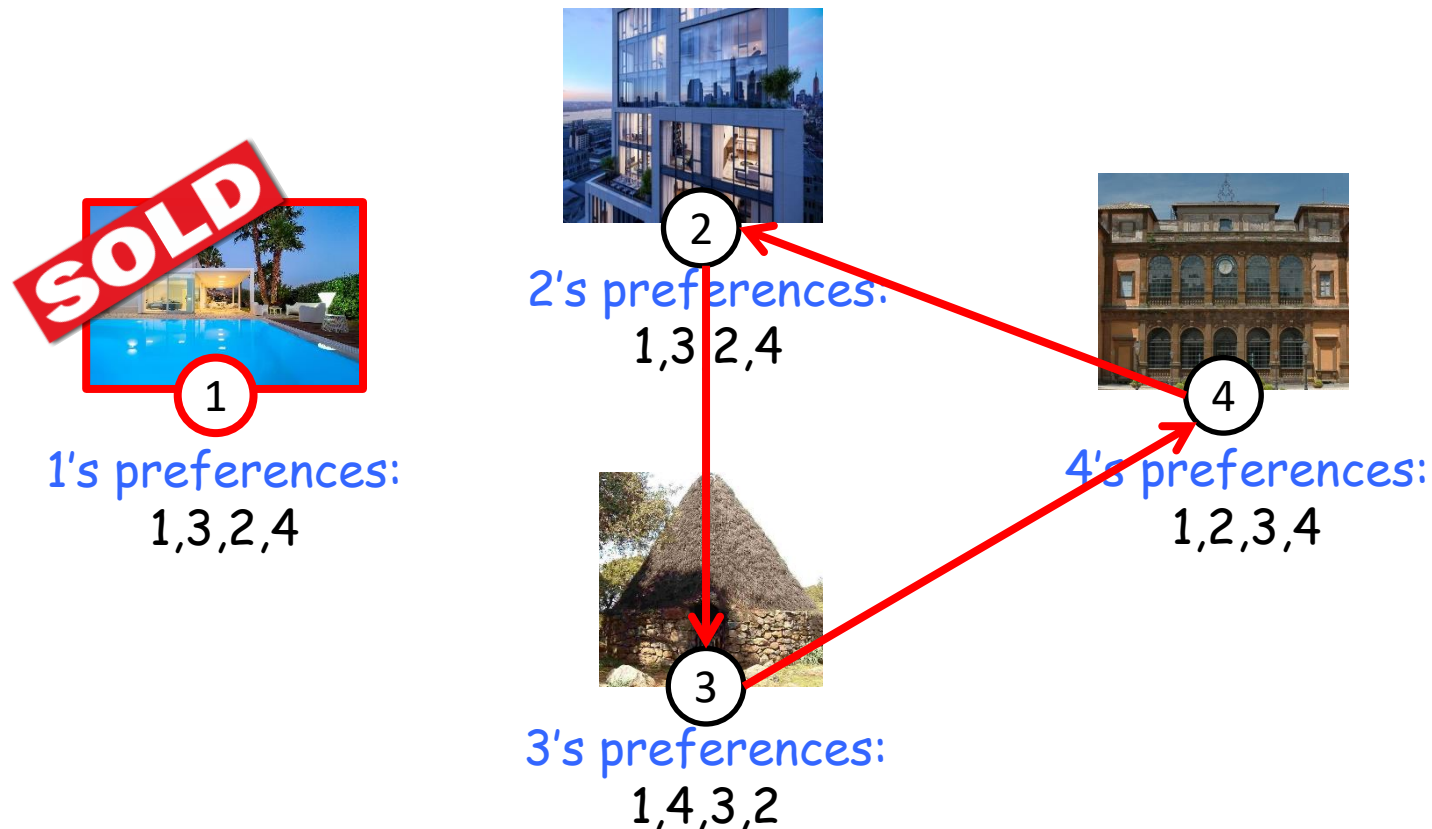


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Top Trading Cycle (TTC) Algorithm

initialize N to the set of all agents

while $N \neq \emptyset$ **do**

form the directed graph G with vertex set N and edge set $\{(i, \ell):$

i 's favorite house within N is owned by $\ell\}$

compute the directed cycles $C_1, \dots, C_h$ of G

// self-loops count as directed cycles

// cycles are disjoint

for each edge (i, ℓ) of each cycle $C_1, \dots, C_h$ **do** reallocate ℓ 's house to agent i

remove the agents of $C_1, \dots, C_h$ from N

- G has at least one directed cycle, since traversing a sequence of outgoing edges must eventually repeat a vertex
- Because all out-degrees are 1, these cycles are disjoint

Some properties of the TTC algorithm

Lemma

Let N_k denote the set of agents removed in the k -th iteration of the TTC algorithm. Every agent of N_k receives her favorite house outside of those owned by $N_1 \cup \dots \cup N_{k-1}$, and the original owner of this house is in N_k .

Theorem

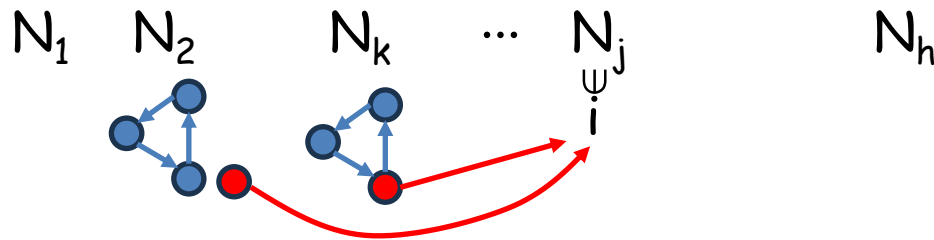
When the TTC algorithm is used for the reallocation, for every agent, it is a dominant strategy to report truthfully.

proof

Fix an agent i and reports by the others.

Assume i reports truthfully.

Let N_k be the set of agents removed in the k -th iteration.



By the previous lemma it suffices to prove:

Claim: no misreport can net i a house of an agent in $N_1 \cup \dots \cup N_{j-1}$.

For each $k < j$:

- no agent in N_k point to i at iteration k
otherwise i would belong to N_k
- no agent in N_k point to i at iteration $< k$
otherwise that agent would point to i at iteration k



Why the previous theorem is nice?

Notice: the mechanism that never reallocates anything is also truthful

Consider a reallocation of the houses. A subset of agents forms a **blocking coalition** for this reallocation if they can internally reallocate their houses to make some member better off while making no member worse off.

A **core allocation** is a reallocation with no blocking coalitions.

Theorem

For every house allocation problem, the allocation computed by the TTC algorithm is the unique core allocation.

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 1

every agent in N_1 receives her first choice

- ➡ N_1 is a blocking coalition for any allocation $\neq$ from the TTC one
- ➡ every core allocation must agree with the TTC one on agents in N_1
- ➡ every agent in N_2 receives her first choice outside N_1
- ➡ N_2 is a blocking coalition for any allocation $\neq$ from the TTC one that agrees with TTC on N_1
- ➡ every core allocation must agree with the TTC one on agents in N_1 and N_2
- ➡ ... 

proof

Claim 1: every allocation that differs from the TTC allocation is not a core allocation.

Claim 2: the TTC allocation is a core allocation.

proof of Claim 2

consider an arbitrary subset S of agents and an internal reallocation of their houses

the reallocation partitions S into directed cycles

consider any such cycle C

- if C contains two agents $i \in N_j$ and $t \in N_k$ with $j < k$

➡ i is worse off than in the TTC allocation

- if C contains agents all in N_k but there is an agent i that does not receives her favorite choice in N_k

➡ i is worse off than in the TTC allocation

➡ the TTC allocation has no blocking coalitions.



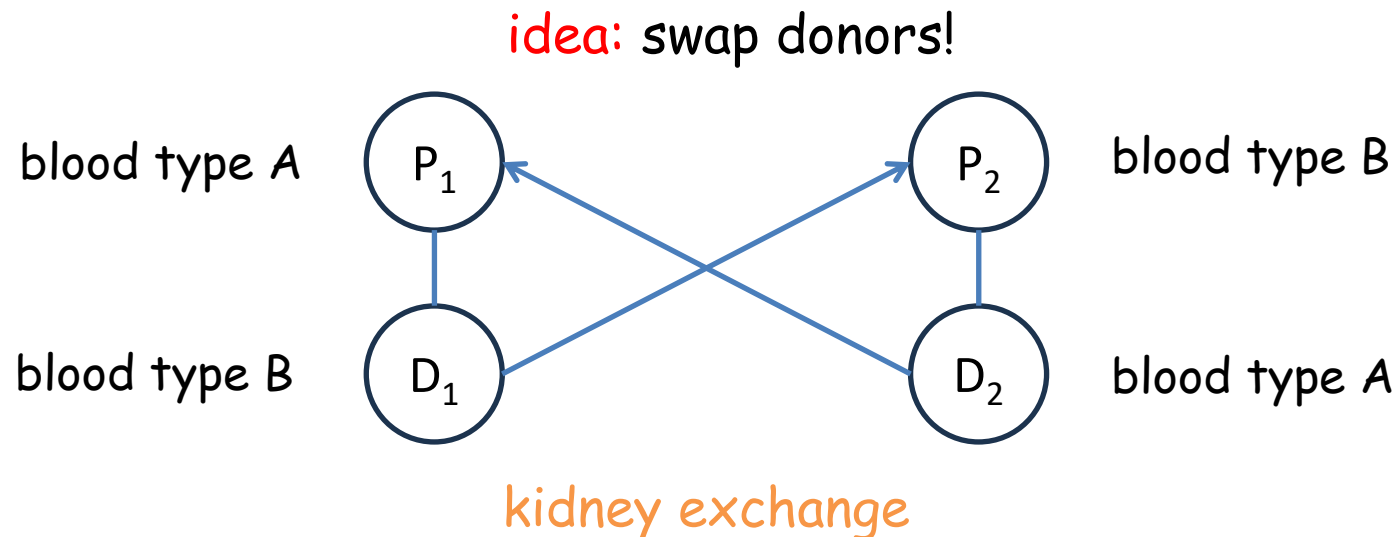
Kidney Exchange

Background

- many people suffer from kidney failure and need a kidney transplant
- In US more than 100.000 people are on the waiting list for such a transplant
- old idea (used also for other organs): deceased donors
- special feature for kidneys: **living** donors
(a healthy person can survive just fine with a single kidney)

compatibility issues:

- having a living kidney donors is not always enough
- a patient-donor pair can be incompatible
(primary culprits for incompatibility: blood and tissue types need to match)



- initially, few kidney exchanges were done on an ad-hoc basis
- This made clear the need of a system to organize kidney exchanges
- a system where patient-donor pairs can register and be matched with others

goal: how such an exchange system can be designed in order to enable as many matches as possible?

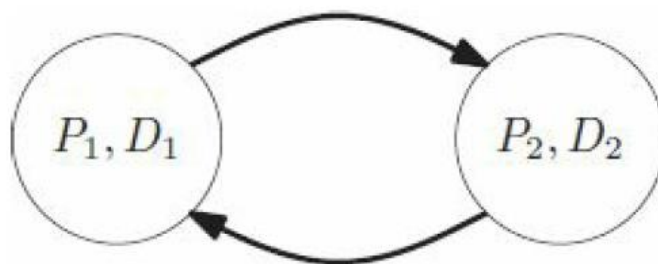
- currently, monetary compensation is illegal in most of the countries



problem naturally modeled as a mechanism
design problem without money

first idea: model the problem as a house allocation problem

- each patient-donor pair treated as a agent-house
 - patient=agent
 - donor=house
- a patient's total ordering over the donors can be defined according to the estimated probability of a successful kidney transplant
- use the **TCC algorithm** to find kidney exchanges
- the reallocation of donors suggested by the TCC algorithm can only improve every patient's probability of a successful transplant



Good case for the TTC algorithm.

- each circle represents an incompatible patient-donor pair
- each arrow represents a kidney transplant from the donor in the first pair to the patient of the second pair.

first idea: model the problem as a house allocation problem

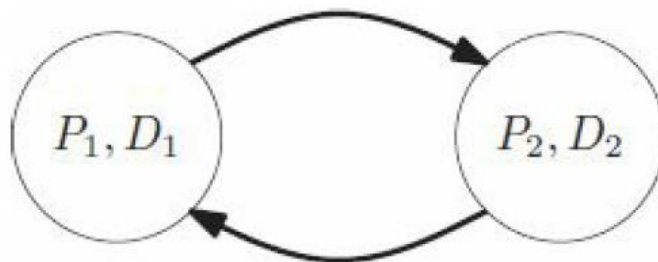
some technical issues:

- need to manage patients without a donor (agent without a house)
- need to manage deceased donors (house without an agent/owner)

} the TTC algorithm & its incentive guarantee can be extended to this more general setting (with some non-trivial extra work)

some more important issues:

- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)



how many surgeries? 4

what if P_1 - D_2 surgeries today and P_2 - D_1 surgeries tomorrow?

D_1 could renege on her offer

- P_1 unfairly got a kidney for free
- P_2 is still sick and can no longer participate in a kidney exchange

first idea: model the problem as a house allocation problem

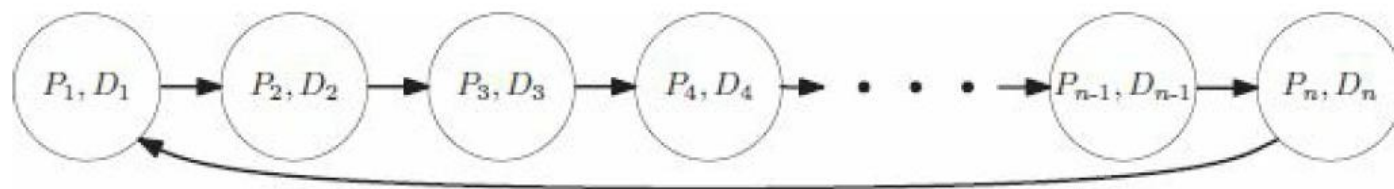
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really bad case

first idea: model the problem as a house allocation problem

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} the TTC algorithm & its incentive guarantee can be extended to this more general setting (with some non-trivial extra work)

some more important issues:

- the TTC algorithm can find very long cycles (the corresponding surgeries must happen *simultaneously*)
- modeling patient's preferences as a total order over donors is overkill (binary preferences over donors are more appropriate)

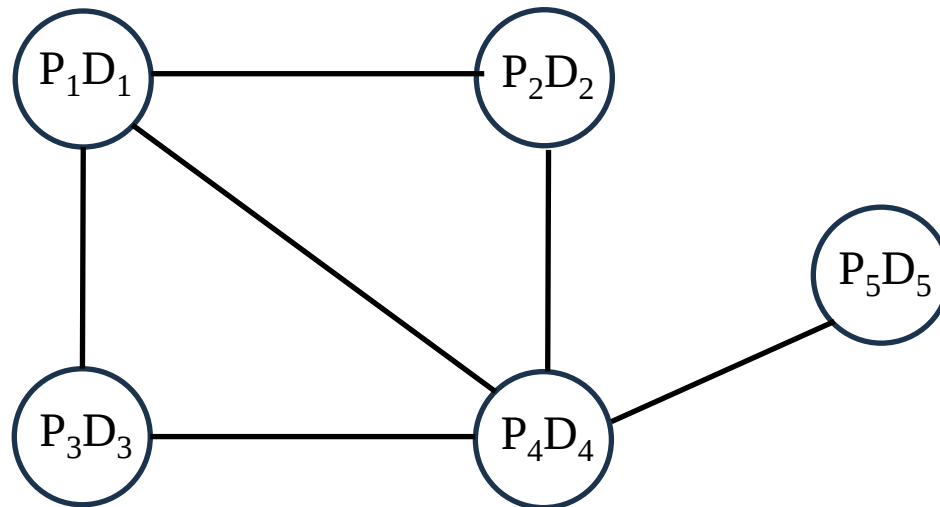


change the model: use graph matching!

A **matching** of an undirected graph is a subset of the edges that share no endpoints.

The relevant **graph** for kidney exchanges:

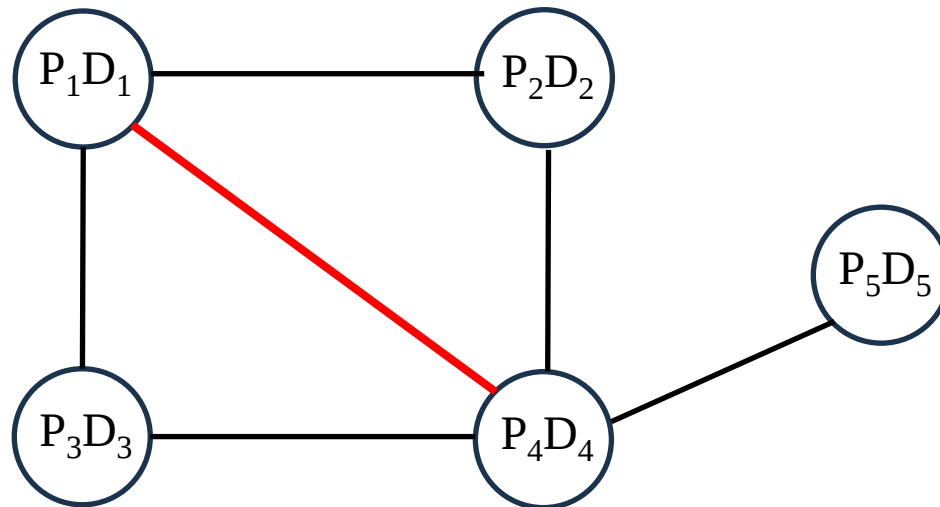
- we have a vertex for each incompatible patient-donor pair
- there is an edge between (P_i, D_i) and (P_j, D_j) if and only if P_i and D_j are compatible & P_j and D_i are compatible



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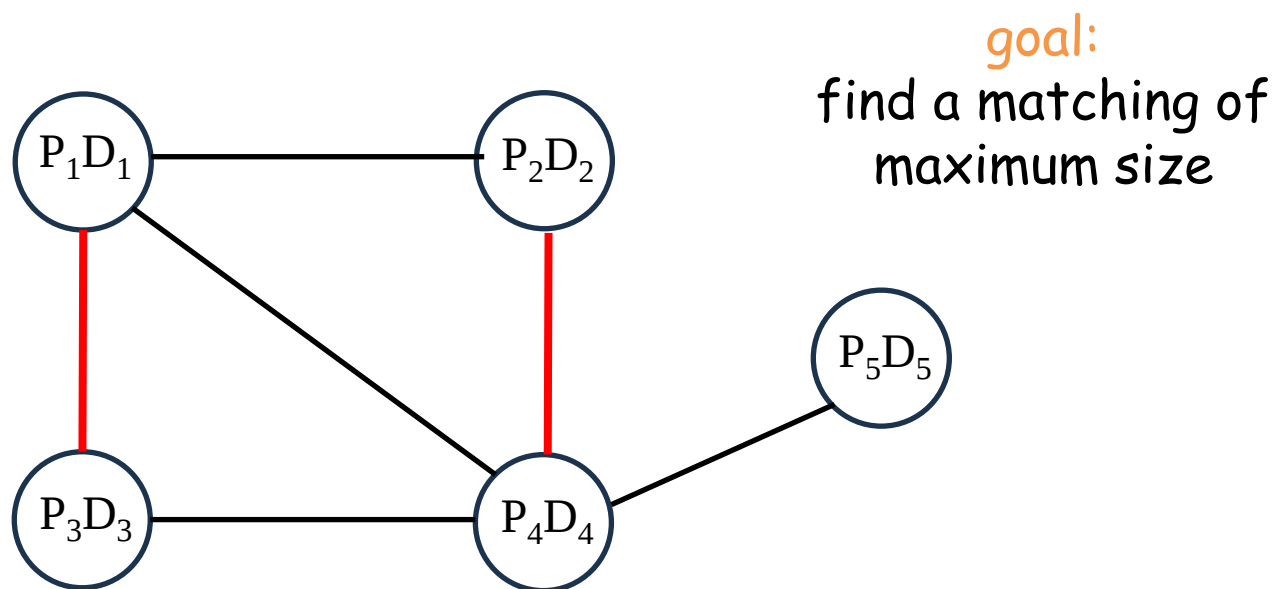
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notice: we are restricting ourselves to 2-length cycles

How do incentives come into play?

we assume that each patient i

- has a set E_i of compatible donors belonging to other patient-donor pairs
- can report any subset $F_i \subseteq E_i$

it makes sense since:

- proposed kidney exchange can be refused by a patient for any reason
- a patient cannot credibly misreport extra donors with whom she is incompatible

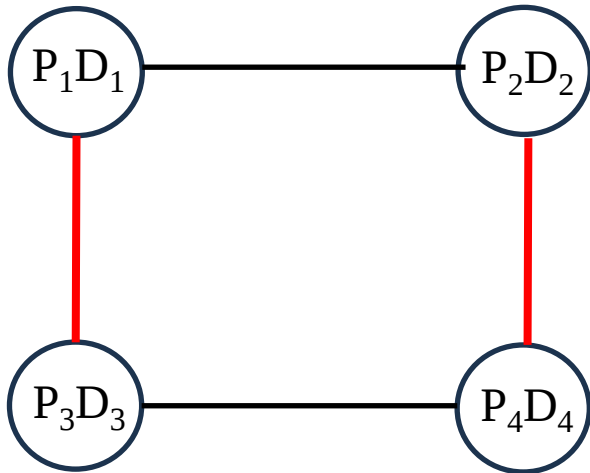
A Mechanism for Pairwise Kidney Exchange

1. Collect a report F_i from each agent i .
2. Form the graph $G = (V, E)$, where V corresponds to agent-donor pairs and $(i, j) \in E$ if and only if the patients corresponding to i and j report as compatible the donors corresponding to j and i , respectively.
3. Return a maximum-cardinality matching of the graph G .

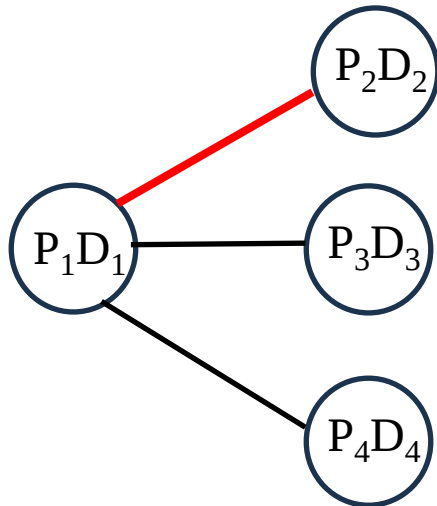
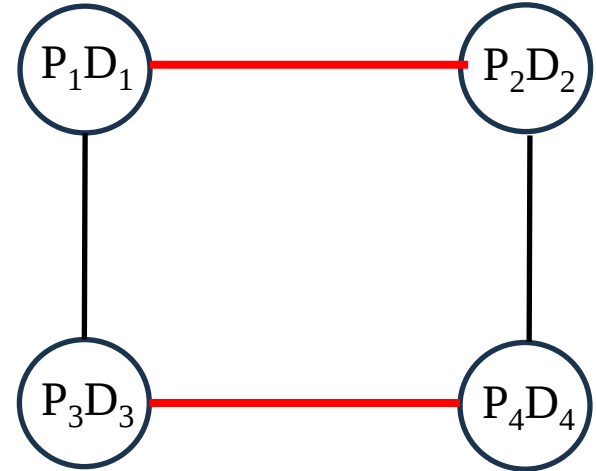
is this mechanism truthful?

It depends on how ties are broken between different maximum matchings

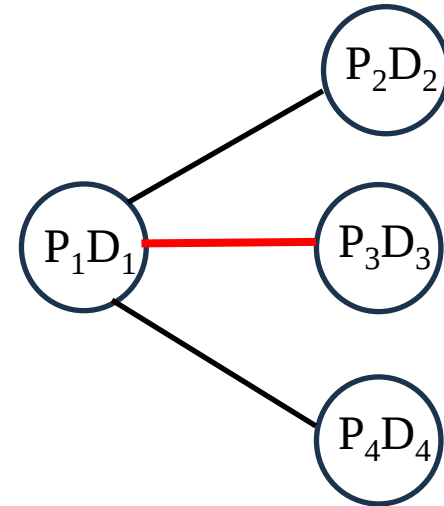
two types of ties



VS



VS



we will manage ties by **prioritizing** the patient-donor pairs

notice: most hospitals already rely on priority schemes to manage their patients

re-index the vertices of G such that $V=\{1,2,\dots,n\}$ are ordered from highest to lowest priority

Priority Mechanism for Pairwise Kidney Exchange

initialize M_0 to the set of maximum matchings of G

for $i = 1, 2, \dots, n$ **do**

let Z_i denote the matchings in M_{i-1} that match vertex i

if $Z_i = \emptyset$ **then**

set $M_i = Z_i$

else if $Z_i = \emptyset$ **then**

set $M_i = M_{i-1}$

return an arbitrary matching of M_n

Theorem

In the priority mechanism for pairwise kidney exchange, for every agent i , it is a dominant strategy to truthfully report E_i .

Exercise 1: Prove it.

Exercise 2:

Exhibit a tie-breaking rule between maximum-cardinality matchings such that the corresponding mechanism is not truthful.

Some other remarks and further directions

length of the cycles:

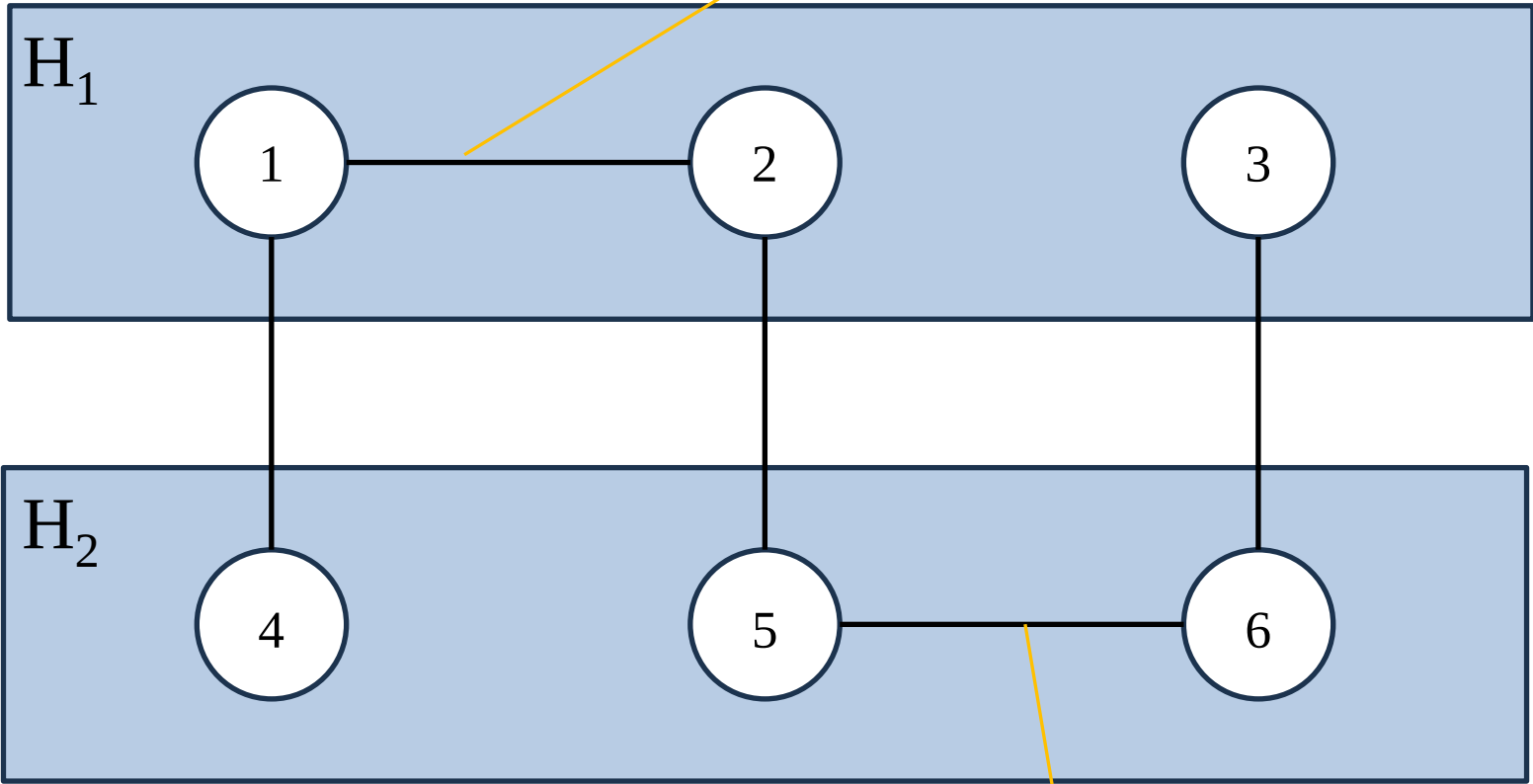
- by using matching we are restricting ourselves to 2-length cycles
- actual algorithms allow 3-way exchanges
(it can significantly increase the number of matched patients)
- 4-way exchanges does not seem to lead to significant further improvements

Incentives for hospitals:

- many patient-donor pairs are reported to national kidney exchanges by hospitals
- the objective of a hospital, to match as many of its patients as possible, is not perfectly aligned with the societal objective of matching as many patients as possible

Example 1

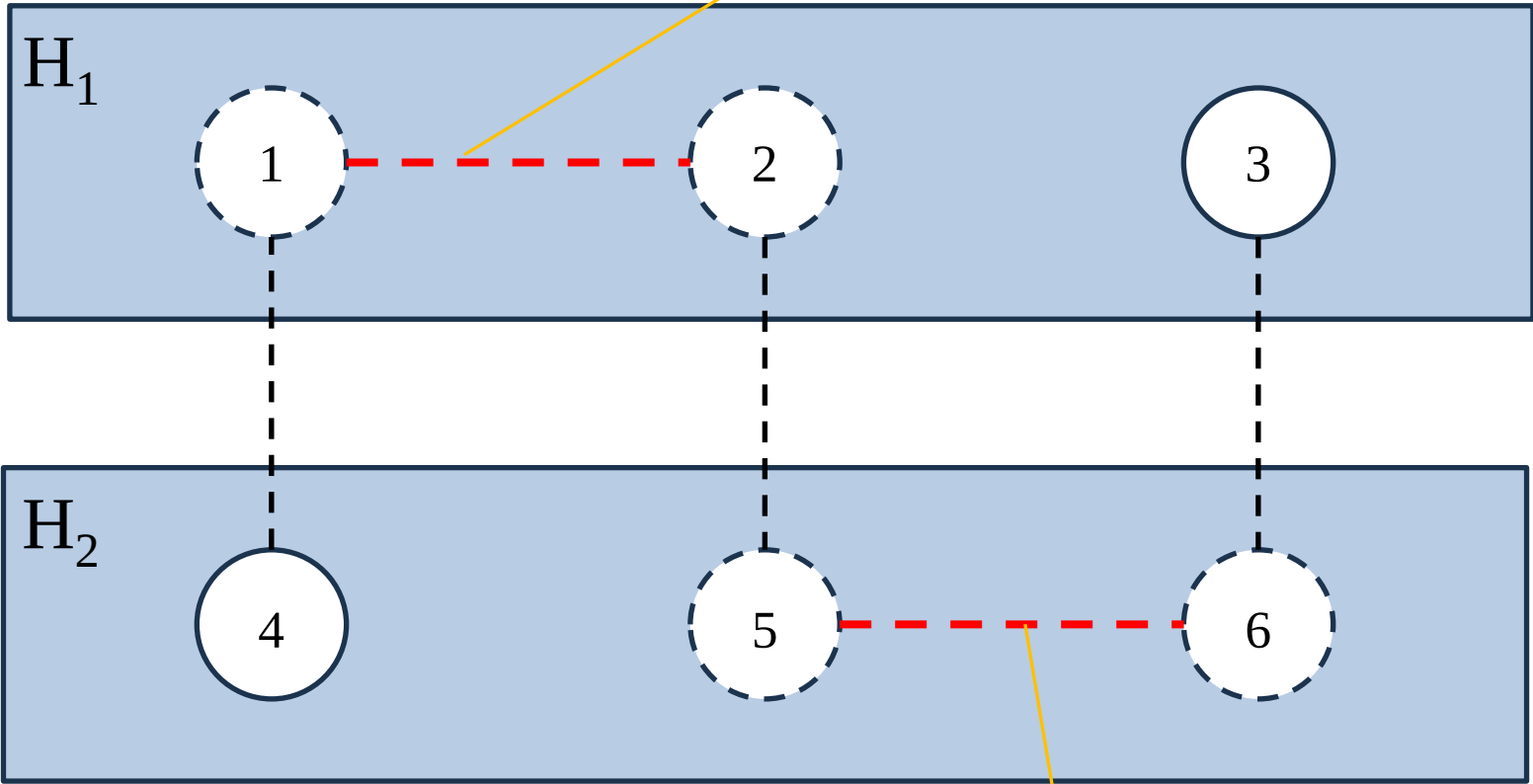
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H_2 could match 5&6 internally without bothering to report them to national kidney exchange

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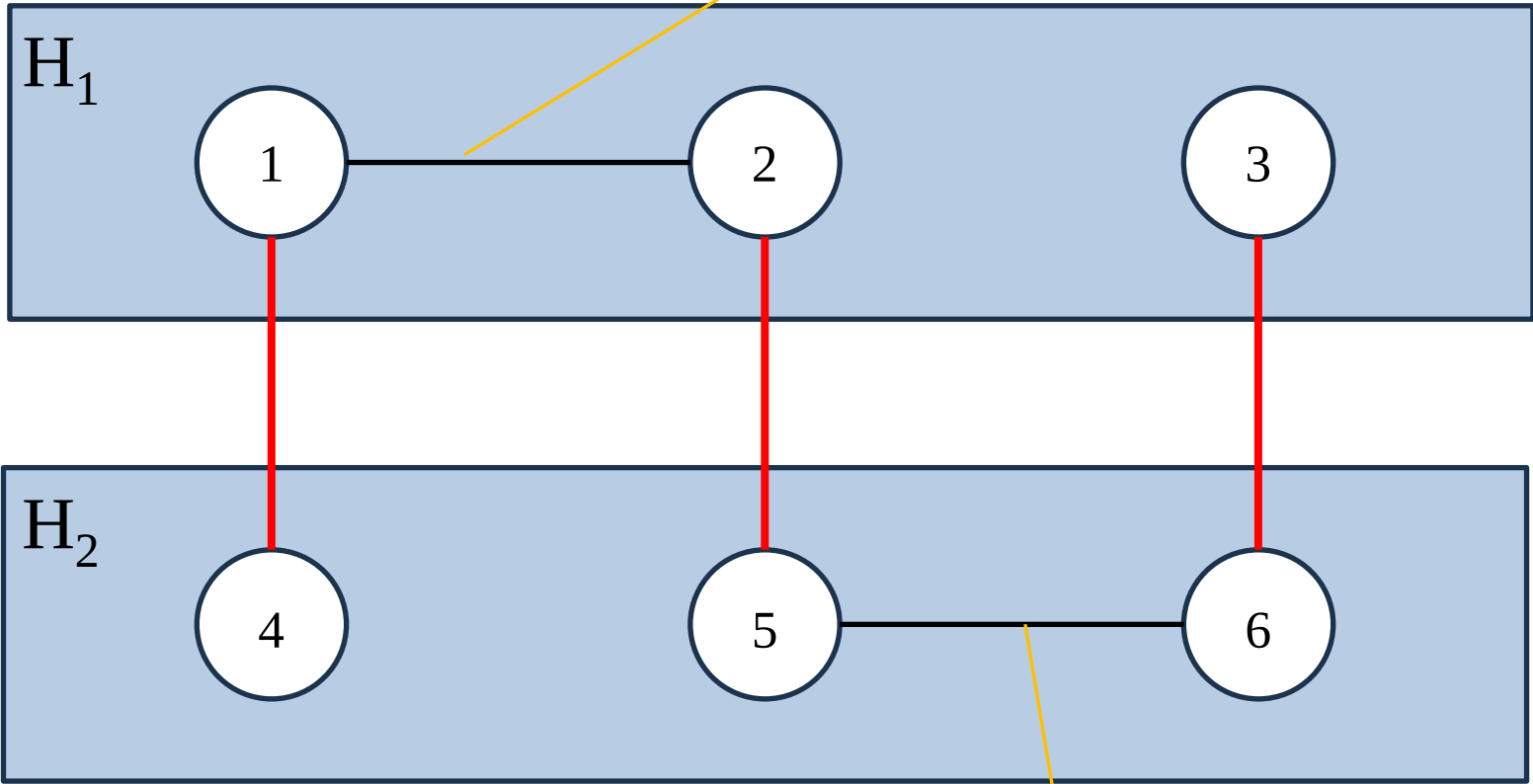
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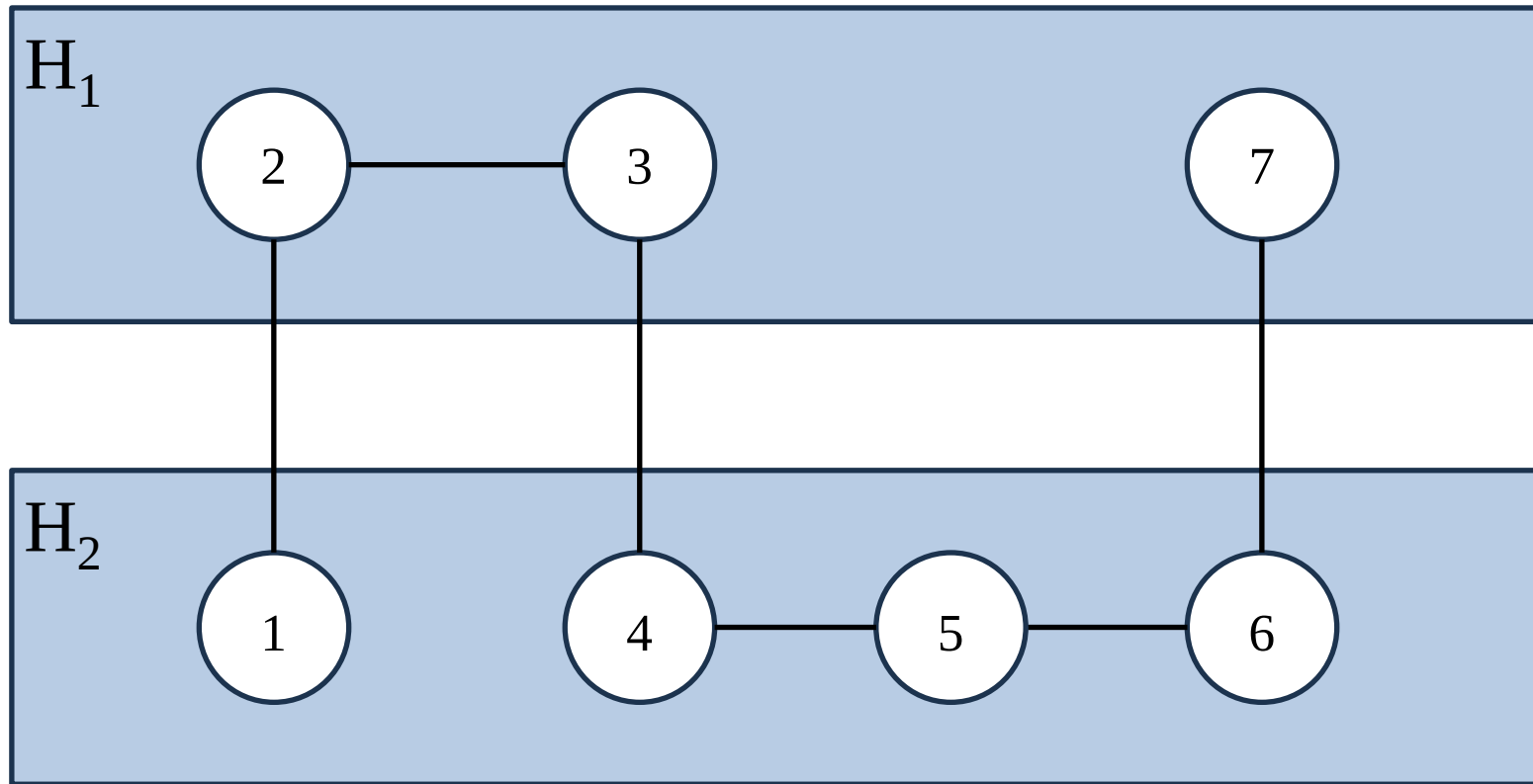
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Full reporting by hospitals leads to more matches

H_2 could match 5&6 internally without bothering to report them to national kidney exchange

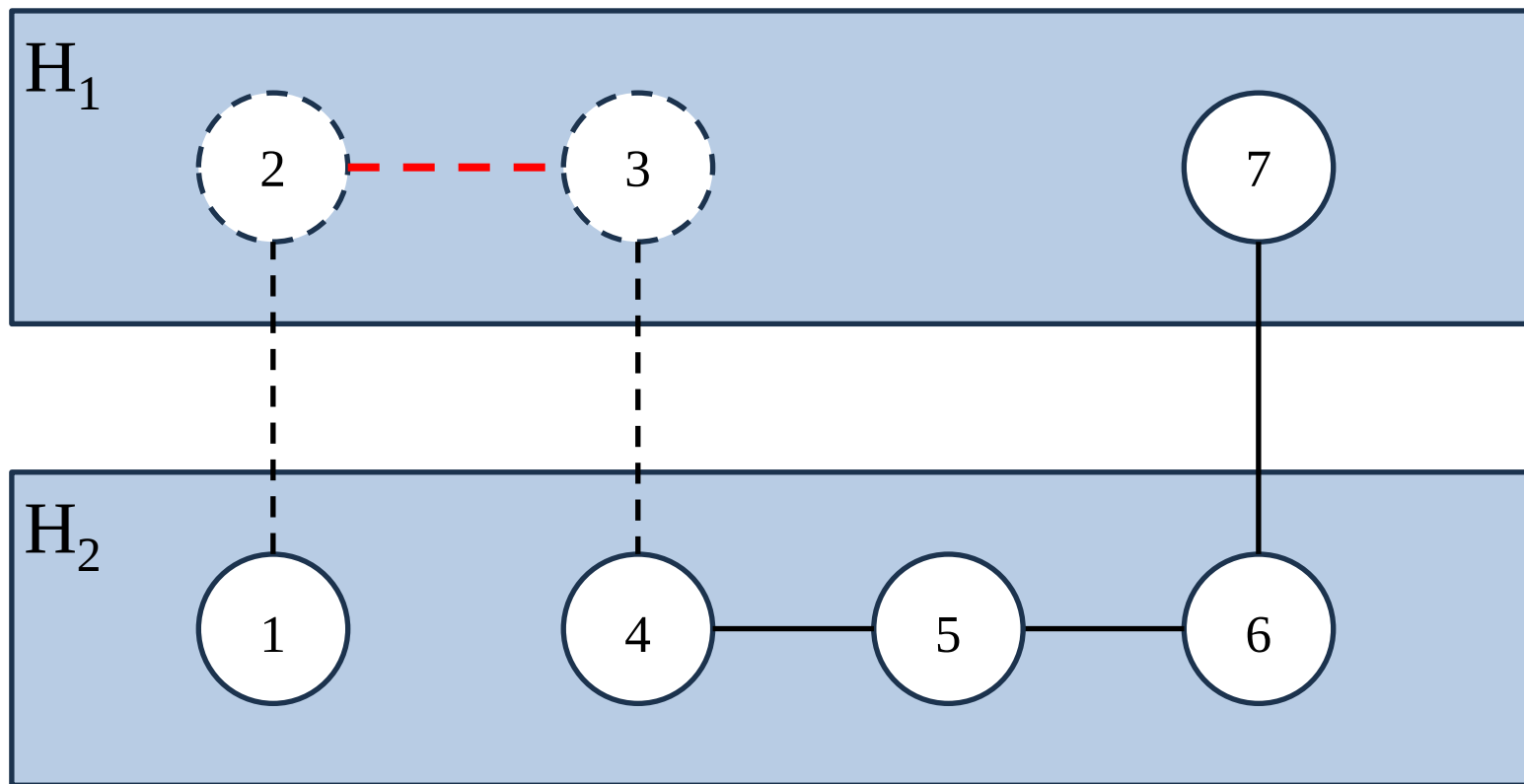
Example 2 (impossibility result)



Assume truthful reporting. A pair x will not be matched.

If $x \in H_1$, H_1 has convenience to not report 2&3 (and get all its pairs matched)

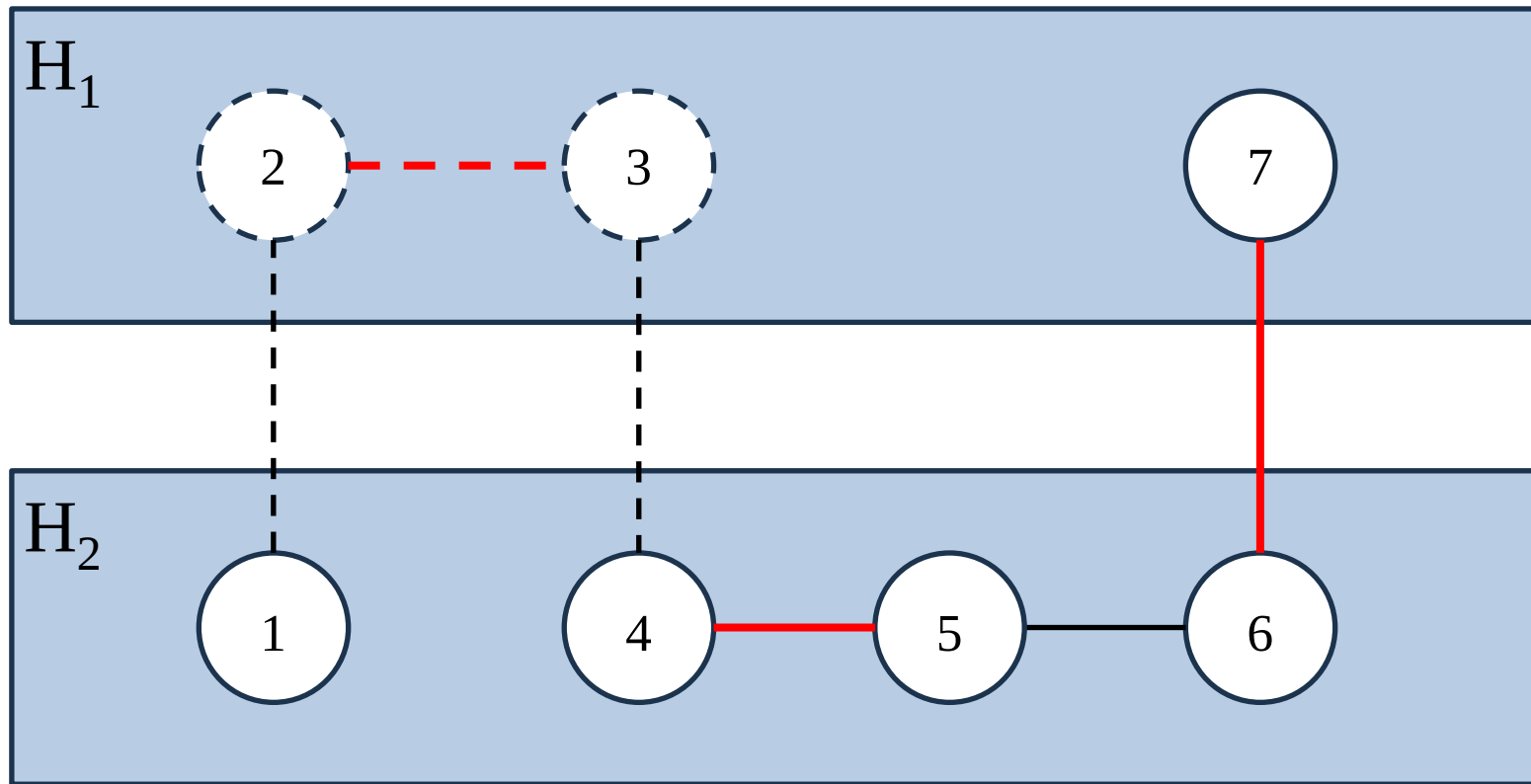
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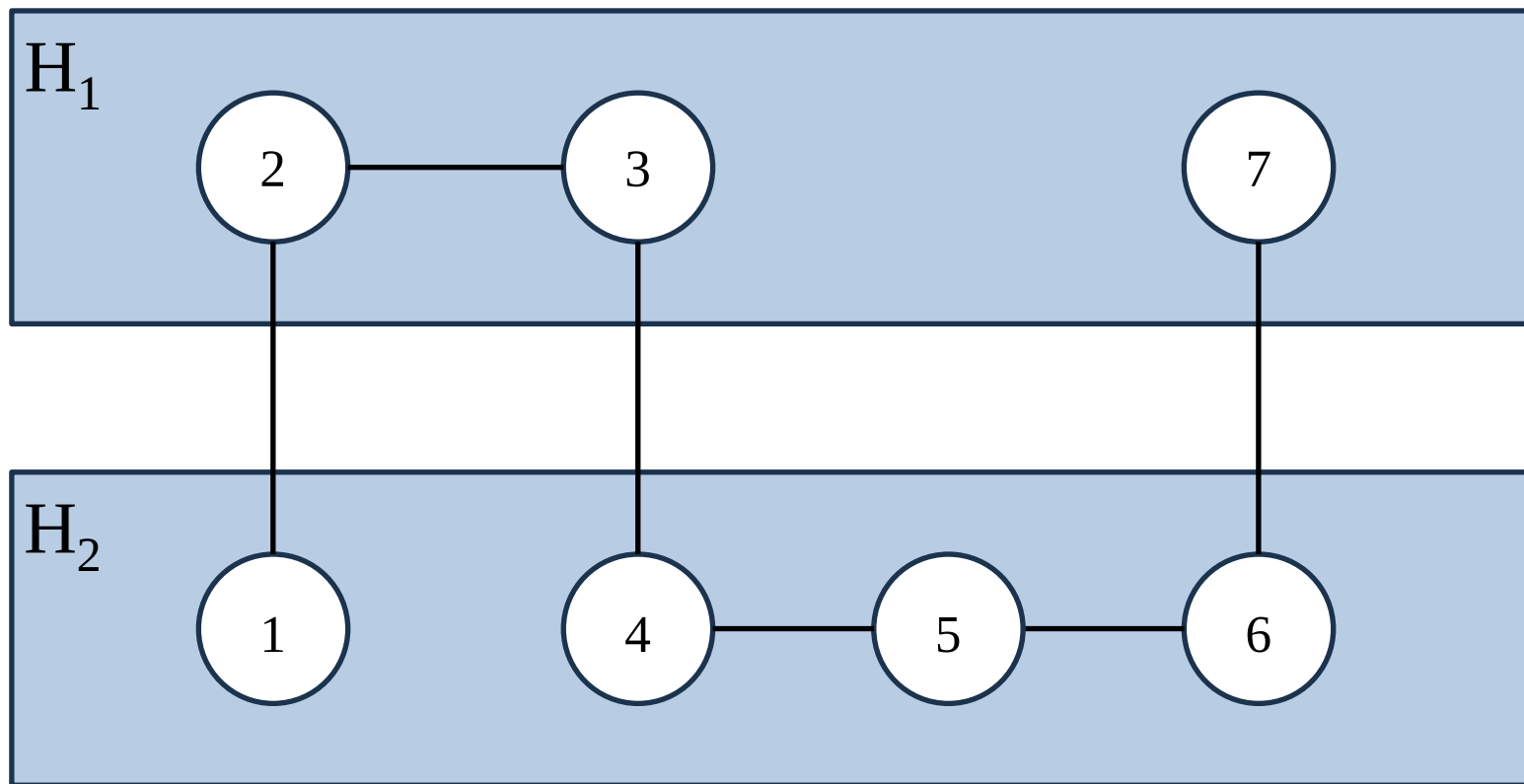
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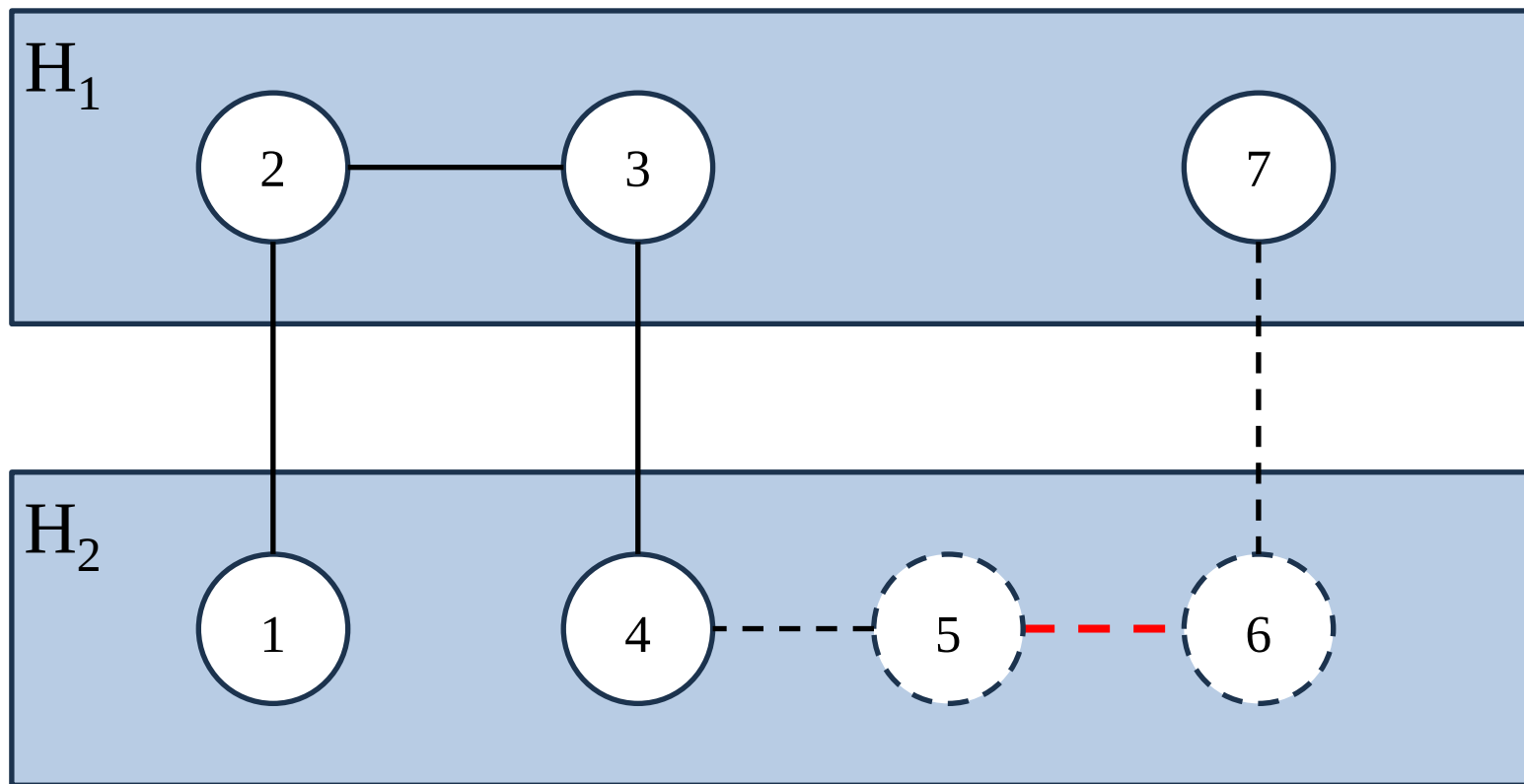


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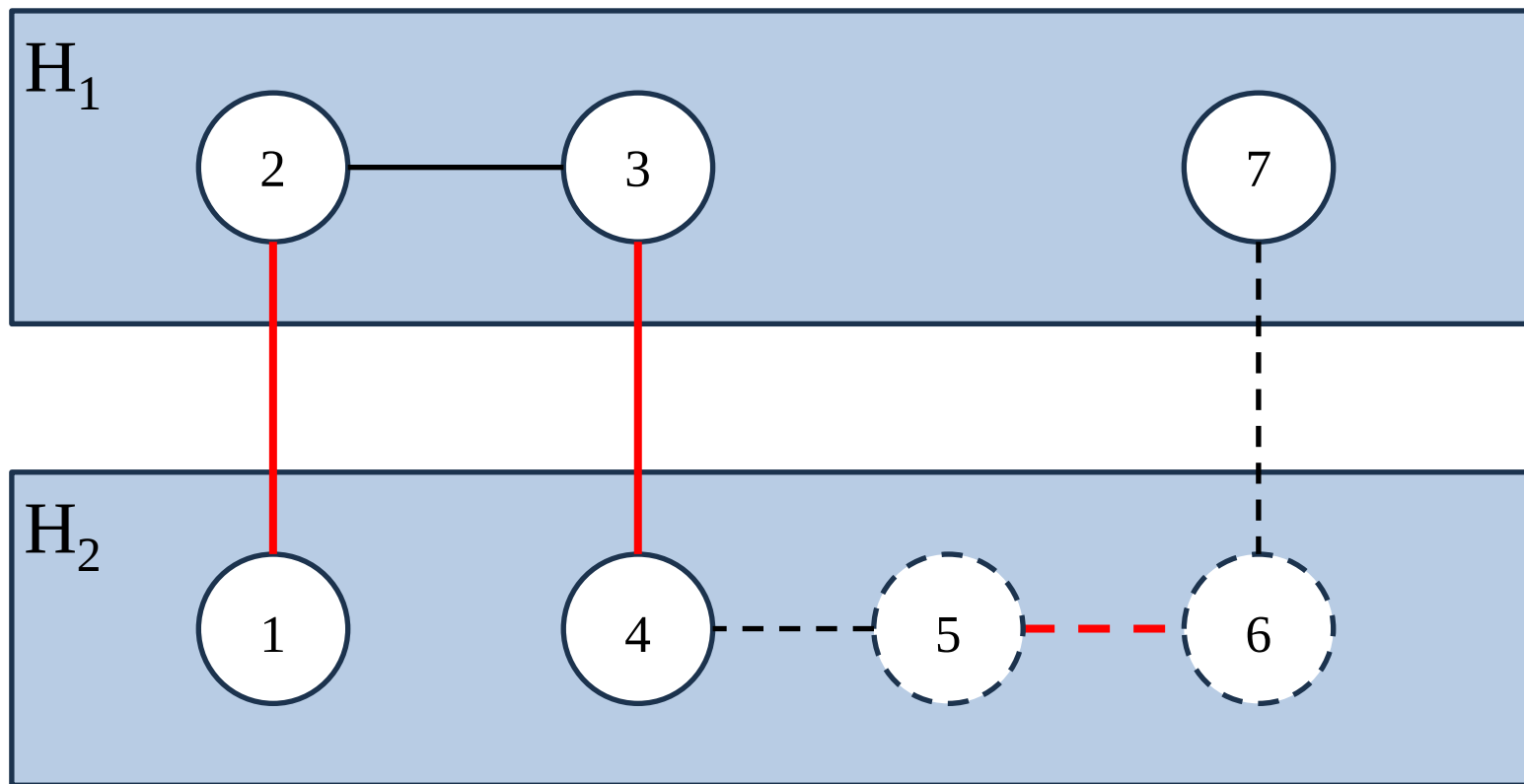


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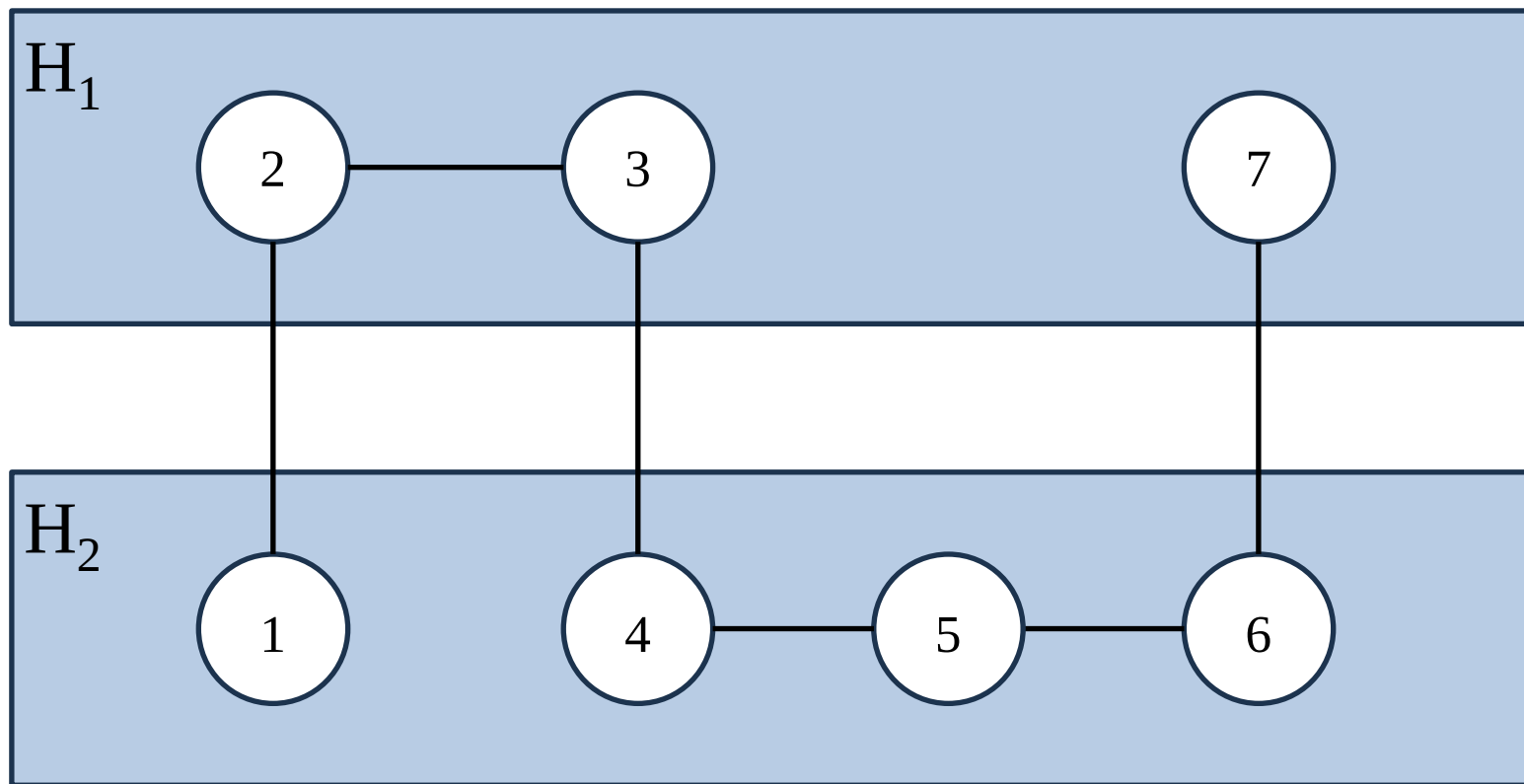


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no truthful maximum matching is possible!